Study of Phononic and Photonic Crystal Problems
by Topological Optimization Method

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Abstract

In this paper, we propose a numerical method for study phononic and photonic crystal problems. The proposed method is based on topological optimization tools. In fact, after modeling crystal photonic and phononic problems, we use topological optimization tools in order to get optimal design. Here the optimal design is the one in which all frequencies near the reference frequency corresponding to the reference wave length $a$ can pass.

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1 Introduction

Phononic crystal
While trying to propagate in a periodically structured media, acoustic waves
may experience situations in which their way forward is totally forbidden, but they can also be confined or constrained to follow the most complicated routes, on a wavelength scale.

**Definition 1.1** *Phononic crystals are synthetic materials that are formed by periodic variation of the acoustic properties of the material.*

One of the main properties of the phononic crystals is the possibility of having a phononic bandgap.

**Applications**

Before going further into our subject, let us introduce some general ideas on acoustic and elastic waves. Such waves are actually part of our everyday experiment and of our closer environment. Sonic waves propagate in the atmosphere, they convey human sound and inform us on what is around. Acoustic waves are used in various areas, such as ultrasound imaging of the human body, detection and location of underwater objects (the sonar), or sismology. The quartz crystals in our wristwatches take advantage of particular resonances of the acoustic waves that propagate within them. Mobile phones and televisions include electronic filters exploiting high frequency acoustic waves in exotic synthetic crystals, such as lithium tantalate (a material with the chemical formula LiTaO3).

**Photonic crystal**

Essentially, photonic crystals contain regularly repeating internal regions of high and low dielectric constant. Photons (behaving as waves) propagate through this structure - or not - depending on their wavelength. Wavelengths of light that are allowed to travel are known as modes, and groups of allowed modes form bands. Disallowed bands of wavelengths are called photonic band gaps. This gives rise to distinct optical phenomena such as inhibition of spontaneous emission, high-reflecting omni-directional mirrors and low-loss-wave-guiding, amongst others.

**Definition 1.2** *Photonic crystals are composed of periodic dielectric or metallo-dielectric nano-structures that affect the propagation of electromagnetic waves (EM) in the same way as the periodic potential in a semiconductor crystal affects the electron motion by defining allowed and forbidden electronic energy bands.*

**Applications**

Photonic crystals are attractive optical materials for controlling and manipulating the flow of light. One dimensional photonic crystals are already in widespread use in the form of thin-film optics with applications ranging from low and high reflection coatings on lenses and mirrors to color changing paints and inks. Higher dimensional photonic crystals are of great interest for both
fundamental and applied research, and the two dimensional ones are beginning to find commercial applications. The first commercial products involving two-dimensionally periodic photonic crystals are already available in the form of photonic-crystal fibers, which use a micro-scale structure to confine light with radically different characteristics compared to conventional optical fiber for applications in nonlinear devices and guiding exotic wavelengths. The three-dimensional counterparts are still far from commercialization but offer additional features possibly leading to new device concepts (e.g. optical computers), when some technological aspects such as manufacturable and principal difficulties such as disorder are under control.

In this paper, we use topological derivative in order to propose a numerical method for solving crystal problems. The topological sensitivity analysis aims to provide an asymptotic expansion of a shape functional on the neighborhood of a small hole created inside the domain. The reported analysis shall be based on the principle that follows.

For a criterion $j(\Omega) = J_\Omega(u_\Omega)$, $\Omega \subset \mathbb{R}^N$ with $u_\Omega$ as the solution of a boundary value problem defined over $\Omega$. The asymptotic expansion of the cost function $j(\Omega)$ can generally be written in the form:

$$j(\Omega \setminus x_0 + \varepsilon \omega) - j(\Omega) = \rho(\varepsilon)g(x_0) + o(\rho(\varepsilon)).$$

In this expression, $\varepsilon$ and $x_0$ denote respectively the radius and the center of the hole, $\omega$ is a reference domain inside $\Omega$ and containing $x_0$. The function $g(x_0)$ is called topological derivative and will be used as descent direction in the optimization process.

The structure of this paper is as follow: In the second section we give the modeling of crystal problems. In the section 3, we give briefly topological optimization tools which will be used in order to get topological derivatives of phononic and photonic problems. The section 4 deals with the proposed algorithm and the numerical results.

2 Modeling of crystal problems

In this section, we present a the modeling of crystal problems. The phononic model is based on the acoustic wave equations, and the photonic one in the electromagnetic equations.

2.1 Modeling of the phononic problem

In this paragraph, we propose a modeling of the acoustic wave problem. The modeling is based on the out-flow of a fluid, barotrope, (for more of details cf [5]) occupying a domain $\Omega$ of the space and immersing a unsettled domain
with a weak amplitude (sound source) occupying himself a domain \( \Omega_1 \) (thus \( \Omega_1 \subset \Omega \)). We study the propagation of these small perturbations in the fluid, and therefore the propagation of the sound.

