Internal control for non-local Schrödinger and wave equations involving the fractional Laplace operator

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We analyse the control problem for the fractional Schrödinger equation

\[ iu_t + (-\Delta)^s u = 0 \]

on a bounded $C^{1,1}$ domain $\Omega \subset \mathbb{R}^n$. We show a controllability result from a neighbourhood of the boundary. As a consequence of that, we obtain the controllability for the fractional wave equation

\[ u_{tt} + (-\Delta)^{2s} u = 0. \]
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\[ u_{tt} + (-\Delta)^{2s} u = 0. \]
**Fractional laplacian**

\[
(-\Delta)^s u(x) := c_{n,s} P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy, \quad s \in (0, 1)
\]

\[
c_{n,s} := \frac{s2^{2s} \Gamma\left(\frac{n+2s}{2}\right)}{\pi^{n/2} \Gamma(1-s)}
\]

**Fractional Sobolev space**

\[
H^s(\Omega) := \left\{ u \in L^2(\Omega) \mid \frac{|u(x) - u(y)|}{|x - y|^{n/2+s}} \in L^2(\Omega \times \Omega) \right\},
\]

\[
\|u\|_{H^s(\Omega)} := \left( \int_{\Omega} |u|^2 dx + \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dxdy \right)^{1/2},
\]

\[
H^s_0(\Omega) := \{ u \in H^s(\mathbb{R}^n) \mid u = 0 \text{ in } \Omega^c \}.\]
Control of finite dimensional systems

Consider the finite dimensional system

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \quad 0 \leq t \leq T \\
x(0) &= x_0
\end{align*}
\]  

(1)

with \( x \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m} \) and \( u \in \mathbb{R}^m \).

System (1) is said to be **controllable** in time \( T \) when every initial datum \( x_0 \in \mathbb{R}^n \) can be driven to any final state \( x_T \in \mathbb{R}^n \) in time \( T \). In other words, for any \((x_0, x_T) \in \mathbb{R}^n \times \mathbb{R}^n\), there exists a control function \( u : [0, T] \to \mathbb{R}^m \) such that the solution of (1) satisfies \( x(T) = x_T \).

**Remark**

In a linear finite dimensional setting it is easy to check that (1) is controllable in any time \( T > 0 \) if and only if it is **null controllable** in time \( T > 0 \), i.e. for any \( x_0 \in \mathbb{R}^n \), there exists a control function \( u : [0, T] \to \mathbb{R}^m \) such that the solution of (1) satisfies \( x(T) = 0 \).
Control of finite dimensional systems

Consider the finite dimensional system

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*In a linear finite dimensional setting it is easy to check that (1) is controllable in any time \( T > 0 \) if and only if it is **null controllable** in time \( T > 0 \), i.e. for any \( x_0 \in \mathbb{R}^n \), there exists a control function \( u : [0, T] \to \mathbb{R}^m \) such that the solution of (1) satisfies \( x(T) = 0 \).*
Fractional Schrödinger equation

Let $\Omega$ be a bounded $C^{1,1}$ domain of $\mathbb{R}^n$. Let consider the fractional Schrödinger equation

$$
\begin{cases}
    iu_t + (-\Delta)^s u = 0 & \text{in } \Omega \times [0, T] := Q \\
    u \equiv 0 & \text{in } \Omega^c \times [0, T] \\
    u(x, 0) = u_0(x) & \text{in } \Omega
\end{cases}
$$

Well posedness

$A : \mathcal{D}(A) \to L^2(\Omega)$ defined as

$$
\mathcal{D}(A) := \{ u \in H^s_0(\Omega) | (-\Delta)^s u \in L^2(\Omega) \}
$$

$Au := -i(-\Delta)^s u$

is a monotone and maximal operator.

