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Counterexample of loss of regularity for fractional order evolution equations with both degenerating and oscillating coefficients

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ABSTRACT

For weak evolution models of fractional order with singularity near the origin, the joint influence from the principal σ-Laplacian operator, degenerating part and oscillating part is of prime concern in the discussion of regularity behavior of the solutions. We apply the techniques from the micro-local analysis to explore the upper bound of loss of regularity. Furthermore, in order to demonstrate the optimality of the estimates, a delicate counterexample with periodic coefficients will be constructed to show the lower bound of loss of regularity by the application of harmonic analysis and instability arguments. This optimality discussion develops the theory in Cicognani and Colombini (2006), Cicognani et al. (2008), Lu and Reissig (2009) and Lu and Reissig (2009) by combining both oscillation and degeneracy of the coefficients.

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1. Introduction

Pseudodifferential operators, especially fractional order operators (also called Riesz fractional derivatives) are very important mathematical models which describe plenty of anomalous dynamic behaviors in our daily life, such as charge carrier transport in amorphous semiconductors, nuclear magnetic resonance diffusometry in percolative and porous media, transport on fractal geometries, diffusion of a scalar tracer in an array of convection rolls, dynamics of a bead in a polymeric network, transport in viscoelastic materials, etc.

Generally speaking, loss of regularity, or loss of derivatives, is a phenomenon in which the solution loses certain regularity in the sense of Sobolev spaces compared with initial Cauchy data. This is a very important aspect of the well-posedness research [1]. First we recall the regularity behavior of the evolutional operator on $[0, T] \times \mathbb{R}$, $\mathscr{L} = \partial_t^2 - \lambda^2(t)\partial_x^2$, where $\lambda(t)$ is the measure function of degeneracy, which is defined as follows.

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- Let $\Lambda(t) \triangleq \int_0^t \lambda(\tau)d\tau$, a measure function of degeneracy $\lambda(t) \in C^2$ is a positive function satisfying:

$$\lambda(0) = 0, \qquad \lambda'(t) > 0, \qquad \frac{\lambda'(t)}{\lambda(t)} \sim \frac{\lambda(t)}{\Lambda(t)}, \qquad |\lambda''(t)| \lesssim \lambda(t)\left(\frac{\lambda(t)}{\Lambda(t)}\right)^2.$$

It is worth noticing that the above model precisely generalizes the weakly hyperbolic operators with infinitely/finitely degenerating coefficients [2,3]. By applying the diagonalization techniques introduced in [4] while considering the propagation of mild singularities for semi-linear weakly hyperbolic equations, we know the fact that there exists no loss of regularity for this kind of operator.

In order to consider the impact of oscillation on the regularity behavior from the principal elliptic operator, we introduced in [5,6] a brand-new weakly hyperbolic operator with both oscillating and degenerating coefficients on $[0, T] \times \mathbb{R}$: $\mathscr{L} = \partial_t^2 - \lambda^2(t)b^2(t)\partial_x^2$, where $\lambda(t)$ is the measure function of degeneracy, $b(t) \in C^2(0, T]$ describes the oscillation of the principal elliptic operator near the origin 0. In physics, $\lambda(t)$ usually describes the degenerating part of the density and $b(t)$ often describes the oscillating part of the density. More precisely,

- $b_0 \triangleq \inf_{t\in(0,T]} b(t) \le b(t) \le b_1 \triangleq \sup_{t\in(0,T]} b(t),\ b_0, b_1 > 0$;
- $|b^{(k)}(t)| \le C\left(\frac{\lambda(t)}{\Lambda(t)}\nu(t)\right)^k$, $C > 0, k = 1, 2$, where $\nu(t)$ is a measure function of oscillation, which is defined as a continuous and decreasing positive function on a finite time interval.

