Boundary controllability for a one-dimensional heat equation with two singular inverse-square potentials

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Outline of the talk

1. Introduction
2. Hardy-Poincaré inequalities
3. Well-posedness
4. Controllability from the boundary
5. Open problems
1 Introduction

2 Hardy-Poincaré inequalities

3 Well-posedness

4 Controllability from the boundary

5 Open problems
Main result

We analyse the control problem for the one-dimensional heat equation with two singular inverse-square potentials

\[ u_t - u_{xx} - \frac{\mu_1}{x^2} u - \frac{\mu_2}{(1-x)^2} u = 0, \]

for \((x, t) \in (0, 1) \times (0, T), \mu_1, \mu_2 \leq 1/4.\)

We show a boundary controllability result.

U.B. - Boundary controllability for a one-dimensional heat equation with two singular inverse-square potentials - Submitted
State of the art

EQUATIONS WITH SINGULAR POTENTIALS

• J. Vancostenoble and E. Zuazua - Null controllability for the heat equation with singular inverse-square potentials (2008)

• S. Ervedoza - Control and stabilization properties for a singular heat equation with an inverse-square potential (2008)

• C. Cazacu - Controllability of the heat equation with an inverse-square potential localized on the boundary (2014)
EQUATIONS WITH DEGENERATE COEFFICIENTS

\[ u_t - (a(x)u_x)_x = 0, \quad (x, t) \in (0, 1) \times (0, T) \]
\[ a(x_0) = 0, \quad \exists x_0 \in [0, 1] \]

• P. Martinez and J. Vancostenoble - Carleman estimates for one-dimensional degenerate heat equations (2006)
• P. Cannarsa, P. Martinez and J. Vancostenoble - Carleman estimates for a class of degenerate parabolic operators (2008)
• G. Fragnelli and D. Mugnai - Carleman estimates and observability inequalities for parabolic equations with interior degeneracy (2013)
• M. Gueye - Exact boundary controllability of 1-d parabolic and hyperbolic degenerate equations (2014)
Let $T > 0$, $\mu_1, \mu_2 \leq 1/4$ and define $Q := (0, 1) \times (0, T)$

\[
\begin{align*}
&u_t - u_{xx} - \frac{\mu_1}{x^2} u - \frac{\mu_2}{(1 - x)^2} u = 0 \quad (x, t) \in Q \\
&u(0, t) = f(t), \quad u(1, t) = 0 \\
&u(x, 0) = u_0(x)
\end{align*}
\]  

(Theorem (Null-controllability))

For any time $T > 0$ and any initial datum $u_0 \in L^2(0, 1)$ there exists a control function $f \in L^2(0, T)$ such that the solution of (1) satisfies $u(x, T) = 0$. 

Formulation of the problem

Let $T > 0$, $\mu_1, \mu_2 \leq 1/4$ and define $Q := (0, 1) \times (0, T)$

\[
\begin{aligned}
&u_t - u_{xx} - \frac{\mu_1}{x^2} u - \frac{\mu_2}{(1 - x)^2} u = 0 \quad (x, t) \in Q \\
&u(0, t) = f(t), \quad u(1, t) = 0 \\
&u(x, 0) = u_0(x)
\end{aligned}
\]

Theorem (Null-controllability)

For any time $T > 0$ and any initial datum $u_0 \in L^2(0, 1)$ there exists a control function $f \in L^2(0, T)$ such that the solution of (1) satisfies $u(x, T) = 0$. 
1 Introduction

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3 Well-posedness

4 Controllability from the boundary

5 Open problems
Hardy-Poincaré inequalities

**Proposition**

Let $\mu_1^*, \mu_2^* \in \mathbb{R}$ be such that $\mu_1^* + \mu_2^* \leq 1/4$. Then, for any $z \in H^1_0(0,1)$ it holds

$$
\int_0^1 z_x^2 \, dx \geq \mu_1^* \int_0^1 \frac{z^2}{x^2} \, dx + \mu_2^* \int_0^1 \frac{z^2}{(1-x)^2} \, dx.
$$


**Proposition**

There exists a constant $M > 0$ such that for any $z \in H^1_0(0,1)$ it holds

$$
\int_0^1 z_x^2 \, dx + M \int_0^1 z^2 \, dx \geq \frac{1}{4} \int_0^1 \frac{z^2}{x^2} \, dx + \frac{1}{4} \int_0^1 \frac{z^2}{(1-x)^2} \, dx. \quad (2)
$$
**Proposition**

Let $\mu_1^*, \mu_2^* \in \mathbb{R}$ be such that $\mu_1^* + \mu_2^* \leq 1/4$. Then, for any $z \in H_0^1(0, 1)$ it holds

$$
\int_0^1 z_x^2 \, dx \geq \mu_1^* \int_0^1 \frac{z^2}{x^2} \, dx + \mu_2^* \int_0^1 \frac{z^2}{(1 - x)^2} \, dx.
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**Proposition**

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$$
Sketch of the proof.

