Localized solutions and filtering mechanisms for the discontinuous Galerkin semi-discretizations of the $1 - d$ wave equation

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Motivation: Control of the wave equation

The Cauchy problem for the $1-d$ wave equation - well posed in the energy space $\dot{H}^1 \times L^2(\mathbb{R})$:

\[
\begin{aligned}
\partial_t^2 u(x, t) - \partial_x^2 u(x, t) &= 0, & x \in \mathbb{R}, & t > 0 \\
u(x, 0) &= u^0(x), & u_t(x, 0) &= u^1(x), & x \in \mathbb{R}.
\end{aligned}
\] (1)

Conservation of the total energy: $E(u^0, u^1) = \frac{1}{2} \int_{\mathbb{R}} (|\partial_x u(x, t)|^2 + |\partial_t u(x, t)|^2) \, dx$.

Observability property

\[
\forall T > 2 \text{ (characteristic time), } \exists C(T) > 0 \text{ s.t. } \forall (u^0, u^1) \text{ of finite energy:}
\]

\[
E(u^0, u^1) \leq C(T) \int_0^T E_\Omega(u^0, u^1, t) \, dt,
\] (2)

where $\Omega = \mathbb{R} \setminus (-1, 1)$ and $E_\Omega(u^0, u^1, t)$ is the energy concentrated in $\Omega$,

\[
E_\Omega(u^0, u^1, t) = \frac{1}{2} \int_{\Omega} (|\partial_x u(x, t)|^2 + |\partial_t u(x, t)|^2) \, dx.
\]


By HUM, the observability problem (2) is equivalent to an exact controllability problem:

**Exact controllability**

\[ \forall T > T^* = 2, \forall (u^0, u^1) \in \dot{H}^1(\mathbb{R}) \times L^2(\mathbb{R}), \text{there exists a control function } f \in L^2(\Omega \times (0, T)) \text{ s.t.} \]

the solution of the inhomogeneous Cauchy problem

\[
\begin{cases}
\partial_t^2 u(x, t) - \partial_x^2 u(x, t) = f(x, t)\chi_\Omega(x), & x \in \mathbb{R}, \ t \in (0, T] \\
u(x, 0) = u^0(x), \ \partial_t u(x, 0) = u^1(x), & x \in \mathbb{R}
\end{cases}
\]

satisfies \( u(x, T) = \partial_t u(x, T) = 0 \) for all \( x \in \mathbb{R} \) (\( \chi_\Omega = \text{the characteristic function of } \Omega \)).

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**Geometric Control Condition (GCC)**

*All rays of Geometric Optics enter the observation set during the observability time.*

*No GCC \( \Rightarrow \) the observability property fails.*

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**References**

The SIPG semi-discretizations of the wave equation

Jumps and averages: \[ f(x) = f(x-) - f(x+), \{ f \}(x) = \frac{f(x+) + f(x-)}{2}. \]

The finite element space \( V_h := U_A^h \oplus U_J^h \), with \( U_A^h = \text{span}\{\phi_i^A, i \in \mathbb{Z}\} \) and \( U_J^h = \text{span}\{\phi_i^J, i \in \mathbb{Z}\} \), where \( \phi_i^A(x) = \left[ 1 - \frac{|x-x_i|}{h} \right]^+ \), \( \phi_i^J(x) = \frac{1}{2} \text{sign}(x_i - x) \left[ 1 - \frac{|x-x_i|}{h} \right]^+ \).

Figure: Typical basis functions for the \( P_1 \)-discontinuous Galerkin methods: \( \phi_i^A \) (left) and \( \phi_i^J \) (right).

In the SIPG method, the Laplacian is discretized by the following bilinear form:

\[ a_h^s(u, v) = \sum_{j \in \mathbb{Z}} \int_{x_j}^{x_{j+1}} u_x(x)v_x(x) \, dx - \sum_{j \in \mathbb{Z}} [u](x_j) \{v_x\}(x_j) - \sum_{j \in \mathbb{Z}} [v](x_j) \{u_x\}(x_j) + \frac{s}{h} \sum_{j \in \mathbb{Z}} [u](x_j)[v](x_j). \]

The semi-discrete wave equation:

\[
\begin{cases}
\bar{u}_h^s(x, t) \in V_h, \text{ for all } t > 0 \\
\partial_t^2 \int_{\mathbb{R}} u_h^s(x, t)v(x) \, dx = a_h^s(u_h^s(\cdot, t), v), \text{ for all } v \in V_h \\
u_h^s(x, 0) = u_0^h(x) \in V_h, \partial_t u_h^s(x, 0) = u_1^h(x) \in V_h.
\end{cases}
\]

