DEFECTS, INTERFACES AND CORRECTORS

dedicated to C. C.

I INTRODUCTION

II DEFECTS : ELLIPTIC PBS

(joint project with X. BLANC and C. LEBRIS)

III DEFECTS : HJ, QUASILINEAR, FULLY NONLINEAR

(joint project with P.E. SOUGANIDIS)

IV INTERFACES AND ELLIPTIC EQS

(joint project with X. BLANC and C. LEBRIS

MORE DETAILS IN The Collège de France Videotapes (13-14 course)

I INTRODUCTION

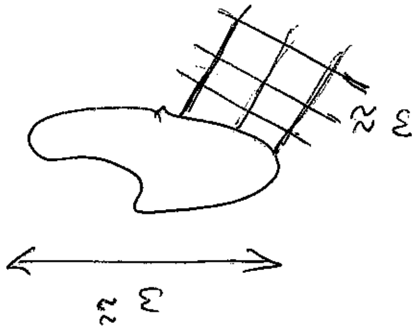
- Systematic study of models involving "localised defects" (or interfaces) with a typical "scale" of order ε (not necessarily "very small") in a background with a typical "scale" of order δ. The cases of interest are $\varepsilon \approx \delta$ or ε small, $\delta \approx 1$. Enough to consider $\varepsilon \approx \delta$ (otherwise background is "locally homogeneous ...)

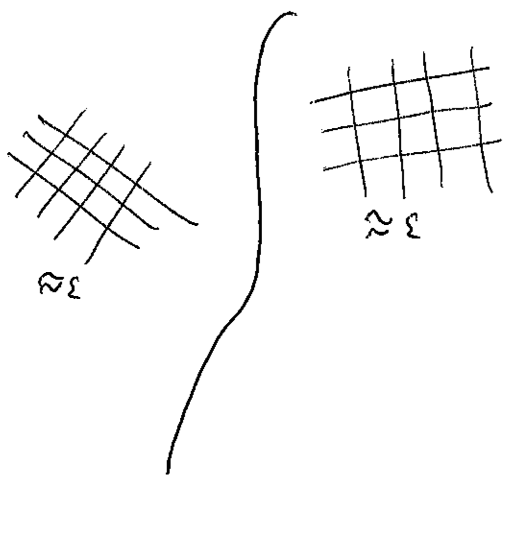
- IF NO DEFECTS $\Rightarrow$ HOMOGENEISATION with an expansion based upon the "correctors" (pbs in a periodic cell

- Q1: IS THE HOMOGENEISATION LIMIT STILL VALID IN THE PRESENCE OF DEFECTS?

  Q2: WHEN IT IS THE CASE, CAN ONE MAKE A MORE PRECISE EXPANSION (in particular "near the defect")

- In some sense, we consider the problems that true multiscale numerical methods "should try to solve".
- RESULTS ARE "VALID" FOR MOST (ALL) CLASSICAL PDE's (including time-dependent problems) but for WAVE PROPAGATION WHEN THE TYPICAL WL IS OF ORDER ε
- FOR INTERFACES, "MORE" PRECISE BEHAVIOR OF THE SOLUTION "NEAR" THE INTERFACE





$$-\partial_i (a_{ij} \partial_j u) = \partial_i f_i \quad \text{in } \mathbb{R}^d, \ u \text{ sublinear}$$

$$a_{ij} = a_{ij}^{per} + b_{ij} \qquad b \sim f \to 0 \text{ at } \infty$$

<u>Th 0</u>
<u>$d \geq 3$</u>: If $f, b \in L^2$, $\exists !$ sol. $Du \in L^2$, $u \in L^{\frac{2d}{d-2}}$ ($C_0 \cdots$)

<u>Th 1</u>
<u>$d \geq 3$</u> If $f, b \in L^p$ $2 \leq p < d$ ($L^{d,1}$ if $p = d$),

$\exists !$ sol $u \in C_0$

(estimate: $(a_{ij}) \geq I \Rightarrow \| G \|_{L^{\frac{d}{d-2},\infty}} \leq K_d^0$

$\| \nabla G \|_{L^{\frac{d}{d-1},\infty}} \leq K_d^1$)

<u>Th 2</u>
<u>$d \geq 1$</u> If $f, b \in L^p$, $p \geq d$, $\exists !$ sol $Du \in L^p$

(uniq. of u up to a constant)

$$|u(x)| \leq C (1 + |x|^\alpha)$$

$$\alpha = 1 - \frac{d}{p} \quad (\log|x| \text{ if } p = d)$$

(delicate estimate for $a_{ij} = a_{ij}^{per} + b_{ij}$

$$\| Du \|_{L^p} \leq K \| f \|_{L^p} \quad (K_p < \infty) \ )$$

<u>Rk</u>: why $p = d$? "Du behaves like f" at ∞

III DEFECTS: HJ, QUASILINEAR, FULLY NON LINEAR

III.1 HJ EQS

Ex. $\varphi^\varepsilon_t + |\nabla \varphi^\varepsilon| = g^{per}\left(\frac{x}{\varepsilon}\right) + f\left(\frac{x}{\varepsilon}\right)$ in $\mathbb{R}^d$

per. $\quad f \to 0$ at ∞.

wlog $\min g^{per} = 0$

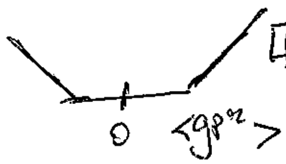
Q1: answer NO! unless $f \geq 0$ (IFF)

