Optimal Control for a Second Grade Fluid System

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- We consider the system of second grade fluid:

$$\frac{\partial}{\partial t}(\mathbf{u} - \alpha\Delta\mathbf{u}) - \nu\Delta\mathbf{u} + \mathrm{curl}(\mathbf{u} - \alpha\Delta\mathbf{u}) \times \mathbf{u} + \nabla q = \mathbf{f} + \mathbf{v}, \ \Omega\times]0, T[$$
$$\mathrm{div}\mathbf{u} = 0, \ \Omega\times]0, T[$$
$$\mathbf{u} = 0, \ \partial\Omega\times]0, T[,$$
$$\mathbf{u}(0) = \mathbf{u}_0, \ \Omega$$

Where **u** is the velocity, **f** and **v** are external forces.

- The problem is to find an externel force **v** such that **u** minimize the functional

$$J(\mathbf{u},\mathbf{v}) = \int_0^T \int_\omega |\mathbf{u} - \mathbf{u}_1|^2 dx + \int_0^T \int_\Omega |\mathbf{v}|^2 dxdt$$

here $\mathbf{u}_1$ defined in $\omega \subset\subset \Omega$ is a given velocity.

- Indeed the system of second grade fluid:

$$\frac{\partial}{\partial t}(\mathbf{u}-\alpha\Delta\mathbf{u})-\nu\Delta\mathbf{u}+\mathrm{curl}(\mathbf{u}-\alpha\Delta\mathbf{u})\times\mathbf{u}+\nabla q=\mathbf{f},\ \Omega\times]0,T[,$$
$$\mathrm{div}\mathbf{u}=0,\ \Omega\times]0,T[$$
$$\mathbf{u}=0,\ \partial\Omega\times]0,T[,$$
$$\mathbf{u}(0)=\mathbf{u}_0,\ \Omega$$

Where **u** is the velocity, **f** is the external forces and q is a kind of "pressure".

- The study of such fluids was initiated by Dunn and Fosdick, and [1] Fosdick and Rajapogal [2] in order to model, for example, ceramics in a fluid state.
- The first successful mathematical analysis was found by Cioranescu and Ouazar[1], and subsequently Cioranescu and Girault [3].
- This work follows the ideas given by Boldrini, Fernández-Cara and Rojas-Medar [2].

- $\Omega \subseteq \mathbb{R}^3$ is a simply-connected bounded domain, with boundary $\partial\Omega$ of class $C^{3,1}$.
- Define:

$$\begin{aligned}
\mathbf{H} &= \{\Psi \in \mathbf{L}^2(\Omega) \ : \ \text{div}\,\Psi = 0, \ \Psi \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\} \\
\mathbf{V} &= \{\mathbf{v} \in \mathbf{H}^1(\Omega) \ : \ \text{div}\,\mathbf{v} = 0, \ \mathbf{v} = \mathbf{0}, \text{ on } \partial\Omega\} \\
H(\text{curl};\Omega) &= \{\mathbf{v} \in \mathbf{L}^2(\Omega) \ : \ \text{curl}\,\mathbf{v} \in \mathbf{L}^2(\Omega)\}
\end{aligned}$$

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\end{aligned}
$$

- For $\alpha \in \mathbb{R}^+$, we introduce the space:

$$\mathbf{V}_2 = \left\{ \mathbf{v} \in \mathbf{V} \ : \ \text{curl}\,(\mathbf{v} - \alpha\Delta\mathbf{v}) \in \mathbf{L}^2(\Omega) \right\}$$

- Equipped with the norm:

$$\|\mathbf{u}\|_{V_2} = \|\mathbf{u}\| + \alpha\|\nabla\mathbf{u}\| + \|\text{curl}\,(\mathbf{u} - \alpha\Delta\mathbf{u})\|$$

