Texas A&M University

Eighth IMACS International Symposium on Iterative Methods in Scientific Computation

A Two-Grid Goal-Oriented Iterative Solver for hp-FE Discretizations with Possibly Elongated Elements of Elliptic Problems

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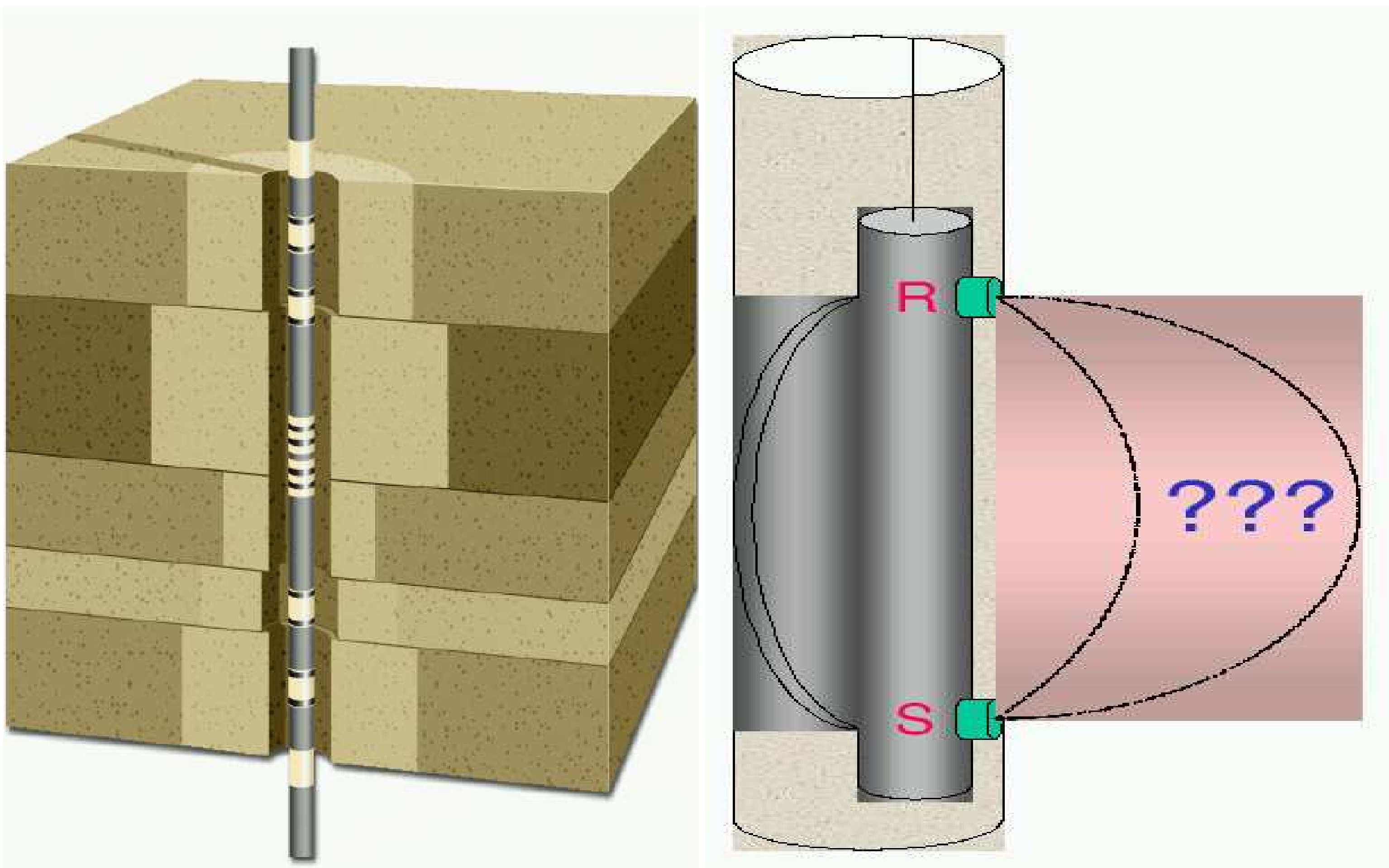
THE UNIVERSITY OF TEXAS AT AUSTIN

OVERVIEW

1. **Motivation**
   - **Simulation of resistivity logging measurements.**
   - **Goal-oriented hp-adaptivity.**
2. **Two-Grid Solver**
   - **Formulation.**
   - **Convergence theory.**
   - **The idea of goal-oriented solver.**
   - **The problem of elongated elements.**
   - **Implementation details.**
3. **Conclusions and future work.**

MOTIVATION (APPLICATIONS)

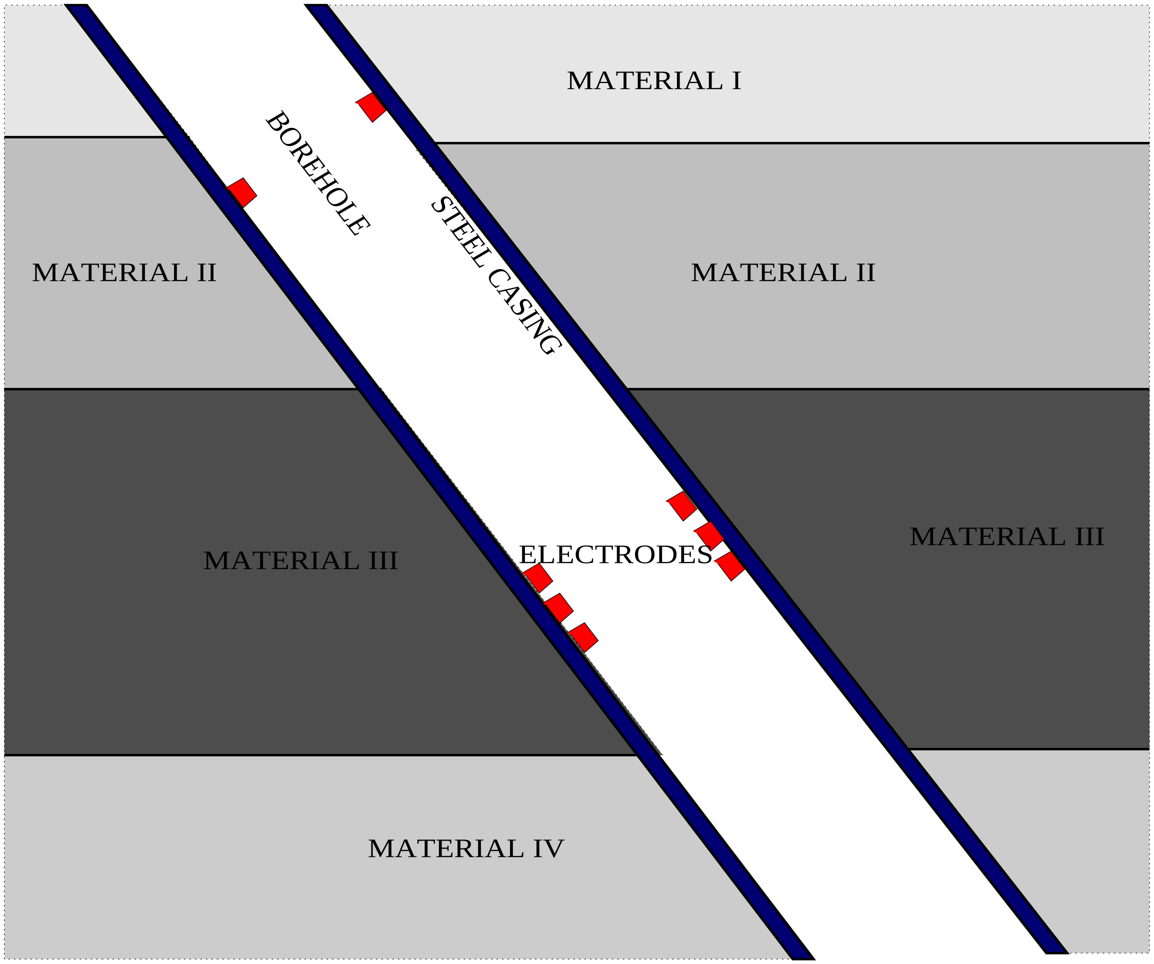
Main Objective: To Solve an Inverse Problem



A software for solving the DIRECT problem is essential in order to solve the INVERSE problem

MOTIVATION (APPLICATIONS)