By two steps of diagonalization procedure, we have an insightful understanding of the impact from the oscillating coefficients. Detailed description of both the loss of regularity and difference of regularity of the initial Cauchy data is given in [6]. In the previous literature [4,6–11], one mainly discussed the wave equation with lower order terms. As a matter of fact, the finite propagation speed holds for this kind of operator. In reality, most of the operators have infinite propagation speed, such as heat equation $u_t - \Delta u = 0$, Petrowsky equation $u_{tt} + \Delta^2 u = 0$, etc. In this manuscript, the weak evolution operator of fractional order with oscillating and degenerating coefficients, which has infinite propagation speed, is of prime concern:

$$\mathscr{L} = \partial_t^2 + A_0(t, \sqrt{-\Delta}), \tag{1}$$

where

$$A_0(t, \sqrt{-\Delta}) \triangleq \lambda^2(t)b^2(t)(-\Delta)^\sigma,$$

with $\sigma > 1$ and $(-\Delta)^\sigma$ defined on the torus $\mathbb{T}^N$. In this model, we call $A_0(t, \sqrt{-\Delta})$ the principal part in the sense of Petrowsky. One typical example of the coefficients on the principal part is $b(t) = 2 + \sin\big((\log(1/t))^\kappa\big)$, $\kappa \in (1, \infty)$, which satisfies the assumptions with $\nu(t) = (\log(1/t))^{\kappa-1}$. Up until now, there is still no complete conclusion about the Levi-condition with oscillation [8]. However, the theory of pseudodifferential operators assures the existence and uniqueness of the solution for the Cauchy problem of (1). As an important application of this model, actually, through Nirenberg's transformation $v = 1 - \exp(-u)$, the problem of the semi-linear Cauchy problem $u_{tt} - a^2(t)\Delta u = u_t^2 - a^2(t)|\nabla u|^2$ can be turned into the linear problem $v_{tt} - a^2(t)\Delta v = 0$. For more discussion in this respect please refer to [12]. In the following, we explore carefully the joint influence upon the regularity behavior of (1) from both oscillation and degeneracy of the principal elliptic operator $A_0(t, \sqrt{-\Delta})$.

Under the above assumptions of $\lambda(t)$ and $b(t)$, one has the following regularity statement.

Theorem 1.1. *Let us consider the Cauchy problem of model* (1) *on* $[0, T] \times \mathbb{T}^N$,

$$\mathscr{L}u = 0, \qquad u(0, x) = u_0(x), \qquad u_t(0, x) = u_1(x). \tag{2}$$

If the initial Cauchy data satisfy

$$u_0 \in H^s(\mathbb{T}^N), \qquad u_1 \in \frac{1}{\Lambda^{-1}\left(\frac{2^{P_1}}{(\sqrt{1-\Delta})^\sigma}\right)} H^s(\mathbb{T}^N),$$

where $P_1 \in \mathbb{N}_+$ is a fixed constant and the Sobolev index s is sufficiently large, then there exists a unique solution u in the following function spaces:

$$u \in C\left([0, T], \exp\left(C_\alpha \nu\left(\left(\frac{\Lambda}{\nu}\right)^{-1}\left(\frac{2^{P_2}}{(\sqrt{1-\Delta})^\sigma}\right)\right)\right) H^s(\mathbb{T}^N)\right),$$

$$u_t \in C\left([0, T], \exp\left(C_\alpha \nu\left(\left(\frac{\Lambda}{\nu}\right)^{-1}\left(\frac{2^{P_2}}{(\sqrt{1-\Delta})^\sigma}\right)\right)\right) H^{s-\sigma}(\mathbb{T}^N)\right);$$

where $C_\alpha \in \mathbb{R}_+$ and $P_2 \in \mathbb{N}_+$ are fixed constants. In this theorem, Λ^{-1} and $\left(\frac{\Lambda}{\nu}\right)^{-1}$ denote respectively the corresponding inverse functions. In fact, according to the monotonicity of $\lambda(t)$ and $\nu(t)$, both inverse functions are well-defined.

