Homogenization in porous media and associated spectral problems: Robin boundary conditions with large adsorption parameters.

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Homogenization problems in perforated domains with nonlinear Robin type boundary conditions on the boundary of the cavities

Joint work: U. de Cantabria– Lomonosov Moscow State U. (1996–2014)

- *M. Lobo, O.A. Oleinik, T.A. Shaposhnikova (1997-1998)* - Linear problems

- *D. Gómez, M. Lobo, A.V. Podolskiy, T.A. Shaposhnikova, V.V. Sukharev, M.N. Zubova (2011 →)*

Asymptotic behavior of the solution u_ε **of** P^ε, $\varepsilon \to 0$

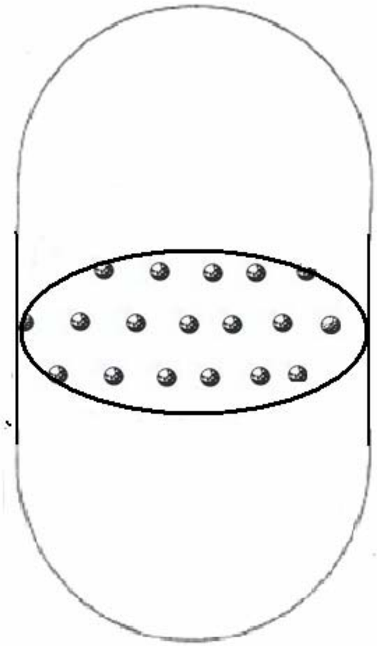
$$P^\varepsilon \begin{cases} -\Delta u_\varepsilon = f \quad \text{in } \Omega^\varepsilon \\ u_\varepsilon = 0 \quad \text{on } \partial\Omega \\ \dfrac{\partial u_\varepsilon}{\partial n} + \beta(\varepsilon)\sigma(x, u_\varepsilon) = 0 \quad \text{on } \bigcup \partial T^\varepsilon \end{cases}$$

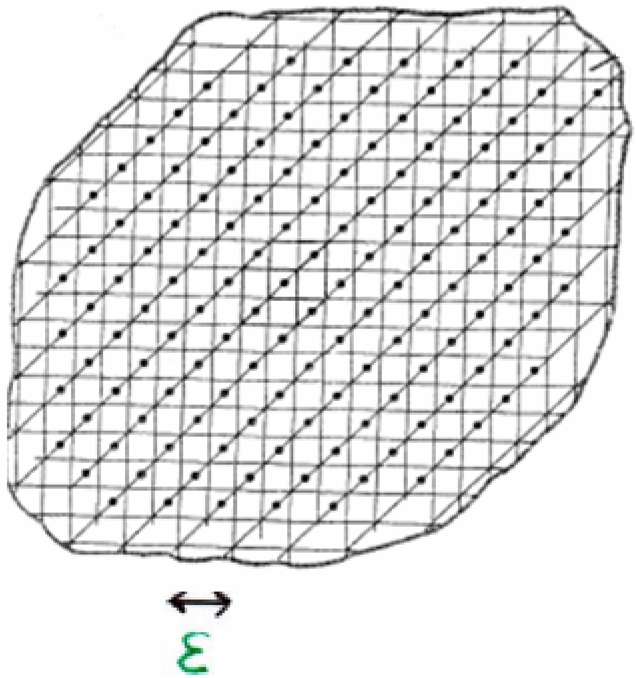
$\Omega \subset \mathbb{R}^n$, $n = 2, 3, \ldots$ & T^ε *perforations* $\subset \Omega$

