Concentrated waves for the finite-difference and discontinuous Galerkin semi-discretizations of the wave equation

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joint work with Enrique Zuazua
Motivation: Control of the wave equation
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2 Finite-differences case
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Discontinuous Galerkin (DG) methods

Conclusions
Motivation: Control of the wave equation

The Cauchy problem for the $1-d$ wave equation - well posed in the energy space $\dot{H}^1 \times L^2(\mathbb{R})$:

\[
\begin{aligned}
  &\left\{ \begin{array}{l}
    u_{tt}(x, t) - u_{xx}(x, t) = 0, \\
    u(x, 0) = u^0(x), \quad u_t(x, 0) = u^1(x),
  \end{array} \right. \\
  &x \in \mathbb{R}, \quad t > 0
\end{aligned}
\]  

(1)

Conservation of the total energy: $E(u^0, u^1) = \frac{1}{2} \int_{\mathbb{R}} (|u_x(x, t)|^2 + |u_t(x, t)|^2) \, dx$.

Observability property

$\forall T > 2$ (characteristic time), $\exists C(T) > 0$ s.t. $\forall (u^0, u^1)$ of finite energy:

\[
E(u^0, u^1) \leq C(T) \int_0^T E_\Omega(u^0, u^1, t) \, dt,
\]

(2)

where $\Omega = \mathbb{R} \setminus (-1, 1)$ and $E_\Omega(u^0, u^1, t)$ is the energy concentrated in $\Omega$,

\[
E_\Omega(u^0, u^1, t) = \frac{1}{2} \int_{\Omega} (|u_x(x, t)|^2 + |u_t(x, t)|^2) \, dx.
\]

**Applications:** boundary control, stabilization, inverse problems...

The observability property - equivalent to the controllability of the wave equation with forces acting on \( \Omega = \mathbb{R} \setminus (-1, 1) \times (0, T) \).

Using the Hilbert Uniqueness Method (HUM), the observability inequality (2) is equivalent to the fact that for each initial data \((y^0, y^1) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})\) and \(T > T^* = 2\), there exists a control function \(f \in L^\infty(0, T, L^2(\Omega))\) such that the solution of the inhomogeneous problem

\[
\begin{aligned}
    y_{tt}(x, t) - y_{xx}(x, t) &= f(x, t)\chi_{\Omega}(x), \\
    y(x, 0) &= y^0(x), \\
    y_t(x, 0) &= y^1(x), \\
    x &\in \mathbb{R}, \\
    t &\in (0, T]
\end{aligned}
\]

satisfies the equilibrium condition at time \(T\):

\[
y(x, T) = y'(x, T) = 0, \quad \forall x \in \mathbb{R}.
\]

\(\chi_{\Omega} = \text{the characteristic function of the set } \Omega.\)

---

Aim: analyze the observability property under numerical discretizations

Actually, it is by now well known that, for classical finite-difference (FD) and finite-element discretizations, the observability constant (OC) diverges because of the presence of high frequency spurious numerical solutions for which the group velocity vanishes.

In this work:

- For the FD scheme, we construct examples of initial data concentrated in the Fourier space for which the OC diverges polynomially.

For basic results in the field, see Zuazua, Propagation, observation, control and numerical approximations of waves approximated by FD method, SIAM Review, 2005.

Micu, S., Uniform boundary controllability of a semi-discrete 1-D wave equation, 2002.
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- We show that the same negative results have to be expected.
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In this work:

- For the FD scheme, we construct examples of initial data concentrated in the Fourier space for which the OC diverges polynomially.
- We perform a Gaussian beam construction at the semi-discrete level showing the existence of exponentially concentrated waves.
- We perform the Fourier analysis of the Discontinuous Galerkin Methods for the wave equation.
- We show that the same negative results have to be expected.
- Our analysis indicates how filtering techniques should be designed to avoid or reduce the action of the high frequencies.