We rewrite $z = z_1 + z_2 + z_3$ with $z_i := z \phi_i$, $i = 1, 2, 3$ and $(\phi_i)_{i=1,2,3}$ a partition of the unity such that

$$\text{supp}(\phi_1) = (1/2, 1), \quad \text{supp}(\phi_2) = (0, 1/2), \quad \phi_3 = 1 - \phi_1 - \phi_2$$

and we apply Hardy inequality.
Proposition

For all $\gamma < 2$ and $n > 0$ there exists a positive constant $C_0 = C_0(\gamma, n)$ such that, for any $z \in H^1_0(0, 1)$ it holds

$$C_0 \int_0^1 z_x^2 \, dx + \frac{2 - \gamma}{2} \int_0^1 z^2 \, dx \geq \frac{(1 - \gamma)^2}{4} \int_0^1 \frac{z^2}{x^2} \, dx + n \int_0^1 \frac{z^2}{(1 - x)^2} \, dx.$$ 

Sketch of the proof.

$$0 \leq \int_0^1 \left( x^{\frac{2-\gamma}{2}} z_x - \frac{\gamma - 1}{2} \frac{z}{x^{\frac{\gamma}{2}}} + \frac{z}{1 - x} \right)^2 \, dx.$$ 

We expand this expression, apply integration by parts and estimate using Hölder inequality, Cauchy-Schwarz inequality and Hardy-Poincaré inequalities.

Proposition

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Sketch of the proof.

$$0 \leq \int_0^1 \left( x^{\frac{2 - \gamma}{2}} z_x - \frac{\gamma - 1}{2} \frac{z}{x^{\frac{3}{2}}} + \frac{z}{1 - x} \right)^2 \, dx.$$

We expand this expression, apply integration by parts and estimate using Hölder inequality, Cauchy-Schwarz inequality and Hardy-Poincaré inequalities.

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Well-posedness

\[
\begin{cases}
    v_t - v_{xx} - \frac{\mu_1}{x^2} v - \frac{\mu_2}{(1 - x)^2} v = h, & (x, t) \in Q \\
v(0, t) = v(1, t) = 0 \\
v(x, 0) = v_0(x)
\end{cases}
\] (3)

The solution of (3) is given in the form \( v(x, t) = e^{Mt} w(x, t) \), where \( w \) solves

\[
\begin{cases}
    w_t - w_{xx} - \frac{\mu_1}{x^2} w - \frac{\mu_2}{(1 - x)^2} w + M w = e^{-Mt} h, & (x, t) \in Q \\
w(0, t) = w(1, t) = 0 \\
w(x, 0) = w_0(x)
\end{cases}
\] (4)
Well-posedness

The solution of (3) is given in the form \( v(x, t) = e^{Mt} w(x, t) \), where \( w \) solves

\[
\begin{align*}
\begin{cases}
  w_t - w_{xx} - \frac{\mu_1}{x^2} w - \frac{\mu_2}{(1 - x)^2} w + Mw &= e^{-Mt} h, \\ w(0, t) &= w(1, t) = 0 \\ w(x, 0) &= w_0(x)
\end{cases}
\end{align*}
\]
Lets consider the operator $A : D(A) \subset L^2(0,1) \to L^2(0,1)$ defined as

$$D(A) := \left\{ w \in H^1_0(0,1) \text{ s.t. } w_{xx} + \frac{\mu_1}{x^2} w + \frac{\mu_2}{(1-x)^2} w \in L^2(0,1) \right\}$$

$$Aw := w_{xx} + \frac{\mu_1}{x^2} w + \frac{\mu_2}{(1-x)^2} w - Mw.$$

Obviously, $A$ is a self-adjoint operator; moreover, using integration by parts and the Hardy-Poincaré inequality (2) we have

$$\langle Aw, w \rangle_{L^2(0,1)} \leq 0.$$

Hence $A$ is an m-dissipative operator.
Theorem

Given $w_0 \in L^2(0, 1)$ and $h \in C([0, T]; L^2(0, 1))$, consider problem (4).

(i) If $h \equiv 0$, then the problem has a unique solution

$$w \in C^0([0, T]; D(A)) \cap C^0((0, T); D(A)) \cap C^1(0, T); L^2(0, 1)).$$

Moreover, if $u_0 \in D(A)$, then

$$w \in C^0([0, T]; D(A)) \cap C^1((0, T); L^2(0, 1)).$$

(ii) If $h \neq 0$, then the problem has a unique solution

$$w \in C^0([0, T]; D(A)) \cap C^1([0, T]; L^2(0, 1)).$$
Finally, we employ transposition techniques and we introduce the solution of (1).

