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GRAVITY WAVES IN TWO-LAYER FLOWS WITH FREE SURFACE

ONDES DE GRAVITÉ DANS UN ÉCOULEMENT À DEUX COUCHES ET À SURFACE LIBRE

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...In graduate school, there is always something that we "should" be doing. But the truth of the matter is that I wouldn’t want to be doing anything else right now. In addition to being dutiful, rigorous, rational scholars, we all have our secret lives. Some of us draw comics strips, others make blown-glass art, play the jazz oboe, or even practice competitive roller-skate dancing. Graduate school is as much about discovering truth in science and culture as it is about discovering truth in ourselves.

PhD Comics
Jorge Cham
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GRAVITY WAVES IN TWO-LAYER FLOWS WITH FREE SURFACE

Abstract

In this work we study the wave propagation in two-layer flows with free surface. Two distinct classes of models are contemplated. First, we consider the “two-layer” version of the shallow water equations (also known by Saint-Venant’s equations). This model is strictly hyperbolic for small relative velocities. It would be natural to consider this model as suitable for the description of hydraulic jumps. However, like most of models describing multi-velocity flows, the system is not presented in conservative form. We present a survey on the number of conservation laws available for the multi-dimensional case that seems to imply that the system is truly nonconservative. Therefore, the impossibility of presenting a complete set of Rankine-Hugoniot conditions enabling the characterization of weak solutions in the classical way.

Then, we obtain a dispersive model suited to the description of large amplitude waves propagating in the same physical system. The model is a “two-layer” generalization of the Green-Naghdi model and can be derived by applying Hamilton’s principle to a Lagrangian that results from the insertion of approximations directly into the Lagrangian for the full water-wave problem. As a consequence, the variational structure of the original problem and the corresponding symmetry properties are preserved. In addition, it is a fully nonlinear model and deals with rotational flows. As in the case of the full problem, the present model captures the resonance between short waves and long waves. In this framework it is shown, by using numerical computations, the existence of homoclinic trajectories embedded into the continuous spectrum. These correspond to true solitary waves having the same velocities at infinity in each layer. Their study reduces to the analysis of a Hamiltonian system with two degrees of freedom. The traveling-wave solutions depend on three parameters: the density ratio, the depth ratio and the Froude number based on the bottom layer. Two wave regimes, characterized by the elevation or depression of the interface between the layers are presented. A critical depth ratio separates these two regimes and it will be shown how it relates to a change of the structure of the potential for the Hamiltonian system. The analysis of the number and nature of critical points turned out to be decisive in this work. It was found that the number of critical points can be four or two, depending on the value of the Froude number (for fixed density and depth ratios). For sets of parameters corresponding to oceanic conditions we have perceived the existence of true solitary waves and their broadening whenever the wave speed increases towards a limit value. Finally, other sets of parameters are considered for which multi-humped solitons exist, highlighting the richness and complexity of the system considered.

Keywords: Two-layer flow, conservation laws, Frobenius’ problem, shallow water equations, Green-Naghdi model, dispersive nonlinear waves, internal waves, embedded solitary waves, Hamiltonian system.
ONDES DE GRAVITÉ DANS UN ÉCOULEMENT À DEUX COUCHES ET À SURFACE LIBRE

Résumé

Nous étudions dans cette thèse la propagation des ondes dans un écoulement à deux couches et à surface libre. Deux classes de modèles sont considérées. Nous nous consacrons d’abord à la version à deux-couches du modèle de Saint-Venant. Ce modèle est strictement hyperbolique pour des faibles vitesses relatives. Il serait naturel de considérer ce système d’équations pour la description des sauts hydrauliques. Il s’avère que, comme pour la plupart des modèles décrivant des systèmes multi-vitesses, les équations ne sont pas présentées sous forme conservative. C’est pourquoi on a réalisé une étude sur le nombre de lois de conservation pour le système qui semble indiquer que celui-ci est en fait non-conservatif. Comme conséquence, on a l’impossibilité de présenter un ensemble complet de conditions de Rankine-Hugoniot permettant ainsi la caractérisation des solutions faibles dans le sens classique.

Nous obtenons ensuite un modèle dispersif adapté à la description de la propagation des ondes de grande amplitude pour le même système physique. Ce modèle est une généralisation naturelle du modèle de Green-Naghdi et sa dérivation est basée sur le principe d’Hamilton. L’idée conduisant au résultat consiste en l’obtention d’un Lagrangien pour le modèle approché en introduisant directement les approximations dans le Lagrangien du modèle complet. Par conséquent, la structure variationnelle ainsi que les propriétés de symétrie correspondantes sont préservées. De plus, aucune restriction du type ondes de faible amplitude ou écoulement potentiel n’est ici considéré.

Comme dans le cas des équations d’Euler complètes, ce modèle reproduit la résonance entre les ondes courtes et les ondes longues. Dans ce cadre on montre, par des calculs numériques, l’existence de trajectoires homoclines qui correspondent aux vraies ondes solitaires ayant les mêmes vitesses à l’infini dans chaque couche. L’étude de ces ondes se réduit à l’étude d’un système Hamiltonien à deux degrés de liberté. Les ondes progressives dépendent de trois paramètres: le rapport de densité des fluides, le rapport des épaisseurs de chaque couche et le nombre de Froude. Deux régimes caractérisés par l’élévation ou la dépression de l’interface entre deux couches sont présentés. Le rapport critique des épaisseurs des couches sépare ces deux régimes et il sera montré comment le relier aux changements de la structure pour le potentiel du système Hamiltonien. L’analyse du nombre et la nature des points d’équilibre se sont montrés décisives pour établir le résultat. On a constaté que leur nombre ne peut être que de quatre ou de deux, selon la vitesse de ces ondes (pour un rapport de densité des fluides et un rapport des épaisseurs fixés). Pour les paramètres qui correspondent aux conditions océaniques, on a observé l’existence des ondes solitaires et leur élargissement (“broadening”) lorsque la vitesse de ces ondes tend vers une certaine valeur limite. Enfin nous considérons différents paramètres pour lesquelles des solutions avec plusieurs bosses (“multihumped shaped profile”) existent, illustrant la richesse et la complexité du système considéré.

Mots clés: Écoulement à deux couches, lois de conservation, problème de Frobenius, équations de Saint-Venant, modèle de Green-Naghdi, ondes nonlinéaires dispersives, ondes internes, ondes solitaires incluses dans le spectre continu, système Hamiltonien.
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1 General Motivation

The theory of water waves has been a source of intriguing – and often difficult – mathematical problems for at least 150 years. Several phenomena can be observed in water waves. Some of them, such as the rings spreading out from a stone thrown in a pool, or breaking waves (whenever the waves approach the beach) are familiar to everyone, others less.

The solitary wave was first observed and described by J. Scott Russell in 1834. The most significant experiment involved the dropping of a weight at one end of a water channel. This produced a wave of \textit{elevation} with a volume that was the same as the volume of water displaced by the weight. He was able to determine a formula for the speed these waves move. Moreover, he found experimentally that corresponding waves of \textit{depression} could not exist. Every attempt to produce depression waves led to a train of oscillatory waves. The mathematical investigations that were initiated by Russell’s observations led to a theory itself, named \textit{soliton theory}, that has completely transformed many aspects of the mathematical description of nonlinear wave propagation, bringing the theory of water waves into the era of modern applied mathematics (for an introduction on this exquisite subject see the book written by Drazin & Johnson [12]). It is now hard to imagine the controversy this wave excited, but just by mentioning that both Airy and Stokes were initially of the opinion that such wave could not exist (see page 167 in [26]) can give the reader an idea.

In England the spectacular distortion of the tidal waveform as it travels up the gradually narrowing Severn estuary generates at high spring tides a hydraulic jump (also called \textit{bore}) as depicted in figure 48 of the famous book written by Lighthill [32]. Stronger, \textit{turbulent bores} with a steeper, violent foaming front and at most a weak wave train behind are generated, however, in certain conditions (see figure 49 in [32]). Approximately 100 rivers around the world are known to produce bores, of which perhaps 20 or so are in United Kingdom. The largest river bore in the world – the Qiantang tidal river bore – can be 4 m high, 3 km wide and travel at speeds in excess of 32 km/h. At certain locations, reflected waves can reach 10 m and the roar can be heard over an hour before its arrival.

Other phenomenon known as \textit{oceanic internal waves} has been known for centuries, but their scientific study is recent. They manifest on the surface of the sea by long isolated stripes of highly agitated features that are defined as audibly breaking waves and white water (widely called \textit{internal wave signatures} or simply \textit{surface signatures}). We reproduce here a passage of [1]:

\begin{quote}
These features propagate past vessels at speeds that are at times in excess of two knots; they are not usually associated with any nearby bottom feature to which one might attribute their origin, but are indeed often seen in quite deep water. In the nautical literature and charts, they are sometimes identified as “tide rips”.
\end{quote}

These waves have amplitudes that can exceed 80 m, wavelengths of 2 or 4 km, and seem to move in slow motion, propagating up onto the continental shelf on the order of 25 to 35 km
from its formative point with a speed of about 0.5 to 0.8 m/s. The unexpectedly large stresses these waves imposed on offshore oil-drilling rigs motivated the study of these phenomena. With the advent of aircraft and satellite imaging sensors, it was possible to recognize their ubiquitous nature. There are more than 75 areas of the world where internal waves have been observed [1].

Theorists frequently regard the ocean as a two-layer fluid with the interface between layers corresponding to the main pycnocline. The basic density gradient may be established by heating or salt content. Pycnocline is known in each particular case by *thermocline* and *halocline*, respectively. A vertical distribution of temperature $T$ and salinity $\chi$ appears to be fully stable when both $T'_{0}(z) \geq 0$ and $\chi'_{0}(z) \leq 0$ (see page 301 of [32]). It is apparent that these conditions happen frequently in coastal regions, especially during summer months. A good example for which this idealization is appropriate is the northwestern subtropical North Atlantic, where the eighteen-degree water forms a nearly homogeneous upper layer (see page 89 of [38]). When only one of these conditions is satisfied, the ocean stratification remains unstable. Namely, in the Gulf of Toulon, for example, cooling of the surface by the Mistral wind has been observed to generate fluid overturning and homogenisation in the whole depth of the gulf.

Figure 1: Air satellite images showing sea surface manifestations of internal solitary waves close to their generation in the center of the Strait of Gibraltar. The picture on the left side shows the signature of a singular internal wave. Pictures taken from ESA’s official web site http://www.esa.int/esaCP/index.html.

As pointed out in [1], the internal solitons in the ocean are usually composed of several oscillations confined to a limited region of space. For these, the term “solitary wave packets” seems then more appropriate. However, the existence of “singular” solitary waves became a clear evidence since Ziegenbein’s observations in 1970 (see [51]). Furthermore, the paper by Duda et al. [13], concerning the isolated wave observed in the South China Sea, illustrates
Figure 2: Air satellite images showing sea surface manifestations of internal solitary waves observed in the South of China. The picture on the left side shows the signature of a singular internal wave. Pictures taken from ESA’s official web site http://www.esa.int/esaCP/index.html.
the tendency for waves with amplitudes that are a significant fraction of the total depth to broaden and develop a flat crest. These are examples of the most beautiful aspect of nature, but beneath we find, sometimes, its most destructive power. We all bear in mind the tragedy caused by the Indian Ocean Tsunami on December 26, 2004.

Figure 3: Isotherms of a single large wave in 340 m of water in the northeast South China. The typical KdV profile (dashed line) is given for comparison. Picture taken from the paper by Duda et al. [13].

To understand these phenomena, simple analytical models giving some of the same physics found in classical water-wave problems are still desirable. The reason for this is that the Euler equations are not easily amenable to analytical investigations and even today, despite the incredible development of computers, we see that numerical simulations for models based on the Euler equations are usually computationally expensive. Long wave models, combining both relative simplicity and full nonlinearity, have been developed and have been found to compare well with experiments. In the context of a homogeneous one-layer fluid, we can see how the models attempt to capture one aspect or another of the classical water-wave problem. The nonlinear shallow water equations (also called the Saint-Venant equations) lead to breaking of the typical hyperbolic kind, with the development of a vertical slope and a multivalued profile. On the other hand, the KdV equation, derived by Korteweg & de Vries in 1885 to describe the propagation of Russell’s solitary wave, cannot describe this breaking phenomenon.

The shallow water equations yield a quasilinear hyperbolic system. Russell’s solitary wave is an example that not every wave motion can be described by hyperbolic equations. These nonhyperbolic wave motions can be grouped largely into a main class called dispersive. This classification is based more on the type of solutions rather than on the equations themselves, in opposition to the classification given for hyperbolic systems. A few special equations exhibit both hyperbolic and dispersive behaviour, but these remain exceptional (see [47]). Nonlinearity and dispersion are two fundamental mechanisms of gravity wave propagation in fluids. As we know, they have opposing effects: nonlinearity tends to steepen a given wave form during the course of its evolution, while dispersion tends to flatten steep free-surface gradients. The single wave of elevation, propagating at the free surface without change in the form, observed by Russell, can be traduced as a possibility of a balance between these two opposing effects.

The shallow water equations neglect dispersion and are suitable to the description of hydraulic jumps. In the class of dispersive models we find the KdV equation and its extensions. Also, we
have the Camassa-Holm equation and the Green-Naghdi equations (see the paper of Johnson [27] relating these models). Each model has its advantages and disadvantages, capturing one or another feature of the water-wave problem. Usually, assumptions such as small amplitude of waves or potential flows are used in their derivation. Obviously, these restrictions can be difficult to maintain in nature, but surprisingly, they can be used with some success to model wave evolution outside their formal range of validity. We could refer e.g., to the eKdV theory, that has been adopted as the phenomenological model of choice, and its capability to capture highly nonlinear waves features (see [23]). Nevertheless, we have seen an increased interest on the Camassa-Holm and the Green-Naghdi models. (In the framework of the GN model we could refer the recent results concerning the linear stability of solitary waves by Li [30], the stability of shear flows by Gavrilyuk & Teshukov [19], unsteady undular bores by El, Grimshaw & Smyth [14] and a class of multi-dimensional solutions and approximate solutions by Teshukov & Gavrilyuk [42].) These models are fully nonlinear and do not use in their derivation such assumptions. We draw attention to the Green-Naghdi model and its relation with the classical shallow water equations. Both are endowed with a variational structure and can be inserted in a general class of Lagrangians

\[ L = \int_{\mathbb{R}} \int_{\mathbb{R}} (T - W) \, dx \, dy \]

where \( T = h|u|^2/2 \) is the kinetic energy and \( W \) is a potential depending on the variables \( h, \dot{h} \). Green-Naghdi’s equations and shallow water equations can be obtained applying Hamilton’s principle to the Lagrangian corresponding to the potentials

\[ W_{\text{GN}} = \frac{1}{2} gh^2 - \frac{1}{6} h^2 \quad \text{and} \quad W_{\text{sw}} = \frac{1}{2} gh^2, \]

respectively. We can say that the Green-Naghdi equations are the higher nonlinear order version of the shallow water equations, including dispersive effects. We present in this thesis the analogous situation in two-layer flows with free surface. In the first part, we consider the two-layer version of the Saint-Venant equations. We will see that, for small relative velocities, the system is strictly hyperbolic. We could think then considering this model to the description of hydraulic jumps (as in the one-layer version). However, by contrast with the one-layer shallow water model, for which an infinite number of conservation laws is available (in the one-dimensional case), the system seems to be nonconservative. Apparently, only the mass of each layer, the total momentum and total energy are conserved. This provides five conservation laws for the six scalar physical variables \( h_1, h_2, u_1 \) and \( u_2 \). Therefore, the impossibility of presenting a complete set of Rankine-Hugoniot conditions enabling the characterization of weak solutions in the classical way. We were not able to prove this assertion, nevertheless a first step was given during this thesis by establishing a complete set of conservation laws, in the one-dimensional case. The result gives a strong indication that the only conservation laws admitted by the two-layer shallow water model are given by those we know already.

In summary, we show that for the two-layer shallow water, meant to describe hydraulic jumps in stratified fluids, a difficulty arises due to the apparent lack of conserved quantities. Furthermore, as expected, this model is not suitable to the description of solitary waves propagating in two-layer flows.

The second and third part of this thesis are devoted to the derivation of a model consisting in a two-layer generalization of the Green-Naghdi equations. We present its general properties
and reveal the existence of large amplitude solitary waves embedded into the continuous spectrum. Some of these solutions will be graphically presented. For parameters close to oceanic conditions, we observe the agreement between computed solutions and real internal waves observed in oceans. In particular, we find that all the features of steady solitary wave solutions of the new model are consistent with those determined by numerical solutions of the full water-wave problem [34] and those obtained for simplified models proposed by Dias & Il’ichev [8] and Fochesato, Dias and Grimshaw [15].

Generalized Green-Naghdi’s equations describing internal waves in two-layered systems were first obtained by Choi & Camassa [6] (rigid lid) and Liska, Margolin & Wendroff [33] (free surface), however using special assumptions on the flow character in their derivation. Unfortunately, due to the complex form in which they are presented, the equations derived by Liska, Margolin & Wendroff [33] are not easily amenable to analytical investigations, creating a serious obstacle for their use in the study of coupled effects of nonlinear dispersion and stratification on the waves. Our aim is a simpler invariant formulation of these equations, based on the variational formulation of the Euler equations.

2 A Hyperbolic Model for Two-Layer Flows with Free Surface

We consider the two-layer generalization of the Saint-Venant equations. The two fluids with constant densities $\gamma_1$, $\gamma_2$ and velocities $u_1$, $u_2$ are bounded by a free surface $z = h_1 + h_2$ over a rigid bottom $z = 0$. The depth of the heavier fluid is denoted by $h_1(x, y, t)$ and the depth of the lighter fluid by $h_2(x, y, t)$. As usual, we denote by $g$ the free-fall acceleration. To serve our purposes, we introduce two new variables $\rho_1$, $\rho_2$ given by $\rho_i = \gamma_i h_i$ for $i = 1, 2$. We are able to rewrite this classical system found e.g., in the books by Baines [2] and Liapidevski & Teshukov [31] as

\[
\begin{align*}
    (\rho_1)_t + \text{div}(\rho_1 u_1) &= 0, \\
    (\rho_2)_t + \text{div}(\rho_2 u_2) &= 0, \\
    \frac{D_1 u_1}{Dt} + \nabla \left( \frac{\partial W}{\partial \rho_1} \right) &= 0, \\
    \frac{D_2 u_2}{Dt} + \nabla \left( \frac{\partial W}{\partial \rho_2} \right) &= 0,
\end{align*}
\]  

with

\[ W(\rho_1, \rho_2) = \frac{g}{2} \left( \frac{\rho_1^2}{\gamma_1} + 2 \frac{\rho_1 \rho_2}{\gamma_1} + \frac{\rho_2^2}{\gamma_2} \right). \]

Here, div stands for the divergence operator defined in two dimensions and the material derivatives $\frac{D_i}{Dt}$ are defined by $\frac{D_i f}{Dt} = f_t + (u_i \cdot \nabla) f$, for every scalar function or vector field $f$. The system can be found in a general setting in the work of Gavrilyuk et al. [17] and it is endowed with a variational structure. Namely, the two last equations of (0.1) are the
Euler-Lagrange equations for the Lagrangian
\[ L = \frac{1}{2} \rho_1 |u_1|^2 + \frac{1}{2} \rho_2 |u_2|^2 - W(\rho_1, \rho_2). \]

The system admits the following conserved quantities: density \( \rho_i \) for each layer, total momentum \( \rho_1 u_1 + \rho_2 u_2 \) and total energy \( \frac{1}{2} \rho_1 |u_1|^2 + \frac{1}{2} \rho_2 |u_2|^2 + W \). If these are, or not, the only available conservation laws still remains an open question. Physically, it is believed that no more conservation laws are admitted for this system. Though, a true proof has not yet been given. An important attempt was made by Montgomery and Moodie [36]. These authors restrained their search to multinomial expressions in the layer variables and were able to prove that all the conservation laws expressed in this particular form were simply the linear combination of the five conservation laws mentioned above. But, what to say about the existence of other kind of conservation laws? These authors were not able to provide an answer to this question. In fact, as mentioned by Montgomery and Moodie, despite their work, we are free to imagine that conserved quantities with tricky expressions can exist – and why not – in infinite number. Our contribution to this problem was to show that, at least in the one-dimensional case, it is possible to put an end to this question. All conservation laws can be obtained by linear combination of the conservation laws for mass and velocity (in each layer), total momentum and total energy. The result for the multi-dimensional case follows from a heuristic argument. In general, whenever we increase the dimension of the space considered, we increase as well the number of constraints for the problem. This means that for the multi-dimensional case, the system has, at most, the conserved scalar quantities exhibited for the one-dimensional case. On the other hand, the fact that the velocities for each fluid are conserved can be seen as a degeneracy of the one-dimensional case, since in this case the motion equations are automatically given in conservative form, which in general can never happen for the multi-dimensional case (with the exception to be made if the flow is potential).

The main difficulty to overcome is described below. In the one-dimensional case, we have the quasilinear system
\[ U_t + AU_x = 0 \]
defined by
\[ U = \begin{bmatrix} \rho_1 \\ \rho_2 \\ u_1 \\ u_2 \end{bmatrix}, \quad A = \begin{bmatrix} u_1 & 0 & \rho_1 & 0 \\ 0 & u_2 & 0 & \rho_2 \\ W_{11} & W_{12} & u_1 & 0 \\ W_{12} & W_{22} & 0 & u_2 \end{bmatrix}, \]
where we denote the constants \( W_{\rho_i \rho_j} \) by \( W_{ij} \). We seek all the pairs \( (\varphi(U), \psi(U)) \) of smooth scalar functions for which every regular solution \( U \) of the system satisfies the conservation law
\[ \varphi_t + \psi_x = 0. \]

The functions \( \varphi \) and \( \psi \) will be called entropy and entropy-flux, respectively. Since the system can be written in conservative form, we have a useful criterion for the existence of an entropy: an entropy \( \varphi \) exists for the system if and only if \( D^2 \varphi \cdot A \) is symmetric. It has been recently applied with success to systems of continuum mechanics by Dimitrova [11] and
Vulkov ([45],[44]), and leads, in our particular case, to:

\[
\begin{align*}
\varphi_{\rho_1 \rho_2} (u_2 - u_1) + \varphi_{\rho_1 u_1} W_{12} + \varphi_{\rho_1 u_2} W_{22} &= \varphi_{\rho_2 u_1} W_{11} + \varphi_{\rho_2 u_2} W_{12}, \\
\varphi_{u_1 u_1} W_{11} + \varphi_{u_1 u_2} W_{12} &= \varphi_{\rho_1 \rho_1}, \\
\varphi_{\rho_1 u_2} (u_2 - u_1) + \varphi_{\rho_1 \rho_2} = \varphi_{u_1 u_2} W_{11} + \varphi_{u_2 u_2} W_{12}, \\
\varphi_{\rho_2 u_1} (u_2 - u_1) + \varphi_{u_1 u_1} W_{12} + \varphi_{u_1 u_2} W_{22} &= \varphi_{\rho_1 \rho_2}, \\
\varphi_{u_1 u_2} W_{12} + \varphi_{u_2 u_2} W_{22} &= \varphi_{\rho_2 \rho_2}, \\
\varphi_{u_1 u_2} (u_2 - u_1) + \varphi_{\rho_2 u_1} = \varphi_{\rho_1 u_2}.
\end{align*}
\]

This system has resisted all our attempts to find constraints for \( \varphi \) that could lead to the result. Hopefully, an alternative approach was found to overcome the problem. It relies on a more algebraic approach relating the mathematical entropies with the solutions of the equation \(XA = AX\). The problem of finding all the matrices \(X\) that commute with a given constant matrix is known as the Frobenius problem. In our case, the matrix \(A\) is a function of the physical variables, but it will be shown in the first part of this thesis how the classical results on the Frobenius problem (the key for the problem) can be used to achieve our goal. It will be seen that the role played by strict hyperbolicity is also decisive to establish the result. We were able to prove the following result:

**Proposition.** For each mathematical entropy \( \varphi \) of the system there exist functions \( \alpha, \beta, \gamma \) and \( \delta \) depending on \(U\) such that

\[
D^2 \varphi = \alpha P + \beta PA + \gamma PA^2 + \delta PA^3.
\]

Here, \(P\) is a certain permutation matrix arising in the following representation of \(A\): in our particular case, \(A\) has a special structure that enables us to write \(A = PD^2E\), where \(E\) stands for the total energy. At the end, we proved that both \(\gamma\) and \(\delta\) must be zero, and then it all summarizes to consider the solutions of the equation

\[
D^2 \varphi = \alpha P + \beta D^2 E,
\]

with \(\alpha\) and \(\beta\) constant. Since its explicit solution is given by

\[
\varphi = \alpha (\rho_1 u_1 + \rho_2 u_2) + \beta E + c_1\rho_1 + c_2\rho_2 + c_3 u_1 + c_4 u_2 + \text{const.},
\]

the result follows.
As mentioned above, it will be proposed in these two last parts of the thesis a model that can be considered as the two-layer generalization of the classical Green-Naghdi model. As stated before, this model is fully nonlinear and deals with rotational flows. For potential two- and three-dimensional water waves, a number of results can be found in the literature (see the recent surveys by Dias & Kharif [10] and Dias & Iooss [9]). Less is known for rotational flows and it is the reason why we focus on this aspect. An elegant and simple derivation of the GN model is found in the paper by Miles & Salmon [35] (in the same year, the same model was derived by Zheleznyak & Pelinovsky [52]). It is based on the variational formulation of the Euler equations and, like in the original work by Green, Laws & Naghdi [20], it is imposed the condition that the vertical component of the velocity is a linear function of the vertical coordinate. In fact, this is equivalent to assume that the fluid moves in vertical columns, i.e., the horizontal components of the velocity vector do not depend on the vertical coordinate. Moreover, the approximations are inserted directly into the fluid Lagrangian as it can also be found in the book by Whitham [47] and, more recently, in the paper by Craig et al. [7]. This will be the strategy followed here to derive the new model. This method has two main advantages over the classical perturbation procedures. First, the approximations do not disturb the corresponding symmetry properties coming from the variational structure of the governing equations. Second, the approximation methods based on Hamilton’s principle suggest transformations to new dependent variables in which the approximate equations take its simplest mathematical form.

In order to establish a precise relation between the approximate theory and the exact three-dimensional theory, we drop the assumption that the fluid moves in vertical columns. Instead we define a class of flows for which an approximate Lagrangian can be calculated with accuracy $O(\varepsilon^2)$, where $\varepsilon$ is the ratio of a typical vertical scale to a typical horizontal scale. Flows in this class will be called weakly sheared flows and defined by

**Definition (Weakly sheared flow).** A flow is called a weakly sheared flow if there exists a real number $s > 1$ such that $u_{\alpha z} = O(\varepsilon^s)$ and $v_{\alpha z} = O(\varepsilon^s)$.