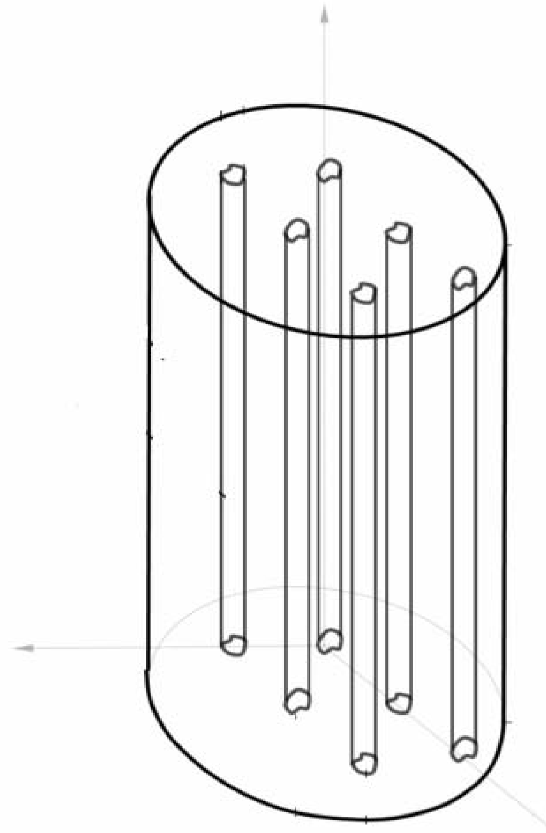
$$\Omega^\varepsilon = \Omega \setminus \bigcup T^\varepsilon$$

Geometrical configurations:

$\Omega \subset \mathbb{R}^n$, $n = 2, 3, ...$
T^ε the perforations contained in Ω







ε measures the periodicity / a_ε measures the size of the perforations

Some authors: *M.Goncharenko (1995), S.Kaizu (1989) & B.Cabarrubias, D.Cioranescu, C.Conca, J.I.Díaz, P.Donato, T.Durante, U.Hornung, W.Jäger, A.Liñan, T.Melnyk, M.Neuss-Radu, C.Timofte, R.Zaki, ...*

The functions:

- $f \in L^2(\Omega)$
- $\sigma \equiv \sigma(x,u)$, $\sigma \in \mathcal{C}^1(\overline{\Omega} \times \mathbb{R})$ /

$$\sigma(x,0) = 0, \quad 0 < k_1 \leq \frac{\partial \sigma}{\partial u}(x,u) \leq k_2, \;\; x \in \overline{\Omega}, \;\; u \in \mathbb{R},$$

...that can be weakened:

$$0 \leq \frac{\partial \sigma}{\partial u}(x,u) \leq k_2(1 + |u|^{\delta}), \;\; \delta \in \left[0, \frac{2}{n-2}\right],$$

... and include some *adsorption lows*: Langmuir function

$$\sigma(u) = \frac{k_1 u}{1 + k_2 u}, \;\; k_1, k_2 > 0, \quad u \in \mathbb{R}_+,$$

and others strictly increasing functions $\sigma : [0,\infty) \to [0,\infty)$

The parameters:

- ε the period: $\varepsilon \to 0$
- a_ε the size of the perforations: $a_\varepsilon = o(\varepsilon)$ or $a_\varepsilon = O(\varepsilon)$
- $\beta(\varepsilon)$ the adsorption parameter: $\beta(\varepsilon) \xrightarrow{\varepsilon \to 0} \infty$

As $\varepsilon \to 0$, we find **very different behaviours of the solution u_ε of P^ε,** depending on the relations between ε, a_ε and $\beta(\varepsilon)$!

Critical relations for parameters: $\beta(\varepsilon)|\bigcup \partial T^\varepsilon| = O(1)$

$\Rightarrow$ a certain relation between ε, a_ε and $\beta(\varepsilon)$

▶ We deal with: *critical sizes*, *strange terms* or *foreign terms*,...

▶ Relations between parameters for which we have a different asymptotic behavior from *extreme cases/behaviors*: it may depend...

V.A.Marchenko & E.Ya. Khruslov (1974), E.Sanchez-Palencia (1982), D.Cioranescu & F.Murat (1982), G.Allaire (1989), ...

C.Conca (1985), D.Cioranescu & P.Donato & H.Ene (1996), M.Lobo & O.A.Oleinik & M.E.Pérez & T.A.Shaposhnikova (1998),...