**Definition**

Let $T > 0$. For any $u_0 \in L^2(0, 1)$ and $f \in L^2(0, T)$, $u \in L^2(0, T; L^2(0, 1))$ is a solution of (1) defined by transposition if it satisfies

$$
\int_Q uh \, dx \, dt = \int_0^T f(t)\phi_x(x, t)|_{x=0} \, dt + \int_0^1 \phi(x, 0)u_0(x) \, dx,
$$

where, for any $h \in L^2(0, T; L^2(0, 1))$, $\phi$ is the solution of the adjoint system

$$
\begin{align*}
\phi_t + \phi_{xx} + \frac{\mu_1}{x^2} \phi + \frac{\mu_2}{(1-x)^2} \phi &= -h(x, t), \quad (x, t) \in Q \\
\phi(0, t) = \phi(1, t) &= 0 \\
\phi(x, T) &= 0
\end{align*}
$$
Regularity of the normal derivative approaching the boundary

Given ε > 0 small enough, let \( \eta \in C^\infty(0, 1) \) be a cut-off function defined as

\[
\begin{align*}
\eta(x) &\equiv 1 \quad x \in (0, \varepsilon/2) \\
\eta(x) &\in (0, 1) \quad x \in [\varepsilon/2, \varepsilon) \\
\eta(x) &\equiv 0 \quad x \in (\varepsilon, 1)
\end{align*}
\]

and lets define \( \psi(x, t) := \eta(x)u(x, t) \). This function satisfies

\[
\begin{cases}
\psi_t - \psi_{xx} - \frac{\mu_1}{x^2}\psi = F(x, t), & (x, t) \in Q \\
\psi(0, t) = \psi(1, t) = 0 \\
\psi(x, 0) = \psi_0(x)
\end{cases}
\]

with

\[
F(x, t) := \frac{\mu_2}{(1 - x)^2}\eta u + \eta u_{xx} + \eta_{xx}u + 2\eta_x u_x.
\]
Regularity of the normal derivative approaching the boundary

Given $\varepsilon > 0$ small enough, let $\eta \in C^\infty(0, 1)$ be a cut-off function defined as

\[
\begin{align*}
\eta(x) &\equiv 1 \quad x \in (0, \varepsilon/2) \\
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\eta(x) &\equiv 0 \quad x \in (\varepsilon, 1)
\end{align*}
\]

and let's define $\psi(x, t) := \eta(x)u(x, t)$. This function satisfies

\[
\begin{aligned}
\psi_t - \psi_{xx} - \frac{\mu_1}{x^2}\psi &= F(x, t), \quad (x, t) \in Q \\
\psi(0, t) &= \psi(1, t) = 0 \\
\psi(x, 0) &= \psi_0(x)
\end{aligned}
\]

with

\[
F(x, t) := \frac{\mu_2}{(1 - x)^2}\eta u + \eta u_{xx} + \eta_{xx}u + 2\eta_x u_x.
\]
$F(x, t) := \frac{\mu^2}{(1-x)^2} \eta u + \eta u_{xx} + \eta_{xx} u + 2\eta_x u_x$

$\implies \text{supp}(F) \subset (0, \varepsilon) + \eta \in C^\infty(0, 1)$

$\implies F$ is a regular function.

The behaviour of $\psi_x$ as $x \to 0^+$ can be obtained analysing the homogeneous problem

\[
\begin{cases}
\psi_t - \psi_{xx} - \frac{\mu_1}{x^2} \psi = 0, & (x, t) \in Q \\
\psi(0, t) = \psi(1, t) = 0 \\
\psi(x, 0) = \psi_0(x)
\end{cases}
\]
The solution of (5) can be expressed in the form

$$
\psi(x, t) = \sum_{k \geq 1} \psi_k e^{-\lambda_k t} \varrho_k(x),
$$

where $\varrho_k(x)$, $k \geq 1$, is the unique solution of the eigenvalues problem

$$
\begin{cases}
-\varrho''_k(x) - \frac{\mu_1}{x^2} \varrho_k(x) = \lambda_k \varrho_k(x), & x \in (0, 1) \\
\varrho_k(0) = \varrho_k(1) = 0.
\end{cases}
$$ (6)
Solution of (6)

\[ \varrho_k(x) = c_1 x^{\frac{1}{2}} J_{\nu_1} \left( \lambda_k^{\frac{1}{2}} x \right) + c_2 x^{\frac{1}{2}} Y_{\nu_1} \left( \lambda_k^{\frac{1}{2}} x \right) \]

\[ \nu_1 := \frac{1}{2} \sqrt{1 - 4\mu_1}, \quad (c_1, c_2) \neq (0, 0) \]