The approximate equations for the description of two-dimensional two-layer flows are the stationary points for the Hamilton action

$$I = \int_{t_1}^{t_2} \tilde{L} \, dt,$$

with the Lagrangian

$$\tilde{L} = \int \int \left\{ \gamma_1 h_1 \left( \frac{1}{2} |\nabla_1|^2 + \frac{1}{6} \left( \frac{D_1 h_1}{Dt} \right)^2 - \frac{1}{2} gh_1 \right) + \frac{1}{2} |\nabla_2|^2 + \frac{1}{2} \left( \frac{D_2 h_1}{Dt} \right)^2 + \frac{1}{2} \left( \frac{D_2 h_2}{Dt} \right)^2 + \frac{1}{6} \left( \frac{D_2 h_2}{Dt} \right)^2 - \frac{1}{2} g(h_2 + 2h_1) \right\} \, dx \, dy$$

submitted to the constraints

$$(h_\alpha)_t + \text{div} (h_\alpha \nabla_\alpha) = 0, \quad \alpha = 1, 2.$$ 

Here, we defined $\mathbf{v}_\alpha = (u_\alpha, v_\alpha)$ as the horizontal velocities and considered the mean velocities $\mathbf{\nabla}_\alpha$ with respect to each layer. It is in this context that we find here and hereafter defined the
material derivatives. Making use of the variables \( \rho_\alpha \) defined above we write

\[
\tilde{L} = \int_{\mathbb{R}} \int_{\mathbb{R}} (T - W) \, dx \, dy,
\]

where \( T \) is the kinetic energy defined by

\[
T = \frac{1}{2} \rho_1 |\nabla_1|^2 + \frac{1}{2} \rho_2 |\nabla_2|^2.
\]

and the potential \( W \) depends on \( \rho_1, \rho_2 \) and their material derivatives. This is what distinguishes the present model from the classical hyperbolic model given above, where the potential only depended on \( \rho_1 \) and \( \rho_2 \). Governing equations consist of

\[
(\rho_\alpha)_t + \text{div}(\rho_\alpha \nabla_\alpha) = 0,
\]

\[
\frac{D_\alpha \nabla_\alpha}{Dt} + \nabla \left( \frac{\delta W}{\delta \rho_\alpha} \right) - \frac{D_\alpha}{Dt} \left( \frac{1}{\rho_\alpha} \frac{\delta W}{\delta \nabla_\alpha} \right) - \left( \frac{\partial \nabla_\alpha}{\partial x} \right)^T \left( \frac{1}{\rho_\alpha} \frac{\delta W}{\delta \nabla_\alpha} \right) = 0.
\]

The expressions for the functional derivatives of \( W \) are not shown here by convenience. For simplicity, we can write \( u_\alpha \) instead of the two-dimensional velocity field \( \nabla_\alpha \) (hopefully with no resulting confusion). It is important to remark that in reality, this is not the original form the equations provided by Hamilton’s principle. In Appendix B of the second part it is found that the Euler-Lagrange equations are originally obtained as

\[
\frac{D_\alpha}{Dt} \left( u_\alpha - \frac{1}{\rho_\alpha} \frac{\delta W}{\delta u_\alpha} \right) + \nabla \left( \frac{\delta W}{\delta \rho_\alpha} - \frac{1}{2} |u_\alpha|^2 \right) + \left( \frac{\partial u_\alpha}{\partial x} \right)^T \left( u_\alpha - \frac{1}{\rho_\alpha} \frac{\delta W}{\delta u_\alpha} \right) = 0,
\]

which enables defining a new variable \( K_\alpha \) by

\[
K_\alpha = u_\alpha - \frac{1}{\rho_\alpha} \frac{\delta W}{\delta u_\alpha}
\]

and obtain for \( 0.2 \) the equivalent form

\[
(K_\alpha)_t + \nabla \left( K_\alpha \cdot u_\alpha + \frac{\delta W}{\delta \rho_\alpha} - \frac{1}{2} |u_\alpha|^2 \right) + (\text{curl} \, K_\alpha) \wedge u_\alpha = 0.
\]

It is natural thus to introduce a generalized vorticity as the curl of the vector field \( K_\alpha \). (It will become clear that in the case when the potential \( W \) does not depend on the material derivatives, this notion coincides with the the classical one.) In addition, we derive analogues of integral of motions, such as Bernoulli integrals, which are well-known in ideal Fluid Mechanics. Conservation laws for the total momentum and total energy are also exhibited. We stress that these results generalize those obtained for the GN model by Miles & Salmon [35] and Gavrilyuk & Teshukov [18].

To validate this new model we started by comparing the dispersion relation with the one given for the full problem. We have verified a particular good agreement for a density ratio \( \gamma = \gamma_2/\gamma_1 \approx 1 \) and a depth ratio \( H = H_2/H_1 \ll 1 \). This is quite surprising, since for example for \( \gamma = 1/10 \), \( H = 7 \) a clear discrepancy is presented. We could try to explain this by saying
that for values of $\gamma$ close to 0 we enter more in the range of validity for one single homogeneous fluid than for a two-layer flow. In this case, as we known, the classical GN model is not completely satisfactory when reproducing the dispersion relation for the Euler equation. Hopefully, this behaviour fits rather well to oceanic conditions. Further, we will show that the present model captures the resonance between short waves and long waves. The resulting solutions are called generalized solitary waves and a number of papers were devoted to this subject within the context of the full problem (see [39], [3], [24], [25], [40], [43], [49], [34]). They are nonlinear long waves characterized by ripples in the far field in addition to the solitary pulse. Mathematically, they correspond to homoclinic connections to periodic orbits close to a saddle-center equilibrium. A special member of this family of waves arises when the ripples vanish. These are called embedded solitary waves and, recently, we have seen an increased interest in these peculiar waves, since the seminal work of Yang et al. [50] (see [5] and [48] for an extensive list of references).

The study for these particular waves summarizes to the study of a Hamiltonian system with two degrees of freedom. An independent study of traveling waves in two-layer flows with free surface was done by Dias and Il’ichev in [8]. The corresponding full non-stationary system was obtained for potential flows by using asymptotic expansion, considering Lagrange’s method, and long waves of small amplitude. In the dynamical system for traveling waves it is assumed, like in our case, that velocities are zero in equilibrium and the solutions depend only on the Froude number, for fixed values of $\gamma$ and $H$. This model provides good qualitative results and has largely influenced our work.

The analysis of the number and nature of critical points turned out to be decisive in the work. It was found that the number of critical points depend on the Froude number (for fixed values of $\gamma$ and $H$), passing from four to only two points whenever attained a certain critical speed denoted by $F_1^C$. In particular, this value gives a bound for the solitary wave speed. Also, there are presented two wave regimes characterized by the elevation or depression of the interface. A critical depth ratio $H^C$ separates both regimes and it will be shown how it relates to a change of the structure of the potential of the present Hamiltonian system. For sets of parameters corresponding to oceanic conditions we have perceived the existence of solitary waves and their broadening whenever the wave speed increases towards a limit value $F_1^L < F_1^C$. The figures 4 and 5 represent two computed solutions. They remind the “singular” oceanic internal wave observed in the Strait of Gibraltar and in the South of China (Fig. 1, 2 and 3). We see that the deformations of the free surface are much smaller than the deformations of the interface in accordance to what can be seen in oceans. Thus, the name internal wave signatures or simply surface signatures. Another observation to point here is the fact that the broadened solution has a larger amplitude than the solitary wave depicted in Fig. 4. This is consistent with the observations made by Duda et al. (see Fig. 3).

These waves are often called in the literature by “table-top” solitons. In the limit as the width of the central core becomes infinite, the wave becomes a front. This phenomenon was predicted for the full problem by Dias and Il’ichev [8] and also verified for other water-wave models (see e.g., [15]). Finally, other sets of parameters are considered for which multi-humped solitons exist. Some of these solutions look somewhat like the numerical solutions presented by Michallet and Dias for the full problem [34].

In summary, the derived model seems to reproduce the main features known for solitary waves by means of numerical computations for the full problem or for approximate models recently proposed. Since it behaves particularly well for $\gamma$ close to 1, it would be interesting to apply
Figure 4: Interfacial wave of depression.

Figure 5: Broadened wave.
this model in real situations such as oceanic internal waves. Some gaps are to be found in this work. Among them, there is the problem of providing a relationship between the critical depth ratio $H^C$, which arises in the model, and the one that is found in the literature (see [34],[8], [28], [46], [37]). Also, it would be important to provide an analytical expression for the limit speed solitary waves can attain. Other questions deserve future investigation. Namely, the question of stability of solitary waves and the analytical proof that these can only exist for special values of the Froude number. We deduce from our numerical investigations that these values probably form a discrete set, but only further analysis would bring a closure to this question.

References for the Introduction

References


REFERENCES


Part I
On the Number of Conservation Laws for the “Two-Layer” Version of the Shallow Water Model
Conservation Laws for One-Dimensional Shallow Water Models for One and Two-Layer Flows


Ricardo Barros*

Abstract

A full set of conservation laws for the two-layer shallow water equations is presented for the one-dimensional case. We prove that all the conservation laws are linear combination of the equations for the conservation of mass and velocity (in each layer), total momentum and total energy. This result generalizes that of Montgomery and Moodie that found the same conserved quantities by restricting their search to the multinomials expressions in the layer variables. Though, the question of whether or not there are only a finite number of these quantities is left as an open question by the authors. Our work puts an end to this: in fact, no more conservation laws are admitted for the two-layer shallow water equations. The key mathematical ingredient of the method proposed leading to the result is the Frobenius problem. Moreover, we present a full set of conservation laws for the classical one-dimensional shallow water model with topography, by using the same techniques.

Keywords: Conservation laws; Entropy; Frobenius’s problem; Shallow water equations.

1 Introduction

Several studies to determine a full set of conservation laws for physical systems were already made. Most of them (see, e.g., [14]) are based on Noether’s theorem which relates conservation laws and symmetry in the equations (if such symmetry exists). It turns out that the existence of conservation laws is not determined by the availability of symmetry in the given differential equations system. Therefore, we cannot assure that all the conservation laws can be provided by Noether’s theorem. Another method was proposed by Lax [10], Rozdestvenskii and Yanenko [17]. It consists in a direct analysis for compatibility of an overdetermined system of partial differential equations of the first order for entropy and entropy-flux of an arbitrary conservation law. The method has been recently applied to systems of continuum mechanics by Dimitrova [6] and Vulkov [19],[20]. We give a summary description of this method: it consists in investigating for a given quasilinear system

\[ U_t + AU_x = 0, \]  

with \( U = (u_1, \ldots, u_n)^T \) and a square matrix \( A \) of dimension \( n \), whenever exists a pair \( (\varphi(U), \psi(U)) \) of smooth scalar functions for which every regular solution \( U \) of (1.1) satisfies the conservation law

\[ \varphi_t + \psi_x = 0. \]
The functions $\varphi$ and $\psi$ will be called *entropy* and *entropy-flux*, respectively (as in Dafermos [5], LeFloch [11] and Serre [18]) \(^1\). In the case when (1.1) can be presented in conservative form, we have a useful criterion for the existence of an entropy: *an entropy $\varphi$ exists for the system if and only if $D^2 \varphi \cdot A$ is symmetric* (see, e.g., [5], [11]). This characterization of the mathematical entropies drives us to a linear system of $n(n - 1)/2$ second-order partial differential equations. This kind of analysis demands many calculations and the difficulty level increases considerably when high dimension systems are considered. Here, we propose a more algebraic approach that enables us to find a structure of the mathematical entropies for certain systems of continuum mechanics. In the two cases presented in the next two sections, we will show an important relation between the mathematical entropies and the solutions of the equation $XA = AX$.

2 A Full Set of Conservation Laws for the One-dimensional Two-layer Shallow Water Model

In this section we present a full set of conservation laws for a two-layer shallow water model.

2.1 Physical model

The following picture

![Figure 1: Two-layer flow](image)

describes a two-layer flow composed by two immiscible fluids with different *constant densities* $\gamma_1$ and $\gamma_2$ and two velocities $u_1$ and $u_2$. The gravity interaction is considered and the free-fall acceleration

\(^1\)This terminology is not quite standard: some authors reserve the term *entropy* for the case when the conserved quantity is a strictly convex function. To avoid any misunderstanding, it will be employed as well, throughout the text, the term *mathematical entropy*. 

is denoted by \( g \). When introduced the long wave approximation, the following model can be derived:

\[
\begin{align*}
(h_1)_t + \text{div} (h_1 \mathbf{u}_1) &= 0 \\
(h_2)_t + \text{div} (h_2 \mathbf{u}_2) &= 0 \\
\frac{D_1 \mathbf{u}_1}{D_t} + \nabla [g(h_1 + \lambda h_2)] &= 0 \\
\frac{D_2 \mathbf{u}_2}{D_t} + \nabla [g(h_1 + h_2)] &= 0
\end{align*}
\] (2.1)

Here, \( \text{div} \) stands for the divergence operator in space dimension 2 and the constant \( \lambda < 1 \) is defined by \( \lambda = \gamma_2 / \gamma_1 \). The material derivative is denoted by \( \frac{D}{D_t} \) and defined by \( \frac{D}{D_t} f = f_t + (\mathbf{u} \cdot \nabla) f \), for every scalar function or vector field \( f \). We note that this model is a direct generalization of the 1-d model presented in [1] and [12].

Introducing the two new variables \( \rho_1 \) and \( \rho_2 \) given by \( \rho_i = h_i \gamma_i \) for \( i = 1, 2 \), one may write (2.1) as:

\[
\begin{align*}
(\rho_1)_t + \text{div} (\rho_1 \mathbf{u}_1) &= 0 \\
(\rho_2)_t + \text{div} (\rho_2 \mathbf{u}_2) &= 0 \\
\frac{D_1 \mathbf{u}_1}{D_t} + \nabla \left( \frac{\partial W}{\partial \rho_1} \right) &= 0 \\
\frac{D_2 \mathbf{u}_2}{D_t} + \nabla \left( \frac{\partial W}{\partial \rho_2} \right) &= 0
\end{align*}
\] (2.2)

with

\[
W(\rho_1, \rho_2) = \frac{g}{2} \left( \frac{\rho_1^2}{\gamma_1} + 2 \frac{\rho_1 \rho_2}{\gamma_1 \gamma_2} + \frac{\rho_2^2}{\gamma_2} \right).
\]

This system of differential equations can also be found in the work of Gavrilyuk et al. [9]. The system is presented there as a model for a two-fluid homogeneous mixture and deduced by the use of a variational approach. The first two equations are the continuity equations for each component, and the last two, called momentum equations, are precisely the Euler-Lagrange equations for a given Lagrangian

\[
L = \frac{1}{2} \rho_1 |\mathbf{u}_1|^2 + \frac{1}{2} \rho_2 |\mathbf{u}_2|^2 - W(\rho_1, \rho_2).
\]

The function \( W \) can be naturally interpreted as the internal energy of the mixture.

The system admits the following conserved quantities: density \( \rho_i \) for each layer, total momentum \( \rho_1 \mathbf{u}_1 + \rho_2 \mathbf{u}_2 \) and total energy \( \frac{1}{2} \rho_1 |\mathbf{u}_1|^2 + \frac{1}{2} \rho_2 |\mathbf{u}_2|^2 + W \). It is not currently known whether or not more conservation laws are admitted. The expectation is that no more conservation laws are admitted for this system. If true, we would have a “genuinely” non-conservative system, since only 5 conservation laws are available for the 6 scalar physical variables: \( \rho_1, \rho_2, \mathbf{u}_1 \) and \( \mathbf{u}_2 \). Therefore, the impossibility of presenting a complete set of Rankine-Hugoniot conditions, enabling the characterization of weak solutions in the classical way. The variational approach was introduced by Gavrilyuk et al. as an attempt to overcome this problem.
The main aim of this work is to make a first step in this direction, presenting a full set of conservation laws for the two-layer shallow water model in the one-dimensional case. It will be shown that the conserved quantities are given by

\[ \alpha_1 \rho_1 u_1 + \alpha_2 \rho_2 u_2 + \alpha_3 \rho_1 + \alpha_4 \rho_2 + \alpha_5 u_1 + \alpha_6 u_2 + \text{const.} \]

for some constants \( \alpha_i, i = 1, \ldots, 6 \). In this expression, \( E \) stands for the total energy of the system. If we consider the fact that the momentum equations in (2.2) reduce, in the one-dimensional case, to conservation laws for the velocity in each layer, we see that our result gives a strong indication that the only conservation laws admitted by the two-layer shallow water model are given by those we know already.

### 2.2 1-d case

We shall study this particular set of equations in the one-dimensional case. The system becomes

\[
\begin{align*}
\left( \rho_1 \right)_t + \left( \rho_1 u_1 \right)_x &= 0 \\
\left( \rho_2 \right)_t + \left( \rho_2 u_2 \right)_x &= 0 \\
\left( u_1 \right)_t + u_1 \left( u_1 \right)_x + \left( W_{\rho_1} \right)_x &= 0 \\
\left( u_2 \right)_t + u_2 \left( u_2 \right)_x + \left( W_{\rho_2} \right)_x &= 0
\end{align*}
\]

and admits a conservative form. Indeed, (2.3) can be rewritten as

\[
\begin{align*}
\left( \rho_1 \right)_t + \left( \rho_1 u_1 \right)_x &= 0 \\
\left( \rho_2 \right)_t + \left( \rho_2 u_2 \right)_x &= 0 \\
\left( u_1 \right)_t + \left( \frac{u_1^2}{2} + W_{\rho_1} \right)_x &= 0 \\
\left( u_2 \right)_t + \left( \frac{u_2^2}{2} + W_{\rho_2} \right)_x &= 0
\end{align*}
\]

We have then the quasilinear system

\[ U_t + AU_x = 0 \]

defined by

\[
U = \begin{bmatrix} \rho_1 \\ \rho_2 \\ u_1 \\ u_2 \end{bmatrix}, \quad A = \begin{bmatrix} u_1 & 0 & \rho_1 & 0 \\ 0 & u_2 & 0 & \rho_2 \\ W_{11} & W_{12} & u_1 & 0 \\ W_{12} & W_{22} & 0 & u_2 \end{bmatrix}
\]

where we denote the constants \( W_{\rho_i \rho_j} \) by \( W_{ij} \).
2.3 Hyperbolicity

We will show that in a certain region, the system is strictly hyperbolic. Let us start to calculate the characteristic polynomial of $A$:

$$
\det(\lambda I - A) = \begin{vmatrix}
\lambda - u_1 & 0 & -\rho_1 & 0 \\
0 & \lambda - u_2 & 0 & -\rho_2 \\
-W_{11} & -W_{12} & \lambda - u_1 & 0 \\
-W_{12} & -W_{22} & 0 & \lambda - u_2
\end{vmatrix}
$$

$$
= (\lambda - u_1)^2(\lambda - u_2)^2 - \rho_2 W_{22}(\lambda - u_1)^2 - \rho_1 W_{11}(\lambda - u_2)^2 + \rho_1 \rho_2 \Delta
$$

where $\Delta = W_{11}W_{22} - W_{12}^2$. If we introduce the relative velocity parameter $w$ defined by $w = u_2 - u_1$, in terms of this parameter, the equation $\det(\lambda I - A) = 0$ can be rewritten as:

$$
(\lambda - u_1)^2(\lambda - (w + u_1))^2 - \rho_2 W_{22}(\lambda - u_1)^2 - \rho_1 W_{11}(\lambda - (w + u_1))^2 + \rho_1 \rho_2 \Delta = 0.
$$

In the particular case when $w = 0$, this equation turns to be

$$
(\lambda - u_1)^4 - (\rho_1 W_{11} + \rho_2 W_{22})(\lambda - u_1)^2 + \rho_1 \rho_2 \Delta = 0. \quad (2.5)
$$

Define $\alpha = (\lambda - u_1)^2$. He have then the following 2nd degree equation for $\alpha$:

$$
\alpha^2 - (\rho_1 W_{11} + \rho_2 W_{22})\alpha + \rho_1 \rho_2 \Delta = 0.
$$

As $(\rho_1 W_{11} + \rho_2 W_{22})^2 - 4\rho_1 \rho_2 \Delta = (\rho_1 W_{11} - \rho_2 W_{22})^2 + 4\rho_1 \rho_2 W_{12}^2 > 0$, and

$$
\rho_1 W_{11} + \rho_2 W_{22} = \sqrt{(\rho_1 W_{11} + \rho_2 W_{22})^2} > \sqrt{(\rho_1 W_{11} + \rho_2 W_{22})^2 - 4\rho_1 \rho_2 \Delta},
$$

we conclude that (2.5) has 4 distinct real solutions. It can be proven, applying the implicit function theorem, that for any fixed $(\rho_1, \rho_2, u_1, w = 0)$ there exists an open neighborhood for which the system is strictly hyperbolic. We address to [2] for a more detailed study of the eigenvalues. Additionally, in this work, a condition giving the range of validity of the model is also presented.

2.4 Reducing the system to a more convenient form

As it was mentioned before, an entropy $\varphi$ for (2.4) exists if and only if $D^2 \varphi \cdot A$ is symmetric.

We go further on this condition in our particular case. In fact, our matrix $A$ has a special structure that is important to remark. Define

$$
E(\rho_1, \rho_2, u_1, u_2) = \frac{1}{2}\rho_1 u_1^2 + \frac{1}{2}\rho_2 u_2^2 + W(\rho_1, \rho_2).
$$

as the total energy for the system and rewrite (2.4) as

$$
\begin{align*}
(\rho_1)_t + (E_{u_1})_x &= 0 \\
(\rho_2)_t + (E_{u_2})_x &= 0 \\
(u_1)_t + (E_{\rho_1})_x &= 0 \\
(u_2)_t + (E_{\rho_2})_x &= 0
\end{align*}
$$

(2.6)
From this, it follows that \( A = PD^2E \), where \( P \) is given by
\[
P = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix},
\]
so that:
\[
D^2\varphi \cdot A \text{ symmetric} \iff (PD^2\varphi)A = A(PD^2\varphi).
\]
If we write \( X = PD^2\varphi \), we see that the problem of seeking an entropy for (2.4) is reduced to the problem of finding the solutions of the equation
\[
XA = AX.
\]

2.5 Some remarks on the Frobenius problem

The problem of finding all the matrices \( X \) that commute with a given constant matrix is known as the Frobenius problem (see [8] for more details). We will present a series of results concerning the solution of this classical problem that will be used in the next subsections.

**Lemma 2.1.** Let \( A \in M_{n \times n}(\mathbb{R}) \). If the minimal polynomial coincides with the characteristic polynomial of \( A \), then the matrices \( I, A, A^2, \ldots, A^{n-1} \) are linearly independent.

**Proof.** Suppose we had \( I, A, A^2, \ldots, A^{n-1} \) linearly dependent. It would exist constants \( \alpha_0, \alpha_1, \ldots, \alpha_{n-1} \in \mathbb{R} \), with \( \alpha_0^2 + \alpha_1^2 + \cdots + \alpha_{n-1}^2 \neq 0 \) such that
\[
\alpha_0 I + \alpha_1 A + \cdots + \alpha_{n-1} A^{n-1} = 0.
\]
If we introduce the polynomial \( q(t) \) defined by:
\[
q(t) = \alpha_0 + \alpha_1 t + \cdots + \alpha_{n-1} t^{n-1}
\]
we see immediately that \( q(A) = 0 \), i.e., \( q(t) \) is an annihilating polynomial of \( A \) and \( \deg q(t) \leq n-1 < n = \deg p(t) \), where \( p(t) \) is the characteristic polynomial of \( A \). This is not possible because \( p(t) \) is the minimal polynomial by hypothesis.

We introduce now the vectorial space formed by the commutators of \( A \). It will be denoted by \( C_A \) and defined by \( C_A = \{ X \in M_{n \times n}(\mathbb{R}) : XA = AX \} \). The following result can be found in [16]:

**Theorem 2.2.** Let \( m \) be the dimension of the space \( C_A \). Then the following conditions are equivalent:

a) \( m = n \);

b) the characteristic polynomial of \( A \) coincides with the minimal polynomial.

**Proof.** See the proof on page 169 of [16].

**Corollary 2.3.** In the case when the minimal polynomial coincides with the characteristic polynomial of \( A \), we have \( C_A = \text{Span} \{ I, A, A^2, \ldots, A^{n-1} \} \).

**Corollary 2.4.** Let \( A \in M_{n \times n}(\mathbb{R}) \). If the characteristic polynomial of \( A \) has \( n \) real distinct roots, then \( C_A = \text{Span} \{ I, A, A^2, \ldots, A^{n-1} \} \).

**Proof.** All the distinct roots of the characteristic polynomial of a given matrix are also roots of the minimal polynomial. Therefore, in the particular case when we have \( n \) real distinct roots for the characteristic polynomial of \( A \), this must coincide with the minimal polynomial of \( A \). One may apply the Corollary above and the result follows.
The equation $XA = AX$ and the entropies for the system

We return now to the problem of finding the solutions of (2.7). For each fixed $U$ we have

$$X(U)A(U) = A(U)X(U),$$

so we may conclude that $X(U) \in C_{A(U)} = \{X \in M_{4 \times 4}(\mathbb{R}) : AXA = A(U)X\}$. As we have seen, there is a region for which the system is strictly hyperbolic. We will denote this region by $\mathcal{R}$. In this region, each matrix $A(U)$ has all the eigenvalues real and distinct. Applying the Corollary 2.4 it follows that $\dim C_{A(U)} = 4, \forall U \in \mathcal{R}$ and its expression is given by

$$C_{A(U)} = \text{Span}(I, A(U), A^2(U), A^3(U)).$$

Therefore, there are scalars $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ such that

$$X(U) = \alpha_0 I + \alpha_1 A(U) + \alpha_2 A^2(U) + \alpha_3 A^3(U).$$

From this holds for an arbitrary value $U \in \mathcal{R}$, we can establish the following Proposition:

**Proposition 2.5.** For each solution $X$ of (2.7), there exists functions $\alpha, \beta, \gamma$ and $\delta$ depending on $U$ such that

$$X = \alpha I + \beta A + \gamma A^2 + \delta A^3.$$

We should be wondering for which functions $B$ becomes an Hessian matrix for a scalar function. Before we give an answer to this, we will concentrate on the two next Lemmas.

**Lemma 2.7.** Let $A$ be a square matrix given by the product of two symmetric matrices $P$ and $S$:

$$A = PS$$

such that $P$ verifies $P^2 = I$. Then, $PA^k$ is symmetric $\forall k \in \mathbb{N}_0$.

**Proof.** The result will be shown by straightforward calculations:

$$PA^k = P(PS)^k$$

$$= P(PS)^kP^2$$

$$= P^2(SP)^kP$$

$$= (SP)^kP$$

$$= (A^T)^kP$$

$$= (A^k)^T P$$

$$= (PA^k)^T.$$
Lemma 2.8. Let \( B = [b_{ij}]_{i,j=1,...,n} \) be a symmetric matrix depending on \( U = (u_1, \ldots, u_n)^T \). \( B \) is an Hessian matrix for a scalar function if and only if
\[
\frac{\partial b_{ij}}{\partial u_k} = \frac{\partial b_{ik}}{\partial u_j}, \quad \forall i, j, k = 1, \ldots, n
\]

Proof. If \( B \) is an Hessian matrix for a scalar function \( f \), then we have
\[
\frac{\partial^2 f}{\partial u_i \partial u_j} = b_{ij}, \quad \forall i, j = 1, \ldots, n
\]
i.e.,
\[
\frac{\partial}{\partial u_j} \left( \frac{\partial f}{\partial u_i} \right) = b_{ij}, \quad \forall i, j
\]
and
\[
\frac{\partial}{\partial u_k} \left( \frac{\partial f}{\partial u_i} \right) = b_{ik}, \quad \forall i, k
\]
The following condition must then be verified:
\[
\frac{\partial b_{ij}}{\partial u_k} = \frac{\partial b_{ik}}{\partial u_j}, \quad \forall i, j, k = 1, \ldots, n
\]
This is then a necessary condition to verify. In fact, this is also sufficient and this can be considered a consequence of the Frobenius theorem (see [4] for more details).

Remark. There are \((n + 1)n(n - 1)/3\) independent relations which form an overdetermined system of first order partial differential equations.