Numerical solution = continuous part + jump part:

\[ u_h^s(x, t) = \sum_{k \in \mathbb{Z}} A_k(t)\phi_k^A(x) + \sum_{k \in \mathbb{Z}} J_k(t)\phi_k^J(x). \]

Matricial form:

\[
\begin{cases}
M_h \bar{U}_{tt}^h(t) + R_h^s \bar{U}^h(t) = 0 \\
\bar{U}^h(0) = \bar{U}_{0}^h, \quad \bar{U}_{t}^h(0) = \bar{U}_{1}^h,
\end{cases}
\]

where \( \bar{U}^h(t) = (\bar{A}^h(t), \bar{J}^h(t)) \) and \( M_h, R_h^s \) are infinite mass and stiffness matrices generated by

\[
m_h = \begin{pmatrix}
\frac{h}{6} & -\frac{h}{12} & \frac{2h}{3} & 0 & \frac{h}{6} & \frac{h}{12} \\
\frac{h}{12} & -\frac{h}{24} & 0 & \frac{h}{6} & -\frac{h}{12} & -\frac{h}{24}
\end{pmatrix}, \quad r_h^s = \begin{pmatrix}
-\frac{1}{h} & 0 & \frac{2}{h} & 0 & -\frac{1}{h} & 0 \\
0 & -\frac{1}{4h} & 0 & \frac{2s-1}{2h} & 0 & -\frac{1}{4h}
\end{pmatrix}.
\]
Fourier analysis of the SIPG method

\[ \Pi_h := [-\pi/h, \pi/h]. \quad \hat{A}^h(\xi, t), \hat{J}^h(\xi, t) = \text{SDFTs of } \hat{A}^h(t), \hat{J}^h(t), \hat{U}^h(\xi, t) := (\hat{A}^h(\xi, t), \hat{J}^h(\xi, t))'. \]

The system (4) can be transformed into the following Cauchy problem:

\[
\begin{cases}
\hat{U}_{tt}^h(\xi, t) + A_s^h(\xi)\hat{U}^h(\xi, t) = 0, & \xi \in \Pi_h, \ t > 0 \\
\hat{U}^h(\xi, 0) = \hat{U}^{h,0}(\xi), & \hat{U}_t^h(\xi, 0) = \hat{U}^{h,1}(\xi), & \xi \in \Pi_h,
\end{cases}
\]

(5)

where \( A_s^h(\xi) = (M_s^h(\xi))^{-1}R_s^h(\xi) \) and \( M_s^h(\xi), R_s^h(\xi) \) are the Fourier symbols of \( M_h(\xi) \) and \( R_h(\xi) \), with

\[
M_h(\xi) = \begin{pmatrix}
-\frac{2+\cos(\xi h)}{3} & \frac{i\sin(\xi h)}{6} \\
\frac{i\sin(\xi h)}{2} & -\frac{6-\cos(\xi h)}{12}
\end{pmatrix}, \quad R_s^h(\xi) = \begin{pmatrix}
\frac{4}{h^2} \sin^2 \left( \frac{\xi h}{2} \right) & 0 \\
0 & \frac{s-\cos^2 \left( \frac{\xi h}{2} \right)}{h^2}
\end{pmatrix}.
\]

The total energy \( E_s^h(\hat{U}^{h,0}, \hat{U}^{h,1}) \) is conserved in time.

Discrete observability inequality (DGOI):

\[
E_s^h(\hat{U}^{h,0}, \hat{U}^{h,1}) \leq C_s^h(T) \int_0^T E_{\Omega, h}^s(\hat{U}^{h,0}, \hat{U}^{h,1}, t) \, dt.
\]

\[
\hat{U}^h(\xi, t) = \sum_{\pm} \frac{1}{2} \left[ P_s^h(\xi) \begin{pmatrix}
\exp(\pm it\lambda_{ph, h}^s(\xi)) & 0 \\
0 & \exp(\pm it\lambda_{sp, h}^s(\xi))
\end{pmatrix} \left( P_s^h(\xi) \right)^{-1} \hat{U}^{h,0}(\xi)
\right.
\]

\[
+ P_s^h(\xi) \begin{pmatrix}
\pm \exp(\pm it\lambda_{ph, h}^s(\xi)) \\
\pm \frac{\exp(\pm it\lambda_{sp, h}^s(\xi))}{i\lambda_{sp, h}^s(\xi)}
\end{pmatrix} \left( P_s^h(\xi) \right)^{-1} \hat{U}^{h,1}(\xi) \right].
\]
Properties of the eigenvalues and eigenvectors of $A_h^s(\xi)$