Since \( J_{\nu_1}(0) = 0 \) and \( Y_{\nu_1}(0) = -\infty \), the boundary condition \( \varrho_k(0) = 0 \) is satisfied choosing \( c_1 = 1 \) and \( c_2 = 0 \). Concerning the condition at \( x = 1 \) we have

\[ \varrho_k(1) = J_{\nu_1} \left( \lambda_k^{\frac{1}{2}} \right) = 0 \quad \Rightarrow \quad \lambda_k := j_{\nu_1, k}^2 \]

Hence, finally,

Solution of (5)

\[ \psi(x, t) = \sum_{k \geq 1} \psi_k e^{-j_{\nu_1, k}^2 t} x^{\frac{1}{2}} J_{\nu_1}(j_{\nu_1, k} x). \]
\( \psi_x(x, t) = \sum_{k \geq 1} \psi_k j_{\nu_1, k} e^{-j_{\nu_1, k}^2 t} x^{\frac{1}{2}} J'_{\nu_1}(j_{\nu_1, k} x) + \frac{1}{2} \sum_{k \geq 1} \psi_k e^{-j_{\nu_1, k}^2 t} x^{-\frac{1}{2}} J_{\nu_1}(j_{\nu_1, k} x) \)

- \( \nu_1 \geq 0 \Rightarrow J_{\nu_1}(x) \sim \frac{1}{\Gamma(\nu_1 + 1)} \left( \frac{x}{2} \right)^{\nu_1} \) as \( x \to 0^+ \)
- \( \psi_x(x, t) \sim \frac{1}{2\Gamma(\nu_1 + 1)} \sum_{k \geq 1} \psi_k \left( \frac{j_{\nu_1, k}}{2} \right)^{\nu_1} e^{-j_{\nu_1, k}^2 t} x^{\nu_1 - \frac{1}{2}} \) as \( x \to 0^+ \)

Hence, we finally conclude that

**Behaviour as \( x \to 0^+ \)**

\[
\left. x^{\frac{1}{2} - \nu_1} \psi_x(x, t) \right|_{x=0} = \frac{1}{2^{\nu_1 + 1} \Gamma(\nu_1 + 1)} \sum_{k \geq 1} \psi_k j_{\nu_1, k} e^{-j_{\nu_1, k}^2 t} < +\infty.
\]

With the same arguments

**Behaviour as \( x \to 1^- \)**

\[
\left. (1 - x)^{\frac{1}{2} - \nu_2} \psi_x(x, t) \right|_{x=1} < +\infty.
\]
\[ \psi_x(x,t) = \sum_{k \geq 1} \psi_k j_{\nu_1,k} e^{-j_{\nu_1,k}^2 t} x^{\frac{1}{2}} J'_{\nu_1}(j_{\nu_1,k} x) + \frac{1}{2} \sum_{k \geq 1} \psi_k e^{-j_{\nu_1,k}^2 t} x^{-\frac{1}{2}} J_{\nu_1}(j_{\nu_1,k} x) \]

\[ \nu_1 \geq 0 \Rightarrow J_{\nu_1}(x) \sim \frac{1}{\Gamma(\nu_1 + 1)} \left( \frac{x}{2} \right)^{\nu_1} \quad \text{as} \quad x \to 0^+ \]

\[ \psi_x(x,t) \sim \frac{1}{2\Gamma(\nu_1 + 1)} \sum_{k \geq 1} \psi_k \left( \frac{j_{\nu_1,k}}{2} \right)^{\nu_1} e^{-j_{\nu_1,k}^2 t} x^{\nu_1 - \frac{1}{2}} \quad \text{as} \quad x \to 0^+ \]

Hence, we finally conclude that

**Behaviour as** \( x \to 0^+ \)

\[ x^{\frac{1}{2} - \nu_1} \psi_x(x,t) \bigg|_{x=0} = \frac{1}{2^{\nu_1+1}\Gamma(\nu_1 + 1)} \sum_{k \geq 1} \psi_k j_{\nu_1,k} e^{-j_{\nu_1,k}^2 t} < +\infty. \]

With the same arguments

**Behaviour as** \( x \to 1^- \)

\[ (1 - x)^{\frac{1}{2} - \nu_2} \psi_x(x,t) \bigg|_{x=1} < +\infty. \]
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Hilbert Uniqueness Method

