Objectives and problem formulation

Wave equation (WE) \((x \in \mathbb{R}, t > 0)\)

\[\partial_t u - \partial_x u = 0, \ u(x, 0) = u^0(x), \ u_t(x, 0) = u^1(x)\]

Observability inequality (OI) \((t \in \mathbb{R} \setminus (-1, 1), T \geq 2)\)

\[|u_0|_{L^2(\mathbb{R})} \leq C(T) \int_0^T \left|\|u_t(x, t)\|_{L^2(\mathbb{R})}\right| dt\]

Applications: control, stabilization and inverse problems

Objectives

- Discrete versions of OI and DE (non-uniformity in \(h\), filtering mechanisms)
- Behaviour of high frequency Gaussian wave packets in complex media (complex schemes, splitting under filtering, non-uniform meshes, etc.)

Discontinuous Galerkin (DG) approximations

Notations: \(\{\cdot\}_{1,2}\) - average jump; \(\gamma > 1\) - penalty parameter; \(p^h, \mathbb{T}^h\) - uniform grid/triangulation of size \(h \in \mathbb{R}\); \(\mathcal{V}^h\) - space of piecewise linear and discontinuous functions

Symmetric interior penalty DG (SIPG) approximation

Bilinear form

\[A^h_i(u, v) = (\partial^h_i u, \partial^h_i v)_{L^2(\Omega)} - ((\partial^h_i u), v)_{L^2(\Omega)} - ([u], [\partial^h_i v])_{L^2(\Omega)} + \gamma h_{i-1/2}(u, v)_{L^2(\Omega)}\]

Discrete wave equation

\[\begin{align*}
\hat{U}^h(\xi, t) & \in \mathbb{R}^{n_t} \text{ s.t.} \quad (u^h_0(t), \xi)_{L^2(\Omega)} + \sum_{i=1}^n A^h_i(u^h_0(t), \xi) = 0, \quad \forall \xi \in \mathbb{R}^n_t \\
\end{align*}\]

Eigenvalues of \(\hat{S}_p^h(\xi)\): physical and spurious \(\lambda_{p,\text{phys}}^h(\xi)\) and \(\lambda_{p,\text{spurious}}^h(\xi)\)

Dispersion relations: \(\lambda = \sqrt{\xi} (\text{see Fig. 1(a)})\)

Group velocities \(\partial_x \lambda_{p,\text{phys}}^h\) and \(\partial_x \lambda_{p,\text{spurious}}^h\)

They vanish for \(\xi = \pi/h, \xi \in \{0, \pi/h\} \implies\) non-uniform OI as \(h \to 0\) (see Fig. 1(b)).

Bi-grid filtering mechanism for the DG method (cf. [5])

Discrete initial data with null jumps + averages obtained by a bi-grid algorithm of mesh ratio 1/2 \(\Rightarrow\) uniform OI as \(h \to 0\) (see Fig. 1(c-d))

Bi-grid algorithms

Gaussian initial data: \(\varphi = \varphi^h_{\gamma,0}\) such that

\[\varphi^h_{\gamma,0}(x) = \frac{2\pi}{\gamma} \exp\left(-\frac{\left|\xi - \xi_0\right|^2}{2\gamma^2}\right) \gamma h(\xi)\]

\(\gamma \leq h^{-2/3}\) for SE and \(\gamma \leq h^{-1/2}\) for WE

Fourier representation of the restriction operator \(F_k^h: \mathbb{C}^h \mapsto \mathbb{C}^{2n_k}\)

\[\hat{F}_k^h \hat{f}^h(\xi) = \sum_{j=-n_k}^{n_k} \hat{f}^h(\xi + 2\pi j/n_k)\]

\[\hat{F}_{2n_k}^h \hat{f}^h(\xi) = \sum_{j=-2n_k}^{2n_k} \hat{f}^h(\xi + 2\pi j/2n_k)\]

High frequency propagation of waves (WE) in discrete heterogeneous media - open problem

- \(x\)-uniform grid of size \(h\) of \(0, 1, \) \(y\)-non-uniform grid of \(0, 1\) and \(p\): \([0, 2\pi/h] \implies \phi^h(\gamma = \gamma_0) = 0\)

References

- Erdogan S., Szafraniec E., The wave equation: control and numerics, Lecture Notes in Mathematics, CIME Subseries, Springer Verlag, to appear
- Lobo, L., Zuazua E., Convergence of a two-grid algorithm for the control of the wave equation, ISEMS, 2009