$P_{ph,h}^s(\xi), P_{sp,h}^s(\xi) =$ physical, spurious eigenvectors

$\Lambda_{ph,h}^s(\xi), \Lambda_{sp,h}^s(\xi) =$ physical, spurious eigenvalue, $\lambda_{ph,h}^s(\xi) = \sqrt{\Lambda_{ph,h}^s(\xi)}, \lambda_{sp,h}^s(\xi) = \sqrt{\Lambda_{sp,h}^s(\xi)}$.

Properties of the eigenvalues and of the group velocities:

- $\forall s > 1, \lim_{\xi \to 0} \partial_\xi \lambda_{ph,h}^s(\xi) = 1$ and $\lim_{\xi \to 0} \partial_\xi \lambda_{sp,h}^s(\xi) = 0$.
- $\forall s \in (1, \infty) \setminus \{3\}, \lim_{\xi \to \pm \pi/h} \partial_\xi \lambda_{ph,h}^s(\xi) = \lim_{\xi \to \pm \pi/h} \partial_\xi \lambda_{sp,h}^s(\xi) = 0$.
- $\lim_{\xi \to \pm \pi/h} \partial_\xi \lambda_{ph,h}^3(\xi) = 1$ and $\lim_{\xi \to \pm \pi/h} \partial_\xi \lambda_{sp,h}^3(\xi) = -1$.

Figure: $\lambda_{ph,1}^s(\xi)$ (black) and $\lambda_{sp,1}^s(\xi)$ (dotted black) for $s = 1, 5, 2, 3, 5$. 
Non-uniform observability inequality

When the vector valued initial data \( \hat{U}^{h,i} \) in (5), \( i = 0, 1 \), are of the form

\[
\hat{U}^{h,i}(\xi) = P_{ph,h}(\xi)\hat{u}^{h,i}(\xi),
\]

the corresponding solutions of (5) involve only the physical dispersion relation:

\[
\hat{U}^{h}(\xi, t) = P_{ph,h}(\xi) \frac{1}{2} \sum_{\pm} \left( \hat{u}^{h,0}(\xi) \pm \frac{\hat{u}^{h,1}(\xi)}{i\lambda_{ph,h}(\xi)} \right) \exp(\pm it\lambda_{ph,h}(\xi)).
\]

Proposition

Let \( T > 0 \) fixed s.t. the ray \( x_{ph}(t) = x^* - t\partial_{\xi}\lambda_{ph,1}(\eta_0) \) that does not enter the observation region before \( T \); \( \gamma := \gamma(h) > 0 \) s.t. \( \gamma \gg 1 \) and \( h\gamma \ll 1 \), \( \phi \in S(\mathbb{R}) \) and (5) with \( \hat{U}^{h,i}(\xi) \) s.t. (6) holds, with

\[
\hat{u}^{h,0}(\xi) = \sqrt{\frac{2\pi}{\gamma}} \hat{\phi} \left( \frac{\xi - \xi_0}{\gamma} \right) \exp(-ix^*(\xi - \xi_0))\chi_{\Pi,h}(\xi) \text{ and } \hat{u}^{h,1}(\xi) = i\lambda_{ph,h}(\xi)\hat{u}^{h,0}(\xi).
\]

Then \( \forall \alpha \in \mathbb{R}_+ \), the observability constant \( C_h^s(T) \) in the DGOI satisfies

\[
C_h^s(T) \geq C_{\alpha}(\phi, T, s)\gamma^{\alpha}.
\]

Stationary phase lemma, L. Evans, PDEs: If \( \hat{\sigma} \in C_c^\infty(\mathbb{R}^d) \), \( \psi \in C^\infty(\mathbb{R}^d) \) s.t. \( \nabla \psi \neq 0 \) in \( \text{supp}(\hat{\sigma}) \), Then

\[
I_\epsilon = \int_{\mathbb{R}^d} \hat{\sigma}(\xi) \exp(i\psi(\xi)/\epsilon) \, d\xi = O(\epsilon^N), \quad \forall N \in \mathbb{N}.
\]
Concentration on the physical mode + Fourier filtering

For $\delta \in (0,1)$, $\Pi^\delta := [-\pi \delta/h, \pi \delta/h]$.