**Adjoint system**

\[
\begin{aligned}
&v_t + v_{xx} + \frac{\mu_1}{x^2} v + \frac{\mu_2}{(1 - x)^2} v = 0 \quad (x, t) \in Q \\
&v(0, t) = v(1, t) = 0 \\
&v(x, T) = v_T(x)
\end{aligned}
\]  

(7)

**Theorem (Observability inequality)**

Let \( T > 0 \). For any \( v_T \in L^2(0, 1) \) the solution of (7) satisfies

\[
\int_0^1 v(x, 0)^2 \, dx \leq C \int_0^T \left[ x^{2\lambda_1} v_x^2 \right]_{x=0}^T \, dt,
\]

(8)

with

\[
\lambda_1 := \frac{1}{2} \left( 1 - \sqrt{1 - 4\mu_1} \right).
\]
Carleman estimate

**Theorem**

There exists a constant $R_0 > 0$ such that, for all $R \geq R_0$, every solution $v$ of (7) satisfies

$$
R^3 C_1 \int_Q \theta^3 \left[ x^{6\lambda_1} (1 - x)^5 \right] v^2 e^{-2R\sigma} \, dx \, dt + RC_2 \int_Q \theta \left( \frac{v^2}{(1 - x)^2} e^{-2R\sigma} \right) \, dx \, dt \\
+ RC_3 \int_Q \theta \frac{v^2}{x^{1-2\lambda_1}} e^{-2R\sigma} \, dx \, dt + RC_4 \int_Q \theta \left[ x^{2\lambda_1} (1 - x) \right] v_x^2 e^{-2R\sigma} \, dx \, dt \\
\leq RC_5 \int_0^T \theta \left[ x^{2\lambda_1} v_x^2 \right] \bigg|_{x=0} \, dt,
$$

where $C_i, \ i = 1, \ldots, 5$ are positive constants and, for $\varpi, \beta > 0$, the weight function $\sigma$ is defined as $\sigma(x, t) := \theta(t)p(x)$ with

$$
\theta(t) := \left( \frac{1}{t(T-t)} \right)^3, \ p(x) := \varpi + \frac{\beta x^{2\lambda_1+1}}{2\lambda_1 + 1} \left[ 1 - \frac{2\lambda_1 + 1}{\lambda_1 + 1} x + \frac{2\lambda_1 + 1}{2\lambda_1 + 3} x^2 \right].
$$
Motivation for the choice of the weight $\sigma$

The function $p$ in the weight $\sigma$ in the Carleman estimate (9) is chosen starting from its first derivative, in order to obtain the boundary term

$$C_5 \int_0^T \theta \left[ x^{2\lambda_1} v_x^2 \right]_{x=0} \, dt$$

For all $R > 0$ we define

$$z(x, t) := v(x, t) e^{-R\sigma(x, t)}$$

This $z$ satisfies

$$\begin{cases} 
P^+_R z + P^-_R z = 0 \\
z(0, t) = z(1, t) = 0, \quad z(x, 0) = z(x, T) = 0
\end{cases}$$

with

$$P^+_R z := z_{xx} + z(R\sigma_t + R^2 \sigma_x^2) + \frac{\mu_1}{x^2} z + \frac{\mu_2}{(1-x)^2} z, \quad P^-_R z := z_t + R\sigma_{xx} z + 2R\sigma_x z_x$$
Motivation for the choice of the weight $\sigma$

The function $p$ in the weight $\sigma$ in the Carleman estimate (9) is chosen starting from its first derivative, in order to obtain the boundary term

$$C_5 \int_0^T \theta \left[ x^{2\lambda_1} v_x^2 \right]_{x=0}^1 dt$$

For all $R > 0$ we define

$$z(x, t) := v(x, t)e^{-R\sigma(x, t)}$$

This $z$ satisfies

$$\begin{cases} 
  P_R^+ z + P_R^- z = 0 \\
  z(0, t) = z(1, t) = 0, \quad z(x, 0) = z(x, T) = 0
\end{cases}$$

with

$$P_R^+ z := z_{xx} + z(R\sigma_t + R^2\sigma_x^2) + \frac{\mu_1}{x^2} z + \frac{\mu_2}{(1 - x)^2} z, \quad P_R^- z := z_t + R\sigma_{xx} z + 2R\sigma_x z_x$$
\[ \langle P_R^+ z; P_R^- z \rangle_{L^2(0,1)} \leq 0 \]

\[ \langle P_R^+ z; P_R^- z \rangle_{L^2(0,1)} = DT + BT \]

with

\[ BT = R \int_0^T \theta(t) \left\{ p'(x) z_x^2 + p''(x) z z_x + p'(x) \left( \frac{\mu_1}{x^2} + \frac{\mu_2}{(1-x)^2} \right) z^2 \right\} \bigg|_{x=0}^{x=1} dt \]

\[ p'(x) = \beta x^{2\lambda_1}(1-x)^2 \Rightarrow \int_0^T \theta \left[ x^{2\lambda_1} v_x^2 \right] \bigg|_{x=0} \ dt \]