One assumes that the strengths of weight \( \vec{f} \) are negligible in front of the strengths of inertia.

\[
\vec{f} = 0
\]  

(1)

Thus, we can suppose that the out-flow irrotational.

\[
\vec{rot} \vec{u} = 0
\]  

(2)

It follows that there exists a function \( \varphi \) such that

\[
\vec{u} = \vec{\nabla} \varphi
\]  

(3)

From equations of mass conservation, we have

\[
\frac{\partial \rho}{\partial t} + \rho \vec{u} \cdot \vec{\nabla} \varphi + \rho \Delta \varphi = 0
\]  

(4)

\[
\frac{\partial}{\partial x_i} \left[ \frac{\partial \varphi}{\partial t} + \frac{u^2}{2} \right] + \frac{1}{\rho} \frac{\partial p}{\partial x_i} = 0, \ i = 1, 2, 3.
\]  

(5)

with

\[
u^2 = |\vec{\nabla} \varphi|^2.
\]  

(6)

We make classical hypotheses of small perturbations cf [5], such that :

✓ The velocity \( u_i \) is enough small as well as their variations \( u_{i,k}, \frac{\partial u_i}{\partial t} \),

✓ The pressure \( p \) and the density \( \rho \) have small variations around the constant initial data \( p_0 \) and \( \rho_0 \). The linearization of the equations (4) and (5) gives:

\[
\frac{\partial \rho}{\partial t} + \rho \Delta \varphi = 0
\]  

(7)

\[
\frac{\partial}{\partial x_i} \left( \frac{\partial \varphi}{\partial t} \right) + \frac{1}{\rho} \frac{dp}{dx_i} \frac{\partial \rho}{\partial x_i} = 0
\]  

(8)
as $\frac{\partial \varphi}{\partial t}$ and $\frac{\partial \rho}{\partial x_i}$ remain small, we can replace $\rho$ and $\frac{dp}{d\rho}$ by the constants $\rho_0$ and $c_0^2$ (see [5]) then:

$$\frac{\partial \rho}{\partial t} + \rho_0 \Delta \varphi = 0 \quad (9)$$

and

$$\frac{\partial}{\partial x_i} (\frac{\partial \varphi}{\partial t} + c_0^2 \frac{\rho}{\rho_0}) = 0 \quad (10)$$

$$\frac{\partial \varphi}{\partial t} + c_0^2 \frac{\rho}{\rho_0}$$

is a function of $t$

Then

$$\frac{\partial \varphi}{\partial t} + c_0^2 \frac{\rho}{\rho_0} = k(t) \quad (11)$$

$\varphi$ can be modify by a function of $t$ without changing the value of $\vec{u}$; indeed if we set

$$\Phi = \varphi - \int_0^t k(s)ds \quad (12)$$

we get

$$\vec{u} = \vec{\nabla} \Phi \quad \text{and} \quad \Delta \varphi = \Delta \Phi.$$  

A combination of the relations (10), (11) and (12) show that $\Phi$ is obtained by the wave equation

$$\frac{1}{c_0^2} \frac{\partial^2 \Phi}{\partial t^2} - \Delta \Phi = 0 \quad (13)$$

In order to get a well posed problem, we add to (13) the following initial conditions

$$\Phi(x, 0) = 0, \quad \frac{\partial \Phi}{\partial t}(x, 0) = 0 \quad (14)$$

Setting $\Phi = \Phi_0 e^{i\omega t}$, it follows $\frac{\partial \Phi}{\partial t} = i\omega \Phi$ and $\frac{\partial^2 \Phi}{\partial t^2} = -\omega^2 \Phi$
the equation (13) becomes:

$$-\frac{\omega^2}{c_0^2} \Phi - \Delta \Phi = 0 \quad (15)$$

and setting $k = \frac{\omega}{c_0}$, we can write the final model under the form:

$$\left\{ \begin{array}{ll} -\Delta \Phi = k^2 \Phi & \text{in} \Omega \\ \Phi = 0 & \text{on} \ \partial \Omega \end{array} \right. \quad (16)$$
2.2 Modeling of the photonic problem

A crystal photonic problem is a periodic dielectric structure that has the feature that there probability frequencies for the propagation of the electromagnetic waves inside.

The photonic problem is based on the electromagnetic equations.

Let $E$ be the electric field, $B$ the magnetic one, $i, j$ the current density vector, $\mu$ the medium magnetic permeability, $\epsilon$, the medium permittivity and $\rho$ the volume density of the electric charge.