$\text{Hille – Yosida} \Rightarrow u \in C(0, T; \mathcal{D}(A)) \cap C^1(0, T; L^2(\Omega))$
Fractional Schrödinger equation

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iu_t + (-\Delta)^s u = 0 & \text{in } \Omega \times [0, T] := Q \\
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Well posedness

\( A : D(A) \to L^2(\Omega) \) defined as

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D(A) := \{ u \in H_0^s(\Omega) \mid (-\Delta)^s u \in L^2(\Omega) \} \\
Au := -i(-\Delta)^s u
\]

is a monotone and maximal operator.

\( Hille - Yosida \Rightarrow u \in C(0, T; D(A)) \cap C^1(0, T; L^2(\Omega)) \)
Controllability result

Theorem

Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$ domain with boundary $\Gamma$, $s \in [1/2, 1)$ and $\Gamma_0 = \{ x \in \Gamma \mid (x \cdot \nu) > 0 \}$. Moreover, let $\omega = \mathcal{O}_\varepsilon \cap \Omega$, where $\mathcal{O}_\varepsilon$ is a neighbourhood of $\Gamma_0$ in $\mathbb{R}^n$. For $u_0 \in L^2(\Omega)$ and $h \in L^2(\omega \times [0, T])$, let $u = u(x, t)$ be the solution of

$$
\begin{cases}
  iy_t + (-\Delta)^s y = h\chi(\omega \times [0, T]) & \text{in } \Omega \times [0, T] := Q \\
y \equiv 0 & \text{in } \Omega^c \times [0, T] \\
y(x, 0) = y_0(x) & \text{in } \Omega
\end{cases}
$$

(i) If $s \in (1/2, 1)$, for any $T > 0$ the control function $h$ is such that the solution of (2) satisfies $u(x, T) = 0$.

(ii) If $s = 1/2$ the same result holds for any $T > 2Pd(\Omega) := T_0$, where $P$ is the Poincaré constant associated to the domain $\Omega$. 
Observability inequality

Proposition

Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$ domain with boundary $\Gamma$, $s \in (1/2, 1)$ and $\Gamma_0 = \{ x \in \Gamma \mid (x \cdot \nu) > 0 \}$. Moreover, let $\omega = \mathcal{O}_\varepsilon \cap \Omega$, where $\mathcal{O}_\varepsilon$ is a neighbourhood of $\Gamma_0$ in $\mathbb{R}^n$. For $u_0 \in L^2(\Omega)$, let $u = u(x, t)$ be the solution of (2) with initial datum $u_0$.

(i) If $s \in (1/2, 1)$, for any $T > 0$ the control function $h$ is such that

$$\|u_0\|_{L^2(\Omega)}^2 \leq C \int_0^T \int_\omega |u|^2 \, dx \, dt \quad (3)$$

(ii) If $s = 1/2$, then (3) holds for any $T > 2Pd(\Omega) := T_0$, where $P$ is the Poincaré constant associated to the domain $\Omega$. 
Pohozaev identity

**Proposition**

Let $\Omega$ be a bounded $C^{1,1}$ domain, $s \in (0, 1)$ and $\delta(x)$ be the distance of a point $x$ from $\partial \Omega$. Moreover, let $\Sigma := \partial \Omega \times [0, T]$. For any solution $u$ of

\[
\begin{cases}
  iu_t + (-\Delta)^s u = 0 & \text{in } \Omega \times [0, T] := Q \\
  u \equiv 0 & \text{in } \Omega^c \times [0, T] \\
  u(x, 0) = u_0(x) & \text{in } \Omega
\end{cases}
\]  

(4)

it holds the identity

\[
\Gamma(1 + s)^2 \int_\Sigma \left( \frac{|u|}{\delta^s} \right)^2 (x \cdot \nu) d\sigma dt \\
= 2s \int_0^T \left\|(-\Delta)^{s/2} u\right\|_{L^2(\Omega)}^2 dt + \mathcal{Q} \int_\Omega \bar{u}(x \cdot \nabla u) dx 
\]

where $\nu$ is the unit outward normal to $\partial \Omega$ at $x$ and $\Gamma$ is the Gamma function.
Proof