Outline

- A general situation:
  critical relations for parameters in P^ε
- The case of perforations by tubes
- Application to the spectral convergence
- The case of perforations by balls along a plane
- Bounds for convergence rates of eigenvalues

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The critical relations for parameters: $\beta(\varepsilon)|\bigcup \partial T^\varepsilon| = O(1)$

$\beta(\varepsilon)$ *very large* or *very small* compared with $|\bigcup \partial T^\varepsilon|^{-1}$

a_ε *large* or *small* compared with the classical critical size

- The most critical case: $\beta(\varepsilon)|\bigcup \partial T^\varepsilon| = O(1)$ + *critical size* T^ε
  $\Rightarrow$ *the strange term* contains a non-linear term $H(x,u)$ solution of the functional equation

$$H = \widetilde{c}\sigma(x, u - H)$$

  where $\widetilde{c} > 0$ depends on n and the geometry of T^ε, and H satisfies the same properties of smoothness of σ

- $\beta(\varepsilon)|\bigcup \partial T^\varepsilon| = O(1)$+ *large sizes of* T^ε
  $\Rightarrow$ *the strange term contains* $\widetilde{c}\sigma(x,u)$

- Other relations: $\beta(\varepsilon)$ *very large* + *critical size of* T^ε
  $\Rightarrow$ *a linear strange term not depending on* σ

- Other extreme relations *do not take into account perforations or adsorption parameters*

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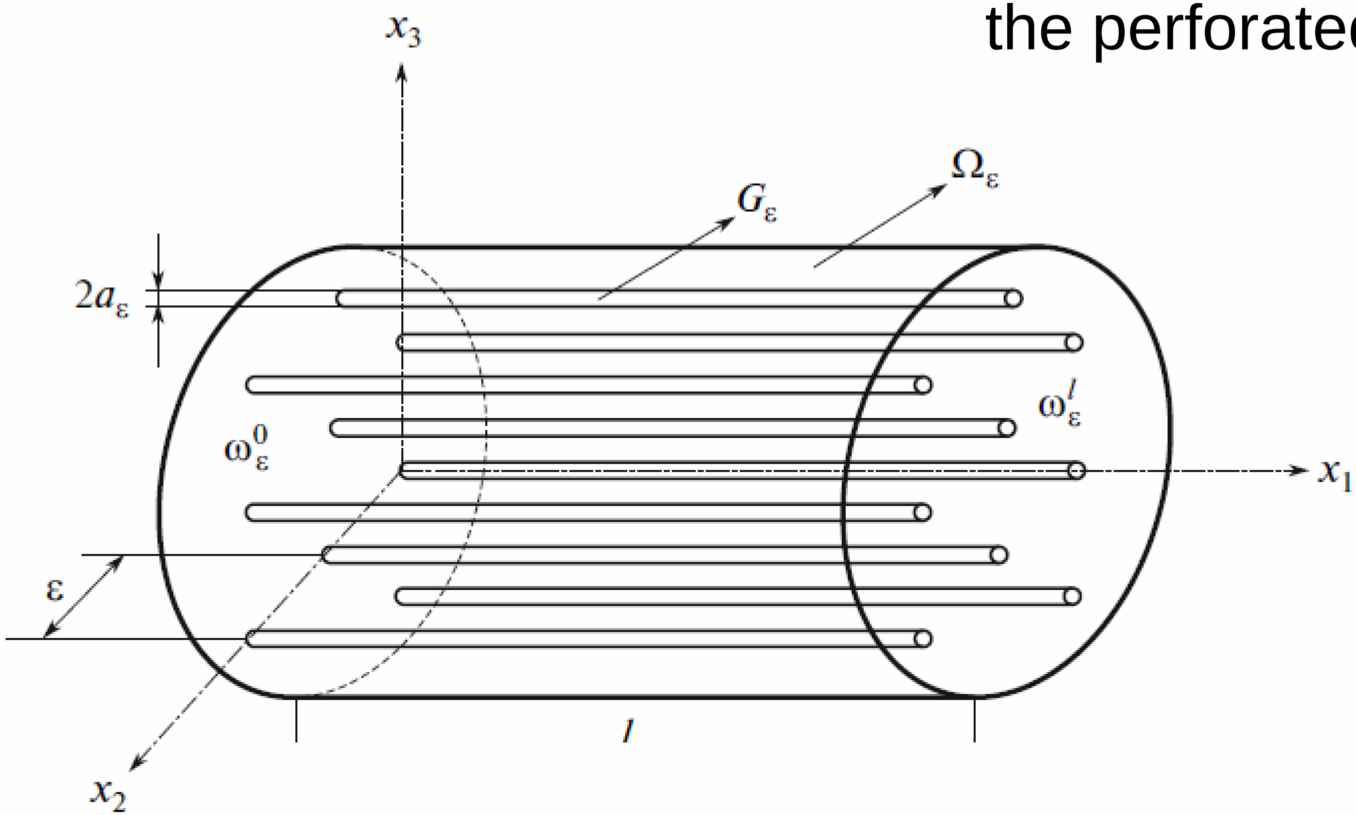
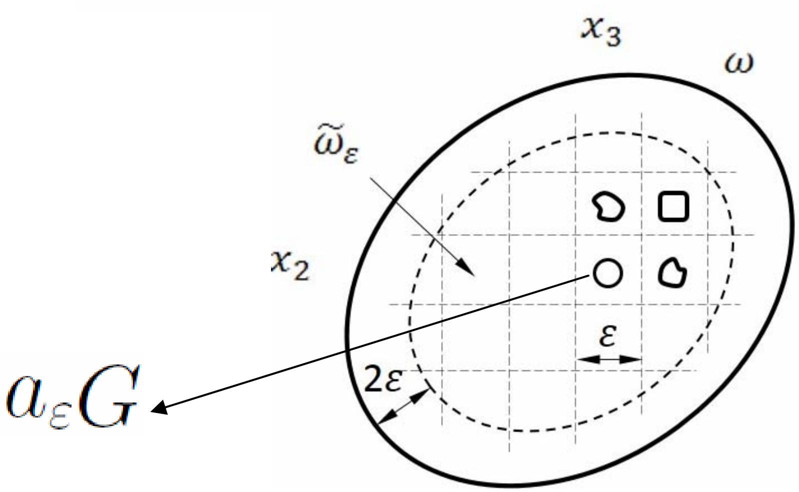
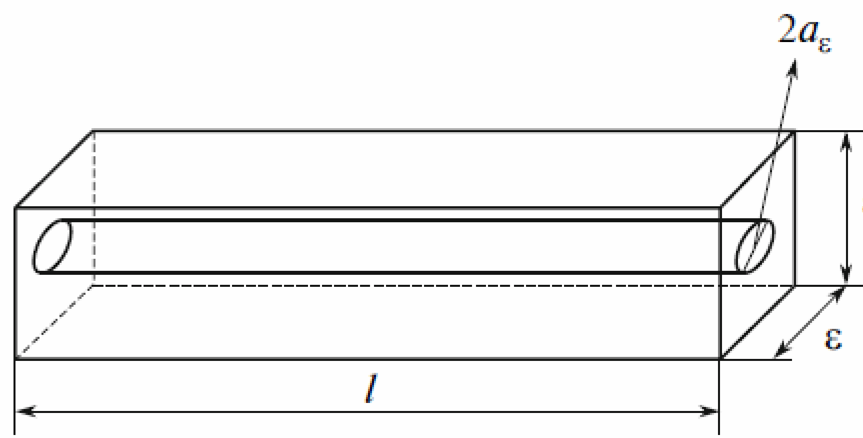
the perforated domain