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- The variational formulation is: Given $\mathbf{f}+\mathbf{v} \in L^2(0,T;H(\mathrm{curl};\Omega)) \cap L^\infty(0,T;\mathbf{L}^2(\Omega))$ and $\mathbf{u}_0 \in \mathbf{V}_2$ to find $\mathbf{u} \in L^\infty(0,T;\mathbf{V}_2)$

$$\begin{aligned} (\mathbf{u}',\mathbf{w}) &+ \alpha(\nabla\mathbf{u}',\nabla\mathbf{w}) + \nu(\nabla\mathbf{u},\nabla\mathbf{w}) + \\ &+ b(\mathbf{u};\mathbf{u},\mathbf{w}) - \alpha b(\mathbf{u};\Delta\mathbf{u},\mathbf{w}) + \alpha b(\mathbf{w},\Delta\mathbf{u},\mathbf{u}) = (\mathbf{f}+\mathbf{v},\mathbf{w}), \\ \mathbf{u}(0) &= \mathbf{u}_0 \end{aligned}$$

for every $\mathbf{w} \in \mathbf{V}$.

- where $b(\mathbf{u};\mathbf{v},\mathbf{w}) = \sum_{i,j=1}^{3} \int_\Omega \mathbf{u}_i \frac{\partial \mathbf{v}_j}{\partial x_i} \mathbf{w}_j dx.$

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- **Theorem**. There exists a constant $\delta > 0$ such that if the following inequalities hold:

$$\|\mathbf{u}_0\|^2_{\mathbf{L}^2(\Omega)} + \alpha|\mathbf{u}_0|^2_{H^1(\Omega)} < \delta$$

$$|\mathbf{u}_0|^2_{\mathbf{V}_2} < \delta,$$

$$\left(\int_0^\infty (\|\mathbf{v}(t)\|^2_{\mathbf{L}^2(\Omega)} + \|\mathrm{curl}(\mathbf{v}(t))\|^2_{\mathbf{L}^2(\Omega)})\right)^{1/2} < \delta,$$

$$\left(\int_0^\infty (\|\mathbf{f}(t)\|^2_{\mathbf{L}^2(\Omega)} + \|\mathrm{curl}\mathbf{f}(t)\|^2_{\mathbf{L}^2(\Omega)})\right)^{1/2} < \delta,$$

then, the above variational formulation has a unique solution for each $t \geq 0$. Moreover $\mathbf{u} \in L^\infty(\mathbb{R}^+, \mathbf{V}_2)$, $\mathbf{u}' \in L^\infty(\mathbb{R}^+, \mathbf{V})$

- Define:

$$\begin{aligned} W_1 &= \{\mathbf{w} \in L^\infty(0,T;\mathbf{V}_2) \;:\; \mathbf{w}' \in L^\infty(0,T;\mathbf{V})\}, \\ W_2 &= L^2(0,T,H(\mathrm{curl};\Omega)) \cap L^\infty(0,T,\mathbf{L}^2(\Omega)), \\ W &= W_1 \times W_2. \end{aligned}$$

- And the admissible controls:

$$\mathcal{U} = \left\{ \mathbf{v} \in W_2 \;:\; \left(\int_0^\infty (\|\mathbf{v}(t)\|^2_{\mathbf{L}^2(\Omega)} + \|\mathrm{curl}(\mathbf{v}(t))\|^2_{\mathbf{L}^2(\Omega)}) \right)^{1/2} < \delta \right\}$$

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- If we define:

$$\mathcal{M}(\mathbf{w}, \mathbf{v}) = (\psi_1, \psi_2)$$

by

$$\begin{cases} \dfrac{\partial}{\partial t}(\mathbf{w} - \alpha A\mathbf{w}) - \nu A\mathbf{w} + P(\mathrm{curl}(\mathbf{w} - \alpha\Delta\mathbf{w}) \times \mathbf{w}) - \mathbf{f} - \mathbf{v} & = & \psi_1 \\ \mathbf{w}(0) - \mathbf{u}_0 & = & \psi_2, \end{cases} \quad (1)$$

where $P \; : \; \mathbf{L}^2(\Omega) \to \mathbf{V}(\Omega)$ is the orthogonal projection and A is the Stokes operator.