Proposition

Let $\Omega$ be a bounded $C^{1,1}$ domain of $\mathbb{R}^n$, $s \in (0, 1)$ and $\delta(x) = \text{dist}(x, \partial \Omega)$, with $x \in \Omega$, be the distance of a point $x$ from $\partial \Omega$. Assume that $u \in H^s(\mathbb{R}^n)$ vanishes in $\Omega^c$ and satisfies:

(i) $u \in C^s(\mathbb{R}^n)$ and, for every $\beta \in [s, 1 + 2s)$, $u$ is of class $C^\beta(\Omega)$ and $[u]_{C^\beta(\{x \in \Omega | \delta(x) \geq \rho\})} \leq C \rho^{s-\beta}$ for all $\rho \in (0, 1)$;

(ii) The function $u/\delta^s|_\Omega$ can be continuously extended to $\overline{\Omega}$. Moreover, there exists $\gamma \in (0, 1)$ such that $u/\delta^s \in C^\gamma(\overline{\Omega})$ and, for all $\beta \in [\gamma, s + \gamma]$, $[u/\delta^s]_{C^\beta(\{x \in \Omega | \delta(x) \geq \rho\})} \leq C \rho^{\gamma-\beta}$ for all $\rho \in (0, 1)$;

(iii) $(-\Delta)^su$ is pointwise bounded in $\Omega$.

Then, the following identity holds

$$\int_\Omega (x \cdot \nabla u)(-\Delta)^s u dx = \frac{2s - n}{2} \int_\Omega u(-\Delta)^s u dx - \frac{\Gamma(1+s)^2}{2} \int_{\partial \Omega} \left( \frac{u}{\delta^s} \right)^2 (x \cdot \nu) d\sigma$$

where $\nu$ is the unit outward normal to $\partial \Omega$ at $x$ and $\Gamma$ is the Gamma function.
STEP 1.

**Lemma**

Let consider the eigenvalues problem for the fractional Laplacian

\[
\begin{cases}
(-\Delta)^s u = \lambda u & \text{in } \Omega \\
u \equiv 0 & \text{in } \Omega^c
\end{cases}
\]

Then \(u \in L^\infty(\Omega)\).

**Proposition**

Let \(\Omega\) be a bounded \(C^{1,1}\) domain, \(f \in C^{0,1}_{\text{loc}}(\overline{\Omega} \times \mathbb{R})\) and let \(u\) be a bounded solution of

\[
\begin{cases}
(-\Delta)^s u = f(x, u) & \text{in } \Omega \\
u \equiv 0 & \text{in } \Omega^c
\end{cases}
\]

Then \(u\) satisfies the hypothesis (I), (II) and (III) above.
STEP 2. Let \( \phi_1, \ldots, \phi_k \) be the first \( k \) eigenfunctions of \((-\Delta)^s\) on \( \Omega \). For any initial datum \( u_{k,0} \in \text{span}(\phi_1, \ldots, \phi_k) \), let \( u_k \) be the corresponding solution of (4). By multiplying the equation by \( x \cdot \nabla \bar{u}_k + \frac{n}{2} \bar{u}_k \), taking the real part and integrating over \( Q \) by using the Pohozaev identity for the fractional Laplacian we get

\[
\Gamma(1 + s)^2 \int_{\Sigma} \left( \frac{|u_k|}{\delta^s} \right)^2 (x \cdot \nu) d\sigma dt
= 2s \int_0^T \left\| (-\Delta)^{s/2} u_k \right\|_{L^2(\Omega)}^2 dt + \mathcal{S} \int_0^T \bar{u}_k (x \cdot \nabla u_k) dx \bigg|_0^T
\]

STEP 3. The identity in the general case is obtained by density, employing the following

