Sonic boom prediction and control

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Partial Differential Equations: Analysis, Control, Numerics and Applications
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2. The sonic boom model
3. Analysis of the model
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1 Motivation

2 The sonic boom model

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Sonic boom prediction and control
Nowadays, one of the main goals in aeronautics is to handle the sonic boom phenomenon produced by supersonic flights.

Historically, linear theory has been used to model this situation, following the works by Witham (1952) and Hayes (1967).

More recently, new nonlinear models have been developed (Rallabhandi, 2011).

CFD techniques can accurately approximate the state in the near field, but it needs new tools for the propagation from that near field to the ground level.

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The sonic boom can be modeled by the Augmented Burgers Equation:

\[
\frac{\partial P}{\partial \sigma} = P \frac{\partial P}{\partial \tau} + \frac{1}{\Gamma} \frac{\partial^2 P}{\partial \tau^2} + \sum_{\nu} C_{\nu} \frac{\partial^2}{\partial \tau^2} \frac{1}{1 + \theta_{\nu} \frac{\partial}{\partial \tau}} P - \frac{1}{2G} \frac{\partial G}{\partial \sigma} P + \frac{1}{2\rho_0 c_0} \frac{\partial (\rho_0 c_0)}{\partial \sigma} P \quad \text{(ABE)}
\]

where:

- The operator in the molecular relaxation term is given by:
  \[
  \frac{\theta_{\nu}}{1 + \theta_{\nu} \frac{\partial}{\partial \tau}} f(\tau) = \int_{-\infty}^{\tau} e^{(s-\tau)/\tau_{\nu}} f(s) ds \quad (1)
  \]

- \( P = P(\sigma, \tau) \) is the dimensionless pressure, \( \sigma \) is the non-dimensional distance and \( \tau \) is the dimensionless time.

- Ambient density \( \rho_0 \), ambient speed of sound \( c_0 \), thermo-viscous parameter \( \Gamma \), dimensionless relaxation time parameter \( \theta_{\nu} \), non-dimensional dispersion parameter \( C_{\nu} \), dimensionless time for each relaxation mode \( \tau_{\nu} \) and ray tube area \( G \).

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Constant parameters model

Assumptions:
- Only one molecular relaxation phenomena.
- Constant $\rho_0 c_0$, $\Gamma$, $\theta_\nu$, $C_\nu$, $\tau_\nu$ and $G$.

Notation:
- $t := \sigma$, $x := \tau$ and $u(x, t) := P(\tau, \sigma)$
- $\nu := \frac{1}{\Gamma}$, $c := C_\nu$ and $\theta := \theta_\nu$.

Hence, we obtain a model in which an integral kernel is added to the classical viscous Burgers Equation:

$$
\begin{cases}
  u_t = uu_x + \nu u_{xx} + \frac{c}{\theta} \int_{-\infty}^{x} e^{(y-x)/\theta} u_{xx}(y, t) dy, & (x, t) \in \mathbb{R} \times (0, \infty) \\
  u(x, t = 0) = u_0(x), & x \in \mathbb{R}
\end{cases}
$$

(CABE)

or, equivalently:

$$
\begin{cases}
  u_t = uu_x + \nu u_{xx} + \frac{c}{\theta} u_x - \frac{c}{\theta^2} u + \frac{c}{\theta^3} K_\theta * u \\
  u(x, t = 0) = u_0(x), & x \in \mathbb{R}
\end{cases}
$$

where $K_\theta(z) := e^{-z} \chi_{\{z > 0\}}(z)$. 
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Theorem (Existence and uniqueness of solution)

For all $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, there exists a unique solution

$$u \in C([0, \infty); L^1(\mathbb{R})) \cap L^\infty(\mathbb{R} \times (0, \infty))$$

for CABE which also satisfies

$$u \in C((0, \infty), W^{2,p}(\mathbb{R})) \cup C^1((0, \infty), L^p(\mathbb{R}))$$

for all $p \in (1, \infty)$. Besides, CABE generates a contractive semigroup in $L^1$.