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4 Conclusions
Preliminaries on the FD semi-discretization

Cauchy problem for the FD semi-discretization:

\[
\begin{cases}
    u_j''(t) - \partial_h^2 u_j(t) = 0, & j \in \mathbb{Z}, t > 0 \\
    u_j(0) = u_j^0, u_j'(0) = u_j^1, & j \in \mathbb{Z}.
\end{cases}
\]

Conservation of the total energy: 
\[ E_h(\overrightarrow{u}^{h,0}, \overrightarrow{u}^{h,1}) = \frac{h}{2} \sum_{j \in \mathbb{Z}} \left( |D_h^1 u_j^h(t)|^2 + |\partial_t u_j^h(t)|^2 \right). \]

But 
\[ \inf_{E_h(\overrightarrow{u}^{h,0}, \overrightarrow{u}^{h,1}) = 1} \int_0^T E_{\Omega,h}(\overrightarrow{u}^{h,0}, \overrightarrow{u}^{h,1}, t) \, dt \to 0, \text{ when } h \to 0. \]

observed that a discrete wave packet for the transport equation moves at the speed predicted by the group velocity \( \frac{\partial \omega_h}{\partial \xi} \):
Importance of the group velocity: Oscillatory integral of the form on \( \mathbb{R} \):

\[
|u(x, t)| = \left| \int_{\mathbb{R}} \exp(-\alpha(\xi - \xi_0)^2) \exp(it\omega(\xi) + i\xi x) d\xi \right| = f(t, \alpha) \left| \exp\left( - \frac{(x + t\omega'(\xi_0))^2}{4(\alpha - \frac{it\omega''(\xi_0)}{2})} \right) \right|,
\]

\[
\omega(\xi) = \omega(\xi_0) + (\xi - \xi_0)\omega'(\xi_0) + \frac{\omega''(\xi_0)}{2} (\xi - \xi_0)^2, \quad f(t, \alpha) = \Gamma(1/2) \left| \left( \alpha - \frac{it\omega''(\xi_0)}{2} \right)^{-1/2} \right|
\]

The Gaussian wave packets propagate along the rays of GO: \( x_h^+(x, t, \xi_0) = x + t \frac{\partial \omega_h}{\partial \xi}(\xi_0) \).

Figure: Dispersion relation for the FD scheme // and group velocity.

Figure: Waves traveling at velocity 1 for the continuous case // and at a very low speed for the FD scheme.
Theorem (Polynomial divergence of the observability constant)

\[ \forall k \in \mathbb{N}, \forall \hat{\phi} \in C^k_c(-1, 1), T \text{ finite and } \overrightarrow{u}^h,i = \overrightarrow{u}^h,i(\hat{\phi}) \text{ (dependence to made precise later), the following lower bound occurs:} \]

\[ E_h(\overrightarrow{u}^h,0, \overrightarrow{u}^h,1) \left\{ T \sum_{|x_j| > 1} \right\} \frac{1}{h} \left| \partial_t u_j^h(t) \right|^2 dt = C(k, \hat{\phi}, T) h^{-k+1/2}. \]

- **Rescaling.** \( u_j^h(t) = u_j^1(t/h) \):
  \[ E_h(\overrightarrow{u}^h,0, \overrightarrow{u}^h,1) \left\{ T \sum_{|x_j| > 1} \right\} \frac{1}{h} \left| \partial_t u_j^h(t) \right|^2 dt = E_1(\overrightarrow{u}^1,0, \overrightarrow{u}^1,1) \left\{ T/h \sum_{|j| > 1/h} \right\} \frac{1}{h} \left| \partial_t u_j^1(t) \right|^2 dt \]

- **Initial data.** \( \hat{\phi} \in C^k_c(-1, 1) \):
  \( \hat{u}^1,0_\varepsilon(\xi) = \frac{1}{\varepsilon^{1/2}} \hat{\phi} \left( \frac{\xi - \xi_0}{\varepsilon} \right), \quad \hat{u}^1,1_\varepsilon(\xi) = i \omega_1(\xi) \hat{u}^1,0_\varepsilon(\xi) \)

- **Time derivative of the solution.** \( \partial_t u_j^1(t) = \frac{1}{2\pi} \int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon} \hat{u}^1,0_\varepsilon(\xi) \exp(it\omega_1(\xi))i \omega_1(\xi) \exp(i\xi j) d\xi \)

