Boundary controllability for a one-dimensional heat equation with two singular inverse-square potentials

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Formulation of the problem

Let $T > 0$, $\mu_1, \mu_2 \leq 1/4$ and define $Q := (0, 1) \times (0, T)$

\[
\begin{align*}
    u_t - u_{xx} - \frac{\mu_1}{x^2} u - \frac{\mu_2}{(1-x)^2} u &= 0 \quad (x, t) \in Q \\
    u(0, t) &= f(t), \quad u(1, t) = 0 \\
    u(x, 0) &= u_0(x)
\end{align*}
\]  

(1)

Theorem (Null-controllability)

For any time $T > 0$ and any initial datum $u_0 \in L^2(0, 1)$ there exists a control function $f \in L^2(0, T)$ such that the solution of (1) satisfies $u(x, T) = 0$. 
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State of the art

SINGULAR POTENTIALS

• J. Vancostenoble and E. Zuazua - Null controllability for the heat equation with singular inverse-square potentials (2008)

• S. Ervedoza - Control and stabilization properties for a singular heat equation with an inverse-square potential (2008)

• C. Cazacu - Controllability of the heat equation with an inverse-square potential localized on the boundary (2014)

DEGENERATE COEFFICIENTS \((u_t - (a(x)u_x)_x = 0)\)

• P. Martinez and J. Vancostenoble - Carleman estimates for one-dimensional degenerate heat equations (2006)

• P. Cannarsa, P. Martinez and J. Vancostenoble - Carleman estimates for a class of degenerate parabolic operators (2008)

• M. Gueye - Exact boundary controllability of 1-d parabolic and hyperbolic degenerate equations (2014)
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Proposition

Let $\mu_1^*, \mu_2^* \in \mathbb{R}$ be such that $\mu_1^* + \mu_2^* \leq 1/4$. Then, for any $z \in H_0^1(0,1)$ it holds

$$
\int_0^1 z_x^2 \, dx \geq \mu_1^* \int_0^1 \frac{z^2}{x^2} \, dx + \mu_2^* \int_0^1 \frac{z^2}{(1-x)^2} \, dx.
$$

(2)

V. Felli and S. Terracini - Elliptic equations with multi-singular inverse-square potentials and critical nonlinearity

Proposition

There exists a constant $M > 0$ such that for any $z \in H_0^1(0,1)$ it holds

$$
\int_0^1 z_x^2 \, dx + M \int_0^1 z^2 \, dx \geq \frac{1}{4} \int_0^1 \frac{z^2}{x^2} \, dx + \frac{1}{4} \int_0^1 \frac{z^2}{(1-x)^2} \, dx.
$$

(3)

Sketch of the proof.

We rewrite $z = z_1 + z_2 + z_3$ with $z_i := z \phi_i$, $i = 1, 2, 3$ and $(\phi_i)_{i=1,2,3}$ a partition of the unity such that

$$
\text{supp}(\phi_1) = (1/2, 1), \quad \text{supp}(\phi_2) = (0, 1/2), \quad \phi_3 = 1 - \phi_1 - \phi_2
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and we apply Hardy inequality.
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Proposition

For all \( \gamma < 2 \) and \( n > 0 \) there exists a positive constant \( C_0 = C_0(\gamma, n) \) such that, for any \( z \in H^1_0(0, 1) \) it holds

\[
C_0 \int_0^1 z_x^2 \, dx + \frac{2 - \gamma}{2} \int_0^1 z^2 \, dx \geq \frac{(1 - \gamma)^2}{4} \int_0^1 \frac{z^2}{x^2} \, dx + n \int_0^1 \frac{z^2}{(1 - x)^2} \, dx. 
\]

(4)

Sketch of the proof.

\[
0 \leq \int_0^1 \left( x^{2-\gamma} z_x - \frac{\gamma - 1}{2} \frac{z}{x^{2}} + \frac{z}{1-x} \right)^2 \, dx.
\]

We expand this expression, apply integration by parts and estimate using Hölder inequality, Cauchy-Schwarz inequality and Hardy-Poincaré inequalities.