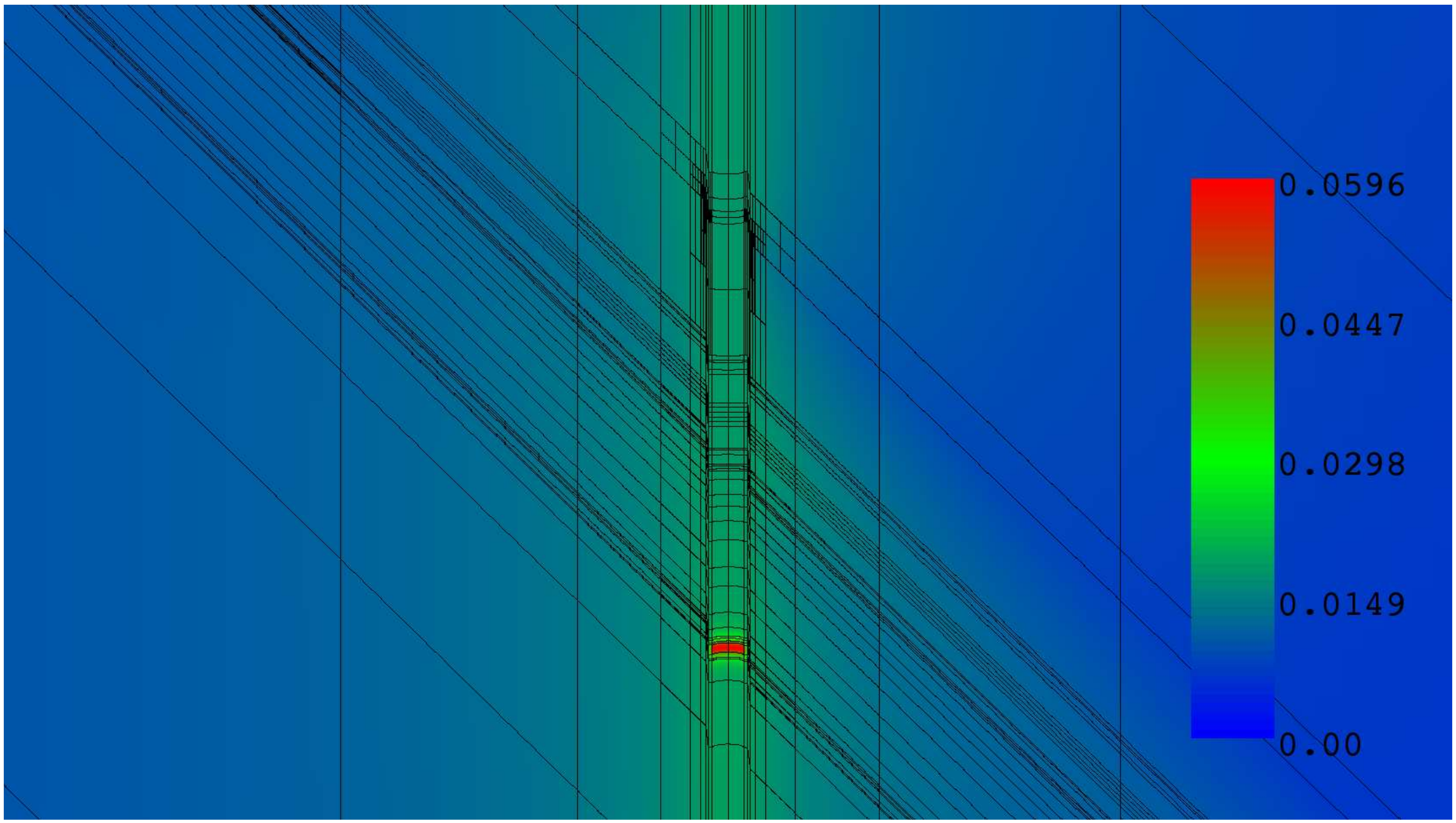
Deviated Cased Wells



Objective: Determine 2nd difference of potential at the receiver antennas.

MOTIVATION (APPLICATIONS)

60 degrees deviated well



MOTIVATION (GOAL-ORIENTED hp-ADAPTIVITY)

Mathematical Formulation (Goal-Oriented Adaptivity)

Let's L be the quantity of interest (Ex.: first vertical difference of electric field).

We consider the following problem (in variational form):

$$\begin{cases} \text{Find } L(\Psi), \text{ where } \Psi \in V \text{ such that :} \\ b(\Psi, \xi) = f(\xi) \quad \forall \xi \in V \ . \end{cases}$$

We define residual $r_e(\xi) = b(e, \xi)$. We seek for solution G of:

$$\begin{cases} \text{Find } G \in V'' \sim V \text{ such that :} \\ G(r_e) = L(e) \ . \end{cases}$$

This is necessarily solved if we find the solution of the *dual* problem:

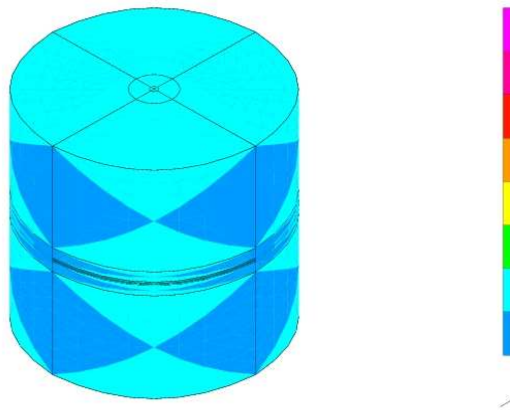
$$\begin{cases} \text{Find } G \in V \text{ such that :} \\ b(\Psi, G) = L(\Psi) \quad \forall \Psi \in V \ . \end{cases}$$

Notice that $L(e) = b(e, G)$.

MOTIVATION (GOAL-ORIENTED hp-ADAPTIVITY)

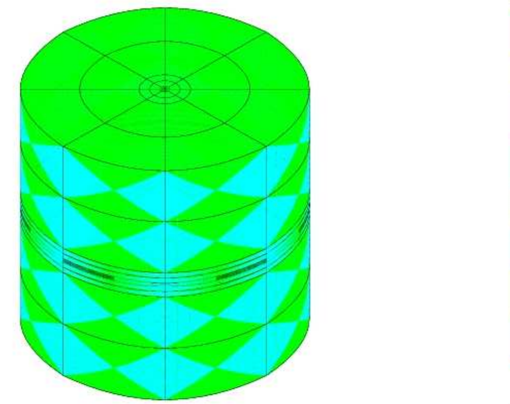
Algorithm for Goal-Oriented Adaptivity - STEP I -

Solve Direct and Dual Problems on Grid hp



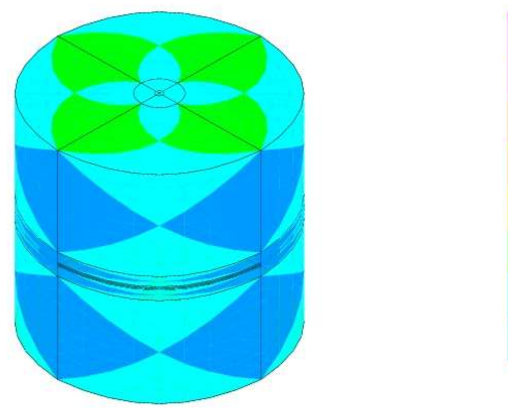
$\longrightarrow$

Solve Direct and Dual Problems on Grid $h/2, p+1$



Use the fine grid solution to estimate the coarse grid error function.
Apply the fully automatic goal-oriented hp-adaptive algorithm.

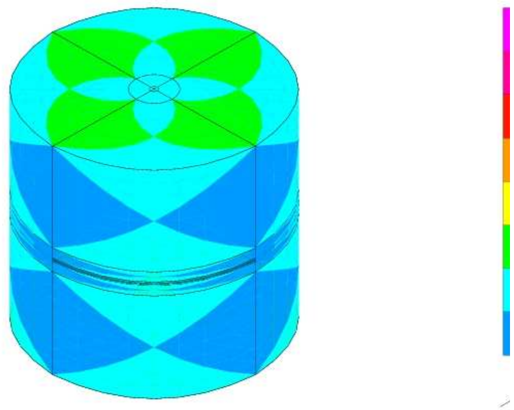
Next optimal hp-grid:



MOTIVATION (GOAL-ORIENTED hp-ADAPTIVITY)

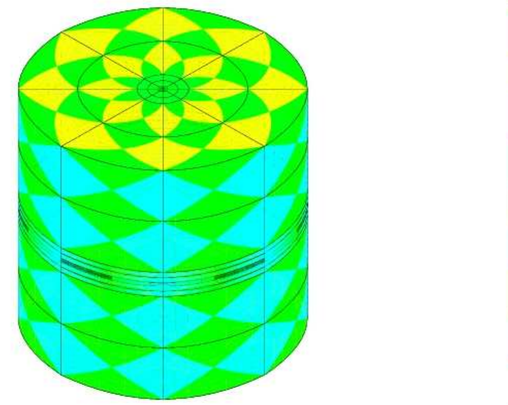
Algorithm for Goal-Oriented Adaptivity - STEP II -

Solve Direct and Dual Problems on Grid hp



$\longrightarrow$

Solve Direct and Dual Problems on Grid $h/2, p+1$



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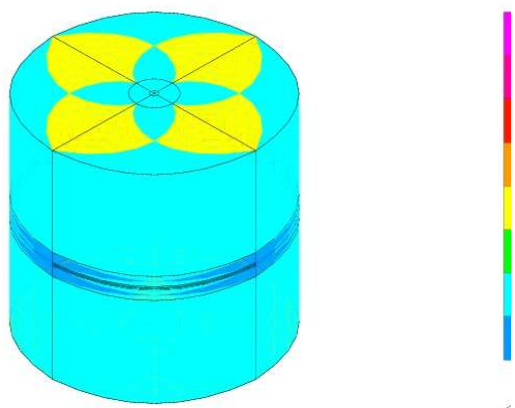
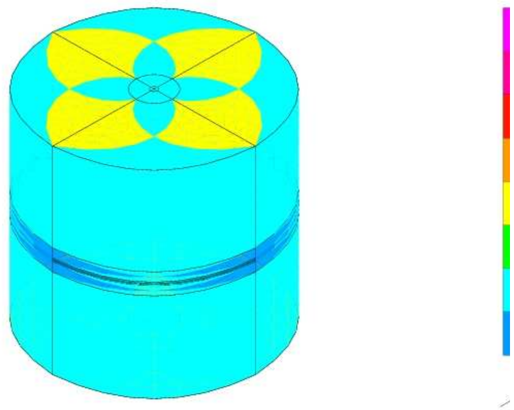


MOTIVATION (GOAL-ORIENTED hp-ADAPTIVITY)