- **Integrate \( k \) times by parts.**
  \[ |\partial_t u_j^1(t)| \leq \frac{1}{2\pi} \frac{1}{|j|^k} \sum_{m=0}^k \sum_{n=0}^{k-m} \binom{k}{m} \binom{k-m}{n} \int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon} |\partial^m \hat{u}^1,0_\varepsilon(\xi)||\partial^n \omega_1(\xi)||\partial^{k-m-n} \exp(it\omega_1(\xi))| d\xi \]
Derivative of the exponential:
\[ \partial^{k-m-n} \exp(it\omega_1(\xi)) = \exp(it\omega_1(\xi)) \sum_{\vec{v} \in A_{k-m-n}} a_{-\vec{v}}(it)|\vec{v}|^{k-m-n} \prod_{l=1}^{\infty} (\partial^{l}\omega_1(\xi))^{v_l}, \]
where
\[ A_l = \{ \vec{v} = (v_1, \ldots, v_l) : v_1 + \cdots + lv_l = l \}. \]
Derivatives of the dispersion relation: \( \partial^n \omega_1(\xi) \sim \omega_1(\xi) \) if \( n \) even, \( \partial^n \omega_1(\xi) \sim \omega'(\xi) \) if \( n \) odd.
Take square in (*) and add in \(|j| > 1/h\).
\[ \sum_{|j|>1/h} \left| \partial_t u_j(t) \right|^2 \leq C(T, k, \hat{\phi}) \sum_{m=0}^{k} \sum_{n=0}^{k-m} \sum_{\vec{v} \in A_{k-m-n}} h^{2k-1-2|\vec{v}|} |\vec{v}|^2 \delta_1(n) + 2|\vec{v}|_o - (2m-1). \]
Optimal choice of \( \epsilon \). \( \epsilon = h^\alpha \):
\[ 2k - 1 - 2|\vec{v}| + \alpha(2\delta_1(n) + 2|\vec{v}|_o - (2m-1)) > 0. \]
\( \alpha = 1/2 \) - optimal:
\[ |\vec{v}| + \frac{1}{2}(2\delta_1(n) + 2|\vec{v}|_o - (2m-1)) \geq k - \frac{1}{2}. \]

Figure: From the left to the right: \( C^1 \), \( C^2 \) and \( C^3 \) - spline functions.
Finite-differences case

Waves concentrated on frequencies where the group velocity vanishes

Derivative of the exponential:

\[ \partial^{k-m-n} \exp(it\omega_1(\xi)) = \exp(it\omega_1(\xi)) \sum_{\vec{v} \in A_{k-m-n}} a_{\vec{v}}(it)|\vec{v}| \prod_{l=1}^{k-m-n} (\partial^l\omega_1(\xi))^{n_l} , \text{ where} \]

\[ A_l = \{ \vec{v} = (v_1, \cdots, v_l) : v_1 + \cdots + lv_l = l \}. \]

Derivatives of the dispersion relation: \( \partial^n\omega_1(\xi) \sim \omega_1(\xi) \) if \( n \) even, \( \partial^n\omega_1(\xi) \sim \omega'(\xi) \) if \( n \) odd.

Take square in (*) and add in \( |j| > 1/h \).

\[ \sum_{|j|>1/h} |\partial_t u_j(t)|^2 \leq C(T, k, \hat{\phi}) \sum_{m=0}^{k} \sum_{n=0}^{k-m} \sum_{\vec{v} \in A_{k-m-n}} h^{2k-1-2|\vec{v}|} |\vec{v}|^{2\delta_1(n)+2|\vec{v}|_o-(2m-1)}. \]

Optimal choice of \( \epsilon \): \( \epsilon = h^{\alpha} \):

\[ 2k - 1 - 2|\vec{v}| + \alpha(2\delta_1(n) + 2|\vec{v}|_o - (2m - 1)) > 0. \]

\[ \alpha = 1/2 - \text{optimal:} \quad |\vec{v}| + \frac{1}{2}(2\delta_1(n) + 2|\vec{v}|_o - (2m - 1)) \geq k - \frac{1}{2}. \]

Figure: The observability constant divided by \( N^{k-1/2} \) for \( k = 1, 2, 3. \)
**Gaussian beams (GB)**

For the continuous case see:

- J. Ralston, *GB and the propagation of singularities*, 1982

<table>
<thead>
<tr>
<th>Ansatz</th>
<th>Continuous</th>
<th>Semi-discrete</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$v_k(x, t) = a(x, t) \exp(ik\phi(x, t))$</td>
<td>$v_j^h(t) = h^{3/4} a_j(t) \exp(i\phi_j(t)/h)$</td>
</tr>
<tr>
<td>Phase</td>
<td>$\phi(x, t) = \xi_0(x - x(t)) + \frac{\beta(t)}{2} (x - x(t))^2$</td>
<td>$\phi_j(t) = t\omega_1(\xi_0) + \xi_0 x_j + \frac{\beta(t)}{2} (x_j - x(t))^2$</td>
</tr>
<tr>
<td>Eikonal</td>
<td>$</td>
<td>\phi_t(x(t), t)</td>
</tr>
<tr>
<td>$\beta(t)$</td>
<td>constant $\beta_0$, $\text{Im}(\beta_0) &gt; 0$</td>
<td>$\beta(t) = \frac{1}{\beta_0 + \frac{\omega_1(\xi_0)}{4} t}$</td>
</tr>
</tbody>
</table>

**Theorem**

- $\exists \phi_j(t)$ and $a_j(t)$ s.t as $h \to 0$: $\|\Box_h \overrightarrow{v}^h(t)\|_{L_2}^2 = h \sum_{j \in \mathbb{Z}} |\Box_h v_j^h(t)|^2 = O(h)$.
- $\forall t \geq 0$ and $a(x, t)$ and interpolation of $a_j(t)$, the following limit hold

$$
\lim_{h \to 0} E_h(\overrightarrow{v}^{h,0}, \overrightarrow{v}^{h,1}, t) = \frac{|a(x(t), t)|^2 \omega_1^2(\xi_0)}{\sqrt{\text{Im}(\beta(t))}} \Gamma(1/2).
$$

- For $r = h^\alpha$, $C > 0$, $1 - 2\alpha > 0$,

$$
\sup_{t \in [0, T]} \left[ h \sum_{|x_j - x(t)| \geq r} (|v_j'(t)|^2 + |\partial_h^+ v_j(t)|^2) \right] \leq C \exp \left( - \frac{\text{min} \text{Im}(\beta(t))}{8h^{1-2\alpha}} \right).
$$
Write \( \partial^2 u_j(t) - \partial^2 h u_j(t) = 0 \) in even \( v_j(t) = u_{2j}(t) \) and odd \( w_j(t) = u_{2j+1}(t) \) components:

\[
\begin{align*}
\partial^2 v_j(t) - \frac{w_j(t) - 2v_j(t) + w_{j-1}(t)}{h^2} &= 0 \\
\partial^2 w_j(t) - \frac{v_{j+1}(t) - 2w_j(t) + v_j(t)}{h^2} &= 0.
\end{align*}
\]

Relation between \( \hat{u}^h \) and \( \hat{\nu}^2h, \hat{\omega}^2h \):

\[
\hat{u}^h(\xi, t) = \frac{\hat{\nu}^2h(\xi, t) + \exp(-i\xi h)\hat{\omega}^2h(\xi, t)}{2}.
\]

In SDFT’s:

\[
\begin{pmatrix}
\hat{\nu}^2h_{tt}(\xi, t) \\
\hat{\omega}^2h_{tt}(\xi, t)
\end{pmatrix}
+ A_h(\xi)
\begin{pmatrix}
\hat{\nu}^2h(\xi, t) \\
\hat{\omega}^2h(\xi, t)
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]

- **Low frequency eigenvalue**: \( \Lambda^lo_h(\xi) = \frac{4}{h^2} \sin^2\left(\frac{\xi h}{2}\right) \), \( P^lo_h(\xi) = \begin{pmatrix}
\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} \exp(i\xi h)
\end{pmatrix} \)

- **High frequency eigenvalue**: \( \Lambda^hi_h(\xi) = \frac{4}{h^2} \cos^2\left(\frac{\xi h}{2}\right) \), \( P^hi_h(\xi) = \begin{pmatrix}
-\frac{1}{\sqrt{2}} \exp(-i\xi h) \\
\frac{1}{\sqrt{2}}
\end{pmatrix} \)

\[\text{Aurora Marica (BCAM)}\]

\[\text{Waves propagation & Numerics}\]

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# Discontinuous Galerkin (DG) methods

## Contents

1. **Motivation: Control of the wave equation**
2. **Finite-differences case**
3. **Discontinuous Galerkin (DG) methods**
4. **Conclusions**
Some historical considerations on the DG methods:

- Introduced for purely hyperbolic problems (Reed-Hill 1973)
- Used for second order elliptic (Arnold-Douglas-Dupont school, mid 70's).
- Abandoned because of the big size of the final system.
- Great revival some 10 years ago (mainly by Cockburn B.-Shu C.-W.) also for applications to problems where the elliptic part is present but it is not dominant. Also Arnold, Brezzi, Marini 2000-2002 - applicability in preconditioning mechanisms and domain decomposition techniques.