Since we have \( A = PD^2E \), we can apply the Lemma 2.7, from which follows that \( B \) is symmetric. We are now in conditions to give an answer to what concerns determining the functions \( \alpha, \beta, \gamma, \delta \) for which \( B \) becomes an Hessian matrix for a scalar function. Applying the Lemma 2.8 we get, for the matrix \( B \) as defined in (2.8), the relations:
\[
\frac{\partial b_{ij}}{\partial v_k} = \frac{\partial b_{ik}}{\partial v_j}, \quad \forall i, j, k = 1, \ldots, 4
\]
where \( v_1 = \rho_1, v_2 = \rho_2, v_3 = u_1 \) and \( v_4 = u_2 \).
We present a list of all the elements of $B$:

\[
\begin{align*}
    b_{11} &= \beta W_{11} + \gamma(2u_1 W_{11}) + \delta(3u_1^2 W_{11} + \rho_1 W_{11}^2 + \rho_2 W_{12}^2) \\
    b_{12} &= \beta W_{12} + \gamma(u_1 + u_2) W_{12} + \delta(u_1^2 + u_1 u_2 + u_2^2 + \rho_1 W_{11} + \rho_2 W_{22}) W_{12} \\
    b_{13} &= \alpha + \beta u_1 + \gamma(u_1^2 + \rho_1 W_{11}) + \delta(3\rho_1 u_1 W_{11} + u_1^3) \\
    b_{14} &= \gamma(\rho_2 W_{12}) + \delta(2\rho_2 u_2 + \rho_2 u_1) W_{12} \\
    b_{22} &= \beta W_{22} + \gamma(2u_2 W_{22}) + \delta(3u_2^2 W_{22} + \rho_1 W_{12}^2 + \rho_2 W_{22}^2) \\
    b_{23} &= \gamma(\rho_1 W_{12}) + \delta(2\rho_1 u_1 + \rho_1 u_2) W_{12} \\
    b_{24} &= \alpha + \beta u_2 + \gamma(u_2^2 + \rho_2 W_{22}) + \delta(3\rho_2 u_2 W_{22} + u_2^3) \\
    b_{33} &= \beta \rho_1 + \gamma(2\rho_1 u_1) + \delta(3\rho_1 u_1^2 + \rho_1^2 W_{11}) \\
    b_{34} &= \delta(\rho_1 \rho_2 W_{12}) \\
    b_{44} &= \beta \rho_2 + \gamma(2\rho_2 u_2) + \delta(3\rho_2 u_2^2 + \rho_2^2 W_{22})
\end{align*}
\]

The compatibility relations give rise to a homogeneous linear system on the variables $\gamma, \delta$ and all the partial derivatives for $\alpha, \beta, \gamma$ and $\delta$ (see Appendix). We can write this system in a condensed form

\[MY = 0, \tag{2.9}\]

where $M$ is a rectangular matrix of dimension $20 \times 18$ depending on $U$, and $Y$ is the vector of dimension $18$ defined by

\[Y = (\alpha_\rho_1, \alpha_\rho_2, \alpha_{u_1}, \alpha_{u_2}, \beta_\rho_1, \beta_\rho_2, \beta_{u_1}, \beta_{u_2}, \gamma, \gamma_\rho_1, \gamma_\rho_2, \gamma_{u_1}, \gamma_{u_2}, \delta, \delta_\rho_1, \delta_\rho_2, \delta_{u_1}, \delta_{u_2})^T.\]

We used the 5th edition of \textsc{Mathematica} by Wolfram Research to compute the nullspace of the matrix $M$ and as a result we obtained a base composed by $4$ vectors $V_1, V_2, V_3$ and $V_4$ with a nice particularity: \textit{the 9th and 14th components, that correspond to the variables $\gamma$ and $\delta$, are both 0} (see Appendix).

Since the solutions of (2.9) satisfy:

\[Y = f_1(U)V_1 + f_2(U)V_2 + f_3(U)V_3 + f_4(U)V_4,\]

for scalar functions $f_1, f_2, f_3, f_4$, this means that if $B$ is an Hessian matrix, then the functions $\gamma$ and $\delta$ are necessarily both zero. We return to the compatibility relations for $B$, that are now enormously simplified. We present some of these relations:
We write
\[ \frac{\partial b_{41}}{\partial u_2} = \frac{\partial b_{44}}{\partial \rho_1} \]
as
\[ 0 = \rho_2 \beta_{\rho_1}, \]
and consequently \( \beta_{\rho_1} = 0 \). The relation
\[ \frac{\partial b_{32}}{\partial u_1} = \frac{\partial b_{33}}{\partial \rho_2}, \]
traded by
\[ 0 = \rho_1 \beta_{\rho_2} \]
implies \( \beta_{\rho_2} = 0 \). Consider now
\[ \frac{\partial b_{43}}{\partial u_2} = \frac{\partial b_{44}}{\partial u_1} \]
and write it as
\[ 0 = \rho_2 \beta_{u_1}. \]
This implies \( \beta_{u_1} = 0 \). Finally, from the relation:
\[ \frac{\partial b_{33}}{\partial u_2} = \frac{\partial b_{34}}{\partial u_1}, \]
written as
\[ \rho_1 \beta_{u_2} = 0, \]
it follows \( \beta_{u_2} = 0 \).

We may conclude that \( \beta = \text{const} \). Similarly, by returning once again to the compatibility relations, we get \( \alpha = \text{const} \). This proves the following Proposition:

**Proposition 2.9.** The matrix \( B \) as defined in (2.8) is an Hessian matrix of a scalar function if and only if

(i) \( \alpha \) and \( \beta \) are constant;

(ii) \( \gamma = \delta = 0 \).

We use this result to find the mathematical entropies of (2.6). It was found that they are the solutions of the equation
\[ D^2 \varphi = \alpha P + \beta D^2 E, \]
with \( \alpha \) and \( \beta \) constant. The explicit solution can be given:
\[ \varphi = \alpha \frac{U^T P U}{2} + \beta E + c \cdot U + \text{const}, \]
where \( c \) represents a constant vector. Alternatively, we can present it as
\[ \varphi = \alpha (\rho_1 u_1 + \rho_2 u_2) + \beta E + c_1 \rho_1 + c_2 \rho_2 + c_3 u_1 + c_4 u_2 + \text{const}. \]

In other words, the conservation laws admitted by the shallow water equations are the linear combination of the equations for the conservation of mass and velocity (in each layer), total momentum and total energy. We note that the same conserved quantities were found by Montgomery and Moodie [13] by restricting their search to the multinomials expressions in the layer variables.
Remark. The main ingredients of the method we have just presented are the following: a conservative system \( U_t + AU_x = 0 \) endowed with a region of strict hyperbolicity; a matrix \( A \) given by the product of two symmetric matrices \( P \) and \( S \), where \( P \) is non-singular matrix. We will present in the next section the system of equations for the shallow water model with topography for which, however not fulfilling completely these requirements, similar techniques can be used to find the complete set of conservation laws.

3 A Full Set of Entropies for the Extended 1-d Shallow Water Model with Topography

3.1 Physical model

Let us consider the following quasilinear system of equations

\[
\begin{cases}
  h_t + (hu)_x = 0 \\
  (hu)_t + \left( hu^2 + \frac{g}{2} \right)_x = -ghb_x , \\
  b_t = 0
\end{cases}
\]

where \( h(x,t) \) is the fluid depth, \( u(x,t) \) the fluid velocity, \( b(x) \) the bottom depth and the free-fall acceleration is denoted by \( g \). These are the equations for the classical shallow water model with topography extended with the obvious condition \( b_t = 0 \) (as in [3], [7] and [15]). It admits the conservative form:

\[
\begin{cases}
  h_t + (hu)_x = 0 \\
  u_t + \left( \frac{u^2}{2} + g(h + b) \right)_x = 0 . \\
  b_t = 0
\end{cases}
\]

We have then the quasilinear system

\[
U_t + AU_x = 0
\]

defined by

\[
U = \begin{bmatrix} h \\ u \\ b \end{bmatrix}, \quad A = \begin{bmatrix} u & h & 0 \\ g & u & g \\ 0 & 0 & 0 \end{bmatrix}.
\]

As in the former section, we will be interested in presenting conservation laws for this system of the form

\[
\varphi_t + \psi_x = 0,
\]

with \( \varphi \) and \( \psi \) depending on the physical variables presented. It will be proved that all the conserved quantities are given by

\[
\alpha_1 \left( \frac{hu^2}{2} + g \left( \frac{h^2}{2} + hb \right) \right) + \alpha_2 h + \alpha_3 u + \Gamma(b)
\]
for some constants $\alpha_i$, $i = 1, 2, 3$ and some function $\Gamma$ depending on $b$. If we take the Froude number $Fr = \frac{|u|}{c}$, where $c = \sqrt{gh}$, we see immediately that the system is strictly hyperbolic if and only if $Fr \neq 1$. This kind of flow is usually called a non-critical flow and it will be the framework considered in this work.

An entropy $\varphi$ exists for this system if and only if $D^2\varphi \cdot A$ is symmetric. As before, our matrix $A$ has a special structure that enables us to go further on this condition. Indeed, if we define

$$ P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} g & u & g \\ u & h & 0 \\ g & 0 & 0 \end{bmatrix}, $$

we see that $A = PB$ and $P$ and $B$ are both symmetric matrices. It follows that

$$ D^2\varphi \cdot A \text{ symmetric } \iff D^2\varphi \cdot PB = BP \cdot D^2\varphi, $$

which implies\(^2\):

$$ (PD^2\varphi)A = A(PD^2\varphi), $$

multiplying both members by the singular matrix $P$. If we write $X = PD^2\varphi$ we are reduced to the problem of finding the solutions of the equation

$$ XA = AX. \quad (3.4) $$

### 3.2 The equation $XA = AX$ and the entropies for the system

As we have seen, the system is strictly hyperbolic for a non-critical flow. As a consequence, the minimal polynomial coincides with the characteristic polynomial of $A$. Making use of the results presented in the previous section, the following Proposition holds:

**Proposition 3.1.** For each solution $X$ of (3.4), there exists functions $\alpha$, $\beta$ and $\gamma$ depending on $U$ such that

$$ X = \alpha I + \beta A + \gamma A^2. $$

We apply this Proposition, obtaining:

$$ \begin{bmatrix} \varphi_{uh} & \varphi_{uu} & \varphi_{ub} \\ \varphi_{hh} & \varphi_{hu} & \varphi_{hb} \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \alpha + \beta u + \gamma(gh + u^2) & \beta h + \gamma(2hu) & \gamma(gh) \\ g\beta + \gamma(2gu) & \alpha + \beta u + \gamma(gh + u^2) & g\beta + \gamma(gu) \\ 0 & 0 & \alpha \end{bmatrix}, $$

from which immediately follows

$$ \alpha = 0. $$

\(^2\)This is no longer an equivalence like in the previous case.
We shall focus now on solving the system of equations:

\[
\begin{cases}
    \varphi_{hh} = g\beta + \gamma(2gu) \\
    \varphi_{hu} = \beta u + \gamma(gh + u^2) \\
    \varphi_{hb} = g\beta + \gamma(gu) \\
    \varphi_{uu} = \beta h + \gamma(2hu) \\
    \varphi_{ub} = \gamma(gh)
\end{cases}
\] (3.5)

Let us define the functions

\[
c_{11} = g\beta + \gamma(2gu),
\]

\[
c_{12} = \beta u + \gamma(gh + u^2),
\]

\[
c_{13} = g\beta + \gamma(gu),
\]

\[
c_{22} = \beta h + \gamma(2hu),
\]

\[
c_{23} = \gamma(gh),
\]

with \(c_{ij} = c_{ji}, \quad \forall i, j = 1, 2, 3\). The compatibility conditions for the existence of \(\varphi\) are given by:

\[
\frac{\partial c_{ij}}{\partial v_k} = \frac{\partial c_{ik}}{\partial v_j}, \quad \forall i, j, k = 1, 2, 3
\]

whenever it makes sense, with \(v_1 = h, v_2 = u, v_3 = b\). This is traduced by the following set of equations:

\[
\beta_h(u) - g\beta_u - g\gamma + \gamma_h(gh + u^2) - \gamma_u(2gu) = 0,
\]

\[
\beta_h - \beta_h + \gamma_h(u) - \gamma_b(2u) = 0,
\]

\[
g\beta_u - \beta_b(u) + g\gamma + \gamma_u(gu) - \gamma_b(gh + u^2) = 0,
\]

\[
\beta_h(h) - \beta_u(u) + \gamma_h(2hu) - \gamma_u(gh + u^2) = 0,
\]

\[
\beta_b(u) - g\gamma - \gamma_h(gh) + \gamma_b(gh + u^2) = 0,
\]

\[
\beta_b(h) - \gamma_u(gh) + \gamma_b(2hu) = 0,
\]
that we present in a condensed form:

\[
\begin{bmatrix}
  u & -g & 0 & -g & gh + u^2 & -2gu & 0 \\
 1 & 0 & -1 & 0 & u & 0 & -2u \\
 0 & g & -u & g & 0 & gu & -gh - u^2 \\
 h & -u & 0 & 0 & 2hu & -gh - u^2 & 0 \\
 0 & 0 & u & -g & -gh & 0 & gh + u^2 \\
 0 & 0 & h & 0 & 0 & -gh & 2hu \\
\end{bmatrix}
\begin{bmatrix}
  \beta_h \\
  \beta_u \\
  \beta_b \\
  \gamma \\
  \gamma_h \\
  \gamma_u \\
  \gamma_b \\
\end{bmatrix} = 0.
\]

For this underdetermined system, we used the 5th edition of MATHEMATICA by Wolfram Research to compute the kernel of this rectangular matrix. The base presented is composed by the two following vectors:

\[
V_1 = \left( \frac{u^3 - gh u}{gh}, \frac{gh - u^2}{g}, -2u, 0, \frac{gh - u^2}{gh}, 0, 1 \right),
\]

\[
V_2 = \left( \frac{gh - u^2}{h}, 0, g, 0, \frac{u}{h}, 1, 0 \right).
\]

As we can see, the 4th component of these two vectors is 0. This means that a necessary condition so that the system can be solved is provided by \(\gamma = 0\).

The compatibility conditions are reduced to

\[
\begin{align*}
  g\beta_u &= u\beta_h \\
  \beta_b &= \beta_h \\
  u\beta_b &= g\beta_u \\
  u\beta_u &= h\beta_h \\
  u\beta_b &= 0 \\
  h\beta_b &= 0
\end{align*}
\]

which enables us to conclude:

\[
\beta_h = \beta_u = \beta_b = 0,
\]

i.e., \(\beta = \text{const}\). Making use of this, we go back to the system (3.5) in order to find the general
solution \( \varphi \). The system is now presented by:

\[
\begin{align*}
\varphi_{hh} &= g\beta \\
\varphi_{hu} &= \beta u \\
\varphi_{hb} &= g\beta \\
\varphi_{uu} &= \beta h \\
\varphi_{ub} &= 0
\end{align*}
\]  

(3.6)

with \( \beta = \text{const} \). Straightforward calculations allow us to determine the general solution:

\[
\varphi = \beta \left( \frac{hu^2}{2} + g \left( \frac{h^2}{2} + hb \right) \right) + c_1 h + c_2 u + \Gamma(b),
\]

(3.7)

where \( c_1 \) and \( c_2 \) are constant and \( \Gamma(b) \) denotes an arbitrary function depending on \( b \).

Until now, we have just seen necessary conditions to be verified so that \( \varphi \) can be a mathematical entropy of (3.2). It is time to verify if the general solution obtained in (3.7) satisfies the condition \( D^2 \varphi \cdot A \text{ is symmetric} \). In fact:

\[
D^2 \varphi \cdot A = \begin{bmatrix}
    g\beta & \beta u & g\beta \\
    \beta u & \beta h & 0 \\
    g\beta & 0 & \Gamma''(b)
\end{bmatrix}
\begin{bmatrix}
    u & h & 0 \\
    g & u & g \\
    0 & 0 & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
    2g\beta u & \beta(gh + u^2) & g\beta u \\
    \beta(gh + u^2) & 2\beta hu & g\beta h \\
    g\beta u & g\beta h & 0
\end{bmatrix}
\]

is symmetric, so we have to conclude that a necessary and sufficient condition to establish a mathematical entropy \( \varphi \) for (3.2) is given by the general formulation in (3.7).

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**A Appendix**

This section is devoted to the presentation of the system of equations arising from the compatibility relations for the matrix \( B \) as defined in (2.8). It is as homogeneous linear system in the variables
\[ \gamma, \delta \text{ and all the partial derivatives for } \alpha, \beta, \gamma \text{ and } \delta. \] It can also be written in a condensed form

\[ MY = 0, \]

where \( M \) is a rectangular matrix of dimension \( 20 \times 18 \) depending on \( U \), and \( Y \) is the vector of dimension 18 defined by

\[ Y = (\alpha_{\rho_1}, \alpha_{\rho_2}, \alpha_{u_1}, \alpha_{u_2}; \beta_{\rho_1}, \beta_{\rho_2}, \beta_{u_1}, \beta_{u_2}, \gamma, \gamma_{\rho_1}, \gamma_{\rho_2}, \gamma_{u_1}, \gamma_{u_2}, \delta, \delta_{\rho_1}, \delta_{\rho_2}, \delta_{u_1}, \delta_{u_2})^T. \]

We list these equations:

\[ \alpha_{\rho_1} + \beta_{\rho_1}u_1 - \beta_{u_1}W_{11} - \gamma_{W_{11}} + \gamma_{u_1}(u_1^2 + \rho_1W_{11}) - \gamma_{u_1}(2u_1W_{11}) - \delta(3u_1W_{11}) + \delta_{\rho_1}(3\rho_1u_1W_{11} + u_1^3) - \delta_{u_1}(3u_1^2W_{11} + \rho_1W_{11}^2 + \rho_2W_{12}^2) = 0 \quad (A.1) \]

\[ \beta_{u_2}W_{11} - \gamma_{\rho_1}(\rho_2W_{12}) + \gamma_{u_2}(2u_1W_{11}) - \delta_{\rho_1}(2\rho_2u_2 + \rho_2u_1)W_{12} + \delta_{u_2}(3u_1^2W_{11} + \rho_1W_{11}^2 + \rho_2W_{12}^2) = 0 \quad (A.2) \]

\[ \beta_{u_2}W_{11} - \gamma_{\rho_1}(\rho_2W_{12}) + \gamma_{u_2}(2u_1W_{11}) - \delta_{\rho_1}(2\rho_2u_2 + \rho_2u_1)W_{12} + \delta_{u_2}(3u_1^2W_{11} + \rho_1W_{11}^2 + \rho_2W_{12}^2) = 0 \quad (A.3) \]

\[ \alpha_{\rho_2} + \beta_{\rho_2}u_1 - \beta_{u_1}W_{12} - \gamma_{W_{12}} + \gamma_{u_1}(u_1^2 + \rho_1W_{11}) - \gamma_{u_1}(u_1 + u_2)W_{12} - \delta(2u_1 + u_2)W_{12} + \delta_{\rho_2}(3\rho_1u_1W_{11} + u_1^3) - \delta_{u_1}(u_1^2 + u_1u_2 + u_2^2 + \rho_1W_{11} + \rho_2W_{22})W_{12} = 0 \quad (A.4) \]

\[ \beta_{u_2}W_{12} - \gamma_{\rho_2}(\rho_2W_{12}) + \gamma_{u_2}(u_1 + u_2)W_{12} - \delta_{\rho_2}(2\rho_2u_2 + \rho_2u_1)W_{12} + \delta_{u_2}(u_1^2 + u_1u_2 + u_2^2 + \rho_1W_{11} + \rho_2W_{22})W_{12} = 0 \quad (A.5) \]

\[ \alpha_{u_2} + \beta_{u_2}u_1 - \gamma_{u_1}(\rho_2W_{12}) + \gamma_{u_2}(u_1^2 + \rho_1W_{11}) - \delta(\rho_2W_{12}) - \delta_{u_1}(2\rho_2u_2 + \rho_2u_1)W_{12} + \delta_{u_2}(3\rho_1u_1W_{11} + u_1^3) = 0 \quad (A.6) \]

\[ \beta_{\rho_1}W_{22} - \beta_{\rho_2}W_{12} + \gamma_{\rho_1}(2u_2W_{22}) - \gamma_{\rho_2}(u_1 + u_2)W_{12} + \delta(W_{12} - W_{12}W_{22}) + \delta_{\rho_1}(3u_2^2W_{22} + \rho_1W_{12}^2 + \rho_2W_{22}^2) - \delta_{\rho_2}(u_1^2 + u_1u_2 + u_2^2 + \rho_1W_{11} + \rho_2W_{22})W_{12} = 0 \quad (A.7) \]

\[ \beta_{\rho_1}W_{12} - \gamma_{\rho_1}(\rho_1W_{12}) + \gamma_{u_1}(u_1 + u_2)W_{12} - \delta_{\rho_1}(2\rho_1u_1 + \rho_1u_2)W_{12} + \delta_{u_1}(u_1^2 + u_1u_2 + u_2^2 + \rho_1W_{11} + \rho_2W_{22})W_{12} = 0 \quad (A.8) \]
\[\alpha_{\rho_1} + \beta_{\rho_1} u_2 - \beta_{u_2} W_{12} - \gamma W_{12} + \gamma_{\rho_1} (u_2^2 + \rho_2 W_{22}) - \gamma_{u_2} (u_1 + u_2) W_{12} - \delta (u_1 + 2u_2) W_{12} + \\
+ \delta_{\rho_1} (3 \rho_2 u_2 W_{22} + u_2^3) - \delta_{u_2} (u_1^2 + u_1 u_2 + u_2^2 + \rho_1 W_{11} + \rho_2 W_{22}) W_{12} = 0 \quad (A.9)\]

\[\beta_{u_1} W_{22} - \gamma_{\rho_2} (\rho_1 W_{12}) + \gamma_{u_1} (2 u_2 W_{22}) - \delta_{\rho_2} (2 \rho_1 u_1 + \rho_1 u_2) W_{12} + \\
+ \delta_{u_1} (3 u_2^2 W_{22} + \rho_1 W_{12}^2 + \rho_2 W_{22}^2) = 0 \quad (A.10)\]

\[\alpha_{\rho_2} + \beta_{\rho_2} u_2 - \beta_{u_2} W_{22} - \gamma W_{22} + \gamma_{\rho_2} (u_2^2 + \rho_2 W_{22}) - \gamma_{u_2} (2 u_2 W_{22}) - \delta (3 u_2 W_{22}) + \\
+ \delta_{\rho_2} (3 \rho_2 u_2 W_{22} + u_2^3) - \delta_{u_2} (3 u_2^2 W_{22} + \rho_1 W_{12}^2 + \rho_2 W_{22}^2) = 0 \quad (A.11)\]

\[\alpha_{u_1} + \beta_{u_1} u_2 + \gamma_{u_1} (u_2^2 + \rho_2 W_{22}) - \gamma_{u_2} (\rho_1 W_{12}) - \delta (\rho_1 W_{12}) + \\
+ \delta_{u_1} (3 \rho_2 u_2 W_{22} + u_2^3) - \delta_{u_2} (2 \rho_1 u_1 + \rho_1 u_2) W_{12} = 0 \quad (A.12)\]

\[\alpha_{u_1} - \beta_{\rho_1} \rho_1 + \beta_{u_1} u_1 - \gamma_{\rho_1} (2 \rho_1 u_1) + \gamma_{u_1} (u_1^2 + \rho_1 W_{11}) + \delta (\rho_1 W_{11}) - \\
- \delta_{\rho_1} (3 \rho_1 u_1^2 + \rho_1^2 W_{11}) + \delta_{u_1} (3 \rho_1 u_1 W_{11} + u_1^3) = 0 \quad (A.13)\]

\[\alpha_{u_2} + \beta_{u_2} u_1 + \gamma_{u_2} (u_1^2 + \rho_1 W_{11}) - \delta (\rho_2 W_{12}) - \delta_{\rho_1} (\rho_1 \rho_2 W_{12}) + \delta_{u_2} (3 \rho_1 u_1 W_{11} + u_1^3) = 0 \quad (A.14)\]

\[\beta_{\rho_2} \rho_1 + \gamma_{\rho_2} (2 \rho_1 u_1) - \gamma_{u_1} (\rho_1 W_{12}) - \delta (2 \rho_1 W_{12}) + \\
+ \delta_{\rho_2} (3 \rho_1 u_1^2 + \rho_1^2 W_{11}) - \delta_{u_1} (2 \rho_1 u_1 + \rho_1 u_2) W_{12} = 0 \quad (A.15)\]

\[\gamma_{u_2} (\rho_1 W_{12}) - \delta_{\rho_2} (\rho_1 \rho_2 W_{12}) + \delta_{u_2} (2 \rho_1 u_1 + \rho_1 u_2) W_{12} = 0 \quad (A.16)\]

\[\beta_{u_2} \rho_1 + \gamma_{u_2} (2 \rho_1 u_1) - \delta_{u_1} (\rho_1 \rho_2 W_{12}) + \delta_{u_2} (3 \rho_1 u_1^2 + \rho_1^2 W_{11}) = 0 \quad (A.17)\]

\[\beta_{\rho_1} \rho_2 + \gamma_{\rho_1} (2 \rho_2 u_2) - \gamma_{u_2} (\rho_2 W_{12}) - \delta (2 \rho_2 W_{12}) + \delta_{\rho_1} (3 \rho_2 u_2^2 + \rho_2^2 W_{22}) - \\
- \delta_{u_2} (2 \rho_2 u_2 + \rho_2 u_1) W_{12} = 0 \quad (A.18)\]
\[ \alpha_{u_2} - \beta_{\rho_2} \rho_2 + \beta_{u_1} u_2 - \gamma_{\rho_2} (2 \rho_2 u_2) + \gamma_{u_2} (u_2^2 + \rho_2 W_{22}) + \delta (\rho_2 W_{22}) - \delta_{\rho_2} (3 \rho_2 u_2^2 + \rho_2^2 W_{22}) + \delta_{u_2} (3 \rho_2 u_2 W_{22} + u_2^3) = 0 \quad (A.19) \]

\[ \beta_{u_1} \rho_2 + \gamma_{u_1} (2 \rho_2 u_2) + \delta_{u_1} (3 \rho_2 u_2^2 + \rho_2^2 W_{22}) - \delta_{u_2} (\rho_1 \rho_2 W_{12}) = 0 \quad (A.20) \]

After computing the nullspace of \( M \), making use of \textit{MATHEMATICA}, we get a base formed by the four following vectors:

\[ V_1 = \begin{bmatrix} u_1 u_2 W_{12} \\ \rho_1 W_{12}^2 + u_1^2 W_{22} - \rho_1 W_{11} W_{22} \\ -\rho_1 u_2 W_{12} \\ -u_1^2 u_2 + \rho_1 u_2 W_{11} \\ -u_1 W_{12} - u_2 W_{12} \\ -2 u_1 W_{22} \\ \rho_1 W_{12} \\ u_1^3 + 2 u_1 u_2 - \rho_1 W_{11} \\ 0 \\ W_{12} \\ W_{22} \\ 0 \\ -2 u_1 - u_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad V_2 = \begin{bmatrix} u_2 W_{11} + \rho_2 W_{12}^2 - \rho_2 W_{11} W_{22} \\ u_1 u_2 W_{12} \\ -u_1^2 u_2 + \rho_2 u_1 W_{22} \\ -\rho_2 u_1 W_{12} \\ -2 u_2 W_{11} \\ -u_1 W_{12} - u_2 W_{12} \\ 2 u_1 u_2 + u_2^2 - \rho_2 W_{22} \\ \rho_2 W_{12} \\ 0 \\ W_{11} \\ W_{12} \\ -u_1 - 2 u_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \]

\[ V_3 = \begin{bmatrix} -\rho_2 u_1 W_{12} \\ -u_1^2 u_2 + \rho_1 u_2 W_{11} \\ \rho_1 \rho_2 W_{12} \\ \rho_2 u_1^2 - \rho_1 \rho_2 W_{11} \\ \rho_2 W_{12} \\ u_1^2 + 2 u_1 u_2 - \rho_1 W_{11} \\ 0 \\ -2 \rho_2 u_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad V_4 = \begin{bmatrix} -u_1^2 u_2 + \rho_2 u_1 W_{22} \\ -\rho_2 u_1 W_{12} \\ \rho_1 u_2^3 - \rho_1 \rho_2 W_{22} \\ \rho_1 \rho_2 W_{12} \\ 2 u_1 u_2 + u_2^2 - \rho_2 W_{22} \\ \rho_1 W_{12} \\ -2 \rho_1 u_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ \rho_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \]
References


Part II
Derivation of a Dispersive Model for Two-Layer Flows with Free Surface
Dispersive Nonlinear Waves in Two-Layer Flows with Free Surface. I. Model derivation and general properties


R. Barros∗ S. L. Gavrilyuk† V.M.Teshukov‡

**Abstract**

In this paper we derive an approximate multi-dimensional model of dispersive waves propagating in a two-layer fluid with free surface. This model is a “two-layer” generalization of the Green-Naghdi model. Our derivation is based on Hamilton’s principle. From the Lagrangian for the full-water problem we obtain an approximate Lagrangian with accuracy $O(\varepsilon^2)$, where $\varepsilon$ is the formal small parameter representing the ratio of a typical vertical scale to a typical horizontal scale. This approach allows us to derive governing equations in a compact and symmetric form. Important properties of the model are revealed. In particular, we introduce the notion of generalized vorticity and derive analogues of integrals of motion, such as Bernoulli integrals, which are well-known in ideal fluid mechanics. Conservation laws for the total momentum and total energy are also obtained.