**Lemma**

Let \( u_k \) the solution of the equation corresponding to the initial data \( u_{k,0} \) we introduced before. Then \( |u_k|/\delta^s \to |u|/\delta^s \) in \( L^2(\Sigma) \) as \( k \to +\infty \).
Boundary observability

Theorem

Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$ domain with boundary $\Gamma$, $s \in [1/2, 1)$ and $\Gamma_0 = \{x \in \Gamma | (x \cdot \nu) > 0\}$. Moreover, let $u_0 \in H^s_0(\Omega)$ and let $u = u(x, t)$ be the corresponding solution of (4) with initial datum $u_0$. Then, there exist two non negative constants $A_1$ and $A_2$, depending only on $n$, $s$, $T$ and $\Omega$, such that

(i) If $s \in (1/2, 1)$, for any $T > 0$ it holds

$$A_1 \|u_0\|_{H^s(\Omega)}^2 \leq \int_{\Sigma} \left( \frac{|u|}{\delta^s} \right)^2 (x \cdot \nu) d\sigma dt \leq A_2 \|u_0\|_{H^s(\Omega)}^2$$  \hspace{1cm} (5)

(ii) If $s = 1/2$, then (5) holds for any $T > 2Pd(\Omega) := T_0$, where $P$ is the Poincaré constant associated to the domain $\Omega$. 


Controllability from a neighbourhood of the boundary

\[ \begin{cases} 
    iu_t + (-\Delta)^s u = 0 \\
    u\big|_{\mathbb{R}^n \setminus \Omega} \equiv 0 \\
    u(x, 0) = u_0(x)
\end{cases} \]

\[ \begin{cases} 
    iy_t + (-\Delta)^s y = u\chi_\omega \times [0, T] \\
    y\big|_{\mathbb{R}^n \setminus \Omega} \equiv 0 \\
    y(x, T) = 0
\end{cases} \]

The function \( y \) is a solution of the backward problem if for any \( \theta \) solution of

\[ \begin{cases} 
    i\theta_t + (-\Delta)^s \theta = f \\
    \theta\big|_{\mathbb{R}^n \setminus \Omega} \equiv 0 \\
    \theta(x, 0) = \theta_0
\end{cases} \]

it holds

Transposition identity

\[ \Re \int_Q y\bar{f} \, dx \, dt - \Re \int_{\Omega} iy(x, 0)\bar{\theta}(x, 0) \, dx = \Re \int_0^T \int_{\omega} u\bar{\theta} \, dx \, dt \] (6)
Controllability from a neighbourhood of the boundary

\[
\begin{aligned}
\left\{
\begin{array}{l}
iu_t + (-\Delta)^s u = 0 \\
\left. u \right|_{\mathbb{R}^n \setminus \Omega} \equiv 0 \\
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\end{array}
\right.
\end{aligned}
\]

\[
\begin{aligned}
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iy_t + (-\Delta)^s y = u \chi_{\omega \times [0, T]} \\
y \left|_{\mathbb{R}^n \setminus \Omega} \equiv 0 \\
y(x, T) = 0
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\right.
\end{aligned}
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\right.
\]

it holds

\[
\Re \int_Q y \overline{f} \, dx \, dt - \Re \int_{\Omega} i y(x, 0) \overline{\theta}(x, 0) \, dx = \Re \int_0^T \int_{\omega} u \overline{\theta} \, dx \, dt \quad \text{(6)}
\]
We introduce the operator $\Lambda : L^2(\Omega) \to L^2(\Omega)$ defined by

$$\Lambda u_0 := -iy(x, 0).$$

Considering (14) for $\theta = u$, it is immediate to check the identity

$$\langle \Lambda u_0; u_0 \rangle = \int_0^T \int_\omega |u|^2 dx dt$$

Thanks to the observability inequality, $\Lambda$ is an isomorphism from $L^2(\Omega)$ to $L^2(\Omega)$. Given $y_0 \in L^2(\Omega)$, let $u$ be the solution of (4) with initial datum $u_0 = \Lambda^{-1}(-iy_0)$. Choosing the control function

$$h(u) = u$$

we finally have $u(x, T) = 0$. 