a basis of the domain

the periodicity cell

Domain of $\mathbb{R}^3$ perforated by tubes. Homogenized problems P^0: $a_\varepsilon << \varepsilon$, $\varepsilon \to 0$

D.Gómez, M.Lobo, E.Pérez, T.A.Shaposhnikova, M.N.Zubova (DM 2013, M2AS 2014, →)

$$P^0 \begin{cases} -\Delta u + \varsigma(x,u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

- If $\beta(\varepsilon) a_\varepsilon \varepsilon^{-2} \to C^2 > 0$ and $\varepsilon^2 \ln(a_\varepsilon) \to -\alpha^2 < 0$

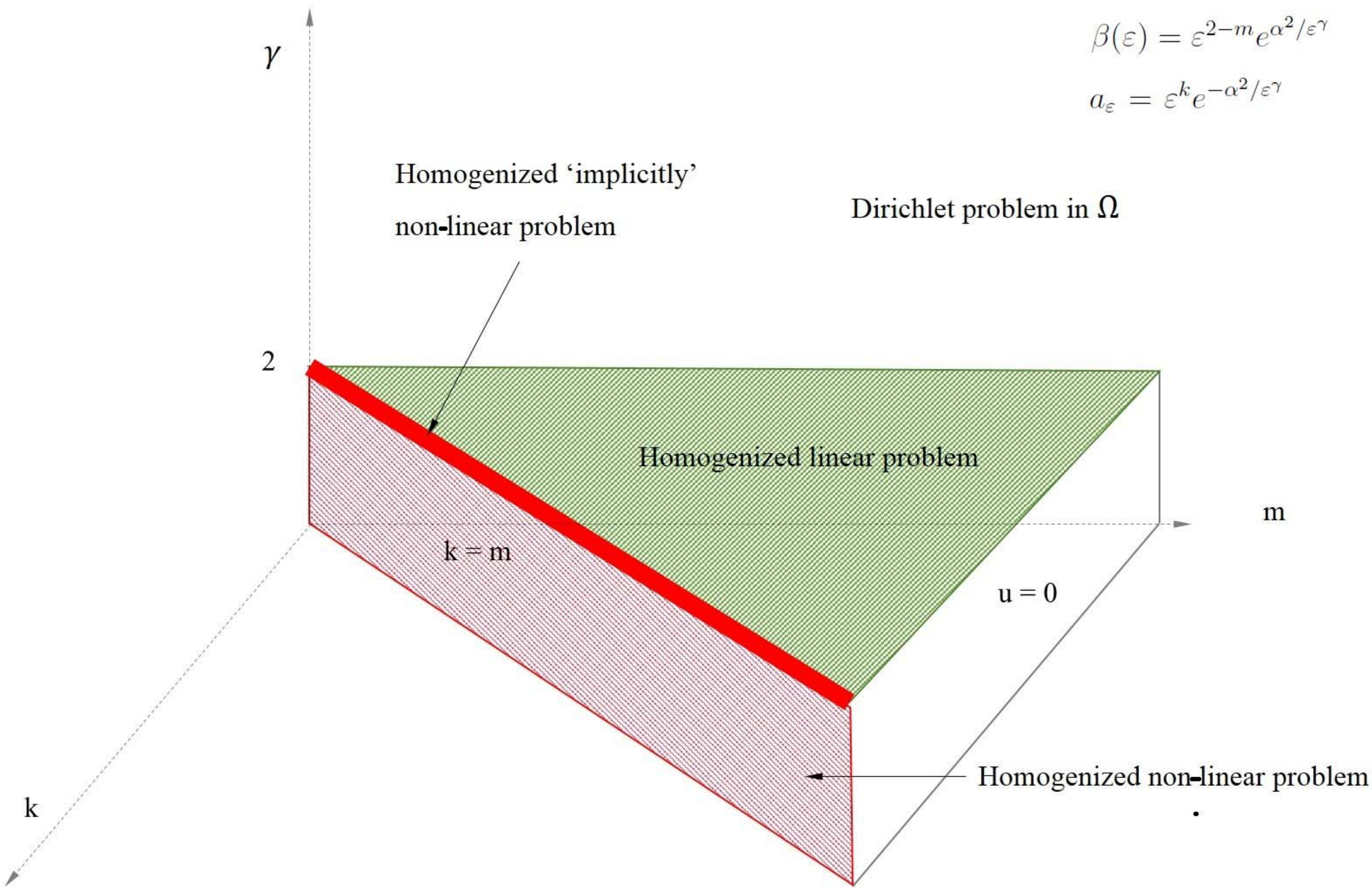
$$\varsigma(x,u) = \frac{2\pi}{\alpha^2} H(x,u), \qquad 2\pi H = |\partial G| \alpha^2 C^2 \sigma(x, u - H)$$