1 **Introduction**

In this article we derive an approximate multi-dimensional model of dispersive water waves propagating in a free boundary two-layer stratified fluid, and study general properties of this system. The model deals with rotational flows and no smallness assumption on the wave amplitude is made in its derivation. This model generalizes the equations derived by Su & Gardner [1], Green, Laws & Naghdhi ([2], [3]) within the context of a homogeneous one-layer fluid. In the literature, this model is usually called Green-Naghdi model (GN model or GN system). A derivation of the GN model based on the variational formulation of the Euler equations was done by Miles & Salmon [4] (see also [5], [6]).

Earlier the solutions of the GN model describing two-dimensional periodic and solitary waves propagating in homogeneous liquid were studied. Solitary wave solutions of the GN model were obtained by Su & Gardner [1]. The linear stability of these waves was recently proved by Li [7]. A criterion of stability of shear flows for the GN model was proposed by Gavrilyuk & Teshukov [8]. A wide class of multi-dimensional solutions and approximate solutions of nonlinear multi-dimensional GN model was found by Gavrilyuk & Teshukov [9] and Teshukov & Gavrilyuk [10]. Unsteady undular bores were described by El, Grimshaw & Smyth [11].

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Generalized Green-Naghdi’s equations describing internal waves in two-layered systems were first obtained by Choi & Camassa [12] (rigid lid) and Liska, Margolin & Wendroff [13] (free surface), however using special assumptions on the flow character in their derivation. Unfortunately, due to the complex form in which they are presented, the equations derived by Liska, Margolin & Wendroff [13] are not easily amenable to analytical investigations, creating a serious obstacle for their use in the study of coupled effects of nonlinear dispersion and stratification on the waves. Our aim is a simpler invariant formulation of these equations, based on the variational formulation of the Euler equations. This method gives governing equations with the accuracy $O(\varepsilon^2)$, where the small parameter $\varepsilon$ is the aspect ratio between a typical fluid depth and a typical wavelength.

2 General statement of the problem

We consider a two-layer free-boundary flow of immiscible fluids with different constant densities $\gamma_1, \gamma_2$ (see Fig. 1).

![Figure 1: Physical model.](image)

The flow is governed by the following system of Euler equations

$$\text{div} \, \mathbf{u}_\alpha = 0$$

$$\gamma_\alpha \frac{D_{\alpha} \mathbf{u}_\alpha}{Dt} + \nabla p_\alpha = \gamma_\alpha \mathbf{g}, \quad \alpha = 1, 2. \hspace{1cm} (1)$$

At the free surface $z = h_1 + h_2$, the interface $z = h_1$ and the flat bottom $z = 0$, the following boundary conditions are prescribed:

$$z = h_1 + h_2 : \quad (h_1 + h_2)_t + u_2(h_1 + h_2)_x + v_2(h_1 + h_2)_y = w_2, \quad p_2 = p_0 = \text{const}$$

$$z = h_1 : \quad (h_1)_t + u_1(h_1)_x + v_1(h_1)_y = w_1, \quad (h_1)_t + u_2(h_1)_x + v_2(h_1)_y = w_2, \quad p_1 = p_2$$

$$z = 0 : \quad w_1 = 0.$$

The notations adopted are conventional: $t$ is the time; the space coordinates are denoted by $\mathbf{x} = (x, y, z)$ and the corresponding components of the velocity vector $\mathbf{u}_\alpha$ by $(u_\alpha, v_\alpha, w_\alpha)$; $h_\alpha(x, y, t)$ denotes the depth of each layer and $p_\alpha$ the pressures; $\mathbf{g} = (0, 0, -g)$, where $g$ is the gravity acceleration. Moreover we suppose that $\gamma_2 < \gamma_1$ i.e., the stratification is stable. Also, the material
The derivative operator $D_\alpha/\text{Dt}$ is defined by

$$
\frac{D_\alpha}{\text{Dt}} = \frac{\partial}{\partial t} + \mathbf{u}_\alpha \cdot \nabla, \quad \alpha = 1, 2.
$$

The Euler equations (1) and boundary conditions can be obtained by applying Hamilton’s principle to the following Lagrangian

$$
\mathcal{L} = \int \int \int_{h_1(x,y,t)}^{h_2(x,y,t)} \gamma_1 \left( \frac{1}{2} |\mathbf{u}_1(x,t)|^2 - gz \right) \, dz \, dx \, dy + \\
+ \int \int \int_{h_1(x,y,t)}^{(h_1+h_2)(x,y,t)} \gamma_2 \left( \frac{1}{2} |\mathbf{u}_2(x,t)|^2 - gz \right) \, dz \, dx \, dy
$$

constrained to the mass conservation laws of each layer

$$
\text{div} \, \mathbf{u}_\alpha = 0.
$$

Our goal is to propose an approximate set of equations following from the three-dimensional Euler equations governing this two-layer flow (also called full-water problem), inserting the approximations directly into the fluid Lagrangian (see e.g., [4], [5], [6], [14], [15]). This method has two main advantages over the classical perturbation procedures. First, the approximations do not disturb the corresponding symmetry properties coming from the variational structure of the governing equations. Second, the approximation methods based on Hamilton’s principle suggest transformations to new dependent variables in which the approximate equations take its simplest mathematical form.

### 3 Derivation of an approximate Lagrangian

We introduce a set of nondimensional variables. Rather than introduce a new notation for all our variables, we write $x \rightarrow Lx$. This is to be read that $x$ is replaced by $Lx$, so that hereafter the symbol $x$ will denote a nondimensional variable. With this understanding, we define:

$$
x \rightarrow Lx, \quad y \rightarrow Ly, \quad z \rightarrow Hz, \quad t \rightarrow U^{-1}Lt \quad u_\alpha \rightarrow U u_\alpha, \quad v_\alpha \rightarrow U v_\alpha, \quad w_\alpha \rightarrow \varepsilon U w_\alpha
$$

with

$$
h_\alpha \rightarrow H h_\alpha, \quad p_\alpha \rightarrow RU^2 p_\alpha \quad \text{and} \quad \gamma_\alpha \rightarrow R \gamma_\alpha.
$$

The corresponding dimensionless Lagrangian for the full-water problem is the following:

$$
\int \int \int_{h_1}^{h_2} \gamma_1 \left( \frac{1}{2} (u_1^2 + v_1^2 + \varepsilon^2 w_1^2) - F^{-2}z \right) \, dz \, dx \, dy + \\
+ \int \int \int_{h_1}^{h_1+h_2} \gamma_2 \left( \frac{1}{2} (u_2^2 + v_2^2 + \varepsilon^2 w_2^2) - F^{-2}z \right) \, dz \, dx \, dy. \quad (2)
$$

Here, $H, L$ are the typical vertical and horizontal scales, $U$ is the characteristic velocity, $R$ is typical density and $F = U/\sqrt{gH}$ denotes the Froude number. The long-wave approximation is considered by introducing a small parameter given by $\varepsilon = H/L \ll 1$. We define next a class of flows for which an approximate Lagrangian can be calculated with accuracy $O(\varepsilon^2)$.

---

1. Here, the given velocity field is $\mathbf{u}_\alpha$ but, more generally, the material derivative will be used throughout the text for different velocity fields (even in two-dimensional case); hopefully with no resulting confusion. Formally, the expression for the operator doesn’t change much (just replace the velocity field and take the gradient $\nabla$ defined in two or three dimensions, depending on whether the velocity field is defined in two or three dimensions, respectively.
Definition (Weakly sheared flow). A flow is called a weakly sheared flow if there exists a real number \( s > 1 \) such that \( u_{\alpha z} = O(\varepsilon^s) \) and \( v_{\alpha z} = O(\varepsilon^s) \).

It will become clear in this work how the introduction of this class enables to establish a precise relation between this approximate theory and the exact three-dimensional theory. In Appendix A, the following statement can be found:

Consider the set of equations (1) for the full-water problem. If the initial data belongs to this class, then its solution remains in this class over time.

Next, we see that, in this class, we are able to perform some approximations that will lead to the calculation of the approximate Lagrangian.

Consider two functions \( f \) and \( g \). Hereafter we will write \( f \approx g \) to denote \( f - g = o(\varepsilon^2) \). Set \( \mathbf{v}_\alpha = (u_\alpha, v_\alpha) \) as the horizontal velocity and consider the mean velocities

\[
\mathbf{v}_1(x, y, t) = \frac{1}{h_1} \int_0^{h_1} \mathbf{v}_1 \, dz, \quad \mathbf{v}_2(x, y, t) = \frac{1}{h_2} \int_{h_1}^{h_1+h_2} \mathbf{v}_2 \, dz.
\]

In this framework, we deduce the following estimates (see proof in Appendix A):

**Estimate 1.** For the class of weakly sheared flows it holds

\[
\int_0^{h_1} |\mathbf{v}_1|^2 \, dz \approx h_1 |\mathbf{v}_1|^2, \quad \int_{h_1}^{h_1+h_2} |\mathbf{v}_2|^2 \, dz \approx h_2 |\mathbf{v}_2|^2.
\]

**Estimate 2.** For the class of weakly sheared flows it holds:

\[
\int_0^{h_1} \varepsilon^2 w_1^2 \, dz \approx \frac{1}{3} \varepsilon^2 h_1 \left( \frac{D_1 h_1}{Dt} \right)^2, \quad 
\int_{h_1}^{h_1+h_2} \varepsilon^2 w_2^2 \, dz \approx \varepsilon^2 h_2 \left( \frac{D_2 h_1}{Dt} \right)^2 + \frac{1}{3} \left( \frac{D_2 h_2}{Dt} \right)^2.
\]

Taking into account these estimates we go back to (2) and obtain an approximate expression for \( \mathcal{L} \) valid with accuracy of \( O(\varepsilon^2) \). It will be denoted by \( \widetilde{\mathcal{L}} \) and reads in dimensional variables

\[
\widetilde{\mathcal{L}} = \int_{\mathbb{R}} \int_{\mathbb{R}} \left\{ \gamma_1 h_1 \left( \frac{1}{2} |\nabla_1|^2 + \frac{1}{6} \left( \frac{D_1 h_1}{Dt} \right)^2 - \frac{1}{2} gh_1 \right) + 
\gamma_2 h_2 \left( \frac{1}{2} |\nabla_2|^2 + \frac{1}{2} \left( \frac{D_2 h_1}{Dt} \right)^2 + \frac{1}{2} \left( \frac{D_2 h_2}{Dt} \right)^2 - \frac{1}{6} \left( \frac{D_2 h_2}{Dt} \right)^2 - \frac{1}{2} gh_2 \right) \right\} \, dx \, dy.
\]

Equations for dynamics will be obtained by using Hamilton’s principle for this Lagrangian. Integrating the continuity equation of Euler equations (1) over each layer, provides

\[
(h_\alpha)_t + \text{div} (h_\alpha \mathbf{v}_\alpha) = 0, \quad \alpha = 1, 2,
\]

that must be considered as constraints to our variational problem.

\textsuperscript{2}Here, the material derivatives are defined with respect to the mean velocity fields \( \mathbf{v}_\alpha \).
4 Derivation of the governing equations for a general class of Lagrangians

In order to write the Lagrangian in a more natural way, one introduces the variables

\[ \rho_\alpha = \gamma_\alpha h_\alpha, \quad \alpha = 1, 2 \]

and defines the kinetic energy

\[ T = \frac{1}{2} \rho_1 |\nabla_1|^2 + \frac{1}{2} \rho_2 |\nabla_2|^2. \]

This way,

\[
\tilde{L} = \int_{\mathbb{R}} \int_{\mathbb{R}} \left\{ T - W \left( \rho_1, \rho_2, \frac{D_1 \rho_1}{Dt}, \frac{D_2 \rho_1}{Dt}, \frac{D_2 \rho_2}{Dt} \right) \right\} dx \, dy
\]

with

\[
W = \frac{g}{2} \left( \frac{\rho_1^2}{\gamma_1} + 2 \frac{\rho_1 \rho_2}{\gamma_1} + \frac{\rho_2^2}{\gamma_2} \right) - \frac{1}{6 \gamma_1} \rho_1 \left( \frac{D_1 \rho_1}{Dt} \right)^2 - \rho_2 \left( \frac{1}{2 \gamma_1} \left( \frac{D_2 \rho_1}{Dt} \right)^2 + \frac{1}{2 \gamma_1 \gamma_2} \left( \frac{D_2 \rho_1}{Dt} \right) \left( \frac{D_2 \rho_2}{Dt} \right) + \frac{1}{6 \gamma_2} \left( \frac{D_2 \rho_2}{Dt} \right)^2 \right).
\]

Note that \( W \) does not depend on \( \frac{D_2 \rho_2}{Dt} \). As a consequence, we lose the symmetry of equations of dynamics, when considering Hamilton’s principle to this Lagrangian. It is therefore convenient to consider a more general class of Lagrangians where the generalized potential energy \( W \) depends on all possible material derivatives.

We are looking for stationary points for the Hamilton action

\[
I = \int_{t_1}^{t_2} L \, dt,
\]

with the general Lagrangian

\[
L = \int_{\Omega} \left\{ \frac{1}{2} \rho_1 |u_1|^2 + \frac{1}{2} \rho_2 |u_2|^2 - W \left( \rho_1, \rho_2, \frac{D_1 \rho_1}{Dt}, \frac{D_2 \rho_1}{Dt}, \frac{D_2 \rho_2}{Dt} \right) \right\} dx
\]

submitted to the constraints

\[
(\rho_\alpha)_t + \text{div} (\rho_\alpha u_\alpha) = 0.
\]

Equations for dynamics are obtained by considering two families of virtual displacements corresponding to each velocity field. In Appendix B it is shown that in these conditions, \( \delta \alpha I = 0 \) reads

\[
\frac{D_\alpha u_\alpha}{Dt} + \nabla \left( \frac{\delta W}{\delta \rho_\alpha} \right) - \frac{D_\alpha}{Dt} \left( \frac{1}{\rho_\alpha} \frac{\delta W}{\delta u_\alpha} \right) - \left( \frac{\partial u_\alpha}{\partial x} \right)^T \left( \frac{1}{\rho_\alpha} \frac{\delta W}{\delta u_\alpha} \right) = 0.
\]

The expressions for the functional derivatives of \( W \) are given below\(^3\):

\[
\frac{\delta W}{\delta \rho_\alpha} = \frac{\partial W}{\partial \rho_\alpha} - \rho_\alpha \frac{D_\alpha}{Dt} \left( \frac{1}{\rho_\alpha} \frac{\partial W}{\partial \rho_\alpha} \right) - \rho_\beta \frac{D_\beta}{Dt} \left( \frac{1}{\rho_\beta} \frac{\partial W}{\partial \rho_\beta} \right),
\]

\[
\frac{\delta W}{\delta u_\alpha} = \frac{\partial W}{\partial \rho_\alpha} \nabla \rho_\alpha + \frac{\partial W}{\partial \rho_\beta} \nabla \rho_\beta,
\]

\(^3\)In what follows, the indexes \( \alpha, \beta \) satisfy \( \alpha, \beta = 1, 2 \) and \( \alpha \neq \beta \).
where for simplicity of writing we have introduced

\[ \rho_{\alpha(\beta)} = \frac{D_{\beta} \rho_{\alpha}}{Dt}, \quad \alpha, \beta = 1, 2. \]

Governing system consists of equations (5), (6) complemented by definitions (7).

## 5 Significant properties of the Euler-Lagrange equations

In this section we illustrate some special properties related to the variational structure of the governing equations. In particular, conservation laws for total momentum and total energy will be derived and a notion of *generalized vorticity* introduced. These properties are important in the qualitative study of solutions of the governing equations.

### 5.1 Conservation laws

The conservation law for the total momentum follows from equations (5) and (6) \(^4\):

\[ (\rho_1 \mathbf{u}_1 + \rho_2 \mathbf{u}_2)_t + \text{div} (\rho_1 \mathbf{u}_1 \otimes \mathbf{u}_1 + \rho_2 \mathbf{u}_2 \otimes \mathbf{u}_2 + p I) = 0 \]

Here “pressure” \( p \) is defined by

\[ p = \rho_1 \frac{\delta W}{\delta \rho_1} + \rho_2 \frac{\delta W}{\delta \rho_2} - W. \]

To write the equation for the total energy we need to define the “internal energy” for the system

\[ E = W - \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} \rho_{\alpha(\beta)} \frac{\partial W}{\partial \rho_{\beta(\alpha)}} \]

and the quantities

\[ F_{\alpha} = \rho_{\alpha} \frac{\delta W}{\delta \rho_{\alpha}} - \sum_{\beta=1}^{2} \rho_{\beta(\alpha)} \frac{\partial W}{\partial \rho_{\beta(\alpha)}}, \quad \alpha = 1, 2. \]

\(^4\)From definitions we can obtain the following algebraic relation

\[ p + E = F_1 + F_2. \]

It is proven in Appendix C that the energy conservation law has the form

\[ \left( \frac{1}{2} \rho_1 |\mathbf{u}_1|^2 + \frac{1}{2} \rho_2 |\mathbf{u}_2|^2 + E \right)_t + \text{div} \left( \mathbf{u}_1 \left( \frac{1}{2} \rho_1 |\mathbf{u}_1|^2 + F_1 \right) + \mathbf{u}_2 \left( \frac{1}{2} \rho_2 |\mathbf{u}_2|^2 + F_2 \right) \right) = 0. \]

\(^4\)See Appendix for detailed calculations.
5.2 Generalized vorticity

Let us rewrite the Euler-Lagrange equations (6) in the form

\[
\frac{D\alpha}{Dt} \left( u_\alpha - \frac{1}{\rho_\alpha} \frac{\delta W}{\delta u_\alpha} \right) + \nabla \left( \frac{\delta W}{\delta \rho_\alpha} \right) + \left( \frac{\partial u_\alpha}{\partial \mathbf{x}} \right)^T \left( u_\alpha - \frac{1}{\rho_\alpha} \frac{\delta W}{\delta u_\alpha} - \mathbf{u} \right) = 0,
\]

and introduce a new variable \( K_\alpha \) defined by

\[
K_\alpha = u_\alpha - \frac{1}{\rho_\alpha} \frac{\delta W}{\delta u_\alpha}.
\]

Then (6) is equivalent to

\[
\frac{D\alpha}{Dt} K_\alpha + \nabla \left( \frac{\delta W}{\delta \rho_\alpha} - \frac{1}{2} |u_\alpha|^2 \right) + \left( \frac{\partial u_\alpha}{\partial \mathbf{x}} \right)^T K_\alpha = 0.
\]

Adding and subtracting the term \( (\partial K_\alpha/\partial \mathbf{x})^T u_\alpha \) in the left-hand side of the equation leads to

\[
(K_\alpha)_t + \nabla \left( K_\alpha \cdot u_\alpha + \frac{\delta W}{\delta \rho_\alpha} - \frac{1}{2} |u_\alpha|^2 \right) + (\text{curl } K_\alpha) \wedge u_\alpha = 0,
\]

We introduce a generalized vorticity

\[
\Omega_\alpha = \text{curl } K_\alpha.
\]

Taking the curl on (9) we obtain

\[
(\Omega_\alpha)_t + \Omega_\alpha \text{ div } u_\alpha + (u_\alpha \cdot \nabla) \Omega_\alpha - \left( \frac{\partial u_\alpha}{\partial \mathbf{x}} \right) \Omega_\alpha = 0,
\]

since \( \Omega_\alpha \) is divergence free. Also,

\[
\frac{D\alpha}{Dt} \left( \frac{\Omega_\alpha}{\rho_\alpha} \right) = \left( \frac{\partial u_\alpha}{\partial \mathbf{x}} \right) \left( \frac{\Omega_\alpha}{\rho_\alpha} \right)
\]

by the mass conservation laws. Equation (10) has the form of the classical Helmholtz equation for the vorticity vector. Another consequence of (9) is the conservation of the circulation

\[
\frac{d}{dt} \Gamma_\alpha = 0, \quad \Gamma_\alpha = \oint_{C_\alpha} K_\alpha \cdot ds
\]

along any material simple closed contour \( C_\alpha \). In the case when \( \text{curl } K_\alpha = 0 \), the equation becomes a conservation law. In particular, the condition \( \text{curl } K_\alpha = 0 \) implies the definition of generalized potential flow. The two-layer flow is potential if there exists two scalar functions \( \varphi_1, \varphi_2 \) such that

\[
K_\alpha = \nabla \varphi_\alpha, \quad \alpha = 1, 2.
\]

For potential flows, a generalized Bernoulli integral holds in each layer

\[
(\varphi_\alpha)_t + \nabla \varphi_\alpha \cdot u_\alpha + \frac{\delta W}{\delta \rho_\alpha} - \frac{1}{2} |u_\alpha|^2 = c_\alpha(t),
\]

where \( c_\alpha(t) \) are arbitrary functions of time. In stationary flows, the quantity

\[
K_\alpha \cdot u_\alpha + \frac{\delta W}{\delta \rho_\alpha} - \frac{1}{2} |u_\alpha|^2
\]

is constant along streamlines. We stress that all these results generalize those obtained for the GN model by Miles & Salmon [4] and Gavrilyuk & Teshukov [9].
6 A dispersive model for a two-layer flow with free surface

In a particular case of two-layer flow, the kinetic energy $T$ and the potential $W$ are:

$$T = \frac{1}{2} \rho_1 |\mathbf{v}_1|^2 + \frac{1}{2} \rho_2 |\mathbf{v}_2|^2,$$

$$W = \frac{g}{2} \left( \frac{\rho_1^2}{\gamma_1} + 2 \frac{\rho_1 \rho_2}{\gamma_1 \gamma_2} + \frac{\rho_2^2}{\gamma_2} \right) - \frac{1}{6 \gamma_1 \gamma_2} \rho_1 \left( \rho_1^{(1)} \right)^2 - \rho_2 \left( \frac{1}{2 \gamma_2^2} \left( \rho_1^{(2)} \right)^2 + \frac{1}{2 \gamma_1 \gamma_2} \rho_1^{(2)} \rho_2^{(2)} + \frac{1}{6 \gamma_2^2} \left( \rho_2^{(2)} \right)^2 \right).$$

where $\rho_\alpha = \gamma_\alpha h_\alpha$ and the material derivatives are defined with respect to the mean horizontal velocity field $\mathbf{v}_1$. It is clear that $\mathcal{L}$ belongs to the general class of Lagrangians previously introduced. We now use the results of the previous section to present the governing equations to our approximate theory. To do so, it suffices to reduce the results obtained previously to two dimensions and to replace the three-dimensional velocity field $\mathbf{u}_\alpha$ by the two-dimensional mean horizontal field $\mathbf{v}_\alpha$. We get

$$\frac{D_t v_\alpha}{\delta \rho_\alpha} + \nabla \left( \frac{\delta W}{\delta \rho_\alpha} \right) - D_t a \left( \frac{1}{\rho_\alpha} \frac{\delta W}{\delta v_\alpha} \right) - \left( \frac{\delta v_\alpha}{\delta \rho_\alpha} \right)^T \left( \frac{1}{\rho_\alpha} \frac{\delta W}{\delta v_\alpha} \right) = 0, \quad \alpha = 1, 2.$$

In the original physical variables $h_\alpha$ and $\mathbf{v}_\alpha$, the following quantities should be replaced in the above equations:

$$\frac{\delta W}{\delta \rho_1} = g(h_1 + \gamma h_2) - \frac{1}{6} \left( \frac{D_1 h_1}{D_t} \right)^2 + \frac{1}{3} \frac{h_1}{h_2} \left( \frac{D_2 h_1}{D_t^2} \right) + \gamma h_2 \left( \frac{D_1^2 h_1}{D_t^2} \right) + \frac{1}{2} \gamma h_2 \left( \frac{D_2^2 h_1}{D_t^2} \right),$$

$$\frac{\delta W}{\delta \rho_2} = g(h_1 + h_2) - \frac{1}{2} \left( \frac{D_2 h_1}{D_t} \right)^2 - \frac{1}{2} \left( \frac{D_1 h_1}{D_t} \right) \left( \frac{D_2 h_2}{D_t} \right) - \frac{1}{6} \left( \frac{D_2 h_2}{D_t^2} \right) + \frac{1}{2} \frac{h_2}{h_1} \left( \frac{D_1^2 h_2}{D_t^2} \right) + \frac{1}{3} \frac{h_2}{h_1} \left( \frac{D_1^2 h_2}{D_t^2} \right),$$

$$\frac{1}{\rho_1} \frac{\delta W}{\delta \nabla} = - \frac{1}{3} \left( \frac{D_1 h_1}{D_t} \right) \nabla h_1,$$

$$\frac{1}{\rho_2} \frac{\delta W}{\delta \nabla} = - \left[ \left( \frac{D_2 h_1}{D_t} \right) + \frac{1}{2} \left( \frac{D_2 h_2}{D_t} \right) \right] \nabla h_1 - \left[ \frac{1}{2} \left( \frac{D_2 h_1}{D_t} \right) + \frac{1}{3} \left( \frac{D_2 h_2}{D_t} \right) \right] \nabla h_2,$$

where the scalar $\gamma < 1$ is defined by $\gamma = \gamma_2 / \gamma_1$. Using these expressions we will obtain, in explicit form, the conservation laws and expressions for a generalized vorticity.