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$$h(u) = u$$

we finally have $u(x, T) = 0$. 
Fourier analysis for the fractional Schrödinger equation

The exponent $s = 1/2$ is the sharp one for the positivity of the controllability result.

**Theorem**

Let us consider the following one dimensional problem for the fractional Schrödinger equation on the interval $(-1, 1)$

\[
\begin{cases}
    iu_t + (-d_x^2)\beta u = 0 & \text{in } (-1, 1) \times [0, T] \\
    u(-1, t) = u(1, t) = 0 & t \in [0, T] \\
    u(x, 0) = u_0(x) & x \in (-1, 1)
\end{cases}
\]  

(7)

with $\beta \in (0, 1)$. Then, (7) is controllable if and only if $\beta \geq 1/2$. 
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\end{cases}
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with $\beta \in (0, 1)$. Then, (7) is controllable if and only if $\beta \geq 1/2$. 

We are interested in getting a control result by applying HUM. This is equivalent to the proof of an observability inequality, following from a boundary estimate which in this case is

\[ C \| u_0 \|_{H^\beta(-1,1)}^2 \leq \int_0^T \left( \frac{|u|}{(1 - |x|)^\beta} \right)^2 \bigg|_{x=1}^{x=-1} dt. \]  

(8)

Since (8) involves the \( H^\beta \) norm of the initial datum, the natural space in which to analyse the problem is \( H^\beta(-1,1) \); this is an Hilbert space, naturally endowed with the inner product

\[ (u, v)_{H^\beta(-1,1)} = \int_{-1}^{1} uv dx + \int_{-1}^{1} (-d^2_x)^{\beta/2} u (-d^2_x)^{\beta/2} v dx. \]  

(9)
Proof

We are interested in getting a control result by applying HUM. This is equivalent to the proof of an observability inequality, following from a boundary estimate which in this case is

\[
C \| u_0 \|^2_{H^\beta(-1,1)} \leq \int_0^T \left( \frac{|u|}{(1 - |x|)^\beta} \right)^2 \bigg|_{x=1} \bigg|_{x=-1} dt. \quad (8)
\]

Since (8) involves the $H^\beta$ norm of the initial datum, the natural space in which to analyse the problem is $H^\beta(-1,1)$; this is an Hilbert space, naturally endowed with the inner product

\[
(u, v)_{H^\beta(-1,1)} = \int_{-1}^1 uv dx + \int_{-1}^1 (-d_x^2)^{\beta/2} u(-d_x^2)^{\beta/2} v dx. \quad (9)
\]
The solution will be given spectrally, i.e., in terms of the eigenvalues and eigenfunctions of the operator \((-d_x^2)\beta\), namely \(\{\lambda_k, \phi_k(x)\}_{k \geq 1}\).

Since the \(\phi_k\) are eigenfunctions, they form an orthonormal basis of \(L^2(-1, 1)\), i.e., \((\phi_k, \phi_j)_{L^2(-1,1)} = \delta_{k,j}\)

\[
(\phi_k, \phi_j)_{H^\beta(-1,1)} \\
= \int_{-1}^{1} \phi_k(x)\phi_j(x)dx + \int_{-1}^{1} (-d_x^2)^{\beta/2} \phi_k(x)(-d_x^2)^{\beta/2} \phi_j(x)dx \\
= (\phi_k, \phi_j)_{L^2(-1,1)} + \int_{-1}^{1} \phi_k(x)(-d_x^2)^{\beta} \phi_j(x)dx \\
= \delta_{k,j} + \int_{-1}^{1} \lambda_j \phi_k(x)\phi_j(x)dx = \delta_{k,j} + \lambda_j (\phi_k, \phi_j)_{L^2(-1,1)} \\
= (1 + \lambda_j)\delta_{k,j}
\]
The solution will be given spectrally, i.e., in terms of the eigenvalues and eigenfunctions of the operator \((-d_x^2)^\beta\), namely \(\{\lambda_k, \phi_k(x)\}_{k \geq 1}\).