The Maxwell, magnetic and electric equations write

$$\epsilon \oint E \cdot ds = Q,$$

$$\oint B \cdot ds = 0$$

**Faraday's Law of electromagnetic induction**

$$\oint E \cdot ds = -\frac{dB}{dt}, \quad \nabla E = -\frac{\partial B}{\partial t}$$

**Ampere's law extended by Maxwell**

$$\oint B \cdot dl = \mu \left( \epsilon \int_s \frac{\partial E}{\partial t} \cdot ds + i \right), \quad \nabla \cdot H = J + \frac{\partial D}{\partial t}$$

where $D = \epsilon E$ and $B = \mu H$.

It follows that the Maxwell equations write in differential form:

$$\begin{cases}
\nabla \cdot E = \frac{\rho}{\epsilon} \\
\nabla \times E = -\mu \frac{\partial H}{\partial t} \\
\nabla \cdot H = 0 \\
\nabla \times H = j + \epsilon \frac{\partial E}{\partial t}
\end{cases} \tag{17}$$

In the special case where $\rho = 0$, $j = 0$, $\mu = \mu_0$, the Maxwell system writes

$$\begin{cases}
\nabla \cdot E = 0 \\
\nabla \times E = -\mu_0 \frac{\partial H}{\partial t} \\
\nabla \cdot H = 0 \\
\nabla \times H = \epsilon \frac{\partial E}{\partial t}
\end{cases} \tag{18}$$

Thus

$$\nabla \times (\nabla \times E) = \nabla \times (- \mu_0 \frac{\partial H}{\partial t}) = -\mu_0 \epsilon \frac{\partial^2 E}{\partial t^2}$$

It is well known that

$$\nabla \times (\nabla \times E) = (\nabla (\nabla \cdot E) - \nabla^2 E)_{=0}$$
It follows that
\[-\nabla^2 E = -\Delta E = -\mu_0 \epsilon \frac{\partial^2 E}{\partial t^2}\]

Without lost of generality, we can set \(\mu_0 \epsilon = 1\), we obtain
\[\frac{\partial^2 E}{\partial t^2} - \Delta E = 0 \quad (19)\]

When the solution wave is monochromatic (and that depends on boundary
and initial conditions), \(E\) is of the form
\[E(x, t) = \mathcal{R} \epsilon (u(x)e^{ikt})\]

where \(u\) is a solution of the Helmholtz equation.

Adding initial conditions, we obtain the final model of the crystal photonic
problem.

\[
\begin{aligned}
\Delta u + k^2 u &= 0 \quad \text{on} \quad \Omega \\
u &= 0 \quad \text{in} \quad \partial\Omega \\
\end{aligned}
\quad (20)
\]

3 Topological optimization

The goal of the topological optimization problem is to find an optimal de-
sign with an a priori poor information on the optimal shape of the initial
domain. The shape optimization problem consists in minimizing a functional
\(j(\Omega) = J(\Omega, u_\Omega)\) where the function \(u_\Omega\) is defined, for example, on an open
and bounded subset \(\Omega\) of \(\mathbb{R}^N\). For \(\varepsilon > 0\), let \(\Omega_\varepsilon = \Omega \setminus (x_0 + \varepsilon \omega)\) be the set
obtained by removing a small part \(x_0 + \varepsilon \omega\) from \(\Omega\), where \(x_0 \in \Omega\) and \(\omega \subset \mathbb{R}^N\)
is a fixed open and bounded subset containing the origin. Then, using general
adjoint method, an asymptotic expansion of the function will be obtained in
the following form
\[j(\Omega_\varepsilon) = j(\Omega) + f(\varepsilon)g(x_0) + o(f(\varepsilon)), \quad \lim_{\varepsilon \to 0} f(\varepsilon) = 0, \quad f(\varepsilon) > 0 \quad (21)\]

The topological sensitivity \(g(x_0)\) provides information when the creating a
small hole located at \(x_0\). Hence, the function \(g\) will be used as descent direction
in the optimization process.

3.1 A generalized adjoint method

The mathematical framework for domain parametrization introduced by the
Murat-Simon work [16] cannot be used here. Alternatively, it is possible how-
ever to invoke the adjoint method, as described in [14], in application to topo-
logical optimization. A basic feature of the adjoint method is yield of an
asymptotic expansion of a functional $J(\Omega, u_{\Omega})$ which depends of a parameter $u_{\Omega}$, using a adjoint state $v_{\Omega}$ which does not depend on the parameter. This implies the need to solve a certain system of equations in order to obtain an approximation of the topological gradient $g(x)$, $\forall x \in \Omega$. Accordingly, let $\mathcal{V}$ be a fixed Hilbert space and $\mathcal{L}(\mathcal{V})$ (resp $\mathcal{L}_2(\mathcal{V})$) denotes the spaces of linear (resp bilinear) forms on $\mathcal{V}$. We are able then to state the following hypotheses:

- **H-1:** There exists a real function $\rho$, a bilinear form $\delta_{a} \in \mathcal{L}_2(\mathcal{V})$ and a linear form $\delta_{l} \in \mathcal{L}(\mathcal{V})$ such that:

$$
\rho(\varepsilon) \longrightarrow 0, \quad \varepsilon \longrightarrow 0^+, 
$$

$$
\| a_{\varepsilon} - a_0 - \rho(\varepsilon)\delta_{a} \|_{\mathcal{L}_2(\mathcal{V})} = o(\rho(\varepsilon)),
$$

$$
\| l_{\varepsilon} - l_0 - \rho(\varepsilon)\delta_{l} \|_{\mathcal{L}(\mathcal{V})} = o(\rho(\varepsilon)).
$$

- **H-2:** The bilinear form $a_0$ is coercive: There exists a constant $\alpha > 0$ such that

$$
a_0(u, u) \geq \alpha\|u\|^2, \quad \forall u \in \mathcal{V}.
$$

According to (23), the bilinear form $a_{\varepsilon}$ depends continuously on $\varepsilon$, hence $\exists \varepsilon_0$ and $\beta > 0$ such that for $\varepsilon \in [0, \varepsilon_0]$ the following uniform coercivity condition holds.

$$
a_{\varepsilon}(u, u) \geq \beta\|v\|^2 \quad \forall v \in \mathcal{V}.
$$

Moreover, according to Lax-Milgram’s theorem, for $\varepsilon \in [0, \varepsilon_0]$, the problem find $u_{\varepsilon} \in \mathcal{V}$, such that

$$
a_{\varepsilon}(u_{\varepsilon}, v) = l_{\varepsilon}(v) \quad \forall v \in \mathcal{V},
$$

has a unique solution.

**Lemma 3.1** [8] If the hypotheses **H-1** and **H-2** hold, then

$$
\|u_{\varepsilon} - u_0\| = O(\rho(\varepsilon)).
$$

- **H-3:** Consider a cost function $j(\varepsilon) = J(u_{\varepsilon})$, where the functional $J$ is differentiable. For $u \in \mathcal{V}$ there exists a linear and continuous form $DJ(u) \in \mathcal{L}(\mathcal{V})$ and $\delta_J$ such that:

$$
J(u) - J(v) = DJ(u)(u - v) + \rho(\varepsilon)\delta_J(u) + o(\|u - v\|_{\mathcal{V}}).
$$
The Lagrangian \( \mathcal{L}_\varepsilon \) is defined by,
\[
\mathcal{L}(u, v) = a(u, v) - l(v) + J(u) \quad \forall u, v \in \mathcal{V},
\]
and its variation is given, for all \( \varepsilon \geq 0 \),
\[
\mathcal{L}_\varepsilon(u, v) = a_\varepsilon(u, v) - l_\varepsilon(v) + J(u) \quad \forall u, v \in \mathcal{V},
\]
to be led to the next theorem which gives an asymptotic expansion for \( j(\varepsilon) \).

**Theorem 3.2** \([11, 9, 24]\) If hypotheses H-1, H-2, and H-3 are satisfied, then
\[
j(\varepsilon) - j(0) = \rho(\varepsilon)\delta \mathcal{L}(u_0, v_0) + o(\rho(\varepsilon)),
\]
where \( u_0 \) is the solution of (25) with \( \varepsilon = 0 \), \( v_0 \) is the solution to the adjoint problem, find \( v_0 \) such that:
\[
a_0(w, v_0) = -DJ(u_0)w \quad \forall w \in \mathcal{V},
\]
and
\[
\delta \mathcal{L}(u, v) = \delta a(u, v) - \delta l(v) + \delta J(u).
\]
In order to get the asymptotic expansion of the cost functional, we will use the fact that variation of the Lagrangian is equal to the one of the cost functional.