6.1 Conservation laws for the present model

Conservation laws for mass, total momentum and total energy for a two-layer flow become

$$(h_1)_t + \text{div}(h_1 \mathbf{v}_1) = 0,$$

$$(h_2)_t + \text{div}(h_2 \mathbf{v}_2) = 0,$$

$$(\gamma_1 h_1 \mathbf{v}_1 + \gamma_2 h_2 \mathbf{v}_2)_t + \text{div}(\gamma_1 h_1 \mathbf{v}_1 \otimes \mathbf{v}_1 + \gamma_2 h_2 \mathbf{v}_2 \otimes \mathbf{v}_2 + pI) = 0,$$

$$\left( \frac{1}{2} \gamma_1 h_1 |\mathbf{v}_1|^2 + \frac{1}{2} \gamma_2 h_2 |\mathbf{v}_2|^2 + E \right)_t + \text{div} \left( \mathbf{v}_1 \left( \frac{1}{2} \gamma_1 h_1 |\mathbf{v}_1|^2 + F_1 \right) + \mathbf{v}_2 \left( \frac{1}{2} \gamma_2 h_2 |\mathbf{v}_2|^2 + F_2 \right) \right) = 0.$$

"Pressure" $p$ is given by

$$p = \frac{g}{2} \left( \gamma_1 h_1^2 + 2 \gamma_2 h_1 h_2 + \gamma_2 h_2^2 \right) + \frac{1}{3} \gamma_1 h_1^2 \left( \frac{D_1^2 h_1}{D_t^2} \right) + \gamma_2 h_2 \left( h_1 + \frac{1}{2} h_2 \right) \left( \frac{D_2^2 h_1}{D_t^2} \right) + \gamma_2 h_2 \left( h_1 + \frac{1}{3} h_2 \right) \left( \frac{D_2^2 h_2}{D_t^2} \right).$$
The quantities $F_1$ and $F_2$ are as follows

$$F_1 = \gamma_1 h_1 \left[ g(h_1 + h_2) + \frac{1}{6} \left( \frac{D_1 h_1}{Dt} \right)^2 + \frac{1}{3} h_1 \left( \frac{D_2^2 h_1}{Dt^2} \right) + \gamma h_2 \left( \frac{D_2^2 h_1}{Dt^2} \right) + \frac{1}{2} \gamma h_2 \left( \frac{D_2^2 h_2}{Dt^2} \right) \right],$$

$$F_2 = \gamma_2 h_2 \left[ g(h_1 + h_2) + \frac{1}{2} \left( \frac{D_2 h_1}{Dt} \right)^2 + \frac{1}{2} \left( \frac{D_2 h_2}{Dt} \right) \left( \frac{D_2 h_2}{Dt} \right) + \frac{1}{6} \left( \frac{D_2 h_2}{Dt} \right)^2 + \right.$$

$$\left. + \frac{1}{2} h_2 \left( \frac{D_2^2 h_1}{Dt^2} \right) + \frac{1}{3} h_2 \left( \frac{D_2^2 h_2}{Dt^2} \right) \right].$$

The “internal energy” takes the form

$$E = \frac{g}{2} (\gamma_1 h_1^2 + 2 \gamma_2 h_1 h_2 + \gamma_2 h_2^2) + \frac{1}{6} \gamma_1 h_1 \left( \frac{D_1 h_1}{Dt} \right)^2 +$$

$$\left. + \frac{1}{2} \gamma_2 h_2 \left( \frac{D_2 h_1}{Dt} \right)^2 + \frac{1}{2} \left( \frac{D_2 h_2}{Dt} \right) \left( \frac{D_2 h_2}{Dt} \right) + \frac{1}{3} \left( \frac{D_2 h_2}{Dt} \right)^2 \right).$$

### 6.2 Generalized vorticity for the present model

In our particular case, the variables $K_\alpha$ defined by (8) are

$$K_1 = \mathbf{v}_1 + \frac{1}{3} \left( \frac{D_1 h_1}{Dt} \right) \nabla h_1,$$

$$K_2 = \mathbf{v}_2 + \frac{1}{2} \left( \frac{D_2}{Dt} (h_2 + \gamma_2 h_1) \right) \nabla h_1 + \frac{1}{3} \left( \frac{D_2}{Dt} (h_2 + \gamma_2 h_1) \right) \nabla h_2.$$

Note that the vectors $K_\alpha$ are two-dimensional. In two dimensions, (11) becomes much simpler. Only the third component (denoted by $\Omega_\alpha$) of the generalized vorticity is non-trivial, and the right-hand side in (11) vanishes. Therefore, we have the following equation for the vorticity:

$$\frac{D_\alpha}{Dt} \left( \frac{\Omega_\alpha}{h_\alpha} \right) = 0.$$

Its conservative form is

$$(\Omega_\alpha)_t + \text{div} (\Omega_\alpha \mathbf{v}_\alpha) = 0,$$

where

$$\Omega_1 = \mathbf{v}_{1x} - \mathbf{v}_{1y} + \frac{1}{3} \mathbf{J} \left( \frac{D_1 h_1}{Dt}, h_1 \right),$$

$$\Omega_2 = \mathbf{v}_{2x} - \mathbf{v}_{2y} + \frac{1}{2} \mathbf{J} \left( \frac{D_2}{Dt} (h_2 + \gamma_2 h_1), h_1 \right) + \frac{1}{3} \mathbf{J} \left( \frac{D_2}{Dt} (h_2 + \gamma_2 h_1), h_2 \right),$$

and $\mathbf{J}$ is an abbreviation for the Jacobian operator in $(x, y)$

$$\mathbf{J}(p, q) = \frac{\partial(p, q)}{\partial(x, y)}, \text{ for any } p, q.$$

In the case of irrotational flows the Bernoulli integral is valid in each layer

$$(\varphi_\alpha)_t + \nabla \varphi_\alpha \cdot \mathbf{v}_\alpha + \frac{\delta W}{\delta \rho_\alpha} - \frac{1}{2} |\mathbf{v}_\alpha|^2 = c_\alpha(t).$$
7 Conclusion

Using Hamilton’s principle, we have derived a mathematical model describing large amplitude long waves in two-layer weakly sheared flows with free surface. In this derivation we did not suppose that the flow was potential. General properties of the governing equations have been revealed. Notions such as generalized vorticity and generalized potential flows have been introduced, and conservation laws and integrals of motion have been obtained.

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Appendix A. Weakly sheared flows

In dimensionless variables equations (1) take the form

$$\gamma_\alpha \frac{D_\alpha u_\alpha}{Dt} + p_\alpha = 0, \quad \gamma_\alpha \frac{D_\alpha v_\alpha}{Dt} + p_\alpha = 0, \quad \varepsilon^2 \gamma_\alpha \frac{D_\alpha w_\alpha}{Dt} + p_\alpha = -\gamma_\alpha F^{-2}, \quad u_{\alpha x} + v_{\alpha y} + w_{\alpha z} = 0.$$

Eliminating the pressure $p_\alpha$, we obtain the Helmholtz equation defining the evolution of non-dimensional vorticity vector $\omega_\alpha = (\omega_{\alpha 1}, \omega_{\alpha 2}, \omega_{\alpha 3}) = (\varepsilon^2 w_{\alpha y} - v_{\alpha z}, u_{\alpha z} - \varepsilon^2 w_{\alpha x}, v_{\alpha x} - u_{\alpha y})$ in $\alpha$-th layer:

$$\omega_{\alpha t} + (u_\alpha \cdot \nabla)\omega_\alpha = (\varepsilon^2 w_{\alpha y} - v_{\alpha z})u_{\alpha z} + (u_{\alpha z} - \varepsilon^2 w_{\alpha x})u_{\alpha y} + (v_{\alpha x} - u_{\alpha y})u_{\alpha z}$$

(12)

Projecting the Helmholtz equation (12) on $x, y$ axes, we obtain

$$\frac{D_\alpha u_{\alpha z}}{Dt} + u_{\alpha y} v_{\alpha z} - v_{\alpha y} u_{\alpha z} = O(\varepsilon^2),$$

$$\frac{D_\alpha v_{\alpha z}}{Dt} + v_{\alpha x} u_{\alpha z} - u_{\alpha x} v_{\alpha z} = O(\varepsilon^2).$$

It follows from these equations that $u_{\alpha z} = O(\varepsilon^\sigma)$, $v_{\alpha z} = O(\varepsilon^\sigma)$ for all $t > 0$, if $u_{\alpha z} = O(\varepsilon^s)$, $v_{\alpha z} = O(\varepsilon^s)$ at $t = 0$. Here $\sigma = \min(2, s)$.

Proof of Estimate 1

Notice that

$$\int_0^{h_1} |v_1|^2 \, dz = h_1 |\nabla_1|^2 + \int_0^{h_1} |v_1 - \nabla_1|^2 \, dz = h_1 |\nabla_1|^2 + O(\varepsilon^{2s}),$$

because

$$|v_1 - \nabla_1| = \left| \int_0^z u_{1 z} \, dz - \frac{1}{h_1} \int_0^{h_1} \left( \int_0^{z'} v_{1 z} \, dz \right) \, dz' \right| \leq \frac{3}{2} h_1 \max |v_{1 z}| = O(\varepsilon^s).$$
Since $s > 1$ by definition of weakly sheared flow, it follows
\[ \int_0^{h_1} |v_1|^2 \, dz \approx h_1 |\nabla v_1|^2. \]

Similarly, we can obtain
\[ \int_{h_1}^{h_1+h_2} |v_2|^2 \, dz \approx h_2 |\nabla v_2|^2. \]

**Proof of Estimate 2**

Observe that in the case of weakly sheared flows, the horizontal velocities admit the representation
\[ v_\alpha(x, t) - \nabla_\alpha(x, y, t) = \varepsilon \delta \left(v_\alpha^*(x, t, \varepsilon) - \nabla_\alpha(x, y, t, \varepsilon)\right). \]

It allows us to state that
\[ \varepsilon v_\alpha(x, t) \approx \varepsilon \nabla_\alpha(x, y, t). \]

Taking into account the constraints given by the mass conservation laws for each fluid, that reads
\[ \text{div } u_\alpha = 0 \]

in dimensionless variables, we integrate with respect to the vertical coordinate, obtaining expressions for the vertical component of $u_\alpha$:
\[ w_1 = -\int_0^z (u_{1x} + v_{1y}) \, dz, \quad w_2 = w_2 \big|_{z=h_1} - \int_{h_1}^z (u_{2x} + v_{2y}) \, dz. \]

This leads to
\[ \int_0^{h_1} \varepsilon^2 w_1^2 \, dz = \int_0^{h_1} \left(- \int_0^z (\varepsilon u_{1x}) + (\varepsilon v_{1y}) \, dz \right)^2 \, dz \]
\[ \approx \int_0^{h_1} \left(- \int_0^z (\varepsilon \nabla_1) + (\varepsilon \nabla_1) \, dz \right)^2 \, dz \]
\[ = \int_0^{h_1} \varepsilon^2 (- z \text{div } \nabla_1)^2 \, dz, \]

where div is defined in two dimensions. Similarly, we get
\[ \int_{h_1}^{h_1+h_2} \varepsilon^2 w_2^2 \, dz \approx \int_{h_1}^{h_1+h_2} \varepsilon^2 \left(- h_1 \text{div } \nabla_1 + (\nabla_2 - \nabla_1) \cdot \nabla h_1 - (z - h_1) \text{div } \nabla_2 \right)^2 \, dz. \]

This can still be simplified. Indeed, from the mass conservation laws we conclude that
\[ (h_\alpha)_t + \text{div } (h_\alpha \nabla_\alpha) = 0. \]

Hence, the following relations hold:
\[ -z \text{div } \nabla_1 = \frac{z}{h_1} \left( \frac{D_1 h_1}{Dt} \right), \]
\[ -h_1 \text{div } \nabla_1 + (\nabla_2 - \nabla_1) \cdot \nabla h_1 - (z - h_1) \text{div } \nabla_2 = \left( \frac{D_2 h_1}{Dt} \right) + \frac{z - h_1}{h_2} \left( \frac{D_2 h_2}{Dt} \right), \]
where the material derivatives are here defined with respect to the mean horizontal velocity fields \( \bar{v}_\alpha \). All these calculations can be summarized as follows:

\[
\int_{0}^{h_1} \varepsilon^2 w_1^2 \, dz \approx \frac{1}{3} \varepsilon^2 h_1 \left( \frac{D_1 h_1}{Dt} \right)^2,
\]

\[
\int_{h_1}^{h_1+h_2} \varepsilon^2 w_2^2 \, dz \approx \varepsilon^2 h_2 \left( \left( \frac{D_2 h_1}{Dt} \right)^2 + \left( \frac{D_2 h_1}{Dt} \right) \left( \frac{D_2 h_2}{Dt} \right) + \frac{1}{3} \left( \frac{D_2 h_2}{Dt} \right)^2 \right).
\]

**Appendix B. Euler-Lagrange equations**

We shall compute the variations of the Hamilton action \( I \) defined by (3) with respect to two families of virtual displacements corresponding to the Lagrangian coordinates related with each fluid velocity. To be precise about which variation we are considering, we use the index notation \( \delta_\alpha \) with \( \alpha = 1, 2 \). Again, the constraints to be considered here are given by the conservation mass of each component

\[ (\rho_\alpha)_t + \text{div} (\rho_\alpha u_\alpha) = 0, \quad \alpha = 1, 2. \]

In this framework, the following relations hold (Serrin [16], Berdishevy [17], Gavrilyuk & Gouin [18])\(^5\):

\[ \delta \rho_\alpha = -\text{div}(\rho_\alpha \delta_\alpha x), \]

\[ \delta u_\alpha = \frac{D_\alpha}{Dt} (\delta_\alpha x) - \left( \frac{\partial u_\alpha}{\partial x} \right) \delta_\alpha x, \quad (13) \]

where \( \delta_\alpha x \) are the virtual displacements of the particles of each component. Assuming that the virtual displacements vanish on the boundary of \( \Omega \times [t_1, t_2] \), and using Green’s formula, we obtain

\[
\delta_\alpha I = \int_{t_1}^{t_2} \int_{\Omega} \left( \frac{1}{2} |u_\alpha|^2 - \frac{\delta W}{\delta \rho_\alpha} \right) \delta \rho_\alpha + \left( \rho_\alpha u_\alpha - \frac{\delta W}{\delta u_\alpha} \right) \cdot \delta u_\alpha \, dx \, dt = 0.
\]

Hence, the stationary points for the action integral \( I \) verify

\[
\frac{D_\alpha u_\alpha}{Dt} + \nabla \left( \frac{\delta W}{\delta \rho_\alpha} \right) - \frac{D_\alpha}{Dt} \left( \rho_\alpha \frac{\delta u_\alpha}{\delta \rho_\alpha} \right) T \left( \frac{\delta \rho_\alpha}{\rho_\alpha} \delta u_\alpha \right) = 0.
\]

To write explicitly the equations for dynamics we have to determine the functional derivatives of \( W \). Consider the functional

\[ W = \int_{t_1}^{t_2} \int_{\Omega} W \left( \rho_1, \rho_2, \rho_1^{(1)}, \rho_2^{(1)}, \rho_1^{(2)}, \rho_2^{(2)} \right) \, dx \, dt. \]

Variations of \( W \) are then given by

\[ \delta_\alpha W = \int_{t_1}^{t_2} \int_{\Omega} \left( \frac{\delta W}{\delta \rho_\alpha} \right) \delta \rho_\alpha + \left( \frac{\delta W}{\delta u_\alpha} \right) \cdot \delta u_\alpha \, dx \, dt. \]

\(^5\)Here, the variations of dependent variables are taken at fixed Eulerian coordinates.
In turn, when performing the variations we obtain

\[ \delta \mathcal{W} = \int_{t_1}^{t_2} \int_{\Omega} \left( \frac{\partial W}{\partial \rho_\alpha} \right) \delta \rho_\alpha + \left( \frac{\partial W}{\partial (\rho_\alpha)} \right) \delta \left( \rho_\alpha^{(\alpha)} \right) + \left( \frac{\partial W}{\partial (\rho_\beta)} \right) \delta \left( \rho_\beta^{(\beta)} \right) \ d\mathbf{x} \ dt. \]

Using (13) and the identities

\[ \delta \left( \rho_\alpha^{(\alpha)} \right) = \frac{D_\alpha}{Dt} \delta \rho_\alpha + \nabla \rho_\alpha \cdot \delta \mathbf{u}_\alpha, \]
\[ \delta \left( \rho_\beta^{(\beta)} \right) = \nabla \rho_\beta \cdot \delta \mathbf{u}_\alpha, \]
\[ \delta \left( \rho_\alpha^{(\beta)} \right) = \frac{D_\beta}{Dt} \delta \rho_\alpha, \]

we write, after applying Green’s formula,

\[ \delta \mathcal{W} = \int_{t_1}^{t_2} \int_{\Omega} \left( \frac{\partial W}{\partial \rho_\alpha} - \rho_\alpha \frac{D_\alpha}{Dt} \left( \frac{1}{\rho_\alpha} \frac{\partial W}{\partial (\rho_\alpha)} \right) - \rho_\beta \frac{D_\beta}{Dt} \left( \frac{1}{\rho_\beta} \frac{\partial W}{\partial (\rho_\beta)} \right) \right) \delta \rho_\alpha + \]
\[ + \left( \frac{\partial W}{\partial (\rho_\alpha \partial (\rho_\alpha))} \rho_\alpha \nabla \rho_\alpha + \frac{\partial W}{\partial (\rho_\beta \partial (\rho_\beta))} \nabla \rho_\beta \right) \cdot \delta \mathbf{u}_\alpha \ d\mathbf{x} \ dt. \]

It follows that

\[ \frac{\delta \mathcal{W}}{\delta \rho_\alpha} = \frac{\partial W}{\partial \rho_\alpha} - \rho_\alpha \frac{D_\alpha}{Dt} \left( \frac{1}{\rho_\alpha} \frac{\partial W}{\partial (\rho_\alpha)} \right) - \rho_\beta \frac{D_\beta}{Dt} \left( \frac{1}{\rho_\beta} \frac{\partial W}{\partial (\rho_\beta)} \right), \]
\[ \frac{\delta \mathcal{W}}{\delta \mathbf{u}_\alpha} = \frac{\partial W}{\partial (\rho_\alpha \partial (\rho_\alpha))} \nabla \rho_\alpha + \frac{\partial W}{\partial (\rho_\beta \partial (\rho_\beta))} \nabla \rho_\beta, \]

so, the functional derivatives are given explicitly. The Euler-Lagrange equations are thus completely determined.

**Appendix C. Conservation laws for the total momentum and total energy**

**Equation for the total momentum**

We go back to the Euler-Lagrange equations

\[ \frac{D_\alpha \mathbf{u}_\alpha}{Dt} + \nabla \left( \frac{\delta W}{\delta \rho_\alpha} \right) - \frac{D_\alpha}{Dt} \left( \frac{1}{\rho_\alpha} \frac{\delta W}{\partial \mathbf{u}_\alpha} \right) - \left( \frac{\partial \mathbf{u}_\alpha}{\partial \mathbf{x}} \right)^T \left( \frac{1}{\rho_\alpha} \frac{\delta W}{\partial \mathbf{u}_\alpha} \right) = 0. \]

Using mass conservation laws, we multiply this equation by \( \rho_\alpha \) and obtain

\[ (\rho_\alpha \mathbf{u}_\alpha)_t + \text{div} (\rho_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha) + \rho_\alpha \nabla \left( \frac{\delta W}{\delta \rho_\alpha} \right) - \rho_\alpha \left[ \frac{D_\alpha}{Dt} \left( \frac{1}{\rho_\alpha} \frac{\delta W}{\partial \mathbf{u}_\alpha} \right) + \left( \frac{\partial \mathbf{u}_\alpha}{\partial \mathbf{x}} \right)^T \left( \frac{1}{\rho_\alpha} \frac{\delta W}{\partial \mathbf{u}_\alpha} \right) \right] = 0. \]
We will show that straightforward calculations show that
\[
\rho_1 \mathbf{u}_1 + \rho_2 \mathbf{u}_2)_t + \text{div} (\rho_1 \mathbf{u}_1 \otimes \mathbf{u}_1 + \rho_2 \mathbf{u}_2 \otimes \mathbf{u}_2) + \rho_1 \nabla \left( \frac{\delta W}{\delta \rho_1} \right) + \rho_2 \nabla \left( \frac{\delta W}{\delta \rho_2} \right) = 0
\]

\[
= \sum_{\alpha=1}^{2} \rho_\alpha \left[ \frac{D_\alpha}{Dt} \left( \frac{1}{\rho_\alpha} \frac{\delta W}{\delta u_\alpha} \right) + \left( \frac{\partial u_\alpha}{\partial x} \right)^T \left( \frac{1}{\rho_\alpha} \frac{\delta W}{\delta u_\alpha} \right) \right].
\]

Straightforward calculations show that
\[
\frac{D_\alpha}{Dt} \left( \frac{1}{\rho_\alpha} \frac{\delta W}{\delta u_\alpha} \right) + \left( \frac{\partial u_\alpha}{\partial x} \right)^T \left( \frac{1}{\rho_\alpha} \frac{\delta W}{\delta u_\alpha} \right) =
\]
\[
= \frac{D_\alpha}{Dt} \left( \frac{1}{\rho_\alpha} \frac{\partial W}{\partial \rho_\alpha^{(a)}} \right) \nabla \rho_\alpha + \frac{D_\alpha}{Dt} \left( \frac{1}{\rho_\alpha} \frac{\partial W}{\partial \rho_\alpha^{(\beta)}} \right) \nabla \rho_\beta + \frac{1}{\rho_\alpha} \frac{\partial W}{\partial \rho_\alpha^{(a)}} \nabla \left( \rho_\alpha^{(a)} \right) + \frac{1}{\rho_\alpha} \frac{\partial W}{\partial \rho_\alpha^{(\beta)}} \nabla \left( \rho_\beta^{(\beta)} \right),
\]

hence, the right-hand side of the equation is equal to
\[
\nabla W - \frac{\delta W}{\delta \rho_1} \nabla \rho_1 - \frac{\delta W}{\delta \rho_2} \nabla \rho_2.
\]

We conclude that
\[
(\rho_1 \mathbf{u}_1 + \rho_2 \mathbf{u}_2)_t + \text{div} (\rho_1 \mathbf{u}_1 \otimes \mathbf{u}_1 + \rho_2 \mathbf{u}_2 \otimes \mathbf{u}_2 + pI) = 0,
\]

where \( p \) is defined by
\[
p = \rho_1 \frac{\delta W}{\delta \rho_1} + \rho_2 \frac{\delta W}{\delta \rho_2} - W.
\]

**Equation for the total energy**

Define the *internal energy* for the system
\[
E = W - \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} \rho_\beta^{(a)} \frac{\partial W}{\partial \rho_\beta^{(a)}}.
\]

We will show that
\[
\frac{1}{2} \rho_1 |\mathbf{u}_1|^2 + \frac{1}{2} \rho_2 |\mathbf{u}_2|^2 + E
\]
is a conserved quantity. We compute the following partial derivative:
\[
\left( \frac{1}{2} \rho_1 |\mathbf{u}_1|^2 + \frac{1}{2} \rho_2 |\mathbf{u}_2|^2 + E \right)_t = (\rho_1)_t \left( \frac{1}{2} |\mathbf{u}_1|^2 \right) + \rho_1 \mathbf{u}_1 \cdot (\mathbf{u}_1)_t + (\rho_2)_t \left( \frac{1}{2} |\mathbf{u}_2|^2 \right) + \rho_2 \mathbf{u}_2 \cdot (\mathbf{u}_2)_t + E_t.
\]

Using mass conservation laws and momentum equations for each component enables us to write the right-hand side of the equation as
\[
- \text{div} \left[ \mathbf{u}_1 \left( \frac{1}{2} \rho_1 |\mathbf{u}_1|^2 + \rho_1 \frac{\delta W}{\delta \rho_1} \right) + \mathbf{u}_2 \left( \frac{1}{2} \rho_2 |\mathbf{u}_2|^2 + \rho_2 \frac{\delta W}{\delta \rho_2} \right) \right] + \left( \frac{\delta W}{\delta \rho_1} \right) \text{div} (\rho_1 \mathbf{u}_1) + \left( \frac{\delta W}{\delta \rho_2} \right) \text{div} (\rho_2 \mathbf{u}_2) +
\]
\[
+ \sum_{\alpha=1}^{2} \rho_\alpha \mathbf{u}_\alpha \cdot \left[ \frac{D_\alpha}{Dt} \left( \frac{1}{\rho_\alpha} \frac{\delta W}{\delta u_\alpha} \right) + \left( \frac{\partial u_\alpha}{\partial x} \right)^T \left( \frac{1}{\rho_\alpha} \frac{\delta W}{\delta u_\alpha} \right) \right] + E_t
\]
Hence,
\[
\left( \frac{1}{2} \rho_1 |u_1|^2 + \frac{1}{2} \rho_2 |u_2|^2 + E \right)_t + \text{div} \left[ u_1 \left( \frac{1}{2} \rho_1 |u_1|^2 + \rho_1 \frac{\delta W}{\delta \rho_1} \right) + u_2 \left( \frac{1}{2} \rho_2 |u_2|^2 + \rho_2 \frac{\delta W}{\delta \rho_2} \right) \right] =
\]
\[
E_t - \left( \frac{\delta W}{\delta \rho_1} \right) (\rho_1)_t - \left( \frac{\delta W}{\delta \rho_2} \right) (\rho_2)_t + \sum_{\alpha=1}^2 \rho_\alpha u_\alpha \cdot \left[ D_\alpha \frac{\partial W}{\partial \rho_\alpha} \left( \frac{1}{\rho_\alpha} \frac{\partial u_\alpha}{\partial x} \right)^T \left( \frac{1}{\rho_\alpha} \frac{\delta W}{\delta \rho_\alpha} \right) \right].
\]
Calculations show that the right-hand side of the equation is equal to
\[
\text{div} \left[ u_1 \left( \frac{\partial W}{\partial \rho_1} \rho_1^{(1)} + \frac{\partial W}{\partial \rho_2} \rho_2^{(1)} \right) + u_2 \left( \frac{\partial W}{\partial \rho_1} \rho_1^{(2)} + \frac{\partial W}{\partial \rho_2} \rho_2^{(2)} \right) \right],
\]
so, defining quantities
\[
F_\alpha = \rho_\alpha \frac{\delta W}{\delta \rho_\alpha} - \sum_{\beta=1}^2 \rho_\beta^{(\alpha)} \frac{\partial W}{\partial \rho_\beta^{(\alpha)}}, \quad \alpha = 1, 2
\]
allows one to establish
\[
\left( \frac{1}{2} \rho_1 |u_1|^2 + \frac{1}{2} \rho_2 |u_2|^2 + E \right)_t + \text{div} \left[ u_1 \left( \frac{1}{2} \rho_1 |u_1|^2 + F_1 \right) + u_2 \left( \frac{1}{2} \rho_2 |u_2|^2 + F_2 \right) \right] = 0.
\]

References


Part III
Large Amplitude Solitary Waves
Embedded into the Continuous Spectrum
Dispersive Nonlinear Waves in Two-Layer Flows with Free Surface. II. Large amplitude solitary waves embedded into the continuous spectrum


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Abstract

In this paper we study the dispersive model derived in Part I, for the description of long wave propagation in two-layer flows with free surface. As in the case of the full water-wave problem, this model reproduces the resonance between short waves and long waves. The resulting wave is a generalized solitary wave, characterized by ripples in the far field in addition to the solitary pulse. In this work we focus on particular members of this family resulting from vanishing ripples. These are called embedded solitary waves and they correspond to true homoclinic orbits embedded into the continuous spectrum. Two wave regimes, characterized by elevation or depression of the interface between the layers, are presented. A critical depth ratio separates these two regimes. It is shown how this relates to a change of the global properties for the potential of the Hamiltonian system derived for traveling waves. In oceanic conditions, solitary waves are presented and their broadening is observed as the wave speed increases. We have observed that, for such waves to exist, their speed cannot exceed a certain limit value depending on the density ratio and thickness of each fluid. Finally, other sets of parameters were considered for which multi-humped solitons exist, showing the richness and complexity of the Hamiltonian system considered here.