- **Not really needed** for purely elliptic problems, but for those where the elliptic part is present, but not dominant (strongly advection-dominated equations)
- Work very well for purely hyperbolic problems, where the continuous solutions are not regular, even discontinuous.

References for DG methods for elliptic problems:

- Brezzi, Manzini, Marini, Pietra, Russo, *Discontinuous Galerkin approximations for elliptic problems*, 2000
- Brezzi, Cockburn, Marini, Süli, *Stabilization mechanisms in Discontinuous Galerkin finite element methods*, 2006
What are the discontinuous Galerkin (DG) methods?

- **Finite elements methods** for approximating the solutions of PDE's
- Based on a partition \( \mathcal{T}_h \) of the spatial domain \( \Omega \) into elements of size \( h \), normally triangles or quadrilaterals
- While the **classical FEM's** use piecewise polynomials continuous along the interfaces, the **DG methods** use piecewise polynomial functions discontinuous along the interfaces

\[
V_h \not\subset \mathcal{V} \quad (= H^1_0(\Omega) \text{ for elliptic problems}) - \text{nonconforming methods}
\]

![Figure: A 2 \( - d \) discontinuous Galerkin function](image-url)
Advantages with respect to the classical FEM's:
- A wide range of PDE's (hyperbolic, elliptic, parabolic)
- Weak approximations of the boundary conditions
- Non-matching (non-overlapping) grids
- Suitable for $hp$-adaptivity
- Flexible choice of the approximation spaces
- Conservative numerical schemes

Drawbacks with respect to the classical FEM's:
- More degrees of freedom. The resulting systems of equations are more difficult to solve. But mass matrices are block-diagonal, easier to invert
- Need for some stabilization mechanisms.

Figure: **Left:** non-overlapping triangulation; **Right:** $h$-adaptivity
The simplest version: the 1D wave equation, uniform grid of size $h > 0 - x_i = hi -$, piecewise linear, but not necessarily continuous on the mesh points deformations.

Figure: Basis functions: $\phi_i$ (left) and $\tilde{\phi}_i$ (right)

Figure: Decomposition of a DG deformation into its continuous and jump components.
Discontinuous Galerkin (DG) methods

Variational formulation

**Relevant notation:**
- **Average:** \( \{ f \}(x_i) = \frac{f(x_i^+) + f(x_i^-)}{2} \)
- **Jump:** \([ f](x_i) = f(x_i^-) - f(x_i^+)\)
- \( V_h = \{ v \in L^2(\mathbb{R}) \mid v|_{(x_j, x_{j+1})} \in P_1, \| v \|_h < \infty \} \)
- \( \| v \|^2_h = \sum_{j \in \mathbb{Z}} \int_{x_j}^{x_{j+1}} |v_x|^2 \, dx + \frac{1}{h} \sum_{j \in \mathbb{Z}} [v]^2(x_j) \)

The bilinear form and the DG Cauchy problem:

\[
 a^s_h(u, v) = \sum_{j \in \mathbb{Z}} \int_{x_j}^{x_{j+1}} u_x v_x \, dx - \sum_{j \in \mathbb{Z}} ([u](x_j)\{v_x\}(x_j) + [v](x_j)\{u_x\}(x_j)) + \frac{s}{h} \sum_{j \in \mathbb{Z}} [u](x_j)[v](x_j),
\]

where \( s > 1 \) is the penalty parameter

\[
 \begin{cases}
 u^s_h(x, t) \in V_h, \ t > 0 \\
 \frac{d^2}{dt^2} \int_{\mathbb{R}} u^s_h(x, t)v(x) \, dx + a^s_h(u^s_h(\cdot, t), v) = 0, \forall v \in V_h, \\
 u^s_h(x, 0) = u^0_h(x), \ u^s_h(t, x, 0) = u^1_h(x) \in V_h.
\end{cases}
\]
Discontinuous Galerkin (DG) methods

