Relaxation to equilibria for a linear kinetic equation at very low temperature

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1. General motivation of the problem: gas of bosons at low temperature.

2. Simplification and presentation of the problem: spatially homogeneous gas of bosons at very low temperature close to equilibrium.

3. New result in collaboration with Tran M.-B (BCAM).

Gaz of bosons at low temperature

• Bosons: quantum particles whose statistics do not restrict the number of them that occupy the same quantum state.

• Low Temperature: there is only one possible equilibrium distribution of particles in a system of N bosons with total momentum P and total energy E:

$$n_0(p) = \frac{1}{e^{\beta(\omega(p)-\mu)} - 1} + \alpha\delta(p - p_0)$$

for some $\beta > 0$, $p_0 \in \mathbb{R}^3$, $\alpha \geq 0$, $\mu \leq 0$ such that $\underline{\alpha\mu = 0}$.

· $\omega(p)$ is the energy of the particle with momentum p; $\beta = \frac{1}{k_B T}$.

·If $\alpha = 0$: supercrit. or crit. temperature. If $\alpha > 0$: subcritical (low) temperature.

·If $\alpha > 0$ non zero density of particles of momentum $p = p_0$: the "condensate".

Particles in the condensate

They are all described by one single wave function $\psi(t, x)$ that satisfies a non linear type Schrödinger type (Gross Pitaevskii) equation.

At zero temperature (all the particles are in the condensate):
Rigorously proved by E. Lieb & al. (PRA '00), for the stationary case and by L. Erdös & al. (Invent. Math. '07) for the time dependent case.

Great number of results for Gross Pitaevskii equation: existence of different types of solutions, stability, numerics, ...

Particles in the dilute gas

At $T > 0$ not all particles are in the condensate. Some are in the gas.

Their density distribution function satisfies a Boltzmann type equation (Nordheim equation, 1929).

No complete rigorous deduction yet.

Partial results by D. Benedetto & al. (M3AS '05), J. Lukkarinen & al. (Invent. Mat. '11), E. Faou & al. preprint '13 & Séminaire Math. Applis. Collège de France

Fewer results on Nordheim equation:

1.- Global well posedness for the Cauchy problem and convergence to equilibrium in the weak sense of measures (X. Lu 2001–2005).

2.- Existence of strong solutions that violate the conservation of particles (M.E. & J. Velázquez '08).

3.- Existence of solutions that blow up and form a condensate in finite time. (M.E. & J. Velázquez '13).

All in the spatially homogeneous and radially symmetric case.

- Many open questions and problems. In particular:

At very low temperature

The system of particles: gas+condensate.

Approximations:

- Below the critical temperature, the evolution of the density function of the particles in the gas and the condensate are considered to be driven by the collisions between the gas and condensate particles.

- At very low temperature: $\omega(p) \approx |p|$ for p small. We take $\omega(p) = |p|$.

- Spatially homogeneous: the density n of particles of momentum p at time t is independent of the space variable x; $n = n(t, p)$.

Problem:

The particle's system is described by a coupled system of equations:

· Boltzmann equation (for the density particles in the gas $n(t, p)$)

· ODE (for condensate's density $n_c(t)$).

Question: Relaxation rates of the solutions $(n(t, p), n_c(t))$?

Interesting because it is known that at low temperature the collision frequency of the particles in the gas is not bounded from below.

Problem similar to the "soft potential" case for classical particles considered by several authors, in particular H. Grad ('63), R. Caflisch ('80) and S. Ukai & K. Asano ('82), Carlen & al. ('99).

Collinear collisions

If P, P' and Q' denote particles with momentum p, p' and and $p - p'$ respectively, and if

$$\omega(p) = |p|$$

the conservation of momentum and energy in the collision:

$$P' + Q \longrightarrow P$$

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$$p = p' + (p - p') \quad \text{and} \quad |p| = |p'| + |p - p'|$$

Then vectors p, p' and $p - p'$ must be parallel.