1 Introduction

It is known that Euler equations are endowed with a variational structure. We used this in [1] as the main ingredient to derive approximate equations by considering Hamilton’s principle for an approximate Lagrangian differing from the original one for the full problem by an error of $O(\varepsilon^2)$, where $\varepsilon$ is the ratio of a typical vertical scale to a typical horizontal scale. Inserting approximations directly into the Lagrangian has two main advantages over the classical perturbation procedures. First, the approximations do not disturb the corresponding symmetry properties coming from the variational structure of the governing equations. Second, the approximation methods based on Hamilton’s principle suggests transformations to new dependent variables in which the approximate equations take its simplest form.

The approximate equations describe long wave propagation. No other common assumptions like small amplitude of waves or potential flows were made to derive the model. The aim of this paper is to, somehow, validate this new model. We start by presenting the dispersion relation for this model and its comparison with the one obtained for the full water-wave problem. The spectrum of the linearized system has two branches. The two possible modes of gravity waves in a two-layer fluid are called the fast and slow modes, according to the magnitude of their velocities [2]. Moreover, it is shown

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the possibility for long waves and short waves to propagate with the same phase velocity. This new model captures therefore the resonance between short waves and long waves. Obviously, this possibility exists for the full problem, but, in the past, models have been derived that could not deal with the resonance. The resulting solutions are called generalized solitary waves and several papers were devoted to this subject in the framework of the full problem (see [3], [4], [5], [6], [7], [8], [9]). They are nonlinear long waves characterized by ripples in the far field in addition to the solitary pulse. Mathematically, generalized solitary waves correspond to homoclinic connections to periodic orbits close to a saddle-center equilibrium. A special member of this family of waves arises when the ripples vanish. These are called embedded solitary waves and, recently, we have seen an increased interest in these peculiar waves, since the seminal work of Yang et al. [10]. An extensive list of references on the subject can be found in [11] and [12].

In this article we will concentrate on these solutions. By using numerical computations we have shown the existence of true solitary waves for our model. The traveling-wave solutions depend on three parameters: the density ratio $\gamma = \gamma_2/\gamma_1 < 1$, the depth ratio $H = H_2/H_1$ and the Froude number $F_1 = D/\sqrt{gH_1}$ based on the bottom layer. It is found that the surface and interface wave are always 180 degrees out of phase. There will be presented two wave regimes of elevation and depression. A critical depth ratio $H_c$ separates these two regimes and it will be shown how it relates to a change of the structure of the potential provided by the Hamiltonian system describing traveling-wave solutions to our model. In oceanic conditions (with $\gamma = 0.99$) we have found true solitary waves in both regimes. Moreover, as the Froude number approaches a limit value $F_1^L$, the amplitude of the central core reaches a maximum amplitude while becoming infinitely wider. These waves are often called “table-top” solitons. In the limit as the width of the central core becomes infinite, the wave becomes a front. The broadening phenomenon is well-known in the case of interfacial waves in a two-fluid system with a rigid lid (see [13] and more recently [14]). In the case of a free surface, this phenomenon was predicted for the full problem by Dias and Il’ichev [15] and also verified for other water-wave models (see e.g., [16]). Furthermore, quite exotic solutions can be obtained for different sets of parameters. For $\gamma = 0.10$, single and multi-humped solitons were found in both wave regimes. Some of these solutions look somewhat like the solutions presented by Michallet and Dias for the full problem with $\gamma = 0.10$ and $H = 3/7$ (see Fig. 17 in [9]). This highlights the complexity and richness of the Hamiltonian system with two degrees of freedom derived for traveling waves.

2 The Dispersive Model for a Two-Layer Flow with Free Surface

The approximate equations for the description of two-dimensional two-layer flows as depicted in the Fig. 1 are the stationary points for the Hamilton action

$$I = \int t_1^{t_2} \mathcal{L} \, dt,$$

with the Lagrangian

$$\mathcal{L} = \int \int \left\{ \gamma_1 h_1 \left( \frac{1}{2} \nabla_1^2 + \frac{1}{6} \left( \frac{D_1 h_1}{Dt} \right)^2 - \frac{1}{2} gh_1 \right) + \right.$$  

$$+ \gamma_2 h_2 \left( \frac{1}{2} \nabla_2^2 + \frac{1}{2} \left( \frac{D_2 h_1}{Dt} \right)^2 + \frac{1}{2} \left( \frac{D_2 h_1}{Dt} \right) \left( \frac{D_2 h_2}{Dt} \right) \right) \right\} \, dx \, dy$$

submitted to the constraints

$$(h_\alpha)_t + \text{div} (h_\alpha \nabla_\alpha) = 0, \quad \alpha = 1, 2.$$
Here, we defined \( \mathbf{v}_\alpha = (u_\alpha, v_\alpha) \) as the horizontal velocities and considered the mean velocities
\[
\mathbf{v}_1(x, y, t) = \frac{1}{h_1} \int_0^{h_1} v_1 dz, \quad \mathbf{v}_2(x, y, t) = \frac{1}{h_2} \int_{h_1}^{h_1+h_2} v_2 dz.
\]
The material derivatives \( \frac{D\alpha}{Dt} \) present in the expression of \( \tilde{\mathcal{L}} \) are defined with respect to the mean velocity fields:
\[
\frac{D\alpha}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}_\alpha \cdot \nabla.
\]
For simplicity of writing, here and hereafter we write \( \mathbf{u}_\alpha \) instead of the two-dimensional velocity field \( \mathbf{v}_\alpha \). In order to write the Lagrangian in a more natural way, one introduces the variables
\[
\rho_\alpha = \gamma_\alpha h_\alpha, \quad \alpha = 1, 2
\]
and defines the kinetic energy
\[
T = \frac{1}{2} \rho_1 |\mathbf{u}_1|^2 + \frac{1}{2} \rho_2 |\mathbf{u}_2|^2.
\]
This way,
\[
\tilde{\mathcal{L}} = \int_{\mathbb{R}} \int_{\mathbb{R}} (T - W) \, dx \, dy,
\]
with
\[
W = \frac{g}{2} \left( \frac{\rho_1^2}{\gamma_1} + 2 \frac{\rho_1 \rho_2}{\gamma_1} + \frac{\rho_2^2}{\gamma_2} \right) - \frac{1}{6 \gamma_1} \rho_1 \left( \frac{\rho_1^{(1)}}{\gamma_1^2} \right)^2 - \rho_2 \left( \frac{1}{2 \gamma_1^2} \right)^2 \rho_2^{(2)} + \frac{1}{2 \gamma_1 \gamma_2} \rho_2^{(2)} + \frac{1}{6 \gamma_2^2} \rho_2^{(2)}.
\]
Here, to simplify the writing we have considered the notation
\[
\rho_\alpha^{(\beta)} = \frac{D\beta}{Dt} \rho_\alpha, \quad \alpha, \beta = 1, 2.
\]
This Lagrangian fits in the general class of Lagrangians admitting the representation
\[
L = \int_{\Omega} \left\{ \frac{1}{2} \rho_1 |\mathbf{u}_1|^2 + \frac{1}{2} \rho_2 |\mathbf{u}_2|^2 - W \left( \rho_1, \rho_2, \rho_1^{(1)}, \rho_2^{(1)}, \rho_1^{(2)}, \rho_2^{(2)} \right) \right\} \, dx.
\]
Governing equations for \( L \) consist of (see [1])
\[
\rho_\alpha_t + \text{div} \left( \rho_\alpha \mathbf{u}_\alpha \right) = 0,
\]
\[
\frac{D_\alpha \mathbf{u}_\alpha}{Dt} + \nabla \left( \frac{\delta W}{\delta \rho_\alpha} \right) - \frac{D_\alpha}{Dt} \left( \frac{1}{\rho_\alpha} \frac{\delta W}{\delta \mathbf{u}_\alpha} \right) - \left( \frac{\partial \mathbf{u}_\alpha}{\partial \mathbf{x}} \right)^T \left( \frac{1}{\rho_\alpha} \frac{\delta W}{\delta \mathbf{u}_\alpha} \right) = 0. \tag{1}
\]
with $\alpha = 1, 2$. The expressions for the functional derivatives of $W$ are given below:

$$\frac{\delta W}{\delta \rho_\alpha} = \frac{\partial W}{\partial \rho_\alpha} - \rho_\alpha \frac{D_\alpha}{Dt} \left( \frac{1}{\rho_\alpha} \frac{\partial W}{\partial \rho_\alpha} \right) - \rho_\beta \frac{D_\beta}{Dt} \left( \frac{1}{\rho_\beta} \frac{\partial W}{\partial \rho_\alpha} \right),$$

$$\frac{\delta W}{\delta u_\alpha} = \frac{\partial W}{\partial \rho_\alpha} \nabla \rho_\alpha + \frac{\partial W}{\partial \rho_\beta} \nabla \rho_\beta.$$

General properties for the Euler-Lagrange equations, related to the Lagrangian symmetries, were also presented in [1]. In particular, conservation laws for total momentum and total energy were given and a notion of generalized vorticity introduced.

It is convenient to write our model in the original variables $h_\alpha$ and $u_\alpha$. This can be done by rewriting the “mass conservation laws” in the equivalent form

$$(h_\alpha)_t + \text{div}(h_\alpha u_\alpha) = 0$$

and replacing in (1) the quantities

$$\frac{\delta W}{\delta \rho_1} = g(h_1 + \gamma h_2) - \frac{1}{6} \left( \frac{D_1 h_1}{Dt} \right)^2 + \frac{1}{3} h_1 \left( \frac{D_1^2 h_1}{Dt^2} \right) + \gamma h_2 \left( \frac{D_2^2 h_1}{Dt^2} \right) + \frac{1}{2} \gamma h_2 \left( \frac{D_2^3 h_2}{Dt^2} \right),$$

$$\frac{\delta W}{\delta \rho_2} = g(h_1 + h_2) - \frac{1}{2} \left( \frac{D_2 h_1}{Dt} \right)^2 - \frac{1}{2} \left( \frac{D_2 h_2}{Dt} \right) \left( \frac{D_2 h_1}{Dt} \right) - \frac{1}{6} \left( \frac{D_2 h_2}{Dt} \right)^2 + \frac{1}{2} \gamma h_2 \left( \frac{D_2^2 h_2}{Dt^2} \right) + \frac{1}{3} \gamma h_2 \left( \frac{D_2^3 h_2}{Dt^2} \right),$$

and

$$\frac{1}{\rho_1} \frac{\delta W}{\delta u_1} = -\frac{1}{3} \left( \frac{D_1 h_1}{Dt} \right) \nabla h_1,$$

$$\frac{1}{\rho_2} \frac{\delta W}{\delta u_2} = -\frac{1}{2} \left( \frac{D_2 h_2}{Dt} \right) \nabla h_1 - \frac{1}{3} \left( \frac{D_2 h_2}{Dt} (h_2 + 3h_1/2) \right) \nabla h_2,$$

where the scalar $\gamma < 1$ is defined by $\gamma = \gamma_2/\gamma_1$.

3 The One-Dimensional Case

Notice that in the 1-d case, conservation laws for mass, total momentum and total energy give four conservation laws for the four physical variables $h_1$, $h_2$, $u_1$ and $u_2$. Thus, governing equations have the equivalent conservative form

$$(h_1)_t + (h_1 u_1)_x = 0,$$

$$(h_2)_t + (h_2 u_2)_x = 0,$$

$$(\gamma_1 h_1 u_1 + \gamma_2 h_2 u_2)_t + (\gamma_1 h_1 u_1^2 + \gamma_2 h_2 u_2^2 + p)_x = 0,$$

$$\left( \frac{1}{2} \gamma_1 h_1 u_1^2 + \frac{1}{2} \gamma_2 h_2 u_2^2 + E \right)_t + \left( u_1 \left( \frac{1}{2} \gamma_1 h_1 u_1^2 + F_1 \right) + u_2 \left( \frac{1}{2} \gamma_2 h_2 u_2^2 + F_2 \right) \right)_x = 0. \quad (2)$$

The expressions for $p$, $E$ and the quantities $F_1$ and $F_2$ are presented below. The “pressure” $p$ is given by

$$p = \frac{g}{2} \left( \gamma_1 h_1^2 + 2 \gamma_2 h_1 h_2 + \gamma_2 h_2^2 \right) + \frac{1}{3} \gamma_1 h_1^2 \left( \frac{D_1^2 h_1}{Dt^2} \right) + \frac{1}{2} \gamma_2 h_2 (h_2 + 2h_1) \left( \frac{D_2^2 h_1}{Dt^2} \right) + \frac{1}{3} \gamma_2 h_2 (h_2 + 3h_1/2) \left( \frac{D_2^3 h_2}{Dt^2} \right).$$

\[^1\text{In what follows, the indexes } \alpha, \beta \text{ satisfy } \alpha, \beta = 1, 2 \text{ and } \alpha \neq \beta.\]
The quantities $F_1$ and $F_2$ are prescribed by

$$F_1 = \gamma_1 h_1 \left[ g(h_1 + \gamma h_2) + \frac{1}{6} \left( \frac{D_1 h_1}{Dt} \right)^2 + \frac{1}{3} h_1 \left( \frac{D_2 h_1}{Dt} \right)^2 + \gamma h_2 \left( \frac{D_3 h_1}{Dt} \right) + \frac{1}{2} \gamma h_2 \left( \frac{D_3 h_2}{Dt} \right) \right],$$

$$F_2 = \gamma_2 h_2 \left[ g(h_1 + h_2) + \frac{1}{2} \left( \frac{D_2 h_1}{Dt} \right)^2 + \frac{1}{2} \left( \frac{D_2 h_1}{Dt} \right) \left( \frac{D_3 h_2}{Dt} \right) + \frac{1}{6} \left( \frac{D_3 h_2}{Dt} \right)^2 + \right.$$  
$$+ \frac{1}{2} h_2 \left( \frac{D_2 h_1}{Dt} \right)^2 + \frac{1}{3} h_2 \left( \frac{D_3 h_2}{Dt} \right)^2 \right],$$

and the “internal energy” takes the form

$$E = \frac{g}{2} \left( \gamma_1 h_1^2 + 2 \gamma_1 h_1 h_2 + \gamma_2 h_2^2 \right) + \frac{1}{6} \gamma_1 h_1 \left( \frac{D_1 h_1}{Dt} \right)^2 +$$  
$$+ \frac{1}{2} \gamma_2 h_2 \left[ \left( \frac{D_2 h_1}{Dt} \right)^2 + \left( \frac{D_2 h_1}{Dt} \right) \left( \frac{D_3 h_2}{Dt} \right) + \frac{1}{3} \left( \frac{D_3 h_2}{Dt} \right)^2 \right].$$

We shall present the dispersion relation for this model and its comparison with the one obtained for the full water-wave problem. For that, we linearize the system taking only into account equilibrium points having the same velocities (i.e., $u_1 = u_2$). By Galilean invariance, we can set $u_1 = u_2 = 0$ in equilibrium. On the other hand, no restrictions concerning the depths for each layer in equilibrium will be considered. They will be denoted by $H_1$ and $H_2$, respectively.

### 3.1 Linearization

To linearize the system it is convenient to rewrite (1) in the form

$$\rho_t + \rho u_x = 0,$$

$$\left( \rho u_x - \frac{1}{\rho} W \right)_t + \frac{1}{2} u^2_x + \frac{\partial W}{\partial u} \left( \frac{1}{\rho} \frac{\partial W}{\partial u} \right)_x = 0. \tag{3}$$

This form results from the fact that, in the 1-d case, the motion equations can be given in conservative form. Moreover, it is a consequence of a wider result concerning “generalized potential flows” (see [1]).

As previously, we will consider the system in terms of the original variables $h_\alpha$ and $u_\alpha$. With this understanding, in order to linearize the system, consider the change of variables

$$h_1 \rightarrow H_1 + \mu h_1, \quad h_2 \rightarrow H_2 + \mu h_2, \quad u_1 \rightarrow \mu u_1, \quad u_2 \rightarrow \mu u_2$$

with $\mu^2 \ll 1$. The linearized system is the following

$$\begin{cases} 
(h_1)_t + H_1(u_1)_x = 0, \\
(h_2)_t + H_2(u_2)_x = 0, \\
(u_1)_t + g(h_1)_x + \left( \frac{1}{3} H_1 + \gamma H_2 \right) (h_1)_{tx} + g\gamma(h_2)_x + \frac{1}{2} \gamma H_2(h_2)_{tx} = 0, \\
(u_2)_t + g(h_2)_x + \frac{1}{3} H_2(h_2)_{tx} + g(h_1)_x + \frac{1}{2} H_2(h_1)_{tx} = 0.
\end{cases} \tag{4}$$
3.2 Dispersion relation

We look for particular solutions of (4) in the form

\[
\begin{bmatrix}
  h_1 \\
  h_2 \\
  u_1 \\
  u_2
\end{bmatrix} =
\begin{bmatrix}
  c_1 \\
  c_2 \\
  c_3 \\
  c_4
\end{bmatrix} e^{i(kx-\omega t)},
\]

where \( \mathbf{c} = (c_1, c_2, c_3, c_4)^T \) is a constant vector, \( k \) is the wave number and \( \omega \) is the wave frequency. This leads to the system \( \mathbf{M} \mathbf{c} = 0 \) where the matrix \( \mathbf{M} \) is defined by

\[
\mathbf{M} =
\begin{bmatrix}
  -\omega & 0 & H_1 k & 0 \\
  0 & -\omega & 0 & H_2 k \\
  gk - \left( \frac{4}{3}H_1 + \gamma H_2 \right) k \omega^2 & g\gamma k - \frac{4}{5} \gamma H_2 k \omega^2 & -\omega & 0 \\
  gk - \frac{1}{2} H_2 k \omega^2 & gk - \frac{1}{3} H_2 k \omega^2 & 0 & -\omega
\end{bmatrix}
\]

In order to have nontrivial solutions for this equation we impose \( \det \mathbf{M} = 0 \). The dispersion relation for the present model is given by

\[
\begin{align*}
1 + \left( \frac{1}{3}H^2 + \gamma H + \frac{1}{3} \right) (kH_1)^2 + \left( \frac{1}{9}H^2 + \frac{1}{12} \gamma H^3 \right) (kH_1)^4 & \omega^4 - \\
& - \frac{g}{H_1} \left( (H + 1)(kH_1)^2 + \frac{1}{3} H(H + 1)(kH_1)^4 \right) \omega^2 + \frac{g^2}{H_1^2} H(1 - \gamma)(kH_1)^4 = 0.
\end{align*}
\]

Here, we introduced the scalar \( H = H_2/H_1 \). In dimensionless form, this becomes

\[
\begin{align*}
1 + \left( \frac{1}{3}H^2 + \gamma H + \frac{1}{3} \right) K^2 + \left( \frac{1}{9}H^2 + \frac{1}{12} \gamma H^3 \right) K^4 & F_1^4 - \left[ H + 1 + \frac{1}{3} H(H + 1)K^2 \right] F_1^2 + H(1 - \gamma) = 0, \\
\end{align*}
\]

where \( F_1 = c/\sqrt{gH_1} \) is the Froude number with respect to the bottom layer, \( c = \omega/k \) is the phase velocity and \( K = kH_1 \) is the dimensionless wave number. This gives the behavior of the wave speed \( F_1 \) in terms of the wave number \( K \). We now present its behavior in terms of the wave frequency. We point out the existence of two critical wave frequencies \( \Omega_+ \) and \( \Omega_- \) corresponding to the fast mode and slow mode, respectively. To define them, let us introduce the dimensionless wave frequency \( \Omega = \sqrt{H_1/g} \omega \). The relation between the Froude number and the wave frequency is given by

\[
\Omega^2 = K^2 F_1^2.
\]

If one writes (5) as

\[
F_1^4 + \left( \frac{1}{3}H^2 + \gamma H + \frac{1}{3} \right) K^2 F_1^4 + \left( \frac{1}{9}H^2 + \frac{1}{12} \gamma H^3 \right) K^4 F_1^4 - (H + 1)F_1^2 - \frac{1}{3} H(H + 1)K^2 F_1^2 + H(1 - \gamma) = 0,
\]

we can use (6) to finally present

\[
F_1^4 + \left[ \left( \frac{1}{3}H^2 + \gamma H + \frac{1}{3} \right) \Omega^2 - (H + 1) \right] F_1^2 + \left( \frac{1}{9}H^2 + \frac{1}{12} \gamma H^3 \right) \Omega^4 - \frac{1}{3} H(H + 1)\Omega^2 + H(1 - \gamma) = 0.
\]

Assigning the value \( F_1 = 0 \) on (7) leads to

\[
\left( \frac{1}{9}H^2 + \frac{1}{12} \gamma H^3 \right) \Omega^4 - \frac{1}{3} H(H + 1)\Omega^2 + H(1 - \gamma) = 0.
\]
The critical wave frequencies are the positive roots of this biquadratic polynomial on $\Omega$. They satisfy
\[
\Omega_{\pm}^2 = \frac{1}{3} H(H + 1) \pm \sqrt{\left(\frac{1}{3} H(H + 1)\right)^2 - 4H(1 - \gamma) \left(\frac{1}{9} H^2 + \frac{1}{12} \gamma H^3\right)}
\]
\[
\div 2 \left(\frac{1}{9} H^2 + \frac{1}{12} \gamma H^3\right).
\]
This is well defined, since the discriminant is positive. Precisely,
\[
\left(\frac{1}{3} H(H + 1)\right)^2 - 4H(1 - \gamma) \left(\frac{1}{9} H^2 + \frac{1}{12} \gamma H^3\right) = \frac{1}{9} H^2 \left(\frac{H^2(1 - 3\gamma + 3\gamma^2) + H(4\gamma - 2) + 1}{\gamma H^3}\right) > 0.
\]
The dispersion relation, in terms of $\Omega$, is given by two branches of decreasing functions with vertical asymptotes $\Omega = \Omega_{-}$ and $\Omega = \Omega_{+}$, respectively (see Fig. 2).

![Figure 2: Dispersion relation curves on $\Omega$ for different set of parameters.](image)

We conclude from the pictures that the group velocity $c_g = \frac{d\omega}{dk}$ is less than the phase velocity. Indeed, it follows from definitions
\[
\frac{1}{c_g} = \frac{1}{c} + \omega \frac{d}{dk} \left(\frac{1}{c}\right)
\]
so, if $c(\omega)$ is a monotonically decreasing function, then $c_g < c$.

### 3.2.1 Comparison with the dispersion relation for the full problem

By “full problem” we mean the Euler equations governing a two-layer flow. The dispersion relation was first investigated by Stokes, according to Lamb [2]. It reads in dimensional variables
\[
(1 + \gamma \tanh(kH_1) \tanh(kH_2)) c^4 - \frac{g}{K} (\tanh(kH_1) + \tanh(kH_2)) c^2 + \frac{g^2}{K^2} (1 - \gamma) \tanh(kH_1) \tanh(kH_2) = 0
\]
and stands in dimensionless form (see [15], [17])
\[
(1 + \gamma \tanh(K \tanh(KH))) F_1^4 - \frac{1}{K} (\tanh K + \tanh(KH)) F_1^2 + \frac{1}{K^2} (1 - \gamma) \tanh K \tanh(KH) = 0.
\]
There are two branches characterizing two different modes that we call fast mode and slow mode. Both branches are well defined for \( K > 0 \) and they are both monotonically decreasing functions on \( K \). The limit \( K \to 0 \) gives the relation
\[
F_1^4 - (H + 1)F_1^2 + H(1 - \gamma) = 0.
\]
This enables us to define wave speeds \( F_1^+ \) and \( F_1^- \) corresponding to fast waves and slow waves, respectively:
\[
(F_1^\pm)^2 = (H + 1) \pm \sqrt{(H - 1)^2 + 4\gamma H}.
\]
(8)

For our model, the same speeds are found by taking the limit \( K \to 0 \) on (5). Alternatively, taking the limit \( \Omega \to 0 \) on (7) would produce the exact same result. Another feature of our model is that it captures the resonance between the solitary waves and periodic waves predicted by the full problem. By resonance we mean a possibility for long waves and short waves to propagate with the same phase velocity. In particular we define \( K_{res} \) as the wave number corresponding to the wave speed \( F_1^- \). Obviously, this feature exists for the full problem, but, in the past, the models that have been derived could not deal with the resonance. Models with this feature were proposed by Dias & Il’ichev [15] and Grimshaw [18]. Below, we find several graphs describing dispersive curves for our model and the full problem.

![Graphs](image)

Figure 3: Dispersion relation curves on \( K \) for different set of parameters. The dashed lines (in black) correspond to the full problem and the solid lines (in blue) corresponds to the present model.

### 4 Traveling Waves for the One-dimensional Case

We start by presenting the study of traveling waves in a general setting. In (2) and (3) we have seen two equivalent systems of conservation laws for our model in the one-dimensional case. Both forms are suitable to the study of traveling waves but, as we will see, there is a great advantage in adopting the
last one. In fact, there is a key feature for this particular set of partial differential equations: it leads — for traveling waves — to a system of two second order ordinary differential equations endowed with a Hamiltonian structure. Once more this major feature has been motivated by using a variational approach. We decided to include here the derivation of the Hamiltonian system for traveling waves by using this approach. We address to Appendix A to show how this system could be obtained directly from the “good set of equations” (3).