Remark

\[ p'(x) = -\beta x^2(1-x)^{2\lambda_2} \Rightarrow \int_0^T \theta \left[ (1-x)^{2\lambda_2} v_x^2 \right] \bigg|_{x=1} \ dt \]
\[ \langle P^+_R z; P^-_R z \rangle_{L^2(0,1)} \leq 0 \]

\[ \langle P^+_R z; P^-_R z \rangle_{L^2(0,1)} = DT + BT \]

with

\[ BT = R \int_0^T \theta(t) \left\{ p'(x)z_x^2 + p''(x)zz_x + p'(x) \left( \frac{\mu_1}{x^2} + \frac{\mu_2}{(1-x)^2} \right) z^2 \right\} \bigg|_{x=0}^{x=1} dt \]

\[ p'(x) = \beta x^{2\lambda_1}(1-x)^2 \Rightarrow \int_0^T \theta \left[ x^{2\lambda_1} v_x^2 \right]_{x=0}^{x=1} dt \]

**Remark**

\[ p'(x) = -\beta x^2(1-x)^{2\lambda_2} \Rightarrow \int_0^T \theta \left[ (1-x)^{2\lambda_2} v_x^2 \right]_{x=1}^{x=0} dt \]
\[
\langle P_R^+ z; P_R^- z \rangle_{L^2(0,1)} \leq 0
\]

\[
\langle P_R^+ z; P_R^- z \rangle_{L^2(0,1)} = DT + BT
\]

with

\[
BT = R \int_0^T \theta(t) \left\{ p'(x)z_x^2 + p''(x)zz_x + p'(x) \left( \frac{\mu_1}{x^2} + \frac{\mu_2}{(1-x)^2} \right) z^2 \right\} \bigg|_{x=0}^{x=1} dt
\]

\[
p'(x) = \beta x^{2\lambda_1} (1-x)^2 \Rightarrow \int_0^T \theta \left[ x^{2\lambda_1} v_x^2 \right] \bigg|_{x=0} \ dt
\]

Remark

\[
p'(x) = -\beta x^2 (1-x)^{2\lambda_2} \Rightarrow \int_0^T \theta \left[ (1-x)^{2\lambda_2} v_x^2 \right] \bigg|_{x=1} \ dt
\]
Proof of the observability inequality

From the Carleman estimate

\[
R^3 C_1 \int_Q \theta^3 \left[ x^{6\lambda_1} (1 - x)^5 \right] v^2 e^{-2R\sigma} \, dxdt + RC_2 \int_Q \theta \frac{v^2}{(1 - x)^2} e^{-2R\sigma} \, dxdt \\
+ RC_3 \int_Q \theta \frac{v^2}{x^{1-2\lambda_1}} e^{-2R\sigma} \, dxdt + RC_4 \int_Q \theta \left[ x^{2\lambda_1} (1 - x) \right] v_x^2 e^{-2R\sigma} \, dxdt \\
\leq RC_5 \int_0^T \theta \left[ x^{2\lambda_1} v_x^2 \right] \bigg|_{x=0} \, dt
\]

we have, in particular

\[
\int_Q \theta \frac{v^2}{x^{1-2\lambda_1}} e^{-2R\sigma} \, dxdt \leq C_1 \int_0^T \theta \left[ x^{2\lambda_1} v_x^2 \right] \bigg|_{x=0} \, dt
\]
Proof of the observability inequality

There exist two positive constants $P_1$ and $P_2$ such that

$$\theta e^{-2R\sigma} \geq P_1 \text{ in } (0, 1) \times \left[ \frac{T}{4}, \frac{3T}{4} \right], \quad \theta e^{-2R\varpi\theta} \leq P_2 \text{ in } (0, T);$$

hence

$$\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_0^1 v^2 \, dx \, dt \leq C_2 \int_0^T \theta \left[ x^{2\lambda_1} v_x^2 \right]_{x=0} \, dt.$$

Using (2)

$$\frac{d}{dt} \int_0^1 v^2 \, dx \geq -M \int_0^1 v^2 \, dx \Rightarrow \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_0^1 v^2 \, dx \geq \frac{T}{2} e^{-2MT} \int_0^1 v(x, 0)^2 \, dx$$

hence

$$\int_0^1 v(x, 0)^2 \, dx \, dt \leq \frac{2}{T} e^{2MT} C_2 \int_0^T \theta \left[ x^{2\lambda_1} v_x^2 \right]_{x=0} \, dt.$$
Proof of the observability inequality