- If $\beta(\varepsilon) a_\varepsilon \varepsilon^{-2} \to C^2 > 0$ and $\varepsilon^2 \ln(a_\varepsilon) \to 0$

$$\varsigma(x,u) = |\partial G| C^2 \sigma(x,u)$$

- If $\beta(\varepsilon) a_\varepsilon \varepsilon^{-2} \to \infty$ and $\varepsilon^2 \ln(a_\varepsilon) \to -\alpha^2 < 0$

$$\varsigma(x,u) = \frac{2\pi}{\alpha^2} u$$


γ
$\beta(\varepsilon) = \varepsilon^{2-m} e^{\alpha^2/\varepsilon^\gamma}$
$a_\varepsilon = \varepsilon^k e^{-\alpha^2/\varepsilon^\gamma}$
Homogenized 'implicitly'
non-linear problem
Dirichlet problem in Ω
2
Homogenized linear problem
m
k = m
u = 0
Homogenized non-linear problem
k

Domain of $\mathbb{R}^3$ perforated by tubes: the spectral problem P^ε

Asymptotics for the eigenelements $\left(\lambda^\varepsilon, u^\varepsilon\right)$ as $\varepsilon \to 0$ depending on the relations between ε, a_ε and $\beta(\varepsilon)$

$a(x) \in C^1(\overline{\Omega})$, $a(x) > 0$

Fixed ε : $0 < \lambda_1^\varepsilon \leq \lambda_2^\varepsilon \leq \cdots \lambda_i^\varepsilon \leq \cdots \xrightarrow{i\to\infty} \infty$

The eigenfunctions $\{u_i^\varepsilon\}_{i=1}^\infty$ basis in $L^2(\Omega^\varepsilon)$ and $H^1(\Omega^\varepsilon)$

$$P^\varepsilon \begin{cases} -\Delta u^\varepsilon = \lambda^\varepsilon u^\varepsilon \quad \text{in } \Omega^\varepsilon \\ u^\varepsilon = 0 \quad \text{on } \partial\Omega \setminus \left(\partial\Omega \cap \bigcup \overline{G}^\varepsilon\right) \\ \dfrac{\partial u^\varepsilon}{\partial n} + \beta(\varepsilon) a(x) u^\varepsilon = 0 \text{ on } \bigcup \partial G^\varepsilon \end{cases}$$

The homogenized spectral problems

- when $\beta(\varepsilon)a_\varepsilon\varepsilon^{-2} \to 0$ or $\varepsilon^2 \ln(a_\varepsilon) \to -\infty$
  P^0 is the spectral Dirichlet problem in Ω,
- when $\beta(\varepsilon)a_\varepsilon\varepsilon^{-2} \to \infty$ and $\varepsilon^2 \ln(a_\varepsilon) \to -\alpha^2 < 0$

$$P^0 \begin{cases} -\Delta u + \dfrac{2\pi}{\alpha^2} u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \Omega, \end{cases}$$

- when $\beta(\varepsilon)a_\varepsilon\varepsilon^{-2} \to C^2 > 0$ and $\varepsilon^2 \ln(a_\varepsilon) \to 0$

$$P^0 \begin{cases} -\Delta u + |\partial G| C^2 a(x) u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \Omega, \end{cases}$$

- when $\beta(\varepsilon)a_\varepsilon\varepsilon^{-2} \to C^2 > 0$ and $\varepsilon^2 \ln(a_\varepsilon) \to -\alpha^2 < 0$

$$P^0 \begin{cases} -\Delta u + \dfrac{2\pi}{\alpha^2} \dfrac{|\partial G|\alpha^2 C^2 a(x)}{2\pi + |\partial G|\alpha^2 C^2 a(x)} u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \Omega, \end{cases}$$

Convergence of $(\lambda^\varepsilon, u^\varepsilon)$ towards the eigenelements of P^0 with conservation of the multiplicity!

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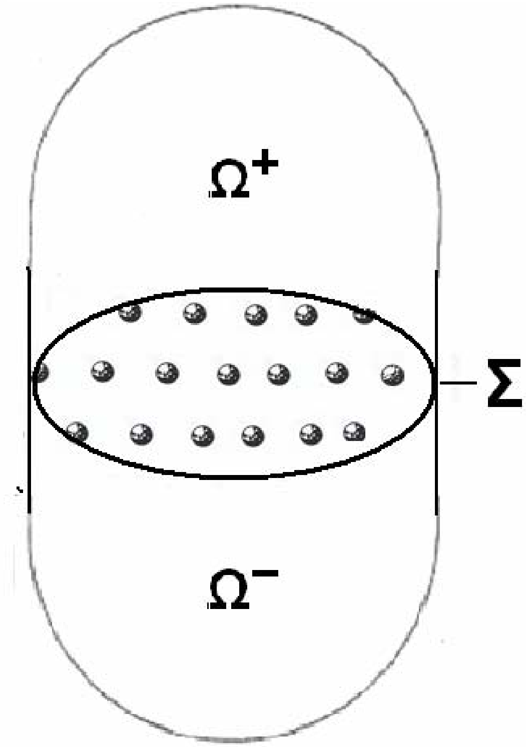
Boundary homogenization problems for perforated domains along planes: Robin type b.c. on the cavities