Theorem (Decay estimates)

For all $p \in [1, \infty]$, there exists a constant $C = C(p) > 0$ such that

$$\|u(t)\|_p \leq C\nu^{-\frac{1}{2}(1-\frac{1}{p})}\|u_0\|_1 t^{-\frac{1}{2}(1-\frac{1}{p})}, \quad \forall t > 0$$

for all solutions of CABE with initial data $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. 
Asymptotic behavior

Theorem (Asymptotic profile)

Let \( u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \). Then

\[
t^{\frac{1}{2}(1-\frac{1}{p})} \|u(t) - u_M(t)\|_p \rightarrow 0, \quad \text{as } t \rightarrow \infty
\]

(2)

for all \( p \in [1, \infty] \), where \( u_M(x, t) = t^{-\frac{1}{2}} f_M \left( \frac{x}{\sqrt{t}} \right) \) is the self-similar solution of the following Burgers-like equation:

\[
\begin{cases}
  u_t = \left( \frac{u^2}{2} \right)_x + (\nu + c) u_{xx}, & x \in \mathbb{R}, \ t > 0 \\
  u(x, 0) = M\delta_0
\end{cases}
\]

(3)

where \( \delta_0 \) denotes the Dirac delta at the origin and \( M \) is given by \( M = \int_{\mathbb{R}} u_0(x)dx \).
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Numerical scheme

Integrating by parts and reordering, we rewrite CABE as:

\[
\begin{cases}
  u_t + \left[f_{c, \theta}(u)\right]_x = \nu u_{xx} + \frac{c}{\theta^3} \int_{-\infty}^{x} e^{(y-x)/\theta} [u(y, t) - u(x, t)] dy, & (x, t) \in \mathbb{R} \times (0, T) \\
  u(x, 0) = u(x), & x \in \mathbb{R}
\end{cases}
\]

with flux \( f_{c, \theta}(u) = -\frac{u^2}{2} - \frac{c}{\theta} u \). This can be discretized as:

\[
\begin{align*}
  u^{n+1}_k &= u^n_k - \frac{\Delta t}{\Delta x} \left[ g^{EO}(u^n_k, u^n_{k+1}) - g^{EO}(u^n_{k-1}, u^n_k) \right] \quad \text{(flux)} \\
  &+ \nu \frac{\Delta t}{\Delta x^2} \left( u^n_{k-1} - 2u^n_k + u^n_{k+1} \right) \quad \text{(laplacian)} \\
  &+ \frac{c}{\theta^3} \Delta t \Delta x \sum_{j=k-N}^{k} e^{\Delta x(j-k)/\theta} \left( u^n_j - u^n_k \right) \quad \text{(integral)} \quad k \in \mathbb{Z}, t > 0 \\
  u^0_k &= \frac{1}{\Delta x} \int_{x_{k-1/2}}^{x_{k+1/2}} u^0(x) dx, \quad k \in \mathbb{Z}
\end{align*}
\]

where:

\[
g^{EO}(u, v) = \frac{1}{2} \left[ f(u) + f(v) - \int_u^v |f'(\xi)| d\xi \right]
\]
Numerical tests

- Norm of the solution to the ABE
- Norm of the solution to the VBE
Numerical tests
Statement of the problem

We want to solve this optimization problem:

\[
\mathcal{J}(u^0) = \min_{u^0 \in \mathcal{U}_{ad}} \frac{1}{2} \int_\mathbb{R} [u(x, T) - u^*(x)]^2 \, dx
\]

subject to

\[
\begin{cases}
  u_t + \left[ f_{c, \theta}(u) \right]_x = \nu u_{xx} + \frac{c}{\theta^3} \left( K_{\theta} \ast u - \theta u \right), & (x, t) \in \mathbb{R} \times (0, T) \\
  u(x, 0) = u(x), & x \in \mathbb{R}
\end{cases}
\]

This leads to the adjoint equation:

\[
\begin{cases}
  -p_t - f'_{c, \theta}(u) p_x - \nu p_{xx} - \frac{c}{\theta^3} \left( \tilde{K}_{\theta} \ast p - \theta p \right) = 0, & (x, t) \in \mathbb{R} \times (0, T) \\
  p(x, T) = u(x, T) - u^*(x), & x \in \mathbb{R}
\end{cases}
\]

where \( \tilde{K}_{\theta}(z) := K_{\theta}(-z) \).
Adjoint method (I)

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Adjoint method (III)

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Conclusions and future work

Briefly:

- We have obtained the decay estimates and the asymptotic profile of the constant-parameter model.
- We built a numerical scheme for the simulation of the solution.
- We have shown the importance and usefulness of an Alternating Descent Method.

Future work:

- Second term of the asymptotic profile (for \( M=0 \))
- Improve the numerical scheme:
  - Asymptotic behavior
  - Splitting methods
  - High order methods
- Adapt the Alternating Descent Method.
Thanks for your attention!