Homog: (Papanicolaou-Varadhan-PL^2): $\forall p \in \mathbb{R}^d$

$\exists!\ \lambda = \overline{H}(p)$ s.t. $\exists\ v^{per}$ (periodic)

$$|p + \nabla v^{per}| = g^{per} + \lambda$$

Ex. $d=1 \qquad \lambda = (|p| - \langle g^{per} \rangle)_+$



THM: If $\overline{H}(p)=0$, f compact support, $f \geq 0$, $\exists$ SOL OF

$$(*) \quad |p + \nabla u| = g^{per} + f + \lambda$$

and $u = v^{per}$ for $|x|$ large

If $\overline{H} > 0$, f decays "fast" $\left(\leq \frac{C}{(1+|x|)^\beta} \quad \beta > 1 \right)$, $\exists$ SOL OF $(*)$ and $u - v^{per}$ IS BDED

Rk $d=1$, $\overline{H}(p)=0$, $f>0$ on $\mathbb{R}$, $\nexists$ SUBLINEAR SOL.

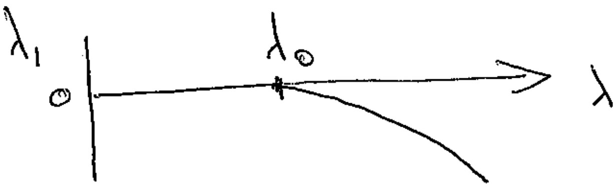
III.2 QUASILINEAR EQS

$$u_\varepsilon - \varepsilon \Delta u_\varepsilon + |\nabla u_\varepsilon|^2 = g^{per}\left(\frac{x}{\varepsilon}\right) + f\left(\frac{x}{\varepsilon}\right)$$

<u>Th1</u> Q1 NO IF "f too negative"

YES IF $f \geq 0$

Ex $g^{per} \equiv 0$, $f \neq 0$ $f \to \lambda f$ $\lambda > 0$, f "fast decay"



$\lambda_1 = \lambda_1(-\Delta + \lambda f)$

$\lambda < \lambda_0$ YES $\lambda \geq \lambda_0$ $\exists$ GD STATE AND NO

<u>Th2</u> <u>$d \geq 3$</u> If $f \geq 0$, $f \in L^q$ $1 < q < \frac{d}{2}$, THEN $\exists!$ SOL. OF

$$-\Delta v + |p + \nabla v|^2 = g^{per} + f \quad \text{on } \mathbb{R}^d$$

$$D v^{p/2} \in L^2 \quad \left(p' = q' / \left(\tfrac{d}{d-2}\right)\right) \ldots$$

III.3 FULLY NONLINEAR EQS

$$u_\varepsilon + F\left(D^2 u^\varepsilon, \frac{x}{\varepsilon}\right) = f\left(\frac{x}{\varepsilon}\right)$$

F per in x, F Lip in "D^2", $F(A+B) \leq F(A) - \operatorname{Tr} B$

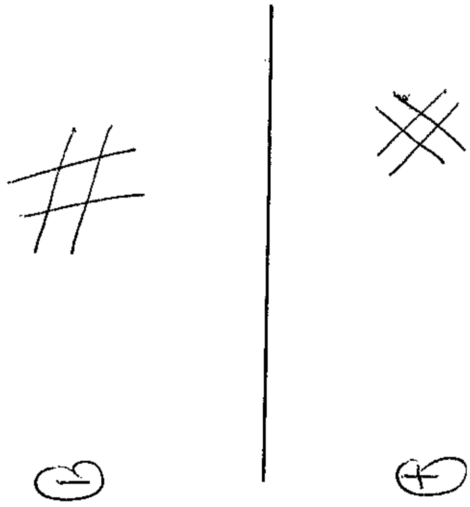
$\geq F(A) - C_0 \operatorname{Tr} B \quad \forall A, B \geq 0$

Q1 YES

<u>THM</u>: f fast decay, $\exists!$ SOL. (UP TO ADD. CST)

- IF $C_0 < d-1$, SOL $u \in C_0$
- If $C_0 \geq d-1$, $|u(x)| \leq C(1+|x|)^\alpha$ $\quad \alpha = -\frac{d-1}{C_0} + 1$

III INTERFACES AND ELLIPTIC EQS



limit as $\varepsilon \to 0$: Hom in ⊖ , Hom in ⊕ and transmission

global corrector on the interface : quasi. periodic pb on the interface (sum of S-Σ operators)

THM : $\exists$ corrector sublinear (bded if periods are rat. dep or if ratio of period is not Liouville)

$\approx \quad -\partial_i \, a_{ij}\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon}\right) \partial_j$, $a_{ij}(x,y)$ per in x T_x per in y T_y

If T_x and T_y are rat. dep. : periodic pb

T_x, T_y rat indt

? $\quad -\partial_i \, a_{ij}(x,x)\,(p_j + \partial_j u) = 0$, u sublinear

THM : $\exists !\, u$ s.t. ∇u QP (x,y) (u smooth...)

And if ratio of periods is not a Liouville number, then u is QP (x,y) and bounded

(λ not L. : $\exists \nu > 0, \exists s \geq 2 \quad |\lambda - \frac{p}{q}| \geq \frac{\nu}{q^s}$)

i) $-\partial_{x_i+y_i} a_{ij}(x,y)\left(P_j + \partial_{x_j+y_j} U\right) = 0$ in $\mathbb{R}^{2d}$

" U per. in (x,y) $(+\delta U \quad \delta \to 0)$ "

ii) $D = \partial_{x+y}$ estimates on DU in H^m $(m \geq 0)$

iii) not Liouville $\Rightarrow$ $\|\Phi\|_{L^2} \leq C \|D\Phi\|_{H^{s-1}}$

(F. series)