Similar as in [6], the above theorem can be proved by applying the comparison lemma and two steps of diagonalization procedure. Interested readers please refer to the doctoral theses [1,13] for a detailed proof. This is a completely new result which considers the coupling influence of both degenerating and oscillating densities for fractional order differential equations. In the following, we give some typical examples to explain the different influence from various kinds of degenerating and oscillating coefficients. First we consider the influence of degenerating coefficients for the Cauchy problem

$$u_{tt} + \lambda^2(t)(-\Delta)^\sigma u = 0, \qquad u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x). \tag{3}$$

According to Theorem 1.1, one has

- $\lambda(t) = \frac{1}{\log^{[n]}\frac{1}{t}}$, $n \geq 1$, (logarithmically degenerating coefficient)

$$(u_0, u_1) \in \left(H^s \times \frac{1}{\Lambda^{-1}\left(\frac{2^{P_1}}{(\sqrt{1-\Delta})^\sigma}\right)} H^s \right) \Rightarrow u \in C([0,T], H^s) \bigcap C^1([0,T], H^{s-\sigma}),$$

  where $\Lambda(t) = O(\frac{t}{\log^{[n]}\frac{1}{t}})$;
- $\lambda(t) = t^\ell$, $\ell \geq 1$, (finitely degenerating coefficient)

$$(u_0, u_1) \in \left(H^s \times H^{s-\frac{\sigma}{\ell+1}} \right) \Rightarrow u \in C([0,T], H^s) \bigcap C^1([0,T], H^{s-\sigma});$$

- $\lambda(t) = \frac{1}{t^2}\exp(-\frac{1}{t})$, (infinitely degenerating coefficient)

$$(u_0, u_1) \in \left(H^s \times \sigma \log(\sqrt{1-\Delta}) H^s \right) \Rightarrow u \in C([0,T], H^s) \bigcap C^1([0,T], H^{s-\sigma});$$

- $\lambda(t) = \frac{d}{dt}\exp\left(-\exp^{[n]}\left(\frac{1}{t}\right)\right)$, $n \geq 1$, (super infinitely degenerating coefficient)

$$(u_0, u_1) \in \left(H^s \times \log^{[n]+1}(\sqrt{1-\Delta})^\sigma H^s \right) \Rightarrow u \in C([0,T], H^s) \bigcap C^1([0,T], H^{s-\sigma}).$$

Next we show the influence of oscillating coefficients for the Cauchy problem

$$u_{tt} + \lambda^2(t) b^2(t)(-\Delta)^\sigma u = 0, \qquad u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x).$$

Suppose that the initial Cauchy data satisfy

$$(u_0, u_1) \in \left(H^s \times \frac{1}{\Lambda^{-1}\left(\frac{2^{P_1}}{(\sqrt{1-\Delta})^\sigma}\right)} H^s \right),$$

then according to Theorem 1.1, we have: ($C_\alpha \in \mathbb{R}_+$)

- No loss of derivatives, $\nu(t) \sim 1$,

$$u \in C([0,T], H^s), \qquad u_t \in C([0,T], H^{s-\sigma});$$

- Finite loss of derivatives, $\nu(t) = \log\frac{1}{\Lambda(t)}$,

$$u \in C([0,T], (\sqrt{1-\Delta})^{\sigma C_\alpha} H^s), \qquad u_t \in C([0,T], (\sqrt{1-\Delta})^{\sigma C_\alpha} H^{s-\sigma});$$

- Arbitrarily small loss of derivatives, $\nu(t) = \left(\log\frac{1}{\Lambda(t)}\right)^\gamma$, $\gamma \in (0,1)$,

$$u \in C\left([0,T], (\sqrt{1-\Delta})^{\sigma C_\alpha \left(\log\frac{(\sqrt{1-\Delta})^\sigma}{2^{P_2}}\right)^{\gamma-1}} H^s \right),$$

$$u_t \in C\left([0,T], (\sqrt{1-\Delta})^{\sigma C_\alpha \left(\log\frac{(\sqrt{1-\Delta})^\sigma}{2^{P_2}}\right)^{\gamma-1}} H^{s-\sigma} \right);$$