Similarly for all the other collisions.

$$\boxed{\frac{\partial n}{\partial t}(t,p) = \mathcal{Q}(f)(t,p) \equiv \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} dp_1 dp_2 \left[R(p,p_1,p_2) - R(p_1,p,p_2) - R(p_2,p_1,p)\right]}$$

- $R(p,p_1,p_2) = \dfrac{|p||p_1||p_2|}{\sqrt{n_c}} \left[\delta(\omega(p) - \omega(p_1) - \omega(p_2))\delta(p - p_1 - p_2)\right] \times$

$$\times \left[n(p_1)n(p_2)(1+n(p)) - (1+n(p_1)(1+n(p_2))n(p)\right]$$

- $n(p) \equiv n(t,p)$: density of quasi particles at time t with momentum p

- $n_c \equiv n_c(t)$ is the condensate's density:

$$\boxed{n_c'(t) = -\int_{\mathbb{R}^3} \mathcal{Q}(f)(t,p)dp}$$

Linearisation

Perturb a radially symmetric equilibrium (without loss of generality):

$$
\begin{aligned}
n(t,p) &= n_0(p) + n_0(p)[1 + n_0(p)]\Omega(t,p) \\
n_0(p) &= \frac{1}{e^{\beta|p|} - 1} \\
n_0(p)[1 + n_0(p)]\frac{\partial \Omega}{\partial t}(t,p) &= -M(p)\Omega(t,p) + \int_{\mathbb{R}^3} \Omega(t,p')\, W(p,p') dp' \\
&\equiv L(\Omega)(t,p)
\end{aligned}
$$

for some explicit W and M.

The linearisation of $n_c(t)$:

$$n_c(t) = 1 + m(t)$$
$$m'(t) = -\int_{\mathbb{R}^3} L(\Omega)(t,p)dp$$

The linear equation for Ω conserves the energy:

$$\frac{d}{dt}\int_{\mathbb{R}^3} n_0(p)(1+n_0(p))\Omega(t,p)\omega(p)dp = 0$$

but not the number of particles:

$$\frac{d}{dt}\int_{\mathbb{R}^3} n_0(p)(1+n_0(p))\Omega(t,p)dp \neq 0.$$

Reduction of the Equation

Look for the solution as: $$\Omega(t,p) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \Omega_{\ell\, m}(t,|p|) Y_{\ell\, m}\left(\frac{p}{|p|}\right).$$

Since collinear collisions, all the modes $\Omega_{\ell\, m}(t,|p|)$ satisfy the same equation:

$$n_0(r)[1+n_0(r)]\frac{\partial \Omega_{\ell\, m}}{\partial t}(t,r) = -M(r)\Omega_{\ell\, m}(t,r) + \int_0^{\infty} G(r,r')\Omega_{\ell\, m}(t,r')dr'$$

with initial data

$$\Omega_{\ell\, m}(0,r) = \int_{\mathbb{S}^2} \Omega_0(p) Y_{\ell\, m}\left(\frac{p}{|p|}\right) d\sigma.$$

Change variables: $$f(t,k) = \frac{r}{2k_BT}\frac{\Omega_{\ell\, m}(t,r)}{\sinh\left(\frac{r}{2k_BT}\right)};\ k = \frac{r}{2k_BT},\ \ r = |p|.$$

The simplified equation

$$\begin{aligned}
\frac{\partial f}{\partial t}(t,k) &= E(f) \equiv -\Gamma(k)\, f(t,k) + T_2[f] \\
T_2[f] &= 2\int_0^\infty K(k,k')f(t,k')dk' \\
\Gamma(k) &= \sinh k \int_0^\infty (\phi(|k-k'|)\phi(k') + \phi(k+k')\phi(k'))\, dk' \\
\phi(k) &= \frac{k^2}{\sinh k}; \quad E(\phi) = 0.
\end{aligned}$$

with: $$\lim_{k\to 0} \frac{\Gamma(k)}{k} = \frac{\pi^4}{15}, \quad \lim_{k\to\infty} \frac{\Gamma(k)}{k^5} = \frac{1}{15}$$

and: $$\int_0^\infty |K(k,k')|^2 dk' < c_1 k^4 + c_2 k^2, \ \forall k > 0.$$

The linearised equation is considered in G.H.Wannier in Bull.Am.Phys.Soc.'69.