### 3.2 Application to phononic crystals

In this section, we use topological optimization tools in order to study the phononic problem. The topological optimization problem consists to minimize the functional
\[
J(u) = J(\lambda_\Omega) = (\lambda_\Omega - a)^2
\]
where \( \lambda_\Omega \) is the first eigenvalue of \(-\Delta\), ”a” a reference wave length and \( u \) is the solution of following boundary value problem:
\[
\begin{cases}
-\Delta u = \lambda_\Omega u & \text{on } \Omega \\
 0 = u & \text{in } \partial \Omega
\end{cases}
\]
\( \Omega \) is a bounded subset of \( \mathbb{R}^N \).
The variational formulation of (31) is
\[
\begin{cases}
\text{Find } (u, \lambda) \in \left[ H^1_0(\Omega) \backslash \{0\} \right] \times \mathbb{R} \text{ such that} \\
\int_\Omega \nabla u \nabla v dx = \lambda_\Omega \int_\Omega uv dx, \\
\forall v \in H^1_0(\Omega)
\end{cases}
\]
For all \( \varepsilon > 0 \), let \( \Omega_\varepsilon \) be the domain obtained by removing a small part \( \omega_\varepsilon \) in \( \Omega \) and \((u_\varepsilon, \lambda_\varepsilon)\) the solution of
\[
\begin{cases}
-\Delta u_\varepsilon = \lambda_\varepsilon u_\varepsilon & \text{in } \Omega_\varepsilon \\
u_\varepsilon = 0 & \text{on } \partial \Omega \\
u_\varepsilon = 0 & \text{on } \partial \omega_\varepsilon 
\end{cases}
\] (33)
where \( u_\varepsilon \) is the eigenfunction and \( \lambda_\varepsilon \) the first eigenvalue of the problem posed in the perturbed domain. The natural question is if \((u, \lambda)\) is the limit of \((u_\varepsilon, \lambda_\varepsilon)\) as \( \varepsilon \to 0 \).

Theorem 3.3 Let \((u, \lambda)\) respectively, \((u_\varepsilon, \lambda_\varepsilon)\) the solution of (31) respectively (33), then we have the following properties:

i) \( \lambda_\varepsilon \to \lambda \)

ii) There exists a subsequence \( \varepsilon' \) such that
\[ u_{\varepsilon'} \rightharpoonup u \text{ weakly } \in H^1_0(\Omega), \]

iii) If the eigenvalue \( \lambda \) is simple, then the hole sequence \( u_\varepsilon \) converges to \( u \).

Variation of the cost function

It is well known, if \( \lambda_\Omega \) is the first eigenvalue of \( \Delta \), it is given by
\[
\lambda_\Omega = \min_{u \in H^1_0(\Omega), \ u \neq 0} \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega |u|^2 \, dx}
\] (34)

Let \( V = \left\{ v \in H^1_0(\Omega) \text{ such that } \int_\Omega |v|^2 \, dx = 1 \right\} \)

it follows that
\[
\lambda_\Omega = \min_{u \in V} \int_\Omega |\nabla u|^2 \, dx
\]
then, there exists \( \varphi \in H^1_0(\Omega) \) such that \( \lambda_\Omega = \int_\Omega |\nabla \varphi|^2 \, dx \), and \( \int_\Omega \varphi^2 \, dx = 1 \).

Let \( h \in H^1_0(\Omega) \),
\[
J(\varphi + h) = \left( \int_\Omega |\nabla(\varphi + h)|^2 \, dx - a \right)^2
\]
\[
= \left( \int_\Omega (|\nabla \varphi|^2 + |\nabla h|^2 + 2|\nabla \varphi| |\nabla h|) \, dx - a \right)^2
\]
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\[ \begin{align*}
= & \left( \int_{\Omega} |\nabla \varphi|^2 \, dx - a + \int_{\Omega} (|\nabla h|^2 + 2(|\nabla \varphi|, |\nabla h|)) \, dx \right)^2 \\
= & \left( \int_{\Omega} |\nabla \varphi|^2 \, dx - a \right)^2 + \left( \int_{\Omega} (|\nabla h|^2 + 2(|\nabla \varphi|, |\nabla h|)) \, dx \right)^2 \\
& + 2 \left( \int_{\Omega} |\nabla \varphi|^2 \, dx - a \right) \left( \int_{\Omega} (|\nabla h|^2 + 2(|\nabla \varphi|, |\nabla h|)) \, dx \right) \\
= & J(\varphi) + \left( \int_{\Omega} (|\nabla \varphi|^2 + 2(|\nabla \varphi|, |\nabla h|)) \, dx \right)^2 + 2 (\lambda_\Omega - a) \left( \int_{\Omega} |\nabla h|^2 \, dx \right) \\
& + 4(\lambda_\Omega - a) \left( \int_{\Omega} (|\nabla \varphi|, |\nabla h|) \, dx \right).
\end{align*} \]

As
\[4(\lambda_\Omega - a) \left( \int_{\Omega} (|\nabla \varphi|, |\nabla h|) \, dx \right) = 4(\lambda_\Omega - a) \int_{\Omega} -\Delta \varphi \cdot h\]
It is well known that \( (\int_{\Omega} |\nabla u|^2 \, dx)^{1/2} \) is a norm in \( H^1_0(\Omega) \), due to the Poincaré inequality, it follows that \( \int_{\Omega} (|\nabla h|^2 \, dx)^{1/2} \) goes to zeros when \( h \) goes to zeros. Then the functional \( J \) can be written as
\[J(\varphi + h) = J(\varphi) + 4(\lambda_\Omega - a) \int_{\Omega} -\Delta \varphi \cdot h + o(\|h\|_{H^1_0(\Omega)}).\]
Let \( v_0 \) be the solution of the adjoint problem in the form: find \( v_0 \) such that
\[\begin{align*}
-\Delta v_0 &= -4(\lambda_\Omega - a) \Delta u & \text{on} & \Omega \\
v_0 &= 0 & \text{in} & \partial \Omega
\end{align*}\]
(35)
Then we have the following result which gives the asymptotic expansion of the cost function.

