Modelling and analysis of structured population dynamics

First meeting BCAM - Inria Bordeaux Sud-Ouest

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What is structured population modelling?

Example: predator $\rightarrow$ *Daphnia*, prey $\rightarrow$ Algal population S

Structure: *Daphnia*'s individual behaviour depends on body size x (and on S)

survival probability $\mathcal{F}(x,S)$,
individual rates: growth $g(x,S)$, birth $\beta(x,S)$, consumption $\gamma(x,S)$

size function $X(a, S|_{[t-a,t]}) \leftarrow x' = g(x,S)$, $x(0) = x_0$

Population dynamics $\rightarrow$ renewal equations $\rightarrow$ population birth rate b

$$
\begin{aligned}
b(t) &= \int_0^\infty \beta(X(a, S|_{[t-a,a]}), S(t))\mathcal{F}(X(a, S|_{[t-a,a]}), S_{[t-a,a]})b(t-a)da \\
S'(t) &= f(S(t)) - \int_0^\infty \gamma(X(a, S|_{[t-a,a]}), S(t))\mathcal{F}(X(a, S|_{[t-a,a]}), S_{[t-a,a]})b(t-a)da
\end{aligned}
$$

Why does structure matter?

$x' = F(x, \alpha_1, \alpha_2)$, α_1, α_2 parameters

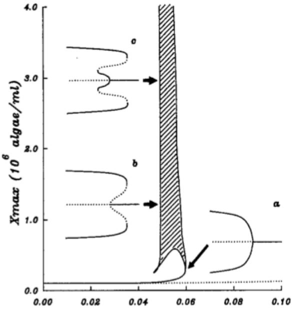
unique equilibrium $F(\overline{x}, \alpha_1, \alpha_2) = 0$

characteristic equation for $\lambda = i\omega \leftarrow$ stability switch

$$Re\,\det(F'(\overline{x}, \alpha_1, \alpha_2) - i\omega) = 0, \quad Im\,\det(F'(\overline{x}, \alpha_1, \alpha_2) - i\omega) = 0$$

$\rightarrow$ 3 equations, 4 variables $\rightarrow$ curve

projection $\sigma \mapsto (\alpha_1, \alpha_2)(\sigma)$ is stability boundary in $\alpha_1 - \alpha_2$-plane



- predator- prey-model: either stable equilibrium or stable limit cycle
- size-structured *Daphnia*-Algae model:
  coexistence of stable equilibrium and stable limit cycle

  found numerically (de Roos et al. *J. Math. Biol.* 1990)
  found in nature (Mc Cauley et al. *Nature* 2008)

  $\longrightarrow$ individual structure matters

Goals

- analyze similar problems for cell populations

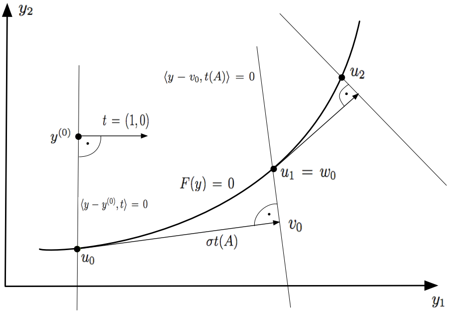
  analogy:
  cell behaviour $\longleftarrow$ cycle phase and stage of maturation

  (Alarcon, Getto, Marciniak, Vivanco, submitted to *AIMS proceedings*)

- develop problem independent software

  to trace stability boundaries in two-parameter space

Numerical curve continuation



- Computations of the Jacobian

- MatCont can continue $H(x, y) = 0$, if H is analytic expression
- for structured models: evaluation of H is the problem $\rightarrow$

Problem: implicitly defined functions

$$x(0) = x_0, \;\; x' = g(x, S), \;\; \text{nonlinear individual growth rate}$$
$$\mathcal{F}(0) = 1, \;\; \mathcal{F}' = -\mu(x, S)\mathcal{F}, \;\; \text{individual mortality rate}$$

- size X, survival $\mathcal{F}$
  - only implicitly defined
  - needed to evaluate curve map H

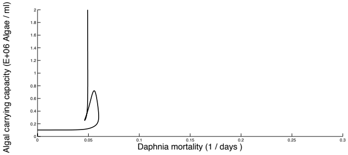
  $\rightarrow$ one evaluation of H $\rightarrow$ integrate ODE in parallel

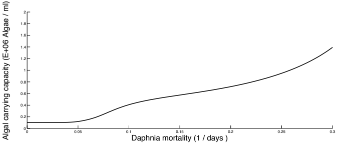
- linearisation of integral operators $\rightarrow$ complex ODE $\rightarrow$ realification

For structured consumer-resource models:
one evaluation $\rightarrow$ integration of 16 ODE

Algorithms have been

- implemented in *C*-code
- tested for *Daphnia*-Algae-models





(de Roos, Diekmann, Getto and Kirkilionis, *Bull. Math. Biol.* 2010)

CONCLUSIONS:

- developed algorithms to compute stability boundaries for class of size-structured consumer resource models
- successfully implemented for *Daphnia*

Future work

- adapt code to models from cell biology

  Stem cell maturation, Quiescent cell populations

- create Matlab interface to make methods more user- friendly

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