4.1 Hamiltonian approach

Consider the Lagrangian density

$$L = \frac{1}{2} \rho_1 u_1^2 + \frac{1}{2} \rho_2 u_2^2 - W\left(\rho_1, \rho_2, \frac{D_1 \rho_1}{D_t}, \frac{D_1 \rho_2}{D_t}, \frac{D_2 \rho_1}{D_t}, \frac{D_2 \rho_2}{D_t}\right).$$

We will be interested in the particular case when all the functions depend only on the variable $\xi = x - D t$, where $D$ is a constant speed. From the mass conservation laws for each layer, we get

$$\rho_\alpha (u_\alpha - D) = \text{const} \overset{\text{def}}{=} m_\alpha, \quad \alpha = 1, 2.$$  \hspace{1cm} (9)

Also, with respect to this new variable, the material derivatives yield

$$\frac{D \alpha f}{D t} = f'(u_\alpha - D) = \frac{f'}{\rho_\alpha} m_\alpha.$$  \hspace{1cm} (10)

Combining these relations, taking into account that in a reference frame moving with constant speed $D$ the velocities $u_\alpha$ become $u_\alpha - D$, we are able to rewrite the Lagrangian density as

$$L = \frac{1}{2} m_1^2 + \frac{1}{2} m_2^2 - W\left(\rho_1, \rho_2, \frac{\rho_1'}{\rho_1} m_1, \frac{\rho_2'}{\rho_2} m_2, \frac{\rho_1'}{\rho_1} m_1, \frac{\rho_2'}{\rho_2} m_2\right).$$

This can still be simplified by defining a new potential $U$ as below:

$$U(\rho_1, \rho_2, \rho_1', \rho_2') = W\left(\rho_1, \rho_2, \frac{\rho_1'}{\rho_1} m_1, \frac{\rho_2'}{\rho_2} m_2, \frac{\rho_1'}{\rho_1} m_1, \frac{\rho_2'}{\rho_2} m_2\right).$$

This way, we can present the Lagrangian density in a more condensed form, namely

$$L = \frac{1}{2} m_1^2 + \frac{1}{2} m_2^2 - U(\rho_1, \rho_2, \rho_1', \rho_2').$$

As usual, we consider the action integral $I$ defined by

$$I = \int_{\xi_1}^{\xi_2} L d\xi$$

and determine the stationary points of this functional when constrained to the mass conservation laws of each fluid. The variation of $\rho_\alpha$, at fixed Eulerian coordinates, is related with the virtual displacement $\delta_\alpha x$ by (see [19], [20], [21])

$$\delta \rho_\alpha = -(\rho_\alpha \delta_\alpha x).$$  \hspace{1cm} (11)

We shall determine the stationary points of $I$ with respect to both fields of variations

$$\delta \alpha I = 0.$$
We have:

$$\delta_\alpha I = \int_{\xi_1}^{\xi_2} \left(- \frac{1}{2} \frac{m^2_\alpha}{\rho^2_\alpha} \delta \rho_\alpha - U_{\rho_\alpha} \delta \rho_\alpha - U_{\rho_\alpha'} \delta \rho_\alpha'\right) d\xi$$

$$= \int_{\xi_1}^{\xi_2} \left(- \frac{1}{2} \frac{m^2_\alpha}{\rho^2_\alpha} + U_{\rho_\alpha'} \right) \delta \rho_\alpha + \left(U_{\rho_\alpha'}\right)' d\xi$$

$$= \int_{\xi_1}^{\xi_2} \left(- \frac{1}{2} \frac{m^2_\alpha}{\rho^2_\alpha} + \frac{d}{d\xi}(U_{\rho_\alpha'}) \delta \rho_\alpha\right) d\xi$$

$$= \int_{\xi_1}^{\xi_2} \left(- \frac{1}{2} \frac{m^2_\alpha}{\rho^2_\alpha} - U_{\rho_\alpha} + \frac{d}{d\xi}(U_{\rho_\alpha'})\right) \delta \rho_\alpha d\xi.$$

The introduction of the relation (11) leads to

$$\delta_\alpha I = - \int_{\xi_1}^{\xi_2} \left(- \frac{1}{2} \frac{m^2_\alpha}{\rho^2_\alpha} - U_{\rho_\alpha} + \frac{d}{d\xi}(U_{\rho_\alpha'})\right) (\rho_\alpha \delta_\alpha x)' d\xi,$$

that becomes

$$\delta_\alpha I = \int_{\xi_1}^{\xi_2} \left(- \frac{1}{2} \frac{m^2_\alpha}{\rho^2_\alpha} - U_{\rho_\alpha} + \frac{d}{d\xi}(U_{\rho_\alpha'})\right)' \rho_\alpha \delta_\alpha x d\xi.$$

Hence, when solving $\delta_\alpha I = 0$ we get the following condition:

$$\frac{1}{2} \frac{m^2_\alpha}{\rho^2_\alpha} - U_{\rho_\alpha} + \frac{d}{d\xi}(U_{\rho_\alpha'}) = \text{const} \stackrel{\text{def}}{=} -c_\alpha,$$

or, equivalently,

$$\left(- \frac{1}{2} \frac{m^2_\alpha}{\rho^2_\alpha} - U_{\rho_\alpha} + c_\alpha\right) - \frac{d}{d\xi}(-U_{\rho_\alpha'}) = 0, \quad \alpha = 1, 2.$$

If we define

$$\mathcal{L} = \sum_{\alpha=1}^{2} \left(\frac{1}{2} \frac{m^2_\alpha}{\rho^2_\alpha} + c_\alpha \rho_\alpha\right) - U,$$

we see that the system of two ordinary equations consists exactly on the Euler-Lagrange equations for $\mathcal{L} = \mathcal{L}(\rho_\alpha, \rho_\alpha')$, i.e.,

$$\mathcal{L}_{\rho_\alpha} - \frac{d}{d\xi} (\mathcal{L}_{\rho_\alpha'}) = 0, \quad \alpha = 1, 2. \quad (12)$$

It is worth to remark that the constants $c_\alpha$ are the Lagrange multipliers for the variational problem, given that

$$\mathcal{L} = L + c_1 \rho_1 + c_2 \rho_2.$$

The physical meaning of these constants is found in [1] where it is proved that they are, in fact, the Bernoulli integrals.

4.2 System of ODEs for traveling waves to the present model

We have presented in a general setting the equations to be verified by traveling-wave solutions for a certain class of models. We have seen that these equations correspond to the Euler-Lagrange equations for a new Lagrangian denoted above by $\mathcal{L}$. The aim of this work is to study the traveling-wave solutions to our dispersive model for a two-layer flow. Let us recall that the potential $W$ is given by

$$W = \frac{q}{2} \left( \frac{\rho_1^2}{\gamma_1} + 2 \frac{\rho_1 \rho_2}{\gamma_1} + \frac{\rho_2^2}{\gamma_2} \right)^2 - \frac{1}{6 \gamma_1^2} \rho_1 \left( \rho_1^{(1)} \right)^2 - \rho_2 \left( \frac{1}{2 \gamma_1^2} \left( \rho_1^{(1)} \right)^2 + \frac{1}{2 \gamma_1 \gamma_2} \rho_1^{(1)} \rho_2^{(2)} + \frac{1}{6 \gamma_2^2} \rho_2^{(2)} \right).$$
Like previously, writing this expression in terms of the variable $\xi$, enables defining a new potential $U$ depending only on the variables $\rho_1, \rho_2$ and their derivatives $\rho'_1$ and $\rho'_2$:

$$U(\rho_1, \rho_2, \rho'_1, \rho'_2) = \frac{g}{2} \left( \frac{\rho_1^2}{\gamma_1} + \frac{2 \rho_1 \rho_2}{\gamma_1} + \frac{\rho_2^2}{\gamma_2} \right) - \frac{1}{6 \gamma_1} \left( \frac{m_1^2}{\rho_1} + \frac{1}{2 \gamma_1} \right) (\rho'_1)^2 + \frac{1}{2 \gamma_1 \gamma_2} \rho'_1 \rho'_2 + \frac{1}{6 \gamma_2} \left( \frac{m_2^2}{\rho_2} + \frac{1}{2 \gamma_2} \right) (\rho'_2)^2.$$  

In order to write the Euler-Lagrange equations we need to perform some basic calculations that we present below:

$$U_{\rho_1} = \frac{g}{\gamma_1} \rho_1 + \frac{g}{\gamma_1} \rho_2 + \frac{m_1^2}{6 \gamma_1} \rho'_1 (\rho'_1)^2,$$

$$U_{\rho_2} = \frac{g}{\gamma_2} \rho_1 + \frac{g}{\gamma_2} \rho_2 + \frac{m_2^2}{2 \gamma_2} \rho'_2 (\rho'_2)^2 + \frac{1}{2 \gamma_1 \gamma_2} \rho'_1 \rho'_2 + \frac{1}{6 \gamma_2} \rho'_2 (\rho'_2)^2,$$

$$U_{\rho'_1} = -\left( \frac{1}{3 \gamma_1^2} \rho_1 + \frac{m_1^2}{\gamma_1} \rho_2 \right) \rho'_1 - \frac{1}{2 \gamma_1 \gamma_2} \rho'_2,$$

$$U_{\rho'_2} = -\frac{1}{2 \gamma_1 \gamma_2} \rho'_1 - \frac{m_2^2}{3 \gamma_2} \rho'_2.$$  

If we attribute the values $\alpha = 1, 2$ and make all necessary replacements we get the equations:

$$\frac{1}{2} \frac{m_1^2}{\rho_1^2} + \frac{g}{\gamma_1} \rho_1 + \frac{g}{\gamma_1} \rho_2 + \frac{m_1^2}{6 \gamma_1} \left( \frac{\rho'_1}{\rho_1} \right)^2 + \frac{m_1^2}{3 \gamma_1} \left( \frac{\rho'_1}{\rho_1} \right)' + \frac{m_2^2}{2 \gamma_1 \gamma_2} \left( \frac{\rho'_1}{\rho_2} \right)' + \frac{m_2^2}{2 \gamma_1 \gamma_2} \left( \frac{\rho'_2}{\rho_2} \right)' = c_1,$$

$$\frac{1}{2} \frac{m_2^2}{\rho_2^2} + \frac{g}{\gamma_2} \rho_1 + \frac{g}{\gamma_2} \rho_2 + \frac{m_2^2}{2 \gamma_2} \left( \frac{\rho'_2}{\rho_2} \right)^2 + \frac{m_2^2}{2 \gamma_2} \left( \frac{\rho'_2}{\rho_2} \right)^2 + \frac{m_2^2}{2 \gamma_1 \gamma_2} \left( \frac{\rho'_1}{\rho_2} \right)' + \frac{m_2^2}{3 \gamma_2} \left( \frac{\rho'_2}{\rho_2} \right)' = c_2.  \quad (13)$$

It is interesting to note that this system of ordinary differential equations appeared first in [22] (see page 68). Ovsyannikov derived a second order approximation model describing the same physical system. He assumed that the flow was potential in each layer and used the Lagrange method for long waves, consisting in developing the flow potentials in series with respect to the vertical coordinate. The full non-stationary system obtained by this author differs from our model (3). However, for traveling waves, the systems of ordinary differential equations coincide. Ovsyannikov did not discover the variational structure of this system, but, even so, he was able to find a conserved quantity that corresponds exactly to the Hamiltonian of the system. No more conservation laws were found by the author. This contributes to our strong belief that we are in the presence of a non-integrable Hamiltonian system of equations with two degrees of freedom. An independent study of traveling waves in a two-layer fluid with free surface was done by Dias et Il’ichev in [15]. The corresponding full non-stationary system was obtained for potential flows by using asymptotic expansion, considering Lagrange’s method, and long waves of small amplitude. As mentioned earlier this is in contrast with our model, since no assumptions such as potential flows or small amplitude of waves were considered in its derivation. In the dynamical system for traveling waves it is assumed, as in our case, that velocities are zero in equilibrium. Also, the Froude number $F_1$, the density ratio $\gamma$ and the relative depth $H$ are parameters for the solutions of this system. This means that for fixed parameters $\gamma$ and $H$, our solutions depend only on the Froude number. Though, their model does not preserve the variational structure of the full problem. This property, shown to exist to our model, will be useful later.

If we define

$$\mathcal{V}(\rho_1, \rho_2) = \frac{g}{2} \left( \frac{\rho_1^2}{\gamma_1} + \frac{2 \rho_1 \rho_2}{\gamma_1} + \frac{\rho_2^2}{\gamma_2} \right) - \frac{1}{2} \frac{m_1^2}{\rho_1^2} + \frac{1}{2} \frac{m_2^2}{\rho_2^2} + c_1 \rho_1 + c_2 \rho_2,$$

we can write the Lagrangian $\mathcal{L}$ in a more natural way

$$\mathcal{L} = \frac{1}{2} \left[ \begin{array}{cc} \rho'_1 & \rho'_2 \end{array} \right] \mathcal{M} \left( \begin{array}{c} \rho'_1 \\ \rho'_2 \end{array} \right) - \mathcal{V}(\rho_1, \rho_2),  \quad (14)$$
with the matrix $\mathcal{M}$ defined by
\[
\mathcal{M}(\rho_1, \rho_2) = \begin{bmatrix}
\frac{1}{3\gamma_1 \rho_1} + \frac{1}{2\gamma_1 \gamma_2 \rho_1} & \frac{1}{2\gamma_1 \gamma_2 \rho_1} \\
\frac{1}{3\gamma_2 \rho_2} & \frac{1}{3\gamma_2 \rho_2} \\
\end{bmatrix}
\]

Adopting a vectorial notation we write (14) in the compact form
\[
\mathcal{L} = \frac{1}{2} \mathbf{q}'^T \mathcal{M} \mathbf{q}' - \mathcal{V},
\]
with $\mathbf{q} = (\rho_1, \rho_2)^T$. If one defines
\[
\mathbf{p} = \frac{\partial \mathcal{L}}{\partial \mathbf{q}'},
\]
we can introduce the Hamiltonian
\[
\mathcal{H} = \mathbf{p} \cdot \mathbf{q}' - \mathcal{L}.
\]
Since $\mathcal{M}$ is symmetric, it follows that
\[
\mathbf{p} = \mathcal{M} \mathbf{q}'.
\]
Additionally, $\mathcal{M}$ is a nonsingular matrix. As a consequence, $(\rho_1', \rho_2') \mapsto (p_1, p_2)$ is a change of variables. We can therefore write $\mathcal{H}$ explicitly in terms of the variables $(\mathbf{q}, \mathbf{p})$:
\[
\mathcal{H} = \frac{1}{2} \mathbf{p}^T \mathcal{M}^{-1} \mathbf{p} + \mathcal{V}.
\]
The Hamilton equations corresponding to (13) are given by:
\[
\begin{cases}
\mathbf{q}' = \frac{\partial \mathcal{H}}{\partial \mathbf{p}}, \\
\mathbf{p}' = -\frac{\partial \mathcal{H}}{\partial \mathbf{q}}.
\end{cases}
\]
The critical points for the system are the points $(\tilde{\rho}_1, \tilde{\rho}_2)$ for which $\nabla \mathcal{V}(\tilde{\rho}_1, \tilde{\rho}_2) = 0$. This enables us to establish the values of the constants $c_1$ and $c_2$:
\[
c_1 = \frac{g}{\gamma_1} \frac{\tilde{\rho}_1}{\rho_1} + \frac{g}{\gamma_1} \frac{\tilde{\rho}_2}{\rho_2} + \frac{m_1^2}{\rho_1},
\]
\[
c_2 = \frac{g}{\gamma_1} \frac{\tilde{\rho}_1}{\rho_1} + \frac{g}{\gamma_2} \frac{\tilde{\rho}_2}{\rho_2} + \frac{m_2^2}{\rho_2}.
\]
We introduce dimensionless depths $\tilde{h}_\alpha$ for each layer, related to the variables $\rho_\alpha$ by
\[
\rho_\alpha = \gamma_\alpha h_\alpha = \gamma_\alpha H_\alpha \tilde{h}_\alpha,
\]
with
\[
\tilde{\rho}_\alpha = \gamma_\alpha H_\alpha.
\]
Hereafter, tildes over the variables $h_\alpha$ will be omitted. From (9) it follows, in particular, that
\[
\tilde{\rho}_\alpha (u_\alpha - D) = m_\alpha
\]
In this work we shall impose the condition
\[
\tilde{u}_1 = \tilde{u}_2 = 0.
\]
This allows us to determine the relation
\[ m_\alpha = -D\gamma_\alpha H_\alpha. \]

If we define now the Froude number \( F_1 \) based on the bottom layer depth
\[ F_1^2 = \frac{D^2}{gH_1} \]
we are able to rewrite the nonlinear system of ordinary differential equations in dimensionless variables. Let us remark that this definition is in agreement with the previous definition of the Froude number given for periodic waves, where the phase velocity \( c \) has been replaced by \( D \). Equations (13) become
\[
\frac{1}{2}F_1^2 \left( \frac{1}{h_1^2} - 1 \right) + (h_1 - 1) + \gamma H (h_2 - 1) + \delta^2 \left[ \frac{1}{6} \left( \frac{h_1'}{h_1} \right)^2 + \frac{1}{3} \left( \frac{h_1'}{h_1} \right)' + \gamma H \left( \frac{h_1'}{h_2} \right)' + \frac{1}{2} \gamma H^2 \left( \frac{h_2'}{h_2} \right)' \right] = 0,
\]
\[
\gamma H \left( \frac{1}{2}F_1^2 \left( \frac{1}{h_2^2} - 1 \right) + (h_1 - 1) + H(h_2 - 1)+ \right.
+ \delta^2 \left[ \frac{1}{2} \left( \frac{h_1'}{h_2} \right)^2 + H \left( \frac{h_1'}{h_2} \right) \left( \frac{h_2'}{h_2} \right) + \frac{H^2}{6} \left( \frac{h_2'}{h_2} \right)^2 + \frac{H}{2} \left( \frac{h_1'}{h_2} \right)' + \frac{H^2}{3} \left( \frac{h_2'}{h_2} \right)' \right] = 0.
\]
We recall that \( \gamma = \gamma_2/\gamma_1 \) and \( H = H_2/H_1 \). Also, we have set
\[ \delta^2 = \frac{F_1^2H_1^2}{L^2}, \]
where \( L \) is a scale parameter
\[ \xi = \left( \frac{\xi}{L} \right) = \tilde{\xi}L. \]
The nonlinear system can be cast into the following form:
\[
\delta^2 \left[ \frac{1}{6} \left( \frac{h_1'}{h_1} \right)^2 + \frac{1}{3} \left( \frac{h_1'}{h_1} \right)' + \gamma H \left( \frac{h_1'}{h_2} \right)' + \frac{1}{2} \gamma H^2 \left( \frac{h_2'}{h_2} \right)' \right] = -\frac{\partial V}{\partial h_1},
\]
\[
\gamma H \delta^2 \left[ \frac{1}{2} \left( \frac{h_1'}{h_2} \right)^2 + H \left( \frac{h_1'}{h_2} \right) \left( \frac{h_2'}{h_2} \right) + \frac{H^2}{6} \left( \frac{h_2'}{h_2} \right)^2 + \frac{H}{2} \left( \frac{h_1'}{h_2} \right)' + \frac{H^2}{3} \left( \frac{h_2'}{h_2} \right)' \right] = -\frac{\partial V}{\partial h_2},
\]
where the potential \( V \) is given by
\[
V(h_1, h_2) = -\frac{1}{2}F_1^2 \left( \frac{h_1 - 1}{h_1} \right)^2 + \frac{1}{2}(h_1 - 1)^2 + \gamma H \left[ -\frac{1}{2}F_1^2 \left( \frac{h_2 - 1}{h_2} \right)^2 + \frac{1}{2}H(h_2 - 1)^2 \right] + \gamma H(h_1 - 1)(h_2 - 1).
\]
This particular potential was chosen in order to verify simultaneously
\[ V(1, 1) = 0, \quad \nabla V(1, 1) = 0. \]

4.3 Linearization at the point (1, 1)

We linearize the nonlinear system at the point (1, 1). This leads to
\[
(1 - F_1^2) h_1 + \gamma H h_2 + \left( \frac{1}{3} + \gamma H \right) \delta^2 h_1'' + \frac{1}{2} \gamma H^2 \delta^2 h_2'' = 0,
\]
\[
\gamma H h_1 + \gamma H(H - F_1^2) h_2 + \frac{1}{2} \gamma H^2 \delta^2 h_1'' + \frac{1}{3} \gamma H^3 \delta^2 h_2'' = 0.
\]
Setting \( h = (h_1, h_2) \) we write the linearized system as

\[
A h'' + B h = 0,
\]

where the matrices \( A \) and \( B \) are given by

\[
A = \delta^2 \begin{bmatrix}
\frac{1}{3} + \gamma H & \frac{1}{3} \gamma H^2 \\
\frac{1}{3} \gamma H & \frac{1}{3} \gamma H^3
\end{bmatrix}, \quad B = \begin{bmatrix}
1 - F_1^2 & \gamma H \\
\gamma H & \gamma H (H - F_1^2)
\end{bmatrix}
\]

Both matrices are symmetric. Further, \( A \) is a positive definite matrix, since \( \det A > 0 \):

\[
\det A = \delta^4 \left( \frac{1}{9} \gamma H^3 + \frac{1}{12} \gamma^2 H^4 \right) > 0.
\]

### 4.4 The eigenvalues for the linearized system in equilibrium \((1, 1)\)

Looking for values \( \lambda \) for which there exists particular solutions \( h = c e^{\lambda t} \), where \( c \) denotes a nonzero vector, is equivalent to finding the eigenvalues for the matrix of the linearized system. If we insert this particular solution in the equation, we get

\[
(B + \lambda^2 A)c = 0.
\]

In order to have nontrivial solutions for this equation we must have

\[
\det (B + \lambda^2 A) = 0.
\]

We state that all the values \( \lambda^2 \) are real numbers. This results from the fact that both matrices \( A \) and \( B \) are symmetric, being \( A \) a positive definite matrix, in addition. We set, for convenience, \( \sigma = -\delta^2 \lambda^2 \) and write the polynomial

\[
\sigma^2 \left( \frac{1}{9} H^2 + \frac{1}{12} \gamma H^3 \right) + \sigma \left( F_1^2 \left( \frac{H^2}{3} + \gamma H + \frac{1}{3} \right) - \frac{H}{3} - \frac{H^2}{3} \right) + (1 - F_1^2)(H - F_1^2) - \gamma H = 0. \tag{17}
\]

Let \( \sigma_1, \sigma_2 \) be the real roots of this polynomial. If we set

\[
a = \frac{1}{9} H^2 + \frac{1}{12} \gamma H^3, \\
b = F_1^2 \left( \frac{H^2}{3} + \gamma H + \frac{1}{3} \right) - \frac{H}{3} - \frac{H^2}{3}, \\
c = (1 - F_1^2)(H - F_1^2) - \gamma H,
\]

then we can write

\[
\begin{cases}
\sigma_1 \cdot \sigma_2 = c/a, \\
\sigma_1 + \sigma_2 = -b/a.
\end{cases}
\]

We can have three different signatures for these roots: \((+, +), (+, -)\) and \((-,-)\). We shall determine conditions allowing us to distinguish them. We proceed as follows:

**Case 1: The roots have different signs**

In this case, we have \( \sigma_1 \cdot \sigma_2 < 0 \). Since \( a > 0 \), we have the inequality

\[
(1 - F_1^2)(H - F_1^2) - \gamma H < 0.
\]

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If we set \( x = F_1^2 \) and consider the polynomial
\[
x^2 - (H + 1)x + H(1 - \gamma),
\]
we see that it has two real roots, since the discriminant \( \Delta = (H - 1)^2 + 4\gamma H \) is positive. We shall denote these roots by \( x_- \) and \( x_+ \). We verify that
\[
\begin{align*}
x_- \cdot x_+ &> 0, \\
x_- + x_+ &> 0,
\end{align*}
\]
so, we conclude that \( x_- \) and \( x_+ \) are both positive. The polynomial for the Froude number has therefore four real roots \( \pm \sqrt{x_-}, \pm \sqrt{x_+} \) where \( x_- \) and \( x_+ \) are given explicitly by
\[
x_{\pm} = \frac{(H + 1) \pm \sqrt{(H - 1)^2 + 4\gamma H}}{2}
\]
and it has the following behaviour:

![Figure 4: The 4th degree polynomial on the Froude number](image)

where we have set
\[
F_1^- = \sqrt{x_-}, \quad F_1^+ = \sqrt{x_+}.
\]

Note that these quantities correspond exactly to the wave speeds \( F_1^{\pm} \) given in (8) corresponding to the fast waves and slow waves characterizing the two branches of the dispersion curves for our model. Since \( F_1 \) is a physical variable and it is positive by definition, we will be only interested in the right-hand side of this symmetric graph. With this, we conclude that the signature \( (+, -) \) corresponds to the set of \( |F_1^-|, F_1^+ \) of values for the Froude number. Physically, this interval corresponds to the resonance interaction between short waves and long waves.

**Case 2: The roots have the same sign**

We know already that in order to have the roots \( \sigma_1 \) and \( \sigma_2 \) with the same sign, the Froude number \( F_1 \) must belong to the set \( ]0, F_1^- [ \cup ]F_1^+, +\infty[ \). At this stage, we must distinguish two cases:

(i) \( \sigma_1 + \sigma_2 > 0 \),

(ii) \( \sigma_1 + \sigma_2 < 0 \).

This corresponds to the cases when \( \sigma_1 \) and \( \sigma_2 \) are both positive and negative, respectively. This will be determined completely by the sign of \( b \) that we recall now:
\[
b = \frac{1}{3}(F_1^2(H^2 + 3\gamma H + 1) - H(H + 1))
\]
We can restrict our analysis to the study of the sign of

\[ F_1^2(H^2 + 3\gamma H + 1) - H(H + 1) \]

In terms of the variable \( F_1 \), this yields a parabola with real roots \( \pm \sqrt{\frac{H(H+1)}{H^2 + 3\gamma H + 1}} \). Bearing in mind that \( F_1 > 0 \), by definition, one introduces the number

\[ F_1^* = \sqrt{\frac{H(H+1)}{H^2 + 3\gamma H + 1}} \]

and get:

\[ \sigma_1 + \sigma_2 > 0 \iff F_1 \in ]0, F_1^*[ , \]

\[ \sigma_1 + \sigma_2 < 0 \iff F_1 \in ]F_1^*, +\infty[ . \]

Define

\[ F_m = \min\{ F_1^-, F_1^* \}, \quad F_M = \max\{ F_1^+, F_1^* \}. \]

Gathering all the information we establish the following:

\( (+, +) \implies F_1 \in ]0, F_m[ , \)

\( (-, -) \implies F_1 \in ]F_M, +\infty[ . \)

We shall see, in fact, that the value of \( F_1^* \) brings no contribution to our problem. The reason for this is that it can be shown that \( F_1^- < F_1^* < F_1^+ \). This is proved in the next following Lemmas.

**Lemma 1.** \( F_1^- < F_1^* \)

*Proof.* This is equivalent to prove that

\[ \frac{(H + 1) - \sqrt{(H - 1)^2 + 4\gamma H}}{2H} < \frac{H + 1}{H^2 + 3\gamma H + 1}, \]

or, in a simpler form,

\[ \frac{\sqrt{(H - 1)^2 + 4\gamma H}}{2H} > \frac{H + 1}{2H(H^2 + 3\gamma H + 1)}((H - 1)^2 + 3\gamma H). \]

We proceed with the calculation, obtaining

\[ (H^2 + 3\gamma H + 1)^2((H - 1)^2 + 4\gamma H) > (H + 1)^2((H - 1)^2 + 3\gamma H)^2, \]

that we can write as

\[ 4H \Pi(\gamma, H) > 0, \]

with \( \Pi(\gamma, H) \) defined below:

\[ H(H - 1)^2 + (1 - 3H + 8H^2 - 3H^3 + H^4)\gamma + 3H(2 - 3H + 2H^2)\gamma^2 + 9H^2 \gamma^3. \]

If we consider it as a polynomial on the variable \( \gamma \) with coefficients depending on \( H \), we see that all these coefficients are positive for \( H > 0 \). Since this is our case and \( \gamma \) is positive, we have \( \Pi \) given by the sum of positive quantities. This proves our claim.

**Lemma 2.** \( F_1^* < F_1^+ \)
Proof. This is equivalent to prove that
\[ \frac{(H - 1)^2 + 4\gamma H}{2H} > (H + 1) \left( \frac{1}{H^2 + 3\gamma H + 1} - \frac{1}{2H} \right), \]
or, in a simpler form,
\[ \frac{(H - 1)^2 + 4\gamma H}{2H} > \frac{H + 1}{2H(H^2 + 3\gamma H + 1)}((H - 1)^2 + 3\gamma H). \]
Notice that the right-hand side of this inequality is always negative for \( \gamma, H > 0 \), therefore this condition is always verified. \( \square \)

All this can be finally summarized in the next figure showing the signatures of \( \sigma \) for different Froude numbers regimes.

\[ \begin{array}{ccc}
0 & (+, +) & (-, -) \\
& F_1^- & F_1^+ \\
\end{array} \]

Figure 5: Signatures for the roots \( \sigma_1, \sigma_2 \).

We can conclude that in the interval \( ]0, F_1^- [ \) all the eigenvalues are purely imaginary. Physically, this corresponds to periodic waves for the system. In the second interval \( ]F_1^-, F_1^+ [ \) we have two real and two purely imaginary eigenvalues. We can expect in this case the existence of generalized solitary waves. Finally, in the interval \( ]F_1^+, +\infty[ \) we have four real eigenvalues, which corresponds to the regime of existence of classical elevation solitary waves. As already mentioned above, we will focus on the study of traveling waves in the second interval. It is, from the mathematical and physical point of view, the most interesting regime to consider.