**Theorem 3.4** Let \( j(\varepsilon) = J_\varepsilon(u_\varepsilon) \) the cost function, where \( J \) is defined by (30) and \( u \) is solution of (31). Then \( j \) has the following asymptotic expansion
\[j(\varepsilon) = j(0) + f(\varepsilon)g(x_0) + o(f(\varepsilon))\]
where the topological derivative at \( x_0 \in \Omega \) is
\[g(x_0) = -2\pi u(x_0)v_0(x_0)\]
\( v_0 \) is solution of the adjoint problem (35).
3.3 Application to photonic crystals

The topological optimization problem for the photonic is the following

$$\min J_\Omega(u_\Omega) = \int_\Omega |\nabla u_\Omega|^2 dx\,dx$$ \hspace{1cm} (36)

where $u_\Omega$ is solution of

$$\begin{cases}
\Delta u_\Omega + k^2 u_\Omega = 0 \quad \text{on } \Omega \\
u_\Omega = 0 \quad \text{in } \partial\Omega 
\end{cases} \hspace{1cm} (37)$$

Let $u_\Omega^\varepsilon$ be solution of the problem in the perturbed domain as follows

$$\begin{cases}
\Delta u_\Omega^\varepsilon + k^2 u_\Omega^\varepsilon = 0 \quad \text{on } \Omega^\varepsilon \\
u_\Omega^\varepsilon = 0 \quad \text{in } \partial\Omega \\
u_{\Omega^\varepsilon} = 0 \quad \text{in } \partial\omega^\varepsilon 
\end{cases} \hspace{1cm} (38)$$

In order to get the topological derivative, we will use the domain truncation in the sense introduced by Masmoudi in [14], but we will not focus in theoretical details of this method widely used in topological optimization theory. First, it allows the analysis to be made in a fixed Hilbert space. Second, it validates the application of the Lagrangian method. The variation of the Lagrangian can then be written as a continuous global bilinear expression.

Let $R > 0$ be an real such that $\omega^\varepsilon \subset B(x_0, R) \subset \Omega$ and $\Omega_R = \Omega \setminus B(x_0, R) \Gamma_R = \partial B(x_0, R)$, be the truncated domain and $u_\Omega^R$ and be the solution of the truncated problem

$$\begin{cases}
\Delta u_{\Omega^R}^\varepsilon + k^2 u_{\Omega^R}^\varepsilon = 0 \quad \text{on } \Omega_R \\
u_{\Omega^R}^\varepsilon = 0 \quad \text{in } \partial\Omega_R \\
u_{\Omega^R}^\varepsilon = T_\varepsilon u_\varepsilon \quad \text{in } \Gamma_R 
\end{cases} \hspace{1cm} (39)$$

where

$$T_\varepsilon : H^{1/2}(\Gamma_R) \to H^{-1/2}(\Gamma_R)$$

$$\phi \to T_\varepsilon \phi = \nabla u_\Omega^\varepsilon$$

where $\varphi = u_\Omega^\varepsilon |\Gamma_R$.

The variational formulation associated to (39) is the following find: $u_{\Omega^R}^\varepsilon$ such that

$$a_\varepsilon(u_{\Omega^R}^\varepsilon, v) = l(v) \quad \forall v \in \mathcal{V}_R$$

where the functional space $\mathcal{V}_R$, the sesquilinear form $a_\varepsilon$ are defined by

$$\mathcal{V}_R = \{ v \in H^1(\Omega), \quad v = 0 \quad \text{on } \Gamma_R \}$$

$$a_\varepsilon(u, v) = \int_{\Omega_R} \nabla u \nabla v dx - \int_{\Omega_R} k^2 u v dx + \int_{\Gamma_R} (T_\varepsilon u) v d\gamma(x)$$
Proposition 3.5 The problem (39) has only one solution in $V_R$ which is the restriction to $\Omega_R$ of the solution to (37).

The proof of the proposition 3.5 is somewhat standard in partial differential equation theory (see for example [8], for the proof).

Asymptotic expansion

Let $j(\varepsilon) = J_\Omega(u_\varepsilon)$ the cost functional, defined by (36), then the following result holds.