- Infinite loss of derivatives,

$$\nu(t) = \left(\log\frac{1}{\Lambda(t)}\right)\left(\log^{[2]}\frac{1}{\Lambda(t)}\right)^{\gamma_2} \cdots \left(\log^{[n]}\frac{1}{\Lambda(t)}\right)^{\gamma_n}, \quad \gamma_i \in (0,1],\ i = 2, \dots, n$$

$$u \in C\left([0,T], (\sqrt{1-\Delta})^{\sigma C_\alpha \left(\log^{[2]}\frac{(\sqrt{1-\Delta})^\sigma}{2^{P_2}}\right)^{\gamma_2} \cdots \left(\log^{[n]}\frac{(\sqrt{1-\Delta})^\sigma}{2^{P_2}}\right)^{\gamma_n}} H^s \right),$$

$$u_t \in C\left([0,T], (\sqrt{1-\Delta})^{\sigma C_\alpha \left(\log^{[2]}\frac{(\sqrt{1-\Delta})^\sigma}{2^{P_2}}\right)^{\gamma_2} \cdots \left(\log^{[n]}\frac{(\sqrt{1-\Delta})^\sigma}{2^{P_2}}\right)^{\gamma_n}} H^{s-\sigma} \right).$$

We call all the above losses incurred by the oscillation measure function ν-loss of derivatives. In the following, we construct a delicate counterexample to demonstrate the existence of ν-loss when the infinite propagation speed appears. For related topics please refer to [7,9,10,14–17].

Let us consider the following Cauchy problem on $[0,T]\times\mathbb{T}$

$$\partial_t^2 u+\lambda^2(t)b^2(t)(-\partial_x^2)^\sigma u=0,\qquad u(0,x)=u_0(x),\qquad \partial_t u(0,x)=u_1(x), \tag{4}$$

with 2π-periodic initial Cauchy data u_0, u_1. Actually, on the torus $\mathbb{T}^N$, one can apply the same procedure.

Definition 1.2. For a 2π-periodic solution $u=u(t,x)$ in the x-variable, we introduce the homogeneous energy

$$\dot{\mathbb{E}}_s(u)(t)\triangleq\lambda^2(t)\|u(t,\cdot)\|^2_{\dot H^s(\mathbb{T})}+\|\partial_t u(t,\cdot)\|^2_{\dot H^{s-\sigma}(\mathbb{T})},\quad s>\sigma, \tag{5}$$

where $\dot H^s(\mathbb{T})$ denotes the homogeneous Sobolev space of exponent s on the torus $\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}$.

Actually, Theorem 1.1 indicates the following conclusion which shows *at most* a ν-loss.

Corollary 1.3. *Let us consider a family of Cauchy problems on* $[0,T]\times\mathbb{T}$,

$$\partial_t^2 u^k+\lambda^2(t)b_k^2(t)(-\partial_x^2)^\sigma u^k=0,\qquad u^k(0,x)=u_0^k(x),\qquad \partial_t u^k(0)=u_1^k(x). \tag{6}$$

Define $\mu(s)$ *as* $\Lambda(s)/\nu(s)$ *and* $\{b_k\}_k$ *satisfy all the assumptions in Theorem 1.1 with constants independent of* k. *If the initial Cauchy data satisfy*

$$u_0^k\in H^s(\mathbb{T}),\qquad u_1^k\in\frac{1}{\Lambda^{-1}\left(\frac{2^{P_1}}{(\sqrt{1-\Delta})^\sigma}\right)}H^s(\mathbb{T}),$$

where $P_1\in\mathbb{N}_+$ *is a fixed constant, then there exists a unique solution* u^k *in the following function spaces:*