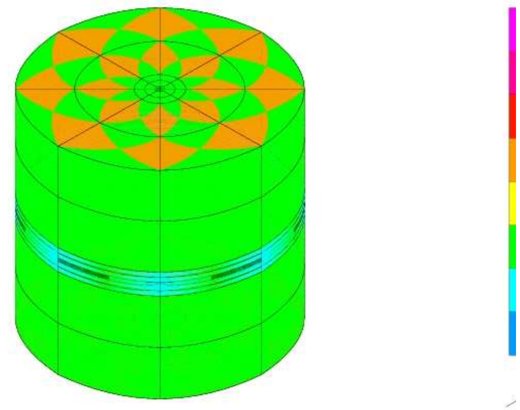
Algorithm for Goal-Oriented Adaptivity - STEP III -

Solve Direct and Dual Problems on Grid hp



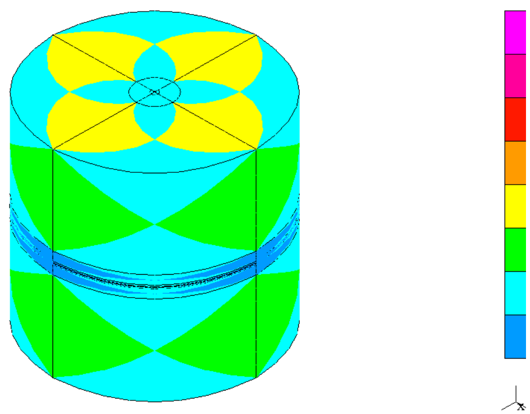
$\longrightarrow$

Solve Direct and Dual Problems on Grid $h/2, p+1$



Use the fine grid solution to estimate the coarse grid error function.
Apply the fully automatic goal-oriented hp-adaptive algorithm.

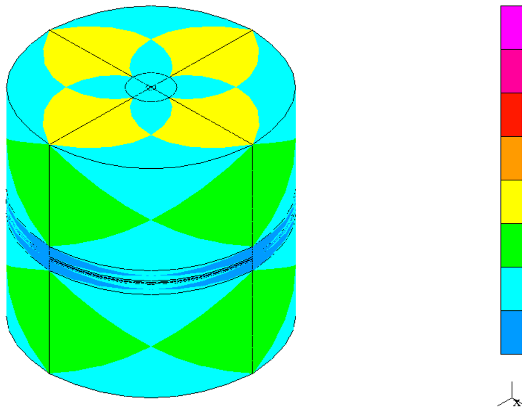
Next optimal hp-grid:



MOTIVATION (GOAL-ORIENTED hp-ADAPTIVITY)

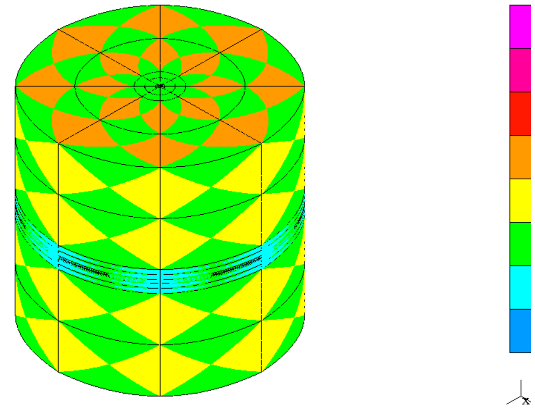
Algorithm for Goal-Oriented Adaptivity - STEP IV -

Solve Direct and Dual Problems on Grid hp



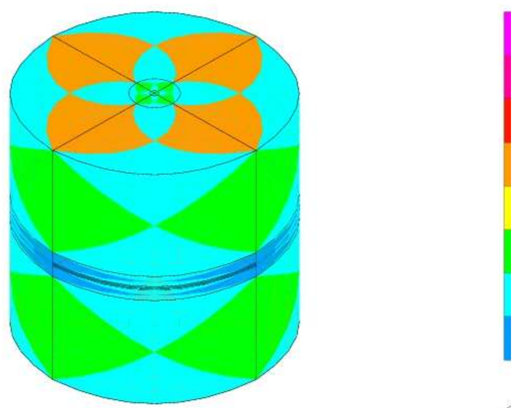
$\longrightarrow$

Solve Direct and Dual Problems on Grid $h/2, p+1$



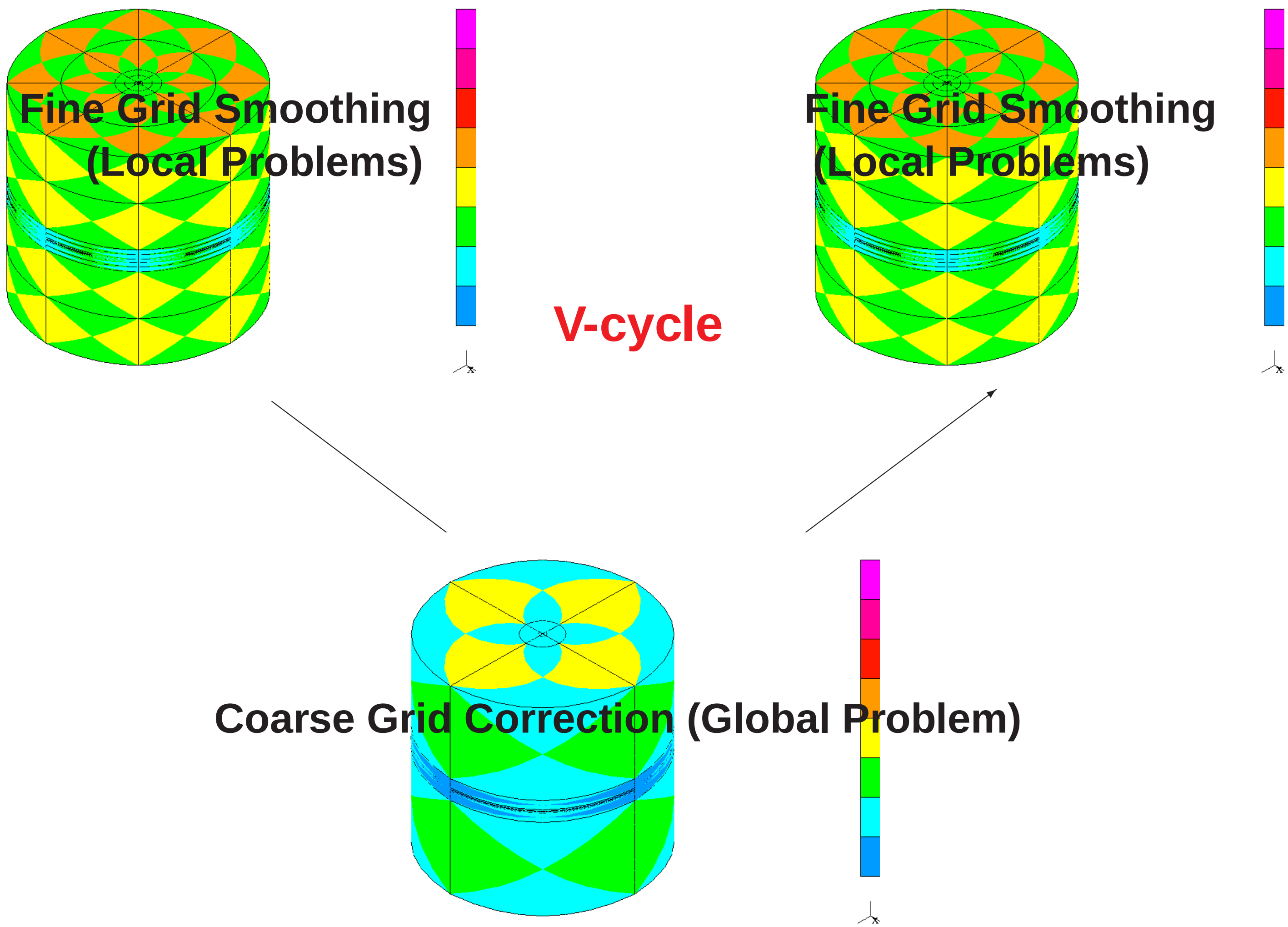
Use the fine grid solution to estimate the coarse grid error function.
Apply the fully automatic goal-oriented hp-adaptive algorithm.

Next optimal hp-grid:



TWO-GRID (TG) SOLVER: FORMULATION

Two-Grid Solver (Ax=b)



TWO-GRID (TG) SOLVER: FORMULATION

We seek x such that $Ax = b$. Consider the following iterative scheme:

$$\boldsymbol{r}^{(n+1)} = [\boldsymbol{I} - \alpha^{(n)} \boldsymbol{A}\boldsymbol{S}]\boldsymbol{r}^{(n)}$$
$$\boldsymbol{x}^{(n+1)} = \boldsymbol{x}^{(n)} + \alpha^{(n)} \boldsymbol{S}\boldsymbol{r}^{(n)}$$

where S is a matrix, and $\alpha^{(n)}$ is a relaxation parameter. $\alpha^{(n)}$ *optimal* if:

$$\alpha^{(n)} = \arg\ \min \| \boldsymbol{x}^{(n+1)} - \boldsymbol{x} \|_A = \frac{(\boldsymbol{A}^{-1}\boldsymbol{r}^{(n)}, \boldsymbol{S}\boldsymbol{r}^{(n)})_A}{(\boldsymbol{S}\boldsymbol{r}^{(n)}, \boldsymbol{S}\boldsymbol{r}^{(n)})_A}$$

Then, we define our two grid solver as:

1 iteration with $S = S_F = \sum A_i^{-1}$ $+$

1 iteration with $S = S_C = P_C A_C^{-1} R_C$

TG SOLVER: CONVERGENCE THEORY

Error reduction and stopping criteria

Error step n **=** $e^{(n)} = x^{(n)} - x$. $\tilde{e}^{(n)} = [I - S_C A]e^{(n)} = [I - P_C]e^{(n)}$. **Then:**

$$\frac{\| e^{(n+1)} \|_A^2}{\| e^{(n)} \|_A^2} = 1 - \frac{| (\tilde{e}^{(n)}, S_F A \tilde{e}^{(n)})_A |^2}{\| \tilde{e}^{(n)} \|_A^2 \| S_F A \tilde{e}^{(n)} \|_A^2} = 1 - \frac{| (\tilde{e}^{(n)}, (P_C + S_F A)\tilde{e}^{(n)})_A |^2}{\| \tilde{e}^{(n)} \|_A^2 \| S_F A \tilde{e}^{(n)} \|_A^2}$$

Then:

$$\frac{\| e^{(n+1)} \|_A^2}{\| e^{(n)} \|_A^2} \leq \sup_e [1 - \frac{| (e, (P_C + S_F A)e)_A |^2}{\| e \|_A^2 \| S_F A e \|_A^2}] \leq C < 1$$

(Error Reduction)

For our stopping criteria, we want: Solver Error $\approx$ **Discretization Error. That is:**

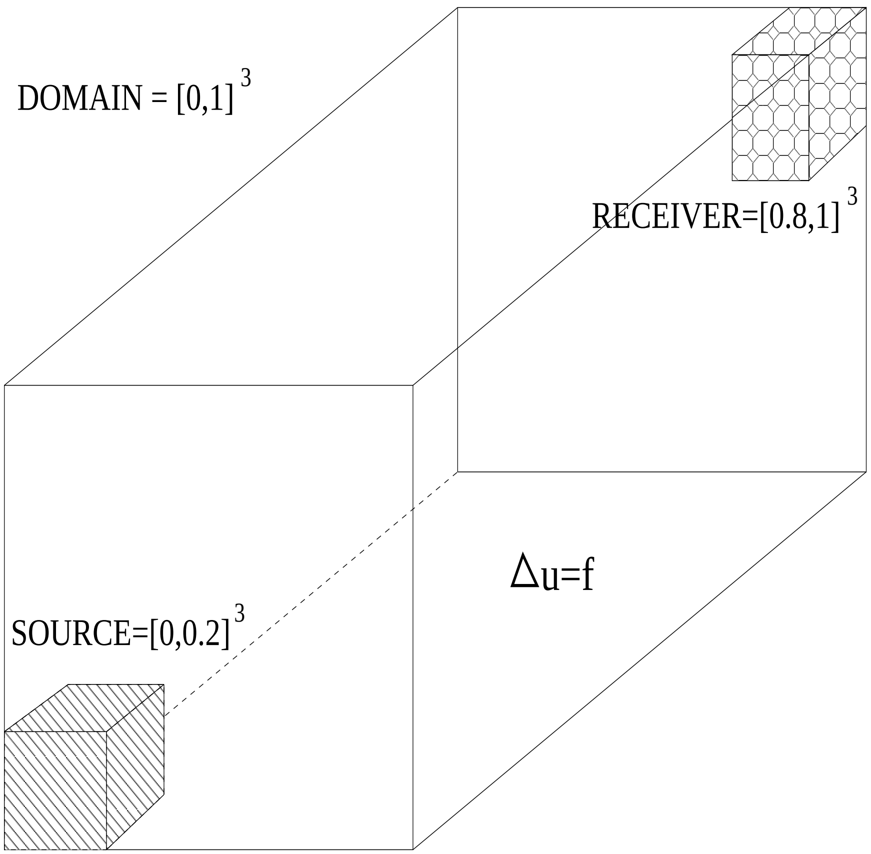
$$\frac{\| e^{(n+1)} \|_A}{\| e^{(0)} \|_A} \leq 0.01$$

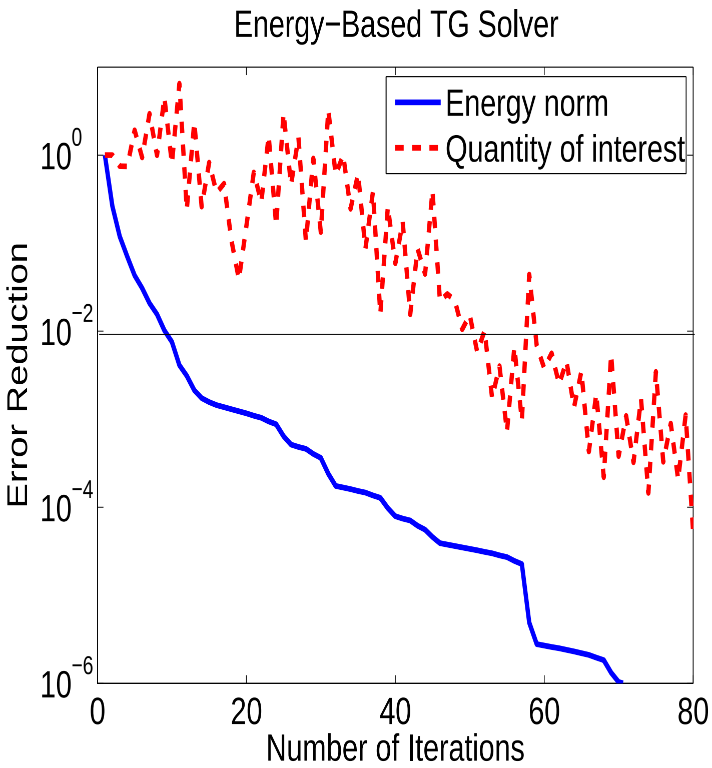
(Stopping Criteria)

TG SOLVER: GOAL-ORIENTED

Goal-Oriented Solver: Motivation

MODEL PROBLEM I



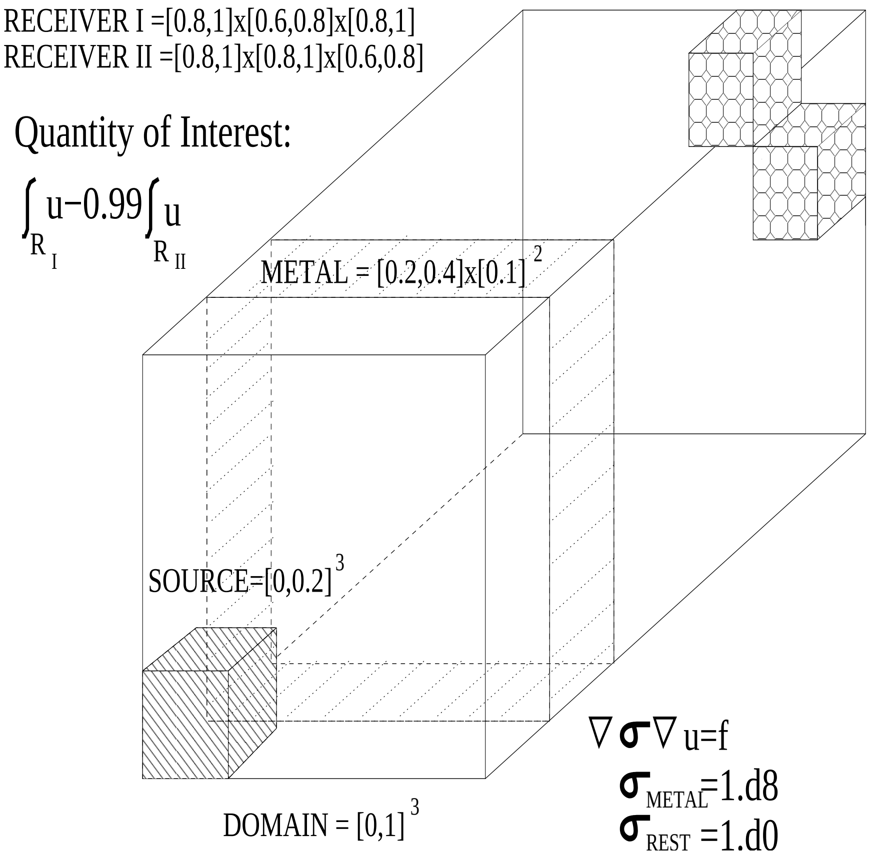


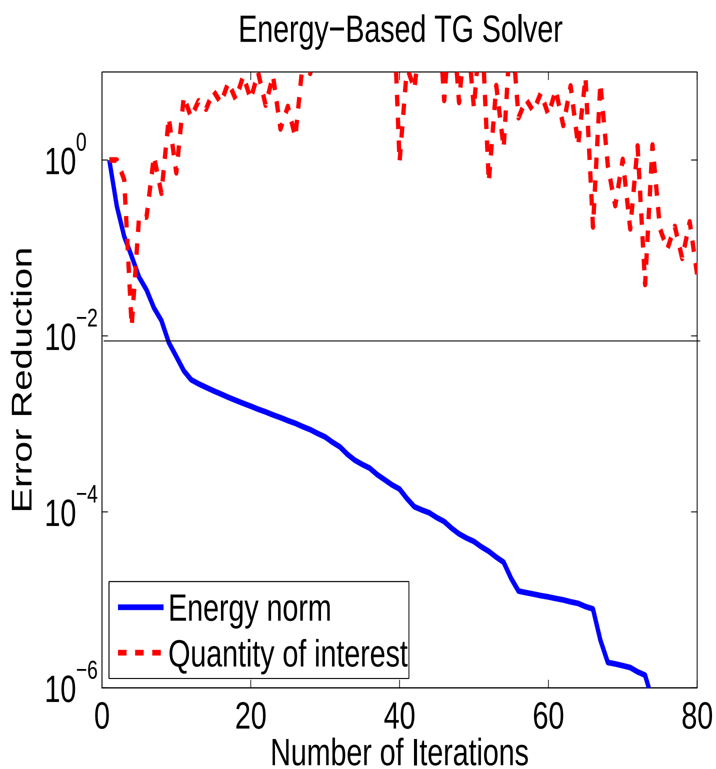
We need 50 iterations to converge!!