- D. Gómez, M. E. Pérez, T. A. Shaposhnikova

AA, 2012/ IJAP, 2013 / IMSE 2013

Notations: $\beta(\varepsilon) \equiv \varepsilon^{-\kappa}$ $(\kappa \in \mathbb{R})$,

$a_\varepsilon \equiv r^\varepsilon \equiv O(\varepsilon^\alpha)$ $(\alpha \geq 1)$



$$P^\varepsilon \begin{cases} -\Delta u^\varepsilon = \lambda^\varepsilon u^\varepsilon \quad \text{in } \Omega^\varepsilon \\ u^\varepsilon = 0 \quad \text{on } \partial\Omega \\ \dfrac{\partial u^\varepsilon}{\partial n} + \varepsilon^{-\kappa} a(x)\, u^\varepsilon = 0 \ \text{ on } \bigcup \partial T^\varepsilon \end{cases}$$

Asymptotics for the eigenelements $(\lambda^\varepsilon, u^\varepsilon)$ **as** $\varepsilon \to 0$ **depending on the relations between** ε, $r^\varepsilon \approx \varepsilon^\alpha$, $\varepsilon^{-\kappa}$

....conservation of the multiplicity!

Critical sizes/relations: $\varepsilon^{-\kappa} | \cup \partial T^{\varepsilon} | = O(1)$

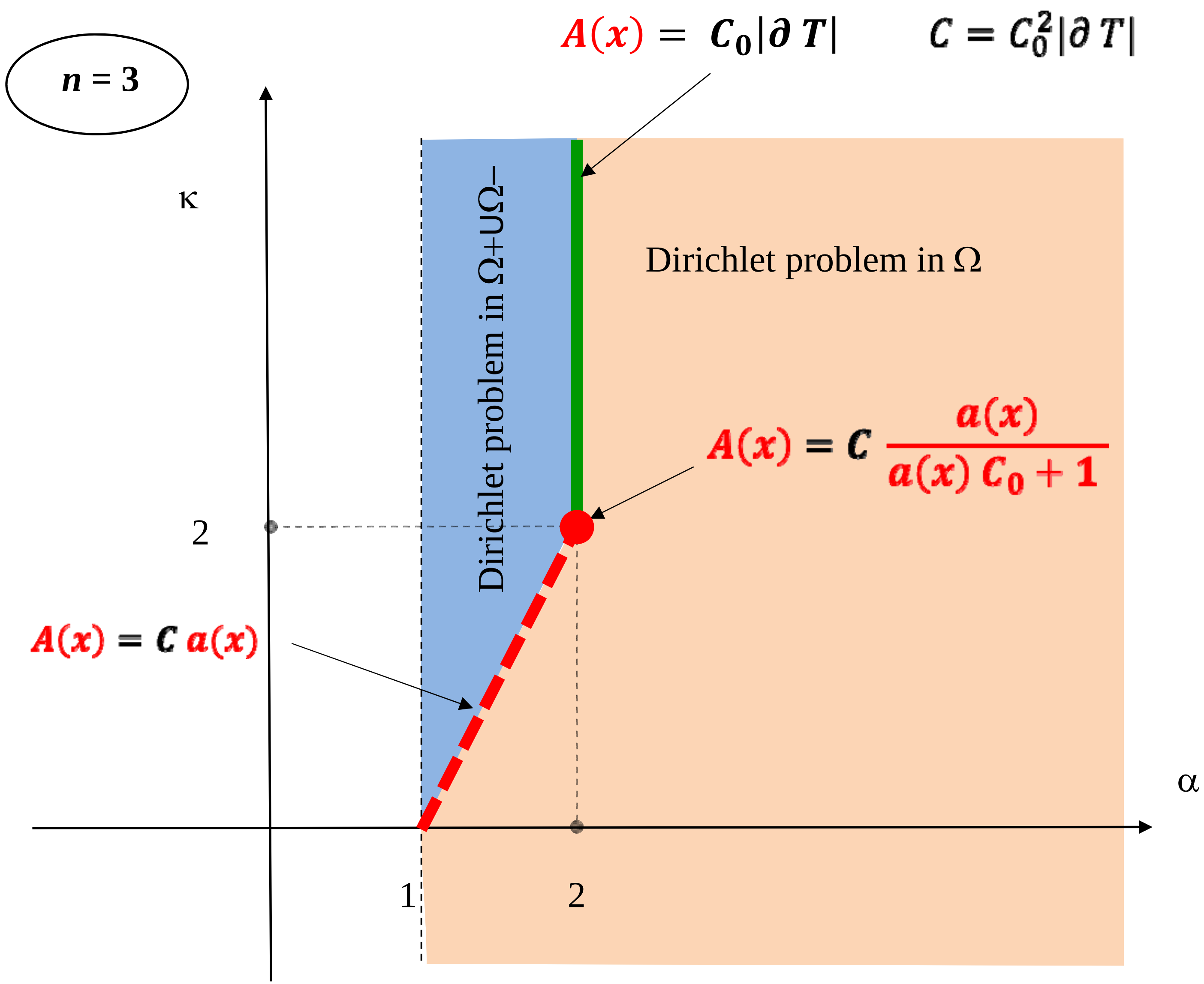
$$P^0 \begin{cases} -\Delta u = \lambda u \quad \text{in } \Omega^+ \cup \Omega^- \\ [u] = 0 \quad \text{on } \Sigma \\ [\dfrac{\partial u}{\partial x_3}] = A(x) u \quad \text{on } \Sigma \\ u^0 = 0 \quad \text{on } \partial\Omega \end{cases}$$