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\[
(\phi_k, \phi_j)_{H^\beta(-1,1)} = \int_{-1}^{1} \phi_k(x)\phi_j(x)dx + \int_{-1}^{1} (-d_x^2)^\beta/2 \phi_k(x)(-d_x^2)^\beta/2 \phi_j(x)dx
\]

\[
= (\phi_k, \phi_j)_{L^2(-1,1)} + \int_{-1}^{1} \phi_k(x)(-d_x^2)^\beta \phi_j(x)dx
\]

\[
= \delta_{k,j} + \int_{-1}^{1} \lambda_j \phi_k(x)\phi_j(x)dx = \delta_{k,j} + \lambda_j (\phi_k, \phi_j)_{L^2(-1,1)}
\]

\[
= (1 + \lambda_j)\delta_{k,j}
\]
Introducing the normalization

\[
\{\theta_k\}_{k \geq 1} = \left\{ \frac{\phi_k}{\sqrt{1 + \lambda_k}} \right\}_{k \geq 1}
\]

we get an orthonormal basis for the space \( H^\beta(-1, 1) \); this is the basis we are going to use for the representation of the solution of the problem

**Solution**

\[
u(x, t) = \sum_{k \geq 1} a_k \theta_k(x) e^{i\lambda_k t}
\]

\[
a_k = \frac{1}{2} \int_{-1}^{1} u_0(x) \theta_k(x) \, dx
\]
Introducing the normalization

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**Solution**

\[ u(x, t) = \sum_{k \geq 1} a_k \theta_k(x) e^{i\lambda_k t} \]

\[ a_k = \frac{1}{2} \int_{-1}^{1} u_0(x) \theta_k(x) dx \]
\[ \| u_0 \|_{H^\beta(-1,1)}^2 = \left( \sum_{k \geq 1} a_k \theta_k, \sum_{k \geq 1} a_k \theta_k \right)_{H^\beta(-1,1)} = \sum_{k \geq 1} |a_k|^2 (\theta_k, \theta_k)_{H^\beta(-1,1)} \]

\[ = \sum_{k \geq 1} |a_k|^2 \]

The boundary estimate becomes

**Boundary estimate**

\[ C_1 \sum_{k \geq 1} |a_k|^2 \leq \int_0^T \left( \sum_{k \geq 1} a_k \frac{\theta_k(x)}{(1 - |x|)^\beta} e^{i \lambda_k t} \right)^2 \left\| \begin{array}{c} x = 1 \\ x = -1 \end{array} \right\| \ dt \]  

(10)
\( \theta_k(x)/(1 - |x|)^\beta \) is continuous up to \( x = \pm 1 \) and for any \( k \geq 1 \) \( \lambda_k > 0 \). Hence, (10) holds only if the vectors \( \{e^{i\lambda_k t}\}_{k \geq 1} \) are linear independent; if not, there will exists at least a coefficient \( a_k \neq 0 \) such that

\[
C_1 \sum_{k \geq 1} |a_k|^2 \leq 0
\]

\( \{e^{i\lambda_k t}\}_{k \geq 1} \) linear independent \( \iff \lambda_k \) all different \( \Rightarrow \lambda_{k+1} - \lambda_k \geq \gamma > 0 \)

T. KULCZYCKI, M. KWAŚNICKI, J. MALECKI and E. STOS - Spectral properties of the Cauchy process on half-line and interval.