TG SOLVER: GOAL-ORIENTED

Goal-Oriented Solver: Motivation

MODEL PROBLEM II





We need more than 80 iterations to converge!!

TG SOLVER: GOAL-ORIENTED

Goal-Oriented Solver: Formulation (Part I)

We seek $L(x)$ such that $Ax = f$ and $Ag = l$. Consider the following iterative scheme:

$$\begin{aligned} \boldsymbol{r}_x^{(n+1)} &= [\boldsymbol{I} - \boldsymbol{\alpha}^{(n)} \boldsymbol{AS}] \boldsymbol{r}_x^{(n)} \quad ; \quad \boldsymbol{r}_g^{(n+1)} = [\boldsymbol{I} - \boldsymbol{\beta}^{(n)} \boldsymbol{AS}] \boldsymbol{r}_g^{(n)} \\ \boldsymbol{x}^{(n+1)} &= \boldsymbol{x}^{(n)} + \boldsymbol{\alpha}^{(n)} \boldsymbol{S} \boldsymbol{r}_x^{(n)} \quad ; \quad \boldsymbol{g}^{(n+1)} = \boldsymbol{g}^{(n)} + \boldsymbol{\beta}^{(n)} \boldsymbol{S} \boldsymbol{r}_g^{(n)} \end{aligned}$$

where S is a matrix, and $\alpha^{(n)}, \beta^{(n)}$ are relaxation parameters. Then, we define our two grid solver as:

1 iteration with $S = S_F = \sum A_i^{-1}$ $+$

1 iteration with $S = S_C = P_C A_C^{-1} R_C$

How do we select $\alpha^{(n)}, \beta^{(n)}$ to be *optimal*?

TG SOLVER: GOAL-ORIENTED

Goal-Oriented Solver: Formulation (Part II)

Selection I: — Recall that $|L(x)| = |b(x,g)| \leq \sum_K |b_K(x,g)| \leq \sum_K ||x||_K ||g||_K$ **—**

$$\alpha^{(n)} = \arg\min \sum_K |b_K(x - x^{(n+1)}, G)| \approx \arg\min \sum_K |b_K(x^{(n+2)} - x^{(n+1)}, G^{(0)})|$$
$$\beta^{(n)} = \arg\min \sum_K |b_K(x, g - g^{(n+1)})| \approx \arg\min \sum_K |b_K(x^{(0)}, g^{(n+2)} - g^{(n+1)})|$$

Selection II:

$$\alpha^{(n)} = \arg\min \sum_K ||x - x^{(n+1)}||_K ||G||_K \approx \arg\min \sum_K ||x^{(n+2)} - x^{(n+1)}||_K ||G^{(0)}||_K$$
$$\beta^{(n)} = \arg\min \sum_K ||g - g^{(n+1)}||_K ||x||_K \approx \arg\min \sum_K ||g^{(n+2)} - g^{(n+1)}||_K ||x^{(0)}||_K$$

Selection III (Goal1):

$$\alpha^{(n)} = \beta^{(n)} = \arg\min \sum_K |b_K(x - x^{(n+1)}, G^{(n+1)})| \approx \arg\min \sum_K |b_K(x^{(n+2)} - x^{(n+1)}, G^{(n+1)})|$$

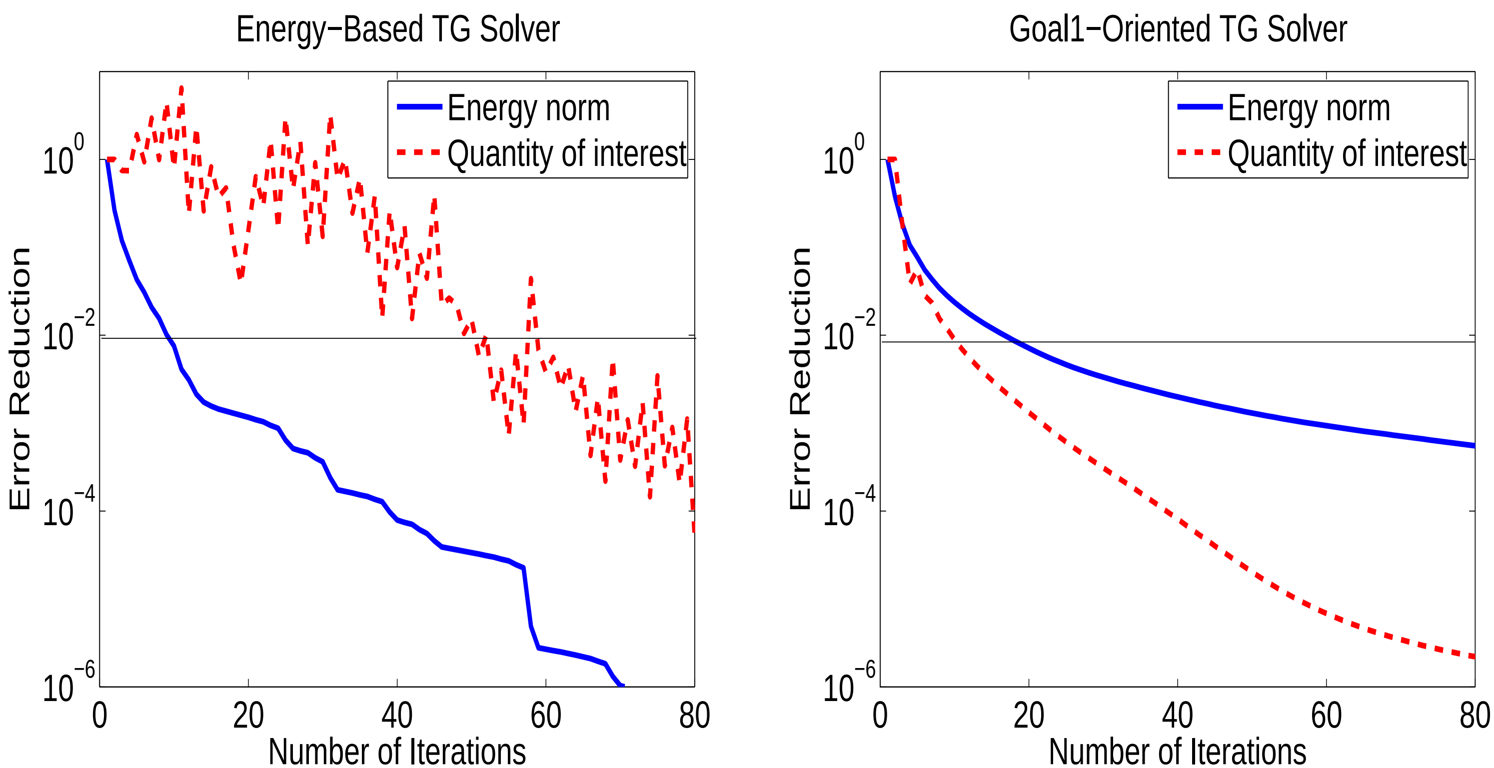
Selection IV (Goal2):

$$\alpha^{(n)} = \beta^{(n)} = \arg\min \sum_K ||x - x^{(n+1)}||_K ||G^{(n+1)}||_K \approx \arg\min \sum_K ||x^{(n+2)} - x^{(n+1)}||_K ||G^{(n+1)}||_K$$

TG SOLVER: GOAL-ORIENTED

Goal-Oriented Solver: Numerical Results

MODEL PROBLEM I



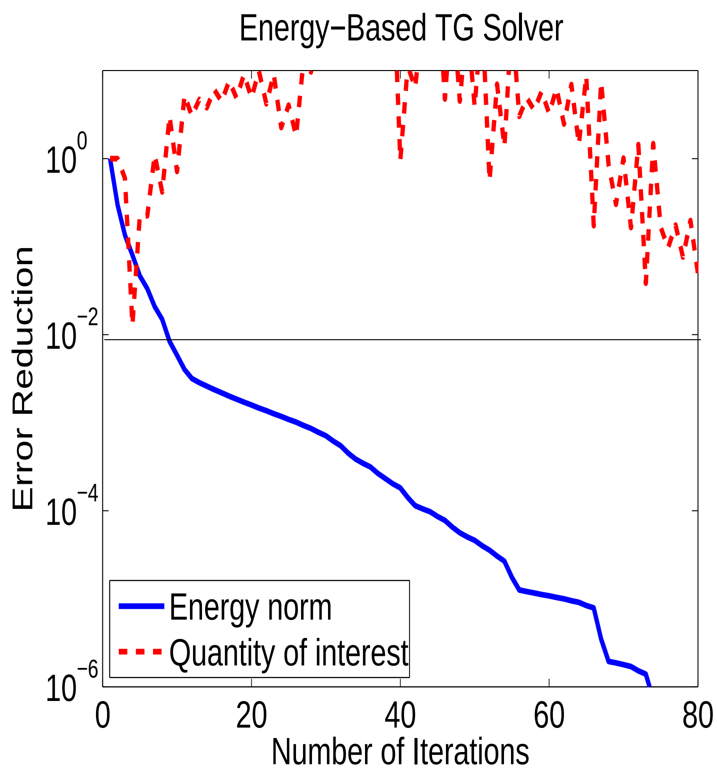
50 iter. to converge **8 iter. to converge**