$$u^k\in C\Big([0,T];\exp\Big(C_\alpha\nu(\mu^{-1}(2^{P_2}/(\sqrt{1-\Delta})^\sigma))\Big)H^s(\mathbb{T})\Big),$$
$$u_t^k\in C\Big([0,T];\exp\Big(C_\alpha\nu(\mu^{-1}(2^{P_2}/(\sqrt{1-\Delta})^\sigma))\Big)H^{s-\sigma}(\mathbb{T})\Big),$$

with positive constants C_α *and* $P_2\in\mathbb{N}_+$ *is a fixed constant. Moreover,* μ^{-1} *denotes the inverse function of* μ.

Now we are ready to introduce the main theorem of this paper, which shows that the ν-loss of derivatives really appears. For the sake of convenience, we only prove the case for homogeneous energy. Similarly, one can define the non-homogeneous energy as

$$\mathbb{E}_s(u)(t)\triangleq\lambda^2(t)\|u(t,\cdot)\|^2_{H^s(\mathbb{T})}+\|\partial_t u(t,\cdot)\|^2_{H^{s-\sigma}(\mathbb{T})},\quad s>\sigma, \tag{7}$$

where $H^s(\mathbb{T})$ denotes the standard Sobolev space of exponent s on the torus $\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}$, and the result also holds for the inhomogeneous energy case. This will be explained during the proof.

Theorem 1.4. *For the Cauchy problem (4), there exists*

- *a sequence of coefficients* $\{b_k(t)\}_k$ *satisfying all assumptions of Theorem 1.1 with constants independent of* k;
- *a sequence of initial Cauchy data* $\{(u_0^k(x),u_1^k(x))\}_k\in\dot H^s(\mathbb{T})\times\frac{1}{\Lambda^{-1}\left(\frac{2^{P_1}}{(\sqrt{1-\Delta})^\sigma}\right)}\dot H^s(\mathbb{T})$;

such that the sequence of corresponding solutions $\{u^k(t,x)\}_k$ *from* $C^\infty([0,T]\times\mathbb{T})$ *satisfies*

$$\sup_k\dot{\mathbb{E}}_\sigma(u^k)(0)\le C(\varepsilon), \tag{8}$$

$$\sup_k\dot{\mathbb{E}}_\sigma(\exp(-c_1(\varepsilon)\nu(\mu^{-1}(2^{P_2}/(\sqrt{1-\Delta})^\sigma)))u^k)(t)=+\infty,\quad \text{for any } t\in(0,T], \tag{9}$$

where $C(\varepsilon)$ *and* $c_1(\varepsilon)$ *depend on the sufficiently small positive constant* ε.

The rest of the paper is organized as follows. Section 2 is a brief introduction of pseudodifferential operators on the torus. In Section 3, we discuss the optimality of the statement for general $\sigma>1$ on the torus $\mathbb{T}$ by the application of instability argument, which develops the discussion in [18] by adding degenerating coefficients. This is an important complement of the discussion in [19] for $\sigma\in(0,1]$, in which case, the finite propagation speed holds.

2. Prerequisite: a brief review of PDOs on the torus

In this section we mainly recall the pseudodifferential operators (PDOs) defined on the torus $\mathbb{T}^N$, which is a typical compact smooth manifold without boundary. The discussion can be applied to general Laplace–Beltrami operators successfully, such as on the sphere $\mathbb{S}^N$, etc. First and foremost, we give the precise definition of PDOs on the torus.