- There exist two positive constants $P_1$ and $P_2$ such that
  \[
  \frac{\theta e^{-2R\sigma}}{x^{1-2\lambda_1}} \geq P_1 \quad \text{in} \quad (0, 1) \times \left[ \frac{T}{4}, \frac{3T}{4} \right], \quad \theta e^{-2R\omega} \leq P_2 \quad \text{in} \quad (0, T);
  \]
  hence
  \[
  \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{0}^{1} v^2 \, dx \, dt \leq C_2 \int_{0}^{T} \theta \left[ x^{2\lambda_1} v_x^2 \right] \bigg|_{x=0} \, dt.
  \]

- Using (2)
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  \]
  hence
  \[
  \int_{0}^{1} v(x, 0)^2 \, dx \, dt \leq \frac{2}{T} e^{2MT} C_2 \int_{0}^{T} \theta \left[ x^{2\lambda_1} v_x^2 \right] \bigg|_{x=0} \, dt.
  \]
1 Introduction

2 Hardy-Poincaré inequalities

3 Well-posedness

4 Controllability from the boundary

5 Open problems
Multi-dimensional problems with distance to the boundary

Let $T > 0$, $\mu \leq 1/4$, and let $\Omega \subset \mathbb{R}^N$ be a regular domain; moreover, let define $d(x) := \text{dist}(x, \partial \Omega)$. Consider the problem

\[
\begin{align*}
    u_t - \Delta u - \frac{\mu}{d^2} u &= 0, & (x, t) \in \Omega \times (0, T) := Q \\
    u &= 0, & (x, t) \in \partial \Omega \times (0, T) := \Sigma \\
    u(x, 0) &= u_0(x)
\end{align*}
\] 

We want to prove a controllability result for (10).
Proposition

Let $\Omega \subset \mathbb{R}^N$ be a bounded regular domain. There exists a constant $\mu \leq 1/4$ such that for any $u \in H^1_0(\Omega)$ it holds

$$\int_{\Omega} |\nabla u|^2 \, dx \geq \mu \int_{\Omega} \frac{u^2}{d^2} \, dx. \quad (11)$$

H. Brezis and M. Marcus - Hardy's inequalities revisited (1997)

Applying (11) we obtain the well-posedness of our problem by classical semigroup theory
Controlability

• INTERNAL CONTROLLABILITY
  We try to adapt the arguments by C. Cazacu for obtaining a Carleman estimate for (10) - work in progress

• BOUNDARY CONTROLLABILITY
  We firstly need to identify the behaviour of the normal derivative of the solution approaching the boundary.

Claim:

$$d^\alpha \left( \frac{\partial u}{\partial \nu} \right)^2 \leq +\infty, \text{ with } \alpha \text{ depending on } \mu \text{ and, possibly, on } \Omega.$$
Controlability

- **INTERNAL CONTROLLABILITY**
  We try to adapt the arguments by C. Cazacu for obtaining a Carleman estimate for (10) - work in progress

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  We firstly need to identify the behaviour of the normal derivative of the solution approaching the boundary.

Claim:

\[ d^\alpha \left( \frac{\partial u}{\partial \nu} \right)^2 \bigg|_\Sigma \leq +\infty, \text{ with } \alpha \text{ depending on } \mu \text{ and, possibly, on } \Omega. \]
Multi-dimensional problem on the unit sphere

Let $T > 0$, $\mu \leq 1/4$, and let $B^N(1)$ be the unit ball in $\mathbb{R}^N$; consider the problem

\[
\begin{cases}
  u_t - \Delta u - \frac{\mu}{d^2} u = 0, & (x, t) \in B^N(1) \times (0, T) := Q \\
  u = f, & (x, t) \in \partial B^N(1) \times (0, T) := \Sigma \\
  u(x, 0) = u_0(x)
\end{cases}
\]

(12)

We want to prove a boundary controllability result for (12).
Hilbert Uniqueness Method

We need to prove an observability inequality for the adjoint problem

\[
\begin{aligned}
\psi_t + \Delta \psi + \frac{\mu}{d^2} \psi &= 0, \quad (x, t) \in B(N)(1) \times (0, T) := Q \\
\psi &= 0, \quad (x, t) \in \partial B(N)(1) \times (0, T) := \Sigma \\
\psi(x, T) &= \psi_T(x)
\end{aligned}
\]

We change variables, going in spherical coordinates
\[ \Phi : \mathbb{R}^N \setminus \{0\} \rightarrow (0, +\infty) \times \mathbb{S}^{N-1} \]

\[ x \mapsto (r, \sigma) := \left( |x|, \frac{x}{|x|} \right) \]