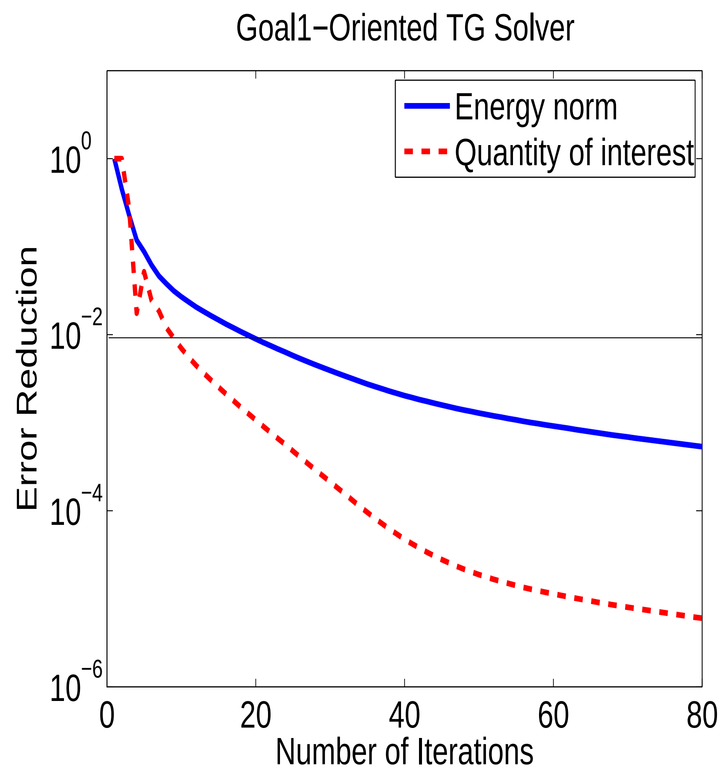
With the GOAL-ORIENTED solver we reduce the number of iterations

TG SOLVER: GOAL-ORIENTED

Goal-Oriented Solver: Numerical Results

MODEL PROBLEM II





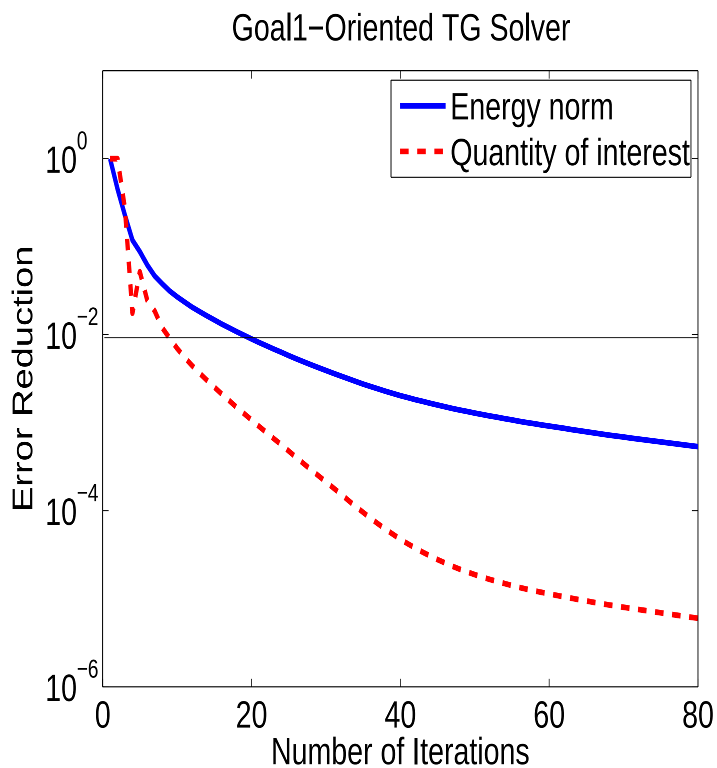
$>$80 iter. to converge **8 iter. to converge**

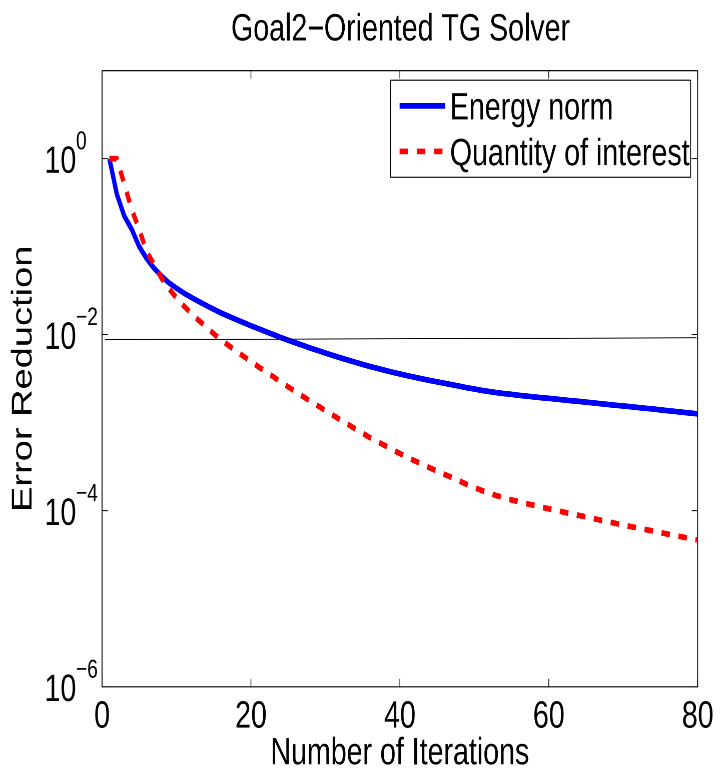
We only converge with the GOAL-ORIENTED solver

TG SOLVER: GOAL-ORIENTED

Goal-Oriented Solver: Numerical Results

MODEL PROBLEM II





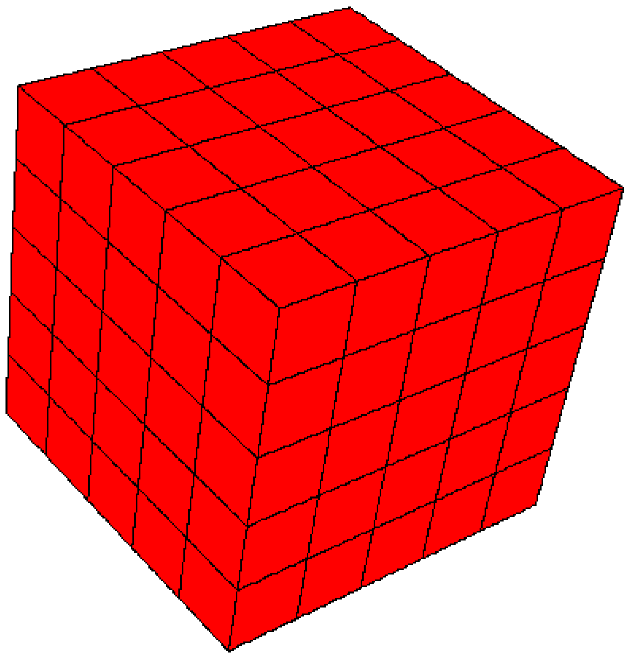
8 iter. to converge **16 iter. to converge**

Goal1 algorithm converges in less iterations than Goal2

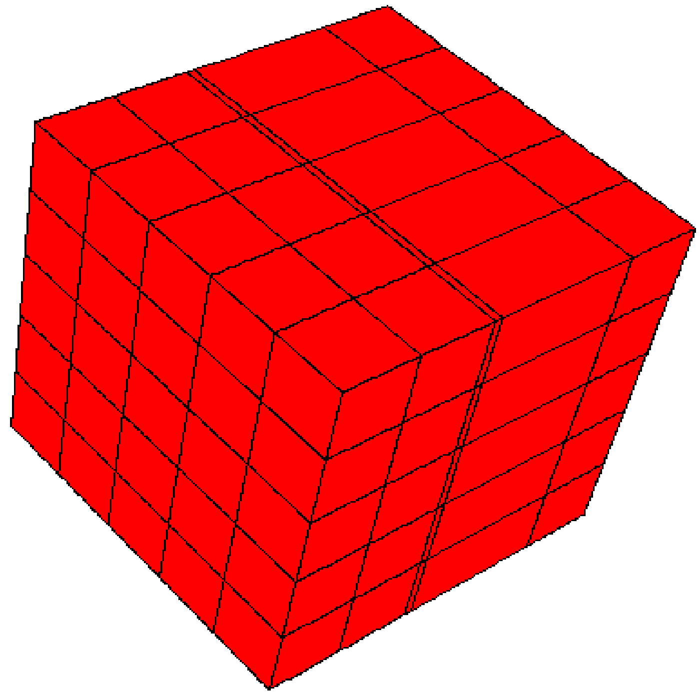
TG SOLVER: ELONGATED ELEMENTS

Elongated Elements

MODEL PROBLEM I (Initial Grid)