The spectrum of a simplified equation, where $\mathcal{M}(p, p_1, p_2)$ is taken to be constant, is studied in F. H. Claro & G. H. Wannier in J. Math. Phys. '71

A rigorous determination of the spectrum and compactness properties of the collision operator is made in F. A. Buot, J. Phys. C, '72.

The spectrum for the non radial linearised operator is studied in D. Benin, Phys. Rev. B, '75.

Theorem 1. *(ME & Tran M.-B) Suppose that* $f_0 \in L^2(\mathbb{R}_+)$ *and denote*

$$c_0 = \int_0^\infty f_0(k)\varphi_0(k)dk, \;\; \varphi_0 = \frac{\phi}{||\phi||_2}, \;\; \left(\phi = \frac{k^2}{\sinh k}\right)$$

There is a unique f satisfying the equation in $L^2((0,\infty), L^2(\Gamma^{-1}))$, such that

$$(f - c_0\varphi_0) \in L^2((0,\infty), L^2(\Gamma)),$$

$$f \in L^\infty((0,\infty), L^2(\mathbb{R}_+)) \cap C([0,\infty), L^2(\mathbb{R}_+)),$$

$$\partial_t f \in L^2((0,\infty), L^2(\Gamma^{-1})),$$

and $$\lim_{t\to 0}\left(||f(t) - f_0||_{L^2(\Gamma^{-1})} + ||f(t) - f_0||_2\right) = 0.$$

It also satisfies : (i) $\forall t > 0 :$ $$\frac{d}{dt}\int_0^\infty f(t,k)\,\varphi_0(k) = 0.$$

(ii) $f_0 \geq 0 \Longrightarrow f(t) \geq 0 \;\; \forall t > 0.$

Theorem 2. *If f_0 also satisfies:*

$$I = \int_0^1 \frac{|f_0(k)|^2}{k} dk < \infty$$

there exists a positive constant $C = C(I)$ such that, for all $t > 0$:

$$\|f(t) - c_0\varphi_0\|_2 \leq C \frac{||f_0 - c_0\varphi_0||_2}{(1+t)^{1/2}}.$$

Remark 1. *Since* $\Gamma(k) \to 0$ *as* $k \to 0$*, there is no* $\gamma > 0$ *such that:*

$$\langle -E(f), f\rangle_{L^2} \geq \gamma ||f||^2_{L^2}, \ \ \forall f \in L^2(\mathbb{R}^+) \cap \mathcal{K}er(E)^{\perp}.$$

If one had such an inequality:

$$\frac{d}{dt}||f||^2_{L^2} = \langle E(f), f\rangle_{L^2} \leq -\gamma ||f||^2_{L^2}$$

and exponential decay of $||f||^2_{L^2}$ *follows.*

The proof of Theorem 2 is based on the two Lemmas:

Lemma 1. There exists a constant $C_* > 0$ such that, for all $h \in L^2(\Gamma)$:

$$-\int_0^\infty E(h)(k)h(k)dk \equiv \langle -Eh, h\rangle_{L^2} \geq C_* \|h - \mathbb{P}h\|^2_{L^2(\Gamma)}$$

$$\text{where : } \mathbb{P}h = c_0(h)\varphi_0, \;\; c_0(h) = \int_0^\infty h(k)\varphi_0(k)dk.$$

Remark 2. If $f_0 \in L^2$ and $\mathbb{P}f_0 = 0$, then by the conservation law: $\mathbb{P}f(t) = 0$ for all $t > 0$. Therefore:

$$\langle Ef(t), f(t)\rangle_{L^2} \leq -C_* \|f(t)\|^2_{L^2(\Gamma)}, \forall t > 0$$

Lemma 1 follows from simple spectral theory.