$I^\delta_h := \{ \hat{f} \in \ell^2(h\mathbb{Z}) : \text{supp}(\hat{f}^h) \subset \Pi^\delta_h \}$ - the space of Fourier filtered data with parameter $\delta$.

$\Gamma^\delta_h \hat{f}_j = \frac{1}{2\pi} \int_{\Pi^\delta_h} \hat{f}^h(\xi) \exp(i\xi x_j) d\xi$ - the projection on $I^\delta_h$ of $\hat{f} \in \ell^2(h\mathbb{Z})$.

Theorem

Set $\Omega = \{ x : |x| > 1 \}$. In (5), consider initial data concentrated on the physical mode, i.e.

$$\hat{U}^{h,i}(\xi) = P_{\text{ph},h}(\xi) \hat{u}^{h,i}(\xi), \quad i = 0, 1,$$ s.t. $\hat{u}^{h,i} \in I^\delta_h$ i.e. filtered with parameter $\delta$. Then for all $T > T_{\text{ph}}^{s,\delta}$, with

$$T_{\text{ph}}^{s,\delta} = \frac{2}{\min_{\xi \in \Pi^\delta_h} \partial_\xi \lambda_{\text{ph},h}^s(\xi)} (1 + C_{\text{ph}}^{s,\delta})$$

and $C_{\text{ph}}^{s,\delta} \in (0,1)$, and all $s > 1$, the following observability inequality holds uniformly as $h \to 0$:

$$E^s_h(\hat{U}^{h,0}, \hat{U}^{h,1}) \leq C_{\text{ph}}^{s,\delta}(T) \int_0^T E^s_{\Omega, h}(\hat{U}^{h,0}, \hat{U}^{h,1}, t) dt,$$

with observability constant

$$C_{\text{ph}}^{s,\delta}(T) = \frac{1}{T - T_{\text{ph}}^{s,\delta}}.$$
**Figure:** The hachured zone corresponds to frequencies eliminated by the filtering mechanism. Concentration on the physical mode + Fourier filtering.
Concentration on the physical mode + bi-grid algorithm

**Theorem**

In (5), consider initial data concentrated on the physical mode, i.e.

\[ \hat{U}^{h,i}(\xi) = P_{ph,h}(\xi)\hat{u}^{h,i}(\xi), \]

\( i = 0, 1, \) s.t. \( \hat{u}^{h,i} \) satisfies the bi-grid condition

\[ u^i_{2j} = \frac{u^i_{2j+1} + u^i_{2j-1}}{2}. \]

For all \( T > T_{ph}^{s,1/2} \) and all \( s > 1 \), there exists a constant \( C_{ph,bigrid}(T) > 0 \) independent of \( h \) s.t. the following observability inequality holds:

\[ E_{h}^{s}(\hat{U}^{h,0}, \hat{U}^{h,1}) \leq C_{ph,bigrid}(T) \int_{0}^{T} E_{\Omega,h}^{s}(\hat{U}^{h,0}, \hat{U}^{h,1}, t) \, dt. \]

Proof based on the following result + dyadic decomposition argument:

**Proposition**

For all \( s > 1 \) and all initial data concentrated on the physical mode s.t. \( \hat{u}^{h,i} \) satisfies the bi-grid condition, there exists a constant \( C_{ph} > 0 \), independent of \( h \) and of \( s \), s.t.

\[ E_{h}^{s}(\hat{U}^{h,0}, \hat{U}^{h,1}) \leq C_{ph} E_{h}^{s}(\Gamma_{h}^{1/2} \hat{U}^{h,0}, \Gamma_{h}^{1/2} \hat{U}^{h,1}). \]
Figure: The hachured zone corresponds to frequencies eliminated by the filtering mechanism Concentration on the physical mode + bi-grid algorithm
Bigrid algorithms

Projection 1

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Projection 2

Graphs showing trends over time.
Physical mode, without bigrid, $s=1.5$

Physical mode, bigrid of ratio 1/2, $s=1.5$

Physical mode, biggrid of ratio 1/4, $s=1.5$
Physical mode, without bigrid, $s=1.5$

Physical mode, bigrid of ratio $1/2$, $s=1.5$

Physical mode, biggrid of ratio $1/4$, $s=1.5$
In (4), consider initial data $\vec{U}^{h,i} = (\vec{A}^{h,i}, \vec{J}^{h,i})'$, $i = 0, 1$, having null jump part, i.e. $\vec{J}^{h,i} = 0$ and s.t. $\vec{A}^{h,i} \in \mathcal{I}_h^\delta$, $\delta \in (0, 1)$. Then for all $T > T_{ph}^{s,\delta}$ and all $s > 1$ s.t.