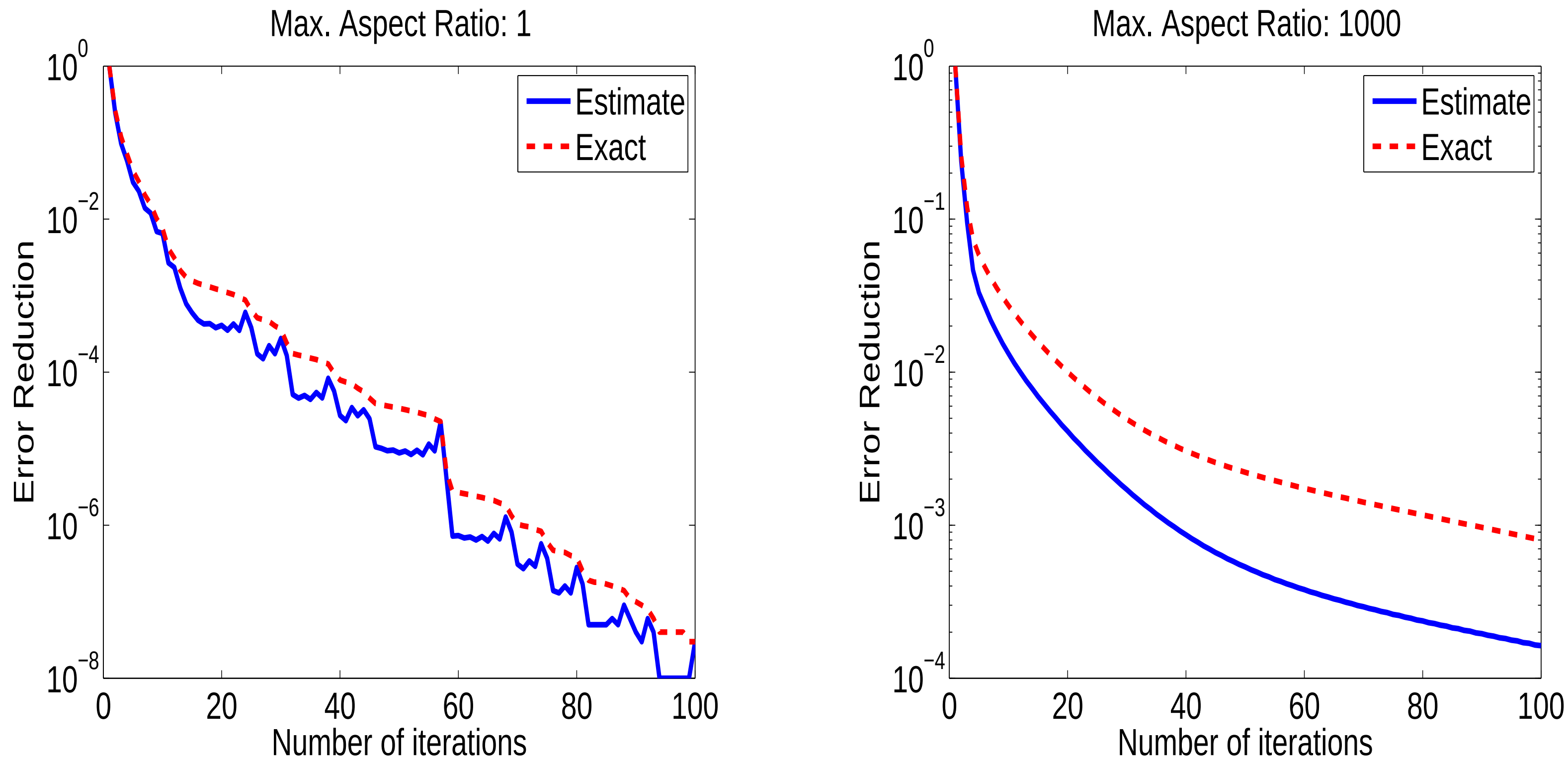
Isotropic



Anisotropic

TG SOLVER: ELONGATED ELEMENTS

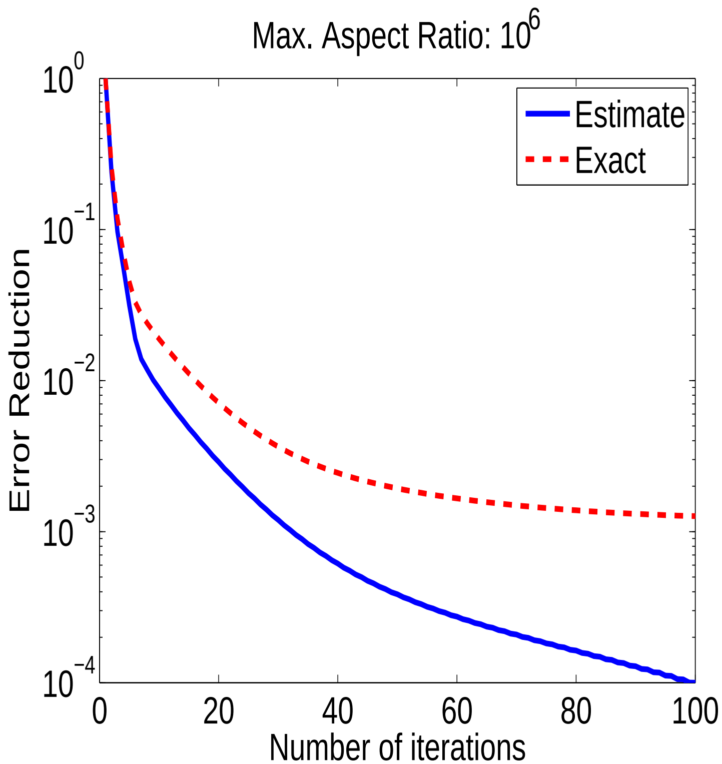
Elongated Elements

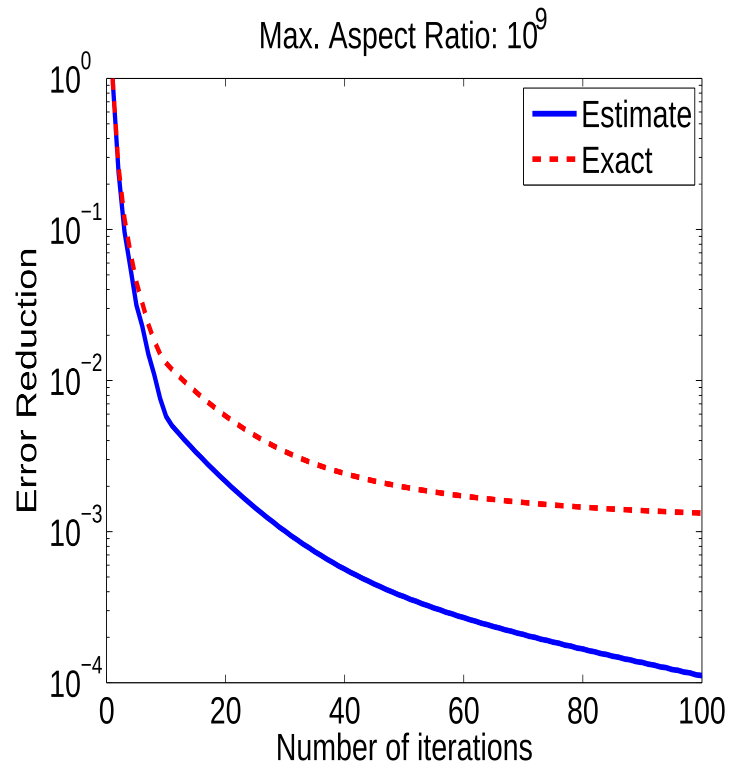


For elongated elements, convergence and error estimation degenerates

TG SOLVER: ELONGATED ELEMENTS

Elongated Elements

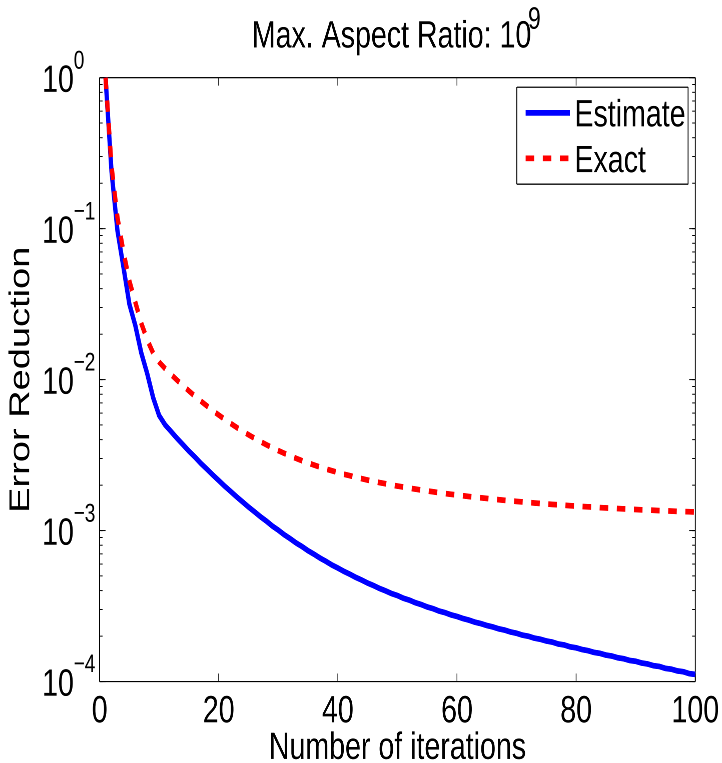


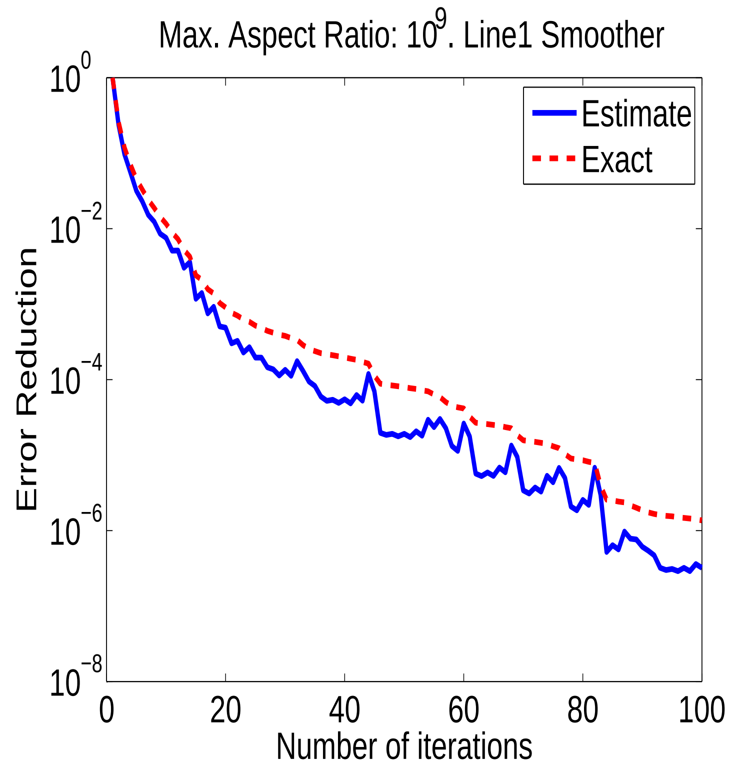


For elongated elements, convergence and error estimation degenerates

TG SOLVER: ELONGATED ELEMENTS

Elongated Elements

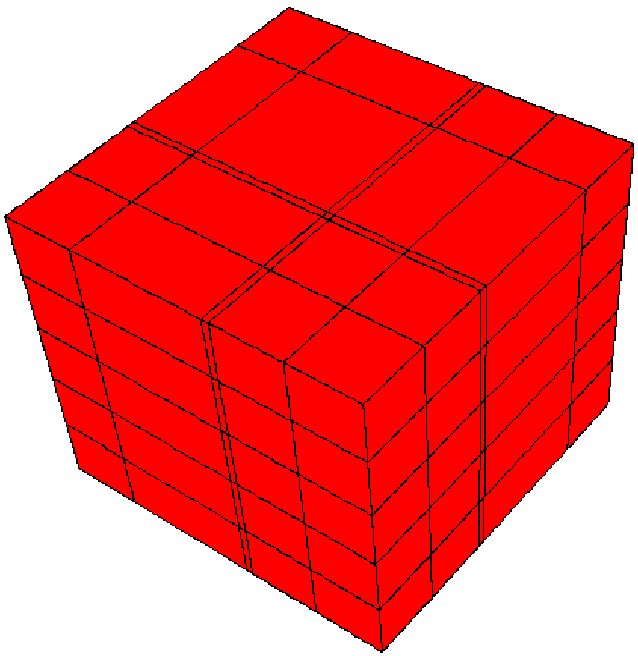


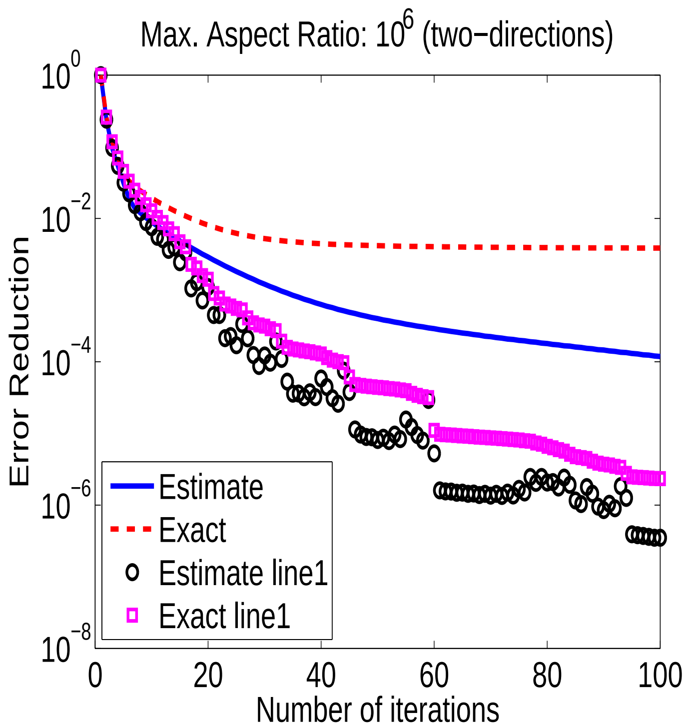


Line1 Smoother=old smoother + additional block composed of all d.o.f. associated to elongated elements

TG SOLVER: ELONGATED ELEMENTS

Elongated Elements

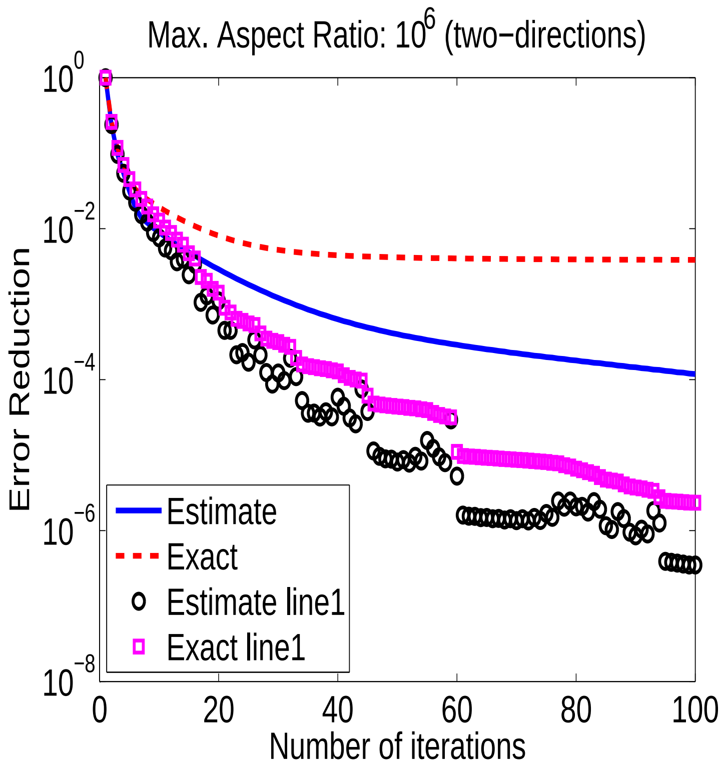


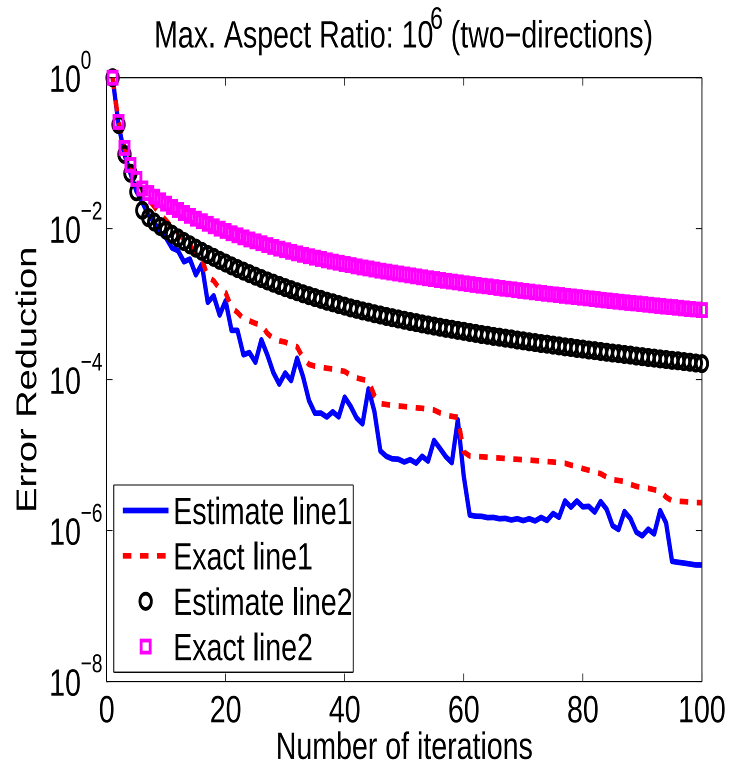


Line smoothers are necessary in presence of elongated elements

TG SOLVER: ELONGATED ELEMENTS

Elongated Elements





Line2 Smoother=old smoother + additional block composed of all EDGE and VERTEX d.o.f. associated to elongated elements

TG SOLVER: IMPLEMENTATION

Implementation Details

Block-Jacobi Smoother: PATCH-LEVEL (UNASSEMBLED) OPERATIONS.

- **Enables flexible (adaptive) smoother selection.**
- **Does not require inversion (only LU factor).**

Stiffness matrix: ASSEMBLED.

- **Facilitates flexible (adaptive) smoother selection.**
- **Minimizes storage (avoids node repetition).**

Transfer operators: ACTING ON RIGHT HAND SIDE, ASSEMBLED.

- **Avoids using matrix-matrix multiplications.**
- **Logic consistent with that of the stiffness matrix.**

Coarse-grid solve: ASSEMBLED.

- **Logic consistent with that of the stiffness matrix.**
- **Logic consistent with that of the smoother.**

CONCLUSIONS AND FUTURE WORK

CONCLUSIONS

- **A goal-oriented adaptive strategy should be accompanied by a goal-oriented iterative solver.**
- **Error estimation may fail in presence of elongated elements.**
- **Elongated elements should be identified before solving the actual problem.**
- **Line smoothers are needed to converge in presence of elongated elements.**

FUTURE WORK

- **New error estimators.**
- **Fast numerical or semi-analytical computation for goal-oriented Krylov subspace optimization methods.**

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