DG as a system of ODE's

Solution = classical FE + jumps:
\[ u^s_h(x, t) = \sum_{j \in \mathbb{Z}} u_j(t) \phi_j(x) + \sum_{j \in \mathbb{Z}} \tilde{u}_j(t) \tilde{\phi}_j(x). \]

Then \( U^s_h(t) = (u_j(t), \tilde{u}_j(t))'_{j \in \mathbb{Z}} \) solves the system of ODE's:
\[ M^s_h \ddot{U}^s_h(t) = R^s_h U^s_h, \]

\( M^s_h \) - mass matrix \( \to \) stencil
\[
\begin{pmatrix}
\frac{h}{6} & -\frac{h}{12} & \frac{2h}{3} & 0 & \frac{h}{6} & \frac{h}{12} \\
\frac{h}{12} & -\frac{2h}{24} & 0 & \frac{h}{6} & -\frac{h}{12} & -\frac{h}{24} \\
\end{pmatrix}
\]

\( R^s_h \) - stiffness matrix \( \to \) stencil
\[
\begin{pmatrix}
-\frac{1}{h} & 0 & \frac{2}{h} & 0 & -\frac{1}{h} & 0 \\
0 & -\frac{1}{4h} & 0 & \frac{2s-1}{2h} & 0 & -\frac{1}{4h} \\
\end{pmatrix}
\]

The Fourier symbols of the mass and stiffness matrices:
\[
M^s_h(\xi) = \begin{pmatrix}
\frac{2+\cos(\xi h)}{3} & \frac{i \sin(\xi h)}{3} \\
\frac{i \sin(\xi h)}{3} & \frac{6-\cos(\xi h)}{12} \\
\end{pmatrix}, \quad R^s_h(\xi) = \begin{pmatrix}
\frac{4}{h^2} \sin^2 \left( \frac{\xi h}{2} \right) & 0 \\
0 & s-\cos^2 \left( \frac{\xi h}{2} \right) \\
\end{pmatrix}.
\]

The system in SDFT's:
\[
\begin{pmatrix}
\hat{u}^h_{tt}(\xi, t) \\
\tilde{u}^h_{tt}(\xi, t) \\
\end{pmatrix} = -A^s_h(\xi) \begin{pmatrix}
\hat{u}^h(\xi, t) \\
\tilde{u}^h(\xi, t) \\
\end{pmatrix}.
\]

- **physical** eigenvalue, eigenvector of \( A^s_h(\xi) \):
  \[ \Lambda^s_{ph,h}(\xi), P^s_{ph,h}(\xi) = \frac{1}{\sqrt{1+|p^s_{ph,h}(\xi)|^2}} \begin{pmatrix} 1 \\ p^s_{ph,h}(\xi) \end{pmatrix} \]

- **spurious** eigenvalue, eigenvector of \( A^s_h(\xi) \):
  \[ \Lambda^s_{sp,h}(\xi), P^s_{sp,h}(\xi) = \frac{1}{\sqrt{1+|p^s_{sp,h}(\xi)|^2}} \begin{pmatrix} p^s_{sp,h}(\xi) \\ 1 \end{pmatrix} \]

Filtering techniques

- initial data related in the Fourier variable - one can eliminate one of exponentials $\exp(\pm it\lambda_{ph,h}^s(\xi))$ or $\exp(\pm it\lambda_{sp,h}^s(\xi))$ + bigrid or Fourier filtering to eliminate the bad high or low frequency components.

- the initial data corresponding to the jump part to be zero + Fourier filtering or bigrid to eliminate the high frequency components. The bad frequency at low wave numbers on the spurious diagram removed by the weight accompanying $\exp(\pm it \exp(it\lambda_{sp,h}^s(\xi)))$. 
Contents

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2 Finite-differences case
3 Discontinuous Galerkin (DG) methods
4 Conclusions
DG provides a rich class of schemes allowing to regulate the physical components of the system, using the penalty parameter $s$, to fit better the behavior of the continuous wave equation.

Despite of this, these schemes generate high frequency spurious oscillations which behave badly, generating possibly wave packets travelling in the wrong sense.

Further work is needed to investigate if preconditioning and/or postprocessing can remove the spurious components.

GD in higher dimensions, other equations (Schrödinger) semi-discretized using DG, etc. Unless the case $s = 3$, the bi-grid method seems not be useful. The group acceleration vanishes at a wave number depending on $s$.

fully discretizations involving DG methods.
¡Thank you!