- \[ \Delta \mapsto \frac{\partial^2}{\partial r^2} + \frac{N - 1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_\sigma \]
- \[ w(r, \sigma, t) := v(r\sigma, t) \]

We obtain the following equation on \((r, \sigma, t) \in (0, 1) \times \mathbb{S}^{N-1} \times (0, T)\)

\[ \begin{aligned}
    w_t + w_{rr} + \frac{N - 1}{r} w_r + \frac{1}{r^2} \Delta_\sigma w + \frac{\mu}{(1 - r)^2} w &= 0, \\
    w(1, \sigma, t) &= 0 \\
    w(r, \sigma, T) &= w_T(r, \sigma)
\end{aligned} \]
• \(\{\lambda_k\}_{k\geq 0}\): eigenvalues of \(\Delta_\sigma\) on \(B^N(1)\) with DBC

• \(\{\Lambda_k\}_{k\geq 0}\): eigenspaces associated; \(\ell_k := \text{dim}(\Lambda_k)\)

\[
L^2 (S^{N-1}) = \bigoplus_{k\geq 0} \Lambda_k
\]

There exists an orthonormal basis of \(L^2 (S^{N-1})\), \(\{f^{k\ell}\}_{1 \leq \ell \leq \ell_k, k\geq 0}\), such that

\[
-\Delta_\sigma f^{k\ell} = \lambda_k f^{k\ell}
\]

\[
\lambda_k = k(N + K - 2)
\]
$$w(r, \sigma, t) = \sum_{k, \ell} \psi^{k\ell}_k(r, t)f^{k\ell}(\sigma)$$

For any $k \geq 0$, and for any $1 \leq \ell \leq \ell_k$ we obtain the following equation on $(r, t) \in (0, 1) \times (0, T)$

$$\begin{cases} 
\psi_{t}^{k\ell} + \psi_{rr}^{k\ell} + \frac{N - 1}{r} \psi_{r}^{k\ell} - \frac{\lambda_k}{r^2} \psi^{k\ell} + \frac{\mu}{(1 - r)^2} \psi^{k\ell} = 0 \\
\psi^{k\ell}(1, \sigma, t) = 0 \\
\psi^{k\ell}(r, T) = \psi^{k\ell}_T(r)
\end{cases}$$
\[
\phi^{k\ell}(r, t) = r^{\frac{N-1}{2}} \psi^{k\ell}(r, t)
\]

\[
\begin{aligned}
\phi_t^{k\ell} + \phi_{r r}^{k\ell} + \frac{\lambda_{kN}}{r^2} \phi^{k\ell} + \frac{\mu}{(1-r)^2} \phi^{k\ell} &= 0, \quad (r, t) \in (0, 1) \times (0, T) \\
\phi^{k\ell}(0, \sigma, t)\phi^{k\ell}(1, \sigma, t) &= 0 \\
\phi^{k\ell}(r, T) &= \phi^T_T(r)
\end{aligned}
\]

\[
\lambda_{kN} := \frac{(1-N)(N-3)}{4} - \lambda_k
\]

**Remark**

*By definition of \( \lambda_k \), for any \( N \geq 1 \) we have*

\[
\lambda_{kN} \leq \frac{(1-N)(N-3)}{4} \leq \frac{1}{4}
\]
Observability inequality

\[ \int_0^1 \phi^{k\ell}(r, 0)^2 \, dr \leq C_1 \int_0^T \left[ (1 - r)^\alpha \phi^{k\ell}_r \right]^2 \bigg|_{r=1} \, dt \]

\[ \alpha = \frac{1}{2} \left( 1 - \sqrt{1 - 4\mu} \right) \]

Applying the inverse change of variables, we finally obtain the following inequality for the original problem

Observability inequality on the disc

\[ \int_{B^N(1)} v(x, 0)^2 \, dx \leq C_2 \int_\Sigma d^{2\alpha} \left| \frac{\partial v}{\partial \nu} \right|^2 \, ds \, dt \]
Observability inequality

\[ \int_{0}^{1} \phi^{k\ell}(r,0)^2 \, dr \leq C_1 \int_{0}^{T} \left[ (1 - r)^{\alpha} \phi^{k\ell}_r \right]^2 \bigg|_{r=1} \, dt \]

\[ \alpha = \frac{1}{2} \left( 1 - \sqrt{1 - 4\mu} \right) \]

Applying the inverse change of variables, we finally obtain the following inequality for the original problem.

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\[ \int_{B^N(1)} v(x,0)^2 \, dx \leq C_2 \int_{\Sigma} d^{2\alpha} \left| \frac{\partial v}{\partial \nu} \right|^2 \, ds dt \]
THANKS FOR YOUR ATTENTION!