4.5 Critical points of the potential \( V \) (a numerical investigation)

The critical points for the potential \( V \) given in (16) are defined as the solutions of \( \nabla V = 0 \), that we write equivalently as
\[ \begin{aligned}
\frac{1}{2} F_1^2 \left( \frac{1}{\gamma_1^2} - 1 \right) + (h_1 - 1) + \gamma H(h_2 - 1) &= 0, \\
\frac{1}{2} F_1^2 \left( \frac{1}{\gamma_2^2} - 1 \right) + (h_1 - 1) + H(h_2 - 1) &= 0. 
\end{aligned} \]

(18)

It represents a coupled system of algebraic equations for unknowns \( h_1, h_2 \) as a function of the Froude number (at fixed values \( \gamma \) and \( H \)). Note that \((1, 1)\) trivially satisfies this system. The other solutions cannot be determined explicitly, since it leads to a ninth degree polynomial on \( h_1 \) and \( h_2 \). Despite this difficulty in presenting analytically the critical points for \( V \), given fixed parameters \( \gamma, H \) and \( F_1 \), we can always determine numerically the complete set of solutions for the system.

Moreover is was confirmed numerically that the nature and number of the critical points only depend on the Froude number. To be precise, consider \( \gamma = 0.99, H = 1/6. \) Below (see Fig. 6), we see several ‘snapshots’ showing how the structure of the critical points for \( V \) evolves with an increasing Froude number.

For any Froude number in the interval \( ]0, F_1^- [ \) there are always four critical points: two saddle points, here denoted by \( S_e \) and \( S_a \); a local minimum \((1, 1)\) denoted by \( m \); a local maximum denoted by \( M \)

\[ \text{The used notation is meant to suggest the kind of wave regime considered: elevation or depression, respectively.} \]

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Figure 6: Critical points evolution for $\gamma = 0.99$, $H = 1/6$. From (a) to (h) it is considered the following sequence of values for $F_1^2$: $\{0.0001, 0.001, 0.002, 0.00314, 0.00315, 1.0, 1.6, 7.0\}$. The wave speeds $F_1^\pm$ are given approximately by $(F_1^-)^2 \approx 0.00143032$ and $(F_1^+)^2 \approx 1.16524$. 
(as in pictures (a), (b) of Fig. 6). Picture (b) shows that for increasing speeds close to \( F_1^- \), the point \( S_d \) approaches \( m \), eventually until coalescence. We see after (picture (c)) the position \((1, 1)\) being now “occupied” by \( S_d \), meaning that we have entered the second interval \([F_1^-, F_1^+]\). It is in this interval where the behaviour becomes more complex. There exists a critical Froude number \( F_1^C \) for which the number of critical points passes from four to only two. The pictures (c), (d) of Fig. 6 show the point \( m \) directing to \( S_e \) until coalescence at \( F_1 = F_1^C \). As a result of this coalescence, only two critical points remain (picture (e)). Point \( M \) starts now directing to \( S_d \) and it gets closer and closer as the Froude number goes towards \( F_1^+ \). After coalescence at \( F_1^+ \) we enter the third interval \([F_1^+, +\infty[\) where we can see that \( M \) has taken the position \((1, 1)\) and it is now the point \( S_d \) that starts to move far from \( M \) (pictures (g), (h) of Fig. 6). We conclude that if we set

\[
i(p) = \begin{cases} 
+1 & \text{if } p \text{ is a local maximum or a local minimum}, \\
-1 & \text{if } p \text{ is a saddle point},
\end{cases}
\]

whenever \( p \) is a non-degenerate critical point of \( V \), then the sum of topological indexes doesn’t change over the real line \( \mathbb{R} \) and it is always zero.

Still conserving the value \( \gamma = 0.99 \), our numerical experiments have shown that the behaviour held for \( H = 1/6 \) holds as well for any \( H < 1 \). On the contrary, if \( H \gg 1 \), a different scenario arises (see Fig. 7). The pictures show that a different type of coalescence holds for \( H = 2 \). We see now the coalescence of \( S_e \) and \( m \) whenever we approach \( F_1^- \), instead of \( S_d \) and \( m \), as previously seen for \( H < 1 \). Already in the interval \([F_1^-, F_1^+]\) we see now the point \( m \) directing to \( S_d \) eventually until coalescence at \( F_1 = F_1^C \). This kind of behaviour was registered for any \( H \gg 1 \). Furthermore, given fixed \( \gamma \) and \( H \) we have always confirmed that there are only two possible scenarios and they can be resumed to the ones we have just described. The physical consequences of this are the following: interfacial waves of depression for \( H < 1 \) and interfacial waves of elevation for \( H \gg 1 \). The prescribed value of \( H \) is the responsible for this change in the structure of the potential. This suggests the existence of a critical depth ratio \( H^C \), dividing the regimes of elevation and depression. An analogous notion was already introduced by several authors. As reported by Walker in [23], this critical depth ratio (that we will denote here by \( H_C \)) is very sensitive to the upper boundary condition, with \( H_C < 1 \) for a rigid upper surface (lid) and \( H_C > 1 \) for a free surface. Analytical expressions for such critical depth ratios were found by Long [24] and Peters & Stoker [25] for rigid surface and free surface, respectively. In the case when the fluid is bounded above by a free surface, the classification of the regimes of interfacial waves are all consistent in the available literature (see [9],[15],[17],[23],[25]). The critical depth ratio \( H_C \) is given by the only real root of the third degree polynomial on \( H \)

\[
H^3 - (3 - 3\gamma - \gamma^2)H^2 + (3 - 4\gamma)H - 1.
\]

We do not know, at present, if this root \( H_C \) corresponds exactly to the value \( H^C \) which appears in our study. We believe that there is a good agreement between both values. For the parameters used in our computed numerical solutions, \( \gamma = 0.99 \) and \( \gamma = 0.10 \), it was verified that \( 1.00 < H^C < 1.05 \) and \( 1.22 < H^C < 1.27 \), respectively. In turn, for these same parameters we have \( H_C = 1.00251 \) and \( H_C = 1.25 \), respectively. We can also refer that \( 1 < H_C < 1.25 \) and it was found for our model that \( H^C \) is close to 1, but further analysis would be required to be more precise. Despite the importance of this problem, it will not be considered here. We hope to address this issue in the near future.

In conclusion, for fixed \( \gamma \) and \( H \) we see that by increasing the Froude number we can pass from four critical points for \( V \) (in the interval \([0, F_1^-]\)) to only two. The passage from one situation to another happens when we approach \( F_1^C \) for which it was found that \( F_1^- < F_1^C < F_1^+ \). Numerical computations
Figure 7: Critical points evolution for $\gamma = 0.99, H = 2$. From (a) to (h) it is considered the following sequence of values for $F_1^2$: \{0.0001, 0.006, 0.0071, 0.007615, 0.0077, 2.0, 4.0, 10.0\}. The wave speeds $F_1^\pm$ are given approximately by $(F_1^-)^2 \approx 0.00668155$ and $(F_1^+)^2 \approx 2.99332$. 
have shown the existence of true solitary waves in the interval $|F_{1}^{-}, F_{1}^{C}|$. Two types of solutions were found corresponding to the two different types of coalescence observed whenever we approach $F_{1}^{-}$:

\[ \text{Coalescence of } S_d \text{ and } m \implies \text{interfacial wave of depression,} \]

\[ \text{Coalescence of } S_e \text{ and } m \implies \text{interfacial wave of elevation.} \]

We have also seen that what determined the type of coalescence, for fixed parameters $\gamma$ and $H$, is the existence of a critical depth ratio $H^C$ for which the structure of the potential changes. We have seen that $H < H^C$ corresponds to (19) and that $H > H^C$ corresponds to (20). We therefore have the following:

\[ H < H^C \implies \text{interfacial wave of depression,} \]
\[ H > H^C \implies \text{interfacial wave of elevation.} \]

4.6 Homoclinic orbits

We are interested in studying the existence of homoclinic orbits for our problem. They correspond to the true solitary waves of the system. As seen above, we have to restrain the Froude number $F_1$ to the set $|F_{1}^{-}, F_{1}^{+}|$. In this interval the linearized system has two real and two purely imaginary eigenvalues.

We will focus on determining conditions to leave from a small neighbourhood of the saddle point $(1, 1)$ of the potential $V$, remaining as close as possible from the true homoclinic orbit. This will be done by prescribing special initial conditions $h(0)$ and $h'(0)$ for the nonlinear system of ODEs (13).

For that we need to calculate the positive eigenvalue and the associated eigenvector. We saw that the eigenvalues for the matrix of the linearized system correspond to the solutions of

\[ \det(B + \lambda^2 A) = 0. \]

We have introduced $\sigma = -\delta^2 \lambda^2$ and wrote this condition in (17) as a quadratic equation for $\sigma$. Denoting by $\lambda_+$ the positive eigenvalue we write

\[ \lambda_+^2 = \frac{b + \sqrt{b^2 - 4ac}}{2a \delta^2}. \]

The associate eigenvector $v$ satisfies

\[ (B + \lambda_+^2 A)v = 0. \]

In order to stay as close as possible from the true solitary wave, consider a small parameter $\kappa$ with $|\kappa| \ll 1$. The initial conditions for (13) should be given by

\[ \begin{cases} h(0) = (1, 1) + \kappa v, \\ h'(0) = \kappa \lambda_+ v. \end{cases} \]

5 Numerical Solutions

In the former subsection we have outlined the main core of the numerical approach used to compute the solitary waves to our model. It all summarizes to considering a classical initial value problem for a system of ODEs. We used the 5th edition of MATHEMATICA to compute the solutions by means of the built-in function NDSolve.

This section is divided in two parts. The first part is devoted to the solutions obtained for oceanic conditions. By oceanic conditions we usually mean $\gamma$ close to 1. Earlier in this paper we have
addressed a particular choice of parameters $\gamma = 0.99$, $H = 1/6$. Indeed these parameters are close to the ones we find in real oceanic conditions such as in the northwestern subtropical North Atlantic with the representative values $H_1 = 3000 \, \text{m}$, $H_2 = 500 \, \text{m}$ and $\gamma = 0.998$ (see pag. 89 of [26]). However, hereafter we write oceanic conditions to refer to $\gamma = 0.99$. We shall present here the two regimes of elevation and depression for solitary waves. Also, it will be shown that as $F_1$ increases towards a limit value $F_1^L$, the solitary waves broaden and eventually become fronts. It is important to refer that no solitary waves were found beyond $F_1^L$. It is clear from the study made for the evolution of the critical points for $V$ that homoclinic orbits to $(1,1)$ cannot exist for $F_1 > F_1^C$, since this speed corresponds to the coalescence of a local minimum and a saddle point, passing this way from four to only two critical points. However, our numerical computations have shown that $F_1^L < F_1^C$ and it can be physically explained by imagining a “particle” moving along the surface of the potential. Then, $F_1^L$ corresponds to the limit speed from which the particle feels no longer the attraction of the local minimum $m$. We see the particle falling incessantly along the surface, hence no return back to the equilibrium is possible for greater speed. This seems to corroborate the conjecture proposed by Dias and Il’ichev, concerning the existence, for fixed values $\gamma$ and $H$, of a maximum speed for generalized solitary waves to the full problem (see Appendix A in [15]).

The second, focuses on more peculiar solutions that can be obtained for different sets of parameters. Single-humped and multi-humped solitary waves will be presented. The passage from single to multi-humped solutions was already observed by Michallet and Dias in a numerical study for the full problem [9]. We pretend, with this last part, to give a glimpse of the enormous complexity and richness of this Hamiltonian system with two degrees of freedom.

5.1 Oceanic conditions

Figures 8 and 9 show computed solutions for different values of $H$ and $F_1$. We have graphically represented the profiles for the interfacial and free-surface wave for the set of parameters $\gamma = 0.99$, $H = 1/6$ and $\gamma = 0.99$, $H = 2$. We illustrate the two possible wave regimes of elevation and depression.

![Figure 8: Interfacial wave of depression for $H = 1/6$.](image)

We see here that we have considered the value $\delta^2 = 10^{-2}$. In fact, this is not a real parameter from which our solutions depend, since it is just a scale parameter. It is always possible to set its value to 1, choosing for that a convenient scale $L$. The reason for which we did not do so is related to the graphical representation of solutions. We draw attention to the fact that the plotted solutions are shown in dimensionless variables $(h_1/H_1, h_2/H_1)$ (despite having used $h_1$ and $h_2$ in axes labels). To recover the original variables just multiply the result by $H_1$.

With the same set of values for $H$ from the examples above, we let increase $F_1$ towards a limit value
Figure 9: Interfacial wave of elevation for $H = 2$.

$F_1^L$. The values chosen for the following solutions are already very close to the maximum speed, for which we have confirmed the existence of solitary waves. The broadening of these waves becomes evident in these pictures (see Fig. 10 and Fig. 12).

Figure 10: Table-top wave of depression for $H = 1/6$.

We also present how the solutions evolve with an increasing Froude number until attaining the table-top wave configuration (see Fig. 11).

If we look to the phase portraits of these solutions, we see how it deforms with increasing Froude number, at fixed $\gamma$ and $H$ (see Fig. 13). At the non-equilibrium point where the curve crosses the axis, the graph becomes a lot less round with increasing values of $F_1$.

Before moving to other sets of parameters, an important observation should be pointed here. As remarked in [15], we see that the deformations of the free surface are much smaller than the deformations of the interface.

5.2 Single and multi-humped solitons

Like previously done for $\gamma = 0.99$, we will set now $\gamma = 0.10$ and present some solutions illustrating both wave regimes. We start with $H = 6/10$ where the interfacial wave is of depression and then move
Figure 11: A sequence of solutions for the interfacial wave with fixed $H = 1/6$.

Figure 12: Table-top wave of elevation for $H = 2$.

Figure 13: The deformation of the phase portrait when attained a table-top configuration for $H = 1/6$. 
to $H = 7$, where the interfacial wave is of elevation. We will see that there are significant discrepancies between the profiles for these waves and the ones obtained for oceanic conditions. For $\gamma$ closer to 0, the complexity of the computed solutions is enormous, comparing to what was seen earlier for $\gamma = 0.99$. A look over the portrait phase for the solutions will help to emphasize this complexity. Also, it was substantially more difficult to find these special waves for $\gamma = 0.10$. It seems that, for this $\gamma$, the set of Froude numbers producing true solitary is milder than the one corresponding to oceanic conditions. The surface of the potential becomes flattened in a neighbourhood of $(1, 1)$ for $\gamma$ close to 0. Imagining a particle moving along the surface of $V$ we can state that, in this case, the movement is affected by the slightest change of the curvature of the surface. If we project on the plane of variables $(h_1, h_2)$, the trajectory of the particle moving along the surface, we would see that single-humped solutions correspond to almost straight lines. By contrast, multi-humped solutions seem to generate much more complex behaviour.

**Interfacial wave of depression**

Here, at fixed $\gamma = 0.10$, $H = 6/10$, we present a set of solutions obtained for an increasing sequence of values $F_1$ (see Figures 14, 15, 16). We pass from the classical configuration when the central core of the surface wave is given by a single hump to a multi-humped configuration. During this process, the interfacial wave keeps the shape of a single hump. We can pass from one to two and successively to ten-peak solitary wave (here graphically represented). This last, corresponds to a very close value to the limit speed for the existence of such waves. This kind of solutions reminds those presented by Michallet and Dias [9] for the full problem, using the parameters $\gamma = 0.10$, $H = 3/7$.

![Figure 14: Single-humped solitary wave for $H = 6/10$.](image)

**Interfacial wave of elevation**

Here, at fixed $\gamma = 0.10$, $H = 7$, we present, like above, a sequence of solutions obtained for increasing Froude numbers. The multi-hump shaped profile is now shared both by the free surface and the interfacial wave and the passage from one configuration to another, with greater speed, is a lot less predictable than before.
Figure 15: Three-humped solitary wave for $H = 6/10$. 

Figure 16: Ten-humped solitary wave for $H = 6/10$. 

$F^2 = \frac{599}{10000}$

$\beta = 10^{-2}$

$\kappa = 10^{-6}$
Figure 17: Single-humped solitary wave for $H = 7$.

Figure 18: Two-humped solitary wave for $H = 7$. 

$F_i^2 = \frac{95}{10000}$
$\delta^2 = 10^{-2}$
$\kappa = -10^{-4}$
Figure 19: Five-humped solitary wave for $H = 7$.

Figure 20: Broadened multi-humped solitary wave for $H = 7$. 
6 Conclusion

We have presented some stationary solutions for the new dispersive model derived in Part I of this work. The model describes long wave propagation in two-layer flows with free surface and it preserves the variational structure of the full water-wave problem. It is worth remembering that the model deals with rotational flows and no smallness assumption on the wave amplitude is made in its derivation.

It was shown, for the model, the possibility of resonance between short waves and long waves. The resulting wave is a generalized solitary wave, characterized by ripples in the far field in addition to the solitary pulse. We have focused in this work on special members of this family known as embedded solitary waves. They appear as true solitary waves embedded into the continuous spectrum and, mathematically, they correspond to the existence of homoclinic orbits at a saddle-centre equilibrium.

By using numerical computations it was shown the existence of such special solutions to our model. Two wave regimes of elevation and depression were presented. We have seen how the critical depth ratio separating both regimes relates to a change of the global properties of the potential of the reduced Hamiltonian system for traveling waves. Characteristic phenomena of highly nonlinear solitary waves such as broadening and multi-hump shaped profiles were shown to be reproduced by the present model.

Some important issues deserve to be addressed in future work. Among them, there is the problem of providing a relationship between the critical depth ratio $H^C$, which arises in the model, and the one that is found in the literature ($H_{cr}$). Also, it would be important to provide an analytical expression for the limit speed solitary waves can attain.

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Appendix A. Direct approach

We recall the system (3):

$$(\rho_\alpha)_t + (\rho_\alpha u_\alpha)_x = 0,
\left( u_\alpha - \frac{1}{\rho_\alpha} \delta W \right)_t + \left( \frac{1}{2} u_\alpha^2 + \frac{\delta W}{\delta \rho_\alpha} - u_\alpha \left( \frac{1}{\rho_\alpha} \delta W \right)_x \right) = 0.$$

We look for particular solutions $(\rho_1(\xi), \rho_2(\xi), u_1(\xi), u_2(\xi))$ where $\xi = x - Dt$. From the mass conservation laws for each layer, we get

$$\rho_\alpha (u_\alpha - D) = \text{const} \equiv m_\alpha, \quad \alpha = 1, 2,$$

as earlier in (9). From the second equation in (3), it follows

$$-D \left( u_\alpha - \frac{1}{\rho_\alpha} \delta W \right)' + \left( \frac{1}{2} u_\alpha^2 + \frac{\delta W}{\delta \rho_\alpha} - u_\alpha \left( \frac{1}{\rho_\alpha} \delta W \right) \right)' = 0,$$

or equivalently

$$\left( \frac{1}{2} u_\alpha^2 - D u_\alpha - \frac{1}{\rho_\alpha} \delta W (u_\alpha - D) + \frac{\delta W}{\delta \rho_\alpha} \right)' = 0.$$
We present a useful algebraic relation. The equations (9) imply
\[ u_\alpha^2 - 2D u_\alpha + D^2 = \frac{m_\alpha}{\rho_\alpha^2}. \]
This relation enables us to rewrite the equation above as
\[ \left( \frac{1}{2} \frac{m_\alpha}{\rho_\alpha^2} - \frac{1}{2} D^2 - \frac{1}{\rho_\alpha} \left( \frac{\delta W}{\delta u_\alpha} \right) \right) \frac{m_\alpha}{\rho_\alpha} + \frac{\delta W}{\delta \rho_\alpha} \right) = 0, \]
from which follows
\[ \frac{1}{2} \frac{m_\alpha}{\rho_\alpha^2} - \frac{m_\alpha}{\rho_\alpha^2} \left( \frac{\delta W}{\delta u_\alpha} \right) + \frac{\delta W}{\delta \rho_\alpha} = \text{const} \left( \frac{\delta W}{\delta \rho_\alpha} \right), \quad \alpha = 1, 2. \] (21)
The reader should not forget that these variational derivatives were originally defined for a potential \( W \) depending on \( \rho_1, \rho_2 \) and all their material derivatives. To interpret this last relation we have to rewrite everything in terms of the variable \( \xi \). Notice that, with respect to this new variable, the material derivatives yield
\[ \frac{D_\alpha f}{D t} = f'(u_\alpha - D) = \frac{f'}{\rho_\alpha} m_\alpha, \]
as in (10). The potential \( W \) is now to be seen as
\[ W = W \left( \rho_1, \rho_2, \frac{\rho_1'}{\rho_1}, \frac{\rho_2'}{\rho_2}, \frac{\rho_1'}{\rho_1} m_1, \frac{\rho_2'}{\rho_2} m_2, \frac{\rho_1'}{\rho_1} m_1, \frac{\rho_2'}{\rho_2} m_2 \right). \]
This suggests the introduction of a new potential \( U \), depending only on the four variables \( \rho_1, \rho_2, \rho_1' \) and \( \rho_2' \)
\[ U(\rho_1, \rho_2, \rho_1', \rho_2') = W \left( \rho_1, \rho_2, \frac{\rho_1'}{\rho_1}, \frac{\rho_2'}{\rho_2}, \frac{\rho_1'}{\rho_1} m_1, \frac{\rho_2'}{\rho_2} m_2, \frac{\rho_1'}{\rho_1} m_1, \frac{\rho_2'}{\rho_2} m_2 \right). \]
We write the variational derivatives with respect to \( \xi \):
\[ \frac{\delta W}{\delta \rho_\alpha} = \frac{\partial W}{\partial \rho_\alpha} - \frac{\rho_\alpha}{\rho_\alpha} \frac{m_\alpha}{\rho_\alpha} \left( \frac{1}{\rho_\alpha} \frac{\partial W}{\partial \left( \frac{\rho_\alpha}{\rho_\alpha} m_\alpha \right)} \right)' - \frac{m_\beta}{\rho_\beta} \left( \frac{1}{\rho_\beta} \frac{\partial W}{\partial \left( \frac{\rho_\alpha}{\rho_\alpha} m_\beta \right)} \right)' \]
\[ = \frac{\partial W}{\partial \rho_\alpha} - \left( \frac{m_\alpha}{\rho_\alpha} \frac{\partial W}{\partial \left( \frac{\rho_\alpha}{\rho_\alpha} m_\alpha \right)} + \frac{m_\beta}{\rho_\beta} \frac{\partial W}{\partial \left( \frac{\rho_\alpha}{\rho_\alpha} m_\beta \right)} \right)' \]
\[ = W_{\rho_\alpha} - \frac{d}{d\xi} (U'_{\rho_\alpha}). \]
Similarly, we have
\[ \frac{\delta W}{\delta u_\alpha} = \frac{\partial W}{\partial \left( \frac{\rho_\alpha}{\rho_\alpha} m_\alpha \right)} \rho_\alpha' + \frac{\partial W}{\partial \left( \frac{\rho_\alpha}{\rho_\alpha} m_\alpha \right)} \rho_\beta', \]
from which follows
\[ \frac{m_\alpha}{\rho_\alpha} \frac{\delta W}{\delta u_\alpha} = \frac{1}{\rho_\alpha} \frac{\partial W}{\partial \left( \frac{\rho_\alpha}{\rho_\alpha} m_\alpha \right)} \frac{\rho_\alpha'}{\rho_\alpha} m_\alpha + \frac{1}{\rho_\alpha} \frac{\partial W}{\partial \left( \frac{\rho_\alpha}{\rho_\alpha} m_\alpha \right)} \frac{\rho_\beta'}{\rho_\alpha} m_\alpha \]
\[ = W_{\rho_\alpha} - U_{\rho_\alpha}. \]
Making the necessary replacements, we obtain from (21) the following:

$$\frac{1}{2} \frac{m^2}{\rho_0^2} + U_{\rho_0} - \frac{d}{d\zeta}(U_{\rho_0}) = c_0,$$

that we rewrite as

$$\left( \frac{1}{2} \frac{m^2}{\rho_0^2} - U_{\rho_0} + c_0 \right) - \frac{d}{d\zeta}(-U_{\rho_0}) = 0.$$

Hence, if we define

$$\mathcal{L} = \sum_{\alpha=1}^{2} \left( \frac{1}{2} \frac{m^2}{\rho_0^2} + c_0 \rho_0 \right) - U,$$

we recover the Euler-Lagrange equations

$$\mathcal{L}_{\rho_0} - \frac{d}{d\zeta}(\mathcal{L}_{\rho_0}) = 0, \quad \alpha = 1, 2,$$

given in (12).

References


Conclusions
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This thesis focuses on the study of gravity waves in two-layer flows with free surface. Two distinct classes of models have been contemplated. We started by considering the “two-layer” version of the shallow water equations (also called the Saint-Venant equations). We have proved that in the one-dimensional case, all conservation laws can be obtained by linear combination of the conservation laws for mass and velocity (in each layer), total momentum and total energy. The number of conservation laws for the multi-dimensional case remains unknown, though we can reasonably expect that all the conservation laws for this model are those we know already. This follows from the following heuristic argument: in general, whenever we increase the dimension of the space considered, we increase as well the number of constraints for the problem. This means that for the multi-dimensional case, the system has, at most, the conserved quantities exhibited for the one-dimensional case. In particular, the conservation laws for the velocities can only be established if the flow is potential in each layer.

As expected, this model is not suitable for the description of solitary waves propagating in two-layer flows. We have derived a dispersive model for the description of large amplitude waves propagating in the same physical system. The model is a “two-layer” generalization of the Green-Naghdi model. It is a fully nonlinear model that can deal with rotational flows. In addition, it preserves the variational structure of the original full water-wave problem. This feature enables us to establish some main properties for this model. We have presented conservation laws for the total momentum and total energy. We have defined a notion of generalized vorticity and derived analogues of integrals of motion, such as Bernoulli integrals.

We have presented a comparison between the dispersion relation for our model and the one given for the full problem. By considering different sets of parameters it was revealed that a particular good agreement holds for oceanic conditions.

We have shown that the equations for traveling waves yield a Hamiltonian system with two degrees of freedom. By means of numerical computations we have proved the existence of true solitary waves for the stationary system. In particular, we have perceived that these waves seem to exist only for a discrete set of wave speeds; two wave regimes, characterized by elevation or depression of the interface were presented; the broadening phenomenon was verified whenever the wave speed increases towards a limit value; multi-humped solutions were found for certain sets of parameters.

As with all research, there remain many open problems and this thesis is very much ‘work still in progress’. We shall now mention some remaining issues for the problems arising from this thesis.

1. Providing a relationship between the critical depth ratio, which arises in the model, and the one that is found in the literature. Also, it would be important to provide an analytical expression for the limit speed solitary waves can attain.
2. The existence of true solitary waves embedded into the continuous spectrum was revealed by numerical computations, but an analytical proof of their existence is still unknown.

3. The question of stability of solitary waves and the analytical proof that these can only exist for special values of the Froude number deserve to be addressed in future work.

4. The Hamiltonian system given for traveling waves is most probably a non-integrable system. If confirmed this conjecture about the discrete set of Froude numbers for which exists solitary waves, is there a possibility for some of these values to present two first integrals and, consequently, an analytical expression for such solitary wave?

5. For real physical applications it would be essential to develop analytical and numerical studies of the non-stationary problem.