- The optimal control problem is the following: Find **u** and **v** such that

$$J(\mathbf{u},\mathbf{v}) = \inf_{(\mathbf{w},\tilde{\mathbf{v}})\in\mathcal{G}} J(\mathbf{w},\tilde{\mathbf{v}})$$

- Where $\mathcal{G}$ is the non-empty set:

$$\mathcal{G} = \{(\mathbf{w},\tilde{\mathbf{v}}) \in W \ : \ \tilde{\mathbf{v}} \in \mathcal{U},\ \mathcal{M}(\mathbf{w},\tilde{\mathbf{v}}) = 0\}.$$

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- Let us see that there exists at least one solution of this problem.
- Let $(\mathbf{w}_n, \tilde{\mathbf{v}}_n)_{n\geq 1}$ be a minimizing sequence in $\mathcal{G}$, i.e.,

$$\lim_{n\to\infty} J(\mathbf{w}_n, \tilde{\mathbf{v}}_n) = \inf_{(\mathbf{w},\tilde{\mathbf{v}})\in\mathcal{G}} J(\mathbf{w}, \tilde{\mathbf{v}}).$$

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- Since $(\mathbf{w}_n, \tilde{\mathbf{v}}_n)_{n\geq 1}$ is uniformly bounded, there exists a subsequence $(\mathbf{w}_{n_k}, \tilde{\mathbf{v}}_{n_k})_{k\geq 1}$ that converge to $(\mathbf{u}, \mathbf{v})$.
- By convexity of J, we have that

$$\liminf_{k\to\infty} J(\mathbf{w}_{n_k}, \tilde{\mathbf{v}}_{n_k}) \geq J(\mathbf{u}, \mathbf{v}).$$

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- Since $\mathcal{M}(\mathbf{w}_{n_k}, \tilde{\mathbf{v}}_{n_k}) = 0$ for every $k \in \mathbb{N}$, we have that $\mathcal{M}(\mathbf{u}, \mathbf{v}) = 0$, then $(\mathbf{u}, \mathbf{v}) \in \mathcal{G}$.
- This conclude the proof.

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- This conclude the proof.

- A set K is a cone with vertex a if for every $\lambda > 0$, $\lambda(K - a) \subseteq K - a$
- Figure, cone with vertex $(0, 0)$:

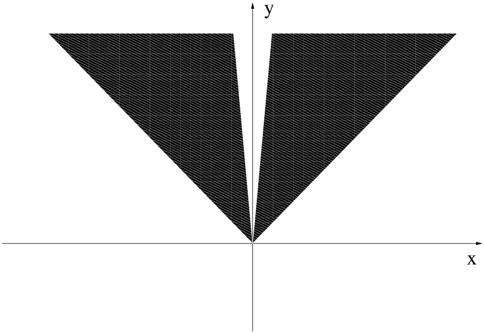

Figure:

- Consider $E \subseteq X$ and $F \subseteq X'$. We define:

$$
\begin{aligned}
E^* &= \{f \in X' : f(x) \geq 0, \forall x \in E\} \\
F_* &= \{x \in X : f(x) \geq 0, \forall f \in F\}
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- If $K \subseteq X$ is a closed convex cone, then $(K^*)_* = K$.

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- If $K \subseteq X$ is a closed convex cone, then $(K^*)_* = K$.

- (Dubovitskii-Milyutin Formalism) Let $K_1, K_2, \ldots, K_n, K_{n+1}$ be convex cones in the real normed space X with $K_1, K_2, \ldots, K_n$ open. Then $\cap_{i=1}^{n+1} K_i = \emptyset$ if and only if there exists $f_i \in K_i^*$, not all zero, such that

$$f_1 + \ldots + f_n + f_{n+1} = 0.$$

- The cone of decreasing directions of J at $(\mathbf{u}, \mathbf{v})$ is given by:

$$DC(J; \mathbf{u}, \mathbf{v}) = \{(\mathbf{w}, \widehat{\mathbf{v}}) \in W : DJ(\mathbf{u}, \mathbf{v})(\mathbf{w}, \widehat{\mathbf{v}}) < 0\}$$