**Eigenvalues of \((-d_x^2)^\beta u\) on \((-1, 1)\)**

\[
\lambda_k = \left( \frac{k\pi}{2} - \frac{(2 - 2\beta)\pi}{8} \right)^{2\beta} + O \left( \frac{1}{k} \right) \quad \text{for} \quad k \geq 1
\]

\[
\lambda_{k+1} - \lambda_k \geq \gamma > 0 \iff \beta \geq 1/2
\]
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T. KULCZYCKI, M. KWAŚNICKI, J. MALECKI and E. STOS - Spectral properties of the Cauchy process on half-line and interval.

**Eigenvalues of** \((-d_x^2)\beta u\) **on** \((-1, 1)\)**

\[ \lambda_k = \left( \frac{k\pi}{2} - \frac{(2 - 2\beta)\pi}{8} \right)^{2\beta} + O\left(\frac{1}{k}\right) \text{ for } k \geq 1 \]

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Eigenvalues of \((-d_x^2)\beta\) on \((-1, 1)\) for \(\beta = 0.1, 0.2, 0.3, 0.4, 0.5\)

![Eigenvalues](image1)

**Figure: Eigenvalues**

Eigenvalues \((-d_x^2)\beta\) on \((-1, 1)\) for \(\beta = 0.6, 0.7, 0.8, 0.9, 1.\)

![Eigenvalues](image2)

**Figure: Eigenvalues**

Eigenvalues \((-d_x^2)\beta\) on \((-1, 1)\) for \(\beta = 0.6, 0.7, 0.8, 0.9, 1.\)

![Gap](image3)

**Figure: Gap**
Introduction Fractional Schrödinger equation Fractional wave equation

Fractional wave equation

\[ u_{tt} + (-\Delta)^{2s} u = 0, \quad s \in [1/2, 1) \]

The equation is controllable if and only if we consider a fractional Laplacian of order \( s \geq 1 \).

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\begin{aligned}
  &u_{tt} + (-d_x^2)^\beta u = 0 & \text{in } (-1, 1) \times [0, T] \\
  &u(-1, t) = u(1, t) = 0 & t \in [0, T] \\
  &u(x, 0) = u_0(x) & x \in (-1, 1) \\
  &u_t(x, 0) = u_1(x) &
\end{aligned}
\] (11)

Solution

\[
u(x, t) = \sum_{k \geq 1} \left\{ a_k \phi_k(x) \cos(\sqrt{\lambda_k} x) + b_k \phi_k(x) \sin(\sqrt{\lambda_k} x) \right\}
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Higher order fractional laplacian

\[ s > 1 \Rightarrow (-\Delta)^s u = (-\Delta)^{s_1}(-\Delta)^{s_2} u \quad s_1 + s_2 = s \]

Definition

\[ (-\Delta)^{2s} u(x) := (-\Delta)^s(-\Delta)^s u(x), \quad s \in (1/2, 1) \]

\[ \mathcal{D}((-\Delta)^{2s}) = \{ u \in H^1_0(\Omega) \mid (-\Delta)^s u|_{\Omega^c} \equiv 0, (-\Delta)^{2s} u \in L^2(\Omega) \} \]

Being defined composing twice a positive, symmetric and self-adjoint operator, \((-\Delta)^{2s}\) inherits all these properties.

The energy associated to the corresponding wave equation is

Energy

\[ E(t) := \frac{1}{2} \int_{\Omega} \left\{ ((-\Delta)^{\frac{s}{2}} u_t)^2 + ((-\Delta)^{\frac{3s}{2}} u)^2 \right\} \, dx \]
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Controllability result

**Theorem**

Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$ domain with boundary $\Gamma$, $s \in [1/2, 1)$ and $\Gamma_0 = \{ x \in \Gamma \mid (x \cdot \nu) > 0 \}$. Moreover, let $\omega = \mathcal{O}_\varepsilon \cap \Omega$, where $\mathcal{O}_\varepsilon$ is a neighbourhood of $\Gamma_0$ in $\mathbb{R}^n$. For any couple of initial data $(u_0, u_1) \in H^{2s}(\Omega) \times L^2(\Omega)$ and $h \in L^2(0, T; H^{2s}(\omega))$, let $u = u(x, t)$ be the solution of

$$
\begin{cases}
  u_{tt} + (-\Delta)^{2s} u = h \chi(\omega \times [0, T]) & \text{in } \Omega \times [0, T] := Q \\
  u \equiv (-\Delta)^s u \equiv 0 & \text{in } \Omega^c \times [0, T] \\
  u(x, 0) = u_0(x) & \text{in } \Omega \\
  u_t(x, 0) = u_1(x) & \text{in } \Omega
\end{cases}
$$