Lemma 2. Let $f_0 \in L^2(\mathbb{R}_+)$ such that $\int_0^\infty f_0(k)\varphi_0(k)dk = 0$. Suppose that there exist $C^* > 0$, $\omega > 0$ and $\tau > 0$ such that the solution f obtained in Theorem 1 satisfies:

$$\|f(t)\|_2 \leq C^* \|f_0\|_2 (t+1)^{-\omega} \quad \forall t \geq \tau.$$

Then, there exist $\theta_1 > 0$, $\kappa_1 > 0$ and $\kappa_2 > 0$, where κ_1 and κ_2 are independent on θ_1, such that, for all $0 < \theta < \theta_1$ and for all $t > \max\{1, \tau\}$

$$\int_0^\infty |f(t,k)|^2 \Gamma(k) dk \geq \kappa_1 \theta \int_0^\infty |f(t,k)|^2 dk - \kappa_2 \left(\frac{\theta^2}{(t+1)^{2\omega}} + \frac{\theta}{(t+1)} \right).$$

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Proposition 1. Let $\{Y_{\ell\, m}\}_{\ell,m}$ be the spherical harmonics on $\mathbb{S}^2$. For any sequence $\{c_{\ell\, m}\}$ such that:

$$\sum_{\ell=0}^{\infty} \sum_{n=-\ell}^{\ell} c_{\ell\, m}^2 < \infty,$$

define the function : $$\Theta(p) = \left(\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} c_{\ell\, m} Y_{\ell\, m} \left(\frac{p}{|p|} \right) \right) |p|.$$

Then:

$$(i) \qquad \Theta \in L^2 \left(\mathbb{R}^3, \frac{dp}{\sinh^2(|p|)} \right)$$

$$(ii) \qquad - M(p)\Theta(p) + \int_{\mathbb{R}^3} \Theta(p')\, W(p,p') dp' = 0.$$

Theorem 2. Suppose that $\Omega_0 \in L^2\left(\mathbb{R}^3, \frac{dp}{\sinh^2(|p|)}\right)$.

- There exists a unique function $\Omega(t,p)$ such that

$$\Omega \in L^\infty\left(0,\infty; L^2\left(\mathbb{R}^3, \frac{dp}{\sinh^2(|p|)}\right)\right) \cap C\left([0,\infty); L^2\left(\mathbb{R}^3, \frac{dp}{\sinh^2(|p|)}\right)\right)$$

$$\Omega - \Theta \in L^2\left(0,\infty; L^2\left(\mathbb{R}^3, M(p)dp\right)\right)$$

$$\frac{\partial \Omega}{\partial t} \in L^2\left(0,\infty; L^2\left(\mathbb{R}^3, \frac{dp}{M(p)\ \sinh^4(|p|)}\right)\right)$$

satisfying the equation in $L^2\left(0,\infty; L^2\left(\mathbb{R}^3, \frac{dp}{M(p)\ \sinh^4(|p|)}\right)\right)$ and:

$$\lim_{t\to 0}\left(||\Omega(t)-\Omega_0||_{L^2\left(\mathbb{R}^3, \frac{dp}{M(p)\ \sinh^4(|p|)}\right)} + ||\Omega(t)-\Omega_0||_{L^2\left(\mathbb{R}^3, \frac{dp}{\sinh^2(|p|)}\right)}\right) = 0.$$

- This solution also satisfies the conservation property:

$$\frac{d}{dt}\int_{\mathbb{R}^3} n_0(p)(1+n_0(p))\Omega(t,p)\omega(p)dp = 0.$$

- If $\displaystyle\int_{|p|<1} \frac{|\Omega_0(p)|^2}{|p|\sinh^2(|p|)}dp < \infty,$ there is $C > 0$:

$$||\Omega(t)-\Theta||_{L^2\left(\mathbb{R}^3,\frac{dp}{\sinh^2(|p|)}\right)} \le \frac{C}{(1+t)^{1/2}}||\Omega_0-\Theta||_{L^2\left(\mathbb{R}^3,\frac{dp}{\sinh^2(|p|)}\right)}.$$

where

$$\Theta(p) = \left(\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} c_{\ell\, m}Y_{\ell\, m}\left(\frac{p}{|p|}\right)\right)|p|$$

$$c_{\ell\, m} = \int_{\mathbb{R}^3}\Omega_0(p)n_0(p)(1+n_0(p))\omega(p)Y_{\ell\, m}\left(\frac{p}{|p|}\right)dp$$