- The corresponding dual cone is:

$$[DC(J; \mathbf{u}, \mathbf{v})]^* = \{-\lambda DJ(\mathbf{u}, \mathbf{v}) : \lambda \geq 0\}$$

- The cone of tangent directions of $\mathcal{G}$ at $(\mathbf{u},\mathbf{v})$, is:

$$TC(\mathcal{M};\mathbf{u},\mathbf{v}) = \{(\mathbf{w},\widehat{\mathbf{v}}) \in W : D\mathcal{M}(\mathbf{u},\mathbf{v})(\mathbf{w},\widehat{\mathbf{v}}) = 0\}$$

- The cone of feasible directions of $\mathcal{U}$ at $(\mathbf{u},\mathbf{v})$ is given by:

$$\begin{aligned} FC(\mathcal{U};\mathbf{u},\mathbf{v}) = \{(\mathbf{w},\widehat{\mathbf{v}}) &\in W_1 \times \mathcal{U} : \exists \varepsilon > 0, \text{ such that} \\ (\mathbf{u},\mathbf{v}) + \lambda(\mathbf{w},\widehat{\mathbf{v}}) &\in W_1 \times \mathcal{U}, \ \forall \lambda \in]0,\varepsilon]\} \end{aligned}$$

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- Or,

$$FC(\mathcal{U};\mathbf{u},\mathbf{v}) = W_1 \times \{\lambda(\mathbf{v}-\widehat{\mathbf{v}}) : \widehat{\mathbf{v}} \in \mathcal{U},\ \lambda > 0\}$$

- And its dual cone:

$$[FC(\mathcal{U};\mathbf{u},\mathbf{v})]^* = \{(0,h) : h \in W_2'\}$$

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- It is known that

$$DC(J;\mathbf{u},\mathbf{v}) \cap TC(\mathcal{M};\mathbf{u},\mathbf{v}) \cap FC(\mathcal{U};\mathbf{u},\mathbf{v}) = \emptyset$$

- Therefore, there exists $f_1 \in [DC(J;\mathbf{u},\mathbf{v})]^*$, $f_2 \in [FC(\mathcal{U};\mathbf{u},\mathbf{v})]^*$ y $f_3 \in [TC(\mathcal{M};\mathbf{u},\mathbf{v})]^*$ not all zero:

$$f_1 + f_2 + f_3 = 0.$$

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- Let us recall that $\mathcal{M}(\mathbf{u},\mathbf{v}) = (M_1(\mathbf{u},\mathbf{v}), M_2(\mathbf{u},\mathbf{v}))$, define:

$$\begin{aligned} M_1(\mathbf{u},\mathbf{v}) &= \frac{\partial}{\partial t}(\mathbf{u} - A\mathbf{u}) - \nu A\mathbf{u} + P(\mathrm{curl}(\mathbf{u} - \Delta\mathbf{u}) \times \mathbf{u}) - \mathbf{f} - \mathbf{v} \\ M_2(\mathbf{u},\mathbf{v}) &= \mathbf{u}(0) - \mathbf{u}_0 \end{aligned}$$

- And its derivative:

$$\begin{aligned} DM_1(\mathbf{u},\mathbf{v})(\mathbf{w},\widehat{\mathbf{v}}) &= \frac{\partial}{\partial t}(\mathbf{w} - A\mathbf{w}) - \nu A\mathbf{w} + P(\mathrm{curl}(\mathbf{u} - \Delta\mathbf{u}) \times \mathbf{w} \\ &+ \mathrm{curl}(\mathbf{w} - \Delta\mathbf{w}) \times \mathbf{u}) - \widehat{\mathbf{v}} \\ DM_2(\mathbf{u},\mathbf{v})(\mathbf{w},\widehat{\mathbf{v}}) &= \mathbf{w}(0) \end{aligned}$$