(i) If $s \in (1/2, 1)$, for any $T > 0$ the control function $h$ is such that the solution of (2) satisfies $u(x, T) = u_t(x, T) = 0$.

(ii) If $s = 1/2$ the same result holds for any $T > 2Pd(\Omega) := T_0$, where $P$ is the Poincaré constant associated to the domain $\Omega$. 
Proposition

Let $A_0$ be a linear, self-adjoint operator such that $A_0^{-1}$ is compact and such that, for some $d \in \mathbb{N}$, its eigenvalues $\{\lambda_k\}_{k \in \mathbb{N}}$ satisfy

$$\sum_{k \in \mathbb{N}} \lambda_k^{-d} < +\infty. \quad (13)$$

Let $H$ be an Hilbert space, $H_1 := \mathcal{D}(A_0)$ and $X := H_1 \times H$. Moreover, let us define $A : \mathcal{D}(A) \to X$ by

$$\mathcal{D}(A) = \mathcal{D}(A_0^2) \times H$$

$$A \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} 0 & I \\ -A_0^2 & 0 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ -A_0^2 f \end{bmatrix}.$$

Let $Y$ be another Hilbert space and let $C_0 \in \mathcal{L}(H_1, Y)$ be such that the pair $(iA_0, C_0)$ is exactly observable in some time $T_0$. Then, the pair $(A, C)$, with $C = [0 \ C_0]$, is exactly observable in any time $T > T_0$. 

Proof.

M. TUCSNAK and G. WEISS - Observation and control for operators semigroups
Proof.

We apply the Proposition presented before with

\[ A_0 := (-\Delta)^s, \quad A_0^2 := (-\Delta)^{2s}, \quad H = Y := L^2(\Omega). \]

The eigenvalue condition (13) is satisfied with \( d = n \).

\[ \lambda_k \sim \frac{n + 2s}{n} \frac{c_{n,s}}{|\Omega|^{2s/n}} k^{2s/n} \quad \text{as} \ k \to +\infty \]

R.M. BLUMENTHAL and R.K. GETOOR - The asymptotic distribution of the eigenvalues for a class of Markov operators

From the observability inequality for the fractional Schrödinger equation, we get

\[ \|u_0\|_{H^{2s}(\Omega)}^2 + \|u_1\|_{L^2(\Omega)}^2 \leq C \int_0^T \|u_t\|_{L^2(\Omega)}^2 \, dt. \]
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1 $\phi(x, t) = (-\Delta)^s u(x, t)$ satisfies

$$\begin{cases}
    \phi_{tt} + (-\Delta)^{2s} \phi = 0 & \text{in } Q \\
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    \phi(x, 0) = (-\Delta)^s u_0(x), \ \phi_t(x, 0) = (-\Delta)^s u_1(x) & \text{in } \Omega
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$$\|u_0\|_{H^{4s}(\Omega)}^2 + \|u_1\|_{H^{2s}(\Omega)}^2 \leq C \int_0^T \|u_t\|_{H^{2s}(\omega)}^2 dt$$

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We conclude applying HUM with the control $h(u) = (-\Delta)^{2s} u$. 

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We conclude applying HUM with the control \( h(u) = (-\Delta)^{2s} u \).
Open problems and perspectives

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  - M. WARMA - The Pohozaev identity for the regional fractional Laplacian with general boundary conditions
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THANKS FOR YOUR ATTENTION!