$$A(x) = C_0^2 \, |\partial T| a(x) \qquad \text{when } \kappa = 2(\alpha - 1)$$

$$A(x) = C_0^2 \, |\partial T| \frac{a(x)}{a(x) C_0 + 1} \qquad \text{when } \kappa = \alpha = 2$$

$$A(x) = C_0 \, |\partial T| \quad \text{for } \kappa > 2 \, , \ \alpha = 2$$

$$A(x) = 0 \quad \text{for } \kappa < 2 \, , \ \alpha = 2$$



n = 3
A(x) = C₀|∂T|
C = C₀²|∂T|
κ
Dirichlet problem in Ω+∪Ω−
Dirichlet problem in Ω
A(x) = C a(x) / (a(x) C₀ + 1)
2
A(x) = C a(x)
α
1
2

Convergence rates for eigenelements: the most critical case $\alpha = \kappa = 2$

Theorem
Let $\{\lambda_k^\varepsilon\}_{k=1}^\infty$ and $\{\lambda_k\}_{k=1}^\infty$ be the eigenvalues of problem P^ε and P^0, respectively. Then, for each fixed k there exists a constant C_k independent of ε such that

$$|\lambda_k^\varepsilon - \lambda_k| \leq C_k \varepsilon^{1/16},$$

holds for sufficiently small ε. In addition, for any eigenvalue λ_k of P^0 with multiplicity s ($\lambda_k = \lambda_{k+1} = \cdots = \lambda_{k+s-1}$), and for any u eigenfunction corresponding to λ_k, with $\|u\|_{L^2(\Omega)} = 1$, there exists $\tilde{u}^\varepsilon$, $\tilde{u}^\varepsilon$ a linear combination of eigenfunctions $\{u_r^\varepsilon\}_{r=k}^{r=k+s-1}$ of P^ε corresponding to $\{\lambda_r^\varepsilon\}_{r=k}^{r=k+s-1}$, such that

$$\|\tilde{u}^\varepsilon - u\|_{L^2(\Omega_\varepsilon)} \leq C_k \varepsilon^{1/16}.$$

... and similar bounds for the rest of α and κ!

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Thank you very much…