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- Thus, $D\mathcal{M}(\mathbf{u},\mathbf{v})(\mathbf{w},\widehat{\mathbf{v}}) = 0$.
- It is equivalent to:

$$\begin{aligned} \frac{\partial}{\partial t}(\mathbf{w} - A\mathbf{w}) - \nu A\mathbf{w} + & \\ + P(\mathrm{curl}(\mathbf{u} - \Delta\mathbf{u}) \times \mathbf{w} + \mathrm{curl}(\mathbf{w} - \Delta\mathbf{w}) \times \mathbf{u}) &= \widehat{\mathbf{v}} \\ \mathbf{w}(0) &= 0 \end{aligned}$$

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- Therefore, $D\mathcal{M}(\mathbf{u},\mathbf{v})(\mathbf{w},\widehat{\mathbf{v}}) = 0$, implies that $f_3(\mathbf{w},\widehat{\mathbf{v}}) = 0$.
- Then,

$$(f_1 + f_2)(\mathbf{w},\widehat{\mathbf{v}}) = 0$$

- Now $f_1(\mathbf{w}, \widehat{\mathbf{v}}) = -\lambda DJ(\mathbf{u}, \mathbf{v})(\mathbf{w}, \widehat{\mathbf{v}})$, for some $\lambda \geq 0$.
- But,

$$DJ(\mathbf{u}, \mathbf{v})(\mathbf{w}, \widehat{\mathbf{v}}) = \int_0^T \int_\omega (\mathbf{u} - \mathbf{u}_1)\mathbf{w} dxdt + \int_0^T \int_\Omega \mathbf{v}\widehat{\mathbf{v}} dxdt$$

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- Now, $f_2(\mathbf{w}, \widehat{\mathbf{v}}) = h(\widehat{\mathbf{v}})$.
- and therefore, by assuming $\lambda = 1$:

$$h(\widehat{\mathbf{v}}) = \int_0^T \int_\omega (\mathbf{u} - \mathbf{u}_1)\mathbf{w} dxdt + \int_0^T \int_\Omega \mathbf{v}\widehat{\mathbf{v}} dxdt$$

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- There exists ξ such that

$$\int_0^T \int_\Omega \xi \widehat{\mathbf{v}} dxdt = -\int_0^T \int_\omega (\mathbf{u} - \mathbf{u}_1)\mathbf{w} dxdt$$

- It follows from the theorem of Riesz. With the right-hand side of the above equality.

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- It follows from the theorem of Riesz. With the right-hand side of the above equality.

- It is solution of the adjoint problem:

$$
\begin{aligned}
-\frac{\partial}{\partial t}(\xi - \alpha A\xi) \quad &- \quad \nu A\xi - \mathrm{P}(\mathrm{curl}(\mathbf{u} - \alpha\Delta\mathbf{u}) \times \xi \\
+\mathrm{curl}[(\mathbf{u} \times \xi) \quad &- \quad \alpha\Delta(\mathbf{u} \times \xi)]) \\
+P(\chi_\omega(\mathbf{u} - \mathbf{u}_1)) \quad &= \quad 0, \ \text{in } \Omega\times]0, T[\\
\xi \quad &= \quad 0, \ \text{on } \partial\Omega\times]0, T[\\
\xi(T) \quad &= \quad 0, \ \text{in } \Omega
\end{aligned}
$$

- So:

$$h(\widehat{\mathbf{v}}) = \int_0^T \int_\Omega (-\xi + \mathbf{v})\widehat{\mathbf{v}} dxdt.$$

- Finally:

$$\int_0^T \int_\Omega (-\xi + \mathbf{v})(\widehat{\mathbf{v}} - \mathbf{v}) dxdt \leq 0.$$

- So:

$$h(\widehat{\mathbf{v}}) = \int_0^T \int_\Omega (-\xi + \mathbf{v})\widehat{\mathbf{v}} dxdt.$$

- Finally:

$$\int_0^T \int_\Omega (-\xi + \mathbf{v})(\widehat{\mathbf{v}} - \mathbf{v}) dxdt \leq 0.$$

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