$$\max_{\xi \in \Pi_h^\delta} |\lambda_{ph,h}^s(\xi)| < \min_{\xi \in \Pi_h} |\lambda_{sp,h}^s(\xi)|$$,

there exists a constant $C_0^s(T) > 0$ uniform as $h \to 0$ s.t.

$$E_h^s(\vec{U}^{h,0}, \vec{U}^{h,1}) \leq C_0^s(T) \int_0^T E_{\Omega,h}^s(\vec{U}^{h,0}, \vec{U}^{h,1}, t) \, dt.$$

Proposition

In (4), consider $\vec{U}^{h,i} = (\vec{A}^{h,i}, \vec{J}^{h,i})'$, $i = 0, 1$, s.t. $\vec{J}^{h,i} = 0$ and $\vec{A}^{h,i} \in \mathcal{I}_h^\delta$. Then $\exists C(\delta) > 0$ s.t.

$$E_h^s(\vec{U}^{h,0}, \vec{U}^{h,1}) \leq C(\delta) E_h^s(\Gamma_{ph} \vec{U}^{h,0}, \Gamma_{ph} \vec{U}^{h,1}).$$

If $\vec{f}^h(t)$ is an evolution process s.t.

$$\hat{f}^h(\xi, t) = \hat{f}_{ph}^h(\xi) \exp(it\lambda_{ph,h}^s(\xi)) + \hat{f}_{sp}^h(\xi) \exp(it\lambda_{sp,h}^s(\xi)),$$

its projection on the physical branch is

$$\Gamma_{ph} f_j(t) = \frac{1}{2\pi} \int_{\Pi_h} \hat{f}_{ph}^h(\xi) \exp(it\lambda_{ph,h}^s(\xi)) \exp(i\xi x_j) \, d\xi.$$
Theorem

In (4) consider $\vec{U}^{h,i} = (\vec{A}^{h,i}, \vec{J}^{h,i})'$, $i = 0, 1$, s.t. $\vec{J}^{h,i} = 0$ and $\vec{A}^{h,i}$ is given by a bi-grid algorithm, i.e. $A_{2j}^i = \frac{A_{2j+1}^i + A_{2j-1}^i}{2}$. For all $T > T_{ph}^{s,1/2}$ and all $s > 1$ s.t.

$$\max_{\xi \in \Pi_{1/2}^{1/2}} |\lambda_{ph,h}^s(\xi)| < \min_{\xi \in \Pi_h} |\lambda_{sp,h}^s(\xi)|$$,

there exists a constant $C_{0,\text{bigrid}}^s(T) > 0$ independent of $h$ s.t.

$$E_h^s(\vec{U}^{h,0}, \vec{U}^{h,1}) \leq C_{0,\text{bigrid}}^s(T) \int_0^T E_{\Omega_h}^s(\vec{U}^{h,0}, \vec{U}^{h,1}, t) \, dt.$$ 

Proposition

In (4), consider initial data $\vec{U}^{h,i} = (\vec{A}^{h,i}, \vec{J}^{h,i})'$, $i = 0, 1$, s.t. $\vec{J}^{h,i} = 0$ and $\vec{A}^{h,i}$ is given by a bi-grid algorithm. For all $s > 1$, the following estimate holds

$$E_h^s(\vec{U}^{h,0}, \vec{U}^{h,1}) \leq 2E_h^s(\Gamma_{h}^{1/2} \vec{U}^{h,0}, \Gamma_{h}^{1/2} \vec{U}^{h,1}).$$
Null jumps, without bigrid, $s=1.5$

Null jumps, bigrid of ratio $1/2$, $s=1.5$

Null jumps, bigrid of ratio $1/4$, $s=1.5$
Conclusions

- DG provides a rich class of schemes allowing to regulate the physical components of the system, using the penalty parameter $s$, to fit better the behavior of the continuous wave equation.
- Despite of this, these schemes generate high frequency spurious oscillations which behave badly, generating possibly wave packets traveling in the wrong sense.
- propagation properties of other DG schemes: LDG.
- fully discrete DG schemes
- DG in higher dimensions on uniform grids
- other equations (Schrödinger) semi-discretized using DG, etc.

Thank you very much!