Theorem 3.6 Let $j(\varepsilon) = J_\Omega(u_\varepsilon)$ the objective functional then $j$ have the following asymptotic expansion:

$$j(\varepsilon) - j(0) = f(\varepsilon)\delta_C(u, v) + o(f(\varepsilon))$$

where $v_0$ is a solution of the adjoint problem, which strong formulation is

$$\begin{aligned}
\Delta v_0 + k^2 v_0 &= -DJ(u_\Omega) & \text{in } \Omega \\
v_0 &= 0 & \text{in } \partial\Omega
\end{aligned}$$

Proof. The proof of the theorem consists to verify that the hypothesis $H_1$, $H_2$ and $H_3$ are satisfied.

Variation of the sesquilinear form

$$a_\varepsilon(u_\varepsilon^{\varepsilon,R}, v) - a_0(u_\Omega, v) = \int_{\Omega_R} (\nabla u_\varepsilon^{\varepsilon,R} - \nabla u_\Omega) \nabla v - k^2 \int_{\Omega_R} (u_\varepsilon^{\varepsilon,R} - u_\Omega) v + \int_{\Gamma_R} (T_\varepsilon - T_0)(u_\varepsilon^{\varepsilon}) v d\gamma(x)$$

$$= \int_{\Gamma_R} (T_\varepsilon - T_0)(u_\varepsilon^{\varepsilon})(v d\gamma(x)$$

The variation of the sesquilinear form writes

$$a_\varepsilon(u_\varepsilon^{\varepsilon}, v) - a_0(u, v) = \int_{\Gamma_R} (T_\varepsilon - T_0)(u_\varepsilon^{\varepsilon})(v d\gamma(x)$$

For $\varphi \in H^{1/2}(\Gamma_R)$, let $u_\varepsilon^\varphi$ be the solution $(\varepsilon > 0)$ to

$$\begin{aligned}
\Delta u_\varepsilon^\varphi + k^2 u_\varepsilon^\varphi &= 0 & \text{in } D_\varepsilon = B(x_0, R) \setminus \omega_\varepsilon \\
u_\varepsilon^\varphi &= 0 & \text{in } \partial\omega_\varepsilon \\
u_\varepsilon^\varphi &= \varphi & \text{in } \Gamma_R
\end{aligned}$$

(41)

For $\varepsilon = 0$, $u_0^\varphi$ be the solution of

$$\begin{aligned}
\Delta u_0^\varphi + k^2 u_0^\varphi &= 0 & \text{in } D_0 = B(x_0, R) \\
u_0^\varphi &= \varphi & \text{in } \Gamma_R
\end{aligned}$$

(42)
For $R < (\sqrt{2}|k|)^{-1}$, it is well known that (41) and (42) are well posed.
Let $v_\omega^\varepsilon$ be solution to the problem
\[
\begin{cases}
\Delta v_\omega^\varepsilon = 0 & \text{in } \mathbb{R}^3 \setminus \overline{\omega} \\
v_\omega^\varepsilon = 0 & \text{in } \omega \\
v_\omega^\varepsilon = u_0^\varepsilon(x_0) & \text{in } \partial \omega
\end{cases}
\]  
(43)
and $P_\omega^\varepsilon(y) = A_\omega(u_0^\varepsilon(x_0))E(y)$ be the dominant part of $v_{\omega\varepsilon}$, $Q_{\omega\varepsilon}$, be the solution to the interior associated problem
\[
\begin{cases}
\Delta Q_\omega^\varepsilon + k^2 Q_\omega^\varepsilon = k^2 P_\omega^\varepsilon & \text{in } D_0 \\
Q_\omega^\varepsilon = P_\omega^\varepsilon|\Gamma_R & \text{in } \Gamma_R
\end{cases}
\]  
(44)
the linear operator $\delta T$ (independent of $\varepsilon$) is yet defined as follow:
\[
\delta T : H^{1/2}(\Gamma_R) \rightarrow H^{-1/2}(\Gamma_R)
\]
\[
\varphi \rightarrow \delta T \varphi = \nabla(Q_\omega^\varepsilon - P_\omega^\varepsilon)) \mathcal{V}'|\Gamma_R
\]
and the operator $T_\varepsilon$ admits the following asymptotic expansion
\[
\|T_\varepsilon - T_0 - f(\varepsilon)\delta T\|_{L^2(\varphi)} = o(f(\varepsilon))
\]
the following result holds
Let $\delta_a(u,v) = \int_{\Gamma_R} \delta_T u.vd\gamma(x)$, $u,v \in \Gamma_R$ the asymptotic expansion of the sesquilinear form $a_\varepsilon$ is given by
\[
\|a_\varepsilon - a_0 - f(\varepsilon)\delta a\|_{L^2(\varphi)} = o(f(\varepsilon))
\]
**Variation of the cost functional**
\[
J_\Omega(u_\Omega^\varepsilon) - J_\Omega(u) = \int_{\Omega_\varepsilon} |\nabla u_\Omega^\varepsilon|^2 dx - \int_{\Omega} |\nabla u_\Omega|^2 dx
\]
\[
= \int_{\Omega} (|\nabla u_\Omega^\varepsilon|^2 - |\nabla u_\Omega|^2)dx - \int_{\omega_\varepsilon} |\nabla u_\Omega|^2 dx
\]
(1) = \int_{\Omega} (|\nabla u_\Omega^\varepsilon|^2 - |\nabla u_\Omega|^2)dx = \int_{\Omega} (\nabla u_\Omega^\varepsilon - \nabla u_\Omega)(\nabla u_\Omega^\varepsilon + \nabla u_\Omega)dx
Let
\[
DJ_u(u_\Omega)\varphi = \int_{\Omega} \nabla(u_\Omega^\varepsilon)\nabla \varphi \quad \text{and} \quad \delta f(u)(\varepsilon) = -\int_{\omega_\varepsilon} |\nabla u_\Omega|^2
\]
its follows that
\[
J_\Omega(u_\Omega^\varepsilon) - J_\Omega(u) = DJ_u(u_\Omega^\varepsilon)(u_\Omega^\varepsilon + u_\Omega) + \delta f(u)(\varepsilon)
\]
We can yet apply the theorem 3.2, to get the topological derivative. When \( \omega = B(0,1) \), the mass matrix \( A_\omega \) can be computed and using Saint-Venant principle, a good approximation of \( P_\omega \) and \( Q_\omega \) can be obtained as follows, with symbolic calculus that