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Sketch of the proof.

$$0 \leq \int_0^1 \left(x^{2-\gamma} z_x - \gamma - 1 \frac{z}{x^{3/2}} + \frac{z}{1 - x}\right)^2 \, dx.$$ We expand this expression, apply integration by parts and estimate using Hölder inequality, Cauchy-Schwarz inequality and Hardy-Poincaré inequalities.

Hilbert Uniqueness Method

Adjoint system

\[
\begin{cases}
    v_t + v_{xx} + \frac{\mu_1}{x^2} v + \frac{\mu_2}{(1-x)^2} v = 0 & (x, t) \in Q \\
    v(0, t) = v(1, t) = 0 \\
    v(x, T) = v_T(x)
\end{cases}
\]  

Theorem (Observability inequality)

Let \( T > 0 \). For any \( v_T \in L^2(0, 1) \) the solution of (5) satisfies

\[
\int_0^1 v(x, 0)^2 \, dx \leq C \int_0^T \left[ x^{2\lambda_1} v_x^2 \right]_{x=0}^T \, dt,
\]  

with

\[
\lambda_1 := \frac{1}{2} \left( 1 - \sqrt{1 - 4\mu_1} \right).
\]
Carleman estimate

**Theorem**

There exists a constant $R_0 > 0$ such that, for all $R \geq R_0$, every solution $v$ of (5) satisfies

$$
R^3 C_1 \int_Q \theta^3 \left[ x^{6\lambda_1} (1 - x)^5 \right] v^2 e^{-2R\sigma} \, dxdt + RC_2 \int_Q \theta \frac{v^2}{(1 - x)^2} e^{-2R\sigma} \, dxdt
$$

$$
+ RC_3 \int_Q \theta \frac{v^2}{x^{1-2\lambda_1}} e^{-2R\sigma} \, dxdt + RC_4 \int_Q \theta \left[ x^{2\lambda_1} (1 - x) \right] v_x^2 e^{-2R\sigma} \, dxdt
$$

$$
\leq RC_5 \int_0^T \theta \left[ x^{2\lambda_1} v_x^2 \right] \bigg|_{x=0} \, dt,
$$

where $C_i$, $i = 1, \ldots, 5$ are positive constants and, for $\varpi, \beta > 0$, the weight function $\sigma$ is defined as $\sigma(x,t) := \theta(t)p(x)$ with

$$
\theta(t) := \left( \frac{1}{t(T-t)} \right)^3, \quad p(x) := \varpi + \frac{\beta x^{2\lambda_1+1}}{2\lambda_1+1} \left[ 1 - \frac{2\lambda_1+1}{\lambda_1+1} x + \frac{2\lambda_1+1}{2\lambda_1+3} x^2 \right].
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$$
Proof of the observability inequality

- From (9) we have
\[ \int_Q \theta \frac{v^2}{x^{1-2\lambda_1}} e^{-2R\sigma} \, dx \, dt \leq C_1 \int_0^T \theta \left[ x^{2\lambda_1} v_x^2 \right] \bigg|_{x=0} \, dt; \]

- There exist two positive constants \( P_1 \) and \( P_2 \) such that
\[ \frac{\theta e^{-2R\sigma}}{x^{1-2\lambda_1}} \geq P_1 \text{ in } (0, 1) \times \left[ \frac{T}{4}, \frac{3T}{4} \right], \quad \theta e^{-2R\varpi} \leq P_2 \text{ in } (0, T); \]

hence
\[ \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_0^1 v^2 \, dx \, dt \leq C_2 \int_0^T \theta \left[ x^{2\lambda_1} v_x^2 \right] \bigg|_{x=0} \, dt. \]

- Using (3)
\[ \frac{d}{dt} \int_0^1 v^2 \, dx \geq -M \int_0^1 v^2 \, dx; \]

hence
\[ \int_0^1 v(x, 0)^2 \, dx \, dt \leq \frac{2}{T} e^{2MT} C_2 \int_0^T \theta \left[ x^{2\lambda_1} v_x^2 \right] \bigg|_{x=0} \, dt. \]
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THANKS FOR YOUR ATTENTION!