\[
\delta_a(u, v) = -2\pi u_{\Omega}(x_0)v_{\Omega}(x_0)
\]

\( v_0 \) is the solution of the adjoint problem, which weak formulation is:

\[
\begin{align*}
\text{Find } v_\Omega \in H_0^1(\Omega) \text{ such that } \\
a(u_\Omega, v_\Omega) = -DJ(u)\varphi; \forall \varphi \in H_0^1(\Omega)
\end{align*}
\]

where

\[
J_\varepsilon(u) = \int_\Omega |\nabla u_\varepsilon|^2 dx, \quad DJ(u) = 2 \int_\Omega \nabla u \nabla (u_\Omega) dx
\]

If follows that the topological derivative in \( x_0 \in \Omega \) is given by

\[
G(x_0) = -2\pi(\nabla u_\Omega(x_0).\nabla v_\Omega(x_0) + u_\Omega(x_0)v_\Omega(x_0))
\]

\[
\blacksquare
\]

4 Numerical simulations

4.1 Numerical results for phononic problem

In this paragraph, we present some numerical results relative to the phononic problem. Let \( \Omega \) be the initial domain, and \( \lambda_1(\Omega) \) the first eigenvalue of \( \Delta \) and "a" the limit wave length.

Our objective is to find the optimal design \( \Omega^* \) in which all admissible frequencies near the limit audible frequency can pass. This limit audible frequency corresponds to the limit wave length \( a = 33, 4 \).

The topological optimization algorithm is summarized in FIG. 1. In all numerical results, we plot the initial domain, the solution of the direct and the adjoint problem and the topological derivative. At each step, we create a small hole, where the topological derivative in the above step.
Step 0: Given an initial domain $\Omega_0$

Step 1: Solve the direct problem

Step 2: Solve the adjoint problem

Step 3: Compute the topological sensitivity

Step 4: Create a small hole where the topological derivative is the most negative

Step 5: The optimal shape is reached

END

Figure 1: The proposed algorithm
Figure 2: step 0: Domain without hole $\lambda_1(\Omega_0) = 5,74$

Figure 3: step 1: Domain with one hole $\lambda_1(\Omega_1) = 8,0887$
Figure 4: step 2: Domain with two holes $\lambda_1(\Omega_2) = 11.7339$

Figure 5: step 3: Domain with three holes $\lambda_1(\Omega_3) = 12,694$
Figure 6: step 4: Domain with four holes \( \lambda_1(\Omega_4) = 14,9164 \)

Figure 7: step 5: Domain with five holes \( \lambda_1(\Omega_5) = 16,0265 \)
Remark 4.1 There is any possibility to put another hole, because, in the
last domain, the topological derivative is equal to zeros, almost everywhere. The optimal shape design is reached after putting seven holes in the initial domain. Then all admissible frequencies near the limit audible one can pass.

4.2 Numerical results for photonic problem

We apply here the same algorithm as above.

![Initial Mesh](image1)

![Direct State](image2)

![Topological derivative](image3)

**Figure 10:** Simulation for the photonic problem

**Remark 4.2** In this case, the topological derivative is equal to zeros, almost everywhere in $\Omega$ except in some small parts of the boundary of the domain. Thus, we can not create a hole centered to $x_0 \in \Omega$.

References


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