Definition 2.1. On the torus $\mathbb{T}^N$, let $u \in C^\infty(\mathbb{T}^N)$, then the sequence of Fourier series $\{\hat{u}(m)\}_{m\in\mathbb{Z}^N}$ defined by

$$\hat{u}(m) \triangleq (2\pi)^{-N} \int_{\mathbb{T}^N} u(x) \exp(-i\langle m, x\rangle)dx, \tag{10}$$

is a rapidly decreasing sequence [20]. By duality, we may produce an extension to the periodic distributions:

$$\mathcal{F} : \mathscr{D}'(\mathbb{T}^N) \to \mathscr{S}'(\mathbb{Z}^N), \qquad \mathcal{F}^{-1} : \mathscr{S}'(\mathbb{Z}^N) \to \mathscr{D}'(\mathbb{T}^N).$$

With the Fourier transform, we define a generalized linear pseudodifferential operator

$$F(\sqrt{-\Delta}) : D(F(\sqrt{-\Delta})) \subset L^2(\mathbb{T}^N) \to L^2(\mathbb{T}^N)$$

as

$$F(\sqrt{-\Delta})u(x) \triangleq \sum_{m\in\mathbb{Z}^N} F(|m|)\mathcal{F}u(m) \exp(i\langle m, x\rangle). \tag{11}$$

The sequence $\{F(|m|) : m \in \mathbb{Z}^N\}$ is referred to as the torus symbol of $F(\sqrt{-\Delta})$, which is also a polynomially bounded sequence. Furthermore, when $F = |x|^s$, then one has the homogeneous Sobolev space $\dot{H}^s(\mathbb{T}^N)$. While when $F = (1 + |x|^2)^{\frac{s}{2}}$, one has the inhomogeneous Sobolev space $H^s(\mathbb{T}^N)$.

Lemma 2.2. *The RHS of (11) converges in the distributional sense. Moreover, when F is a real-valued functional, then the operator $F(\sqrt{-\Delta})$ in Definition 2.1 is a self-adjoint operator.*

Proof. (I) Actually, in Definition 2.1, $F\left(\sqrt{-\Delta}\right)$ is defined in the distributional sense. Indeed, for $\forall \eta \in \mathscr{D}(\mathbb{T}^N)$, since $-\Delta(\mathscr{D}(\mathbb{T}^N)) = \mathscr{D}(\mathbb{T}^N)$, then there exists a unique $\eta_k \in \mathscr{D}(\mathbb{T}^N)$ such that $\underbrace{-\Delta \cdots - \Delta}_{k} \eta = \eta_k$ for each $k \in \mathbb{N}$. As a result,

$$\begin{aligned}\Big(\exp(i\langle m, x\rangle), \eta_k\Big)_{L^2(\mathbb{T}^N)} &= \Big(\exp(i\langle m, x\rangle), \underbrace{-\Delta \cdots - \Delta}_{k} \eta\Big)_{L^2(\mathbb{T}^N)} \\ &= \Big(-\Delta \exp(i\langle m, x\rangle), \underbrace{-\Delta \cdots - \Delta}_{k-1} \eta\Big)_{L^2(\mathbb{T}^N)} \\ &= |m|^2\Big(\exp(i\langle m, x\rangle), \underbrace{-\Delta \cdots - \Delta}_{k-1} \eta\Big)_{L^2(\mathbb{T}^N)} \\ &= |m|^{2k}\Big(\exp(i\langle m, x\rangle), \eta\Big)_{L^2(\mathbb{T}^N)}.\end{aligned}$$

Hölder's inequality tells that

$$|(\exp(i\langle m, x\rangle), \eta_k)_{L^2(\mathbb{T}^N)}| \le \| \exp(i\langle m, x\rangle)\|_{L^2(\mathbb{T}^N)} \|\eta_k\|_{L^2} = \sqrt{\int_{\mathbb{T}^N} 1dx} \|\eta_k\|_{L^2(\mathbb{T}^N)}.$$

As a result, $\{|(\exp(i\langle m, x\rangle), \eta)_{L^2(\mathbb{T}^N)}|\}_m$ is a rapidly decreasing sequence with respect to $|m|$. On the other hand, $\{\hat{u}(m)\}_m$ is a polynomially bounded sequence with respect to $|m|$. Since F is also a polynomially bounded function, consequently, the series on the RHS converges. i.e.

$$\sum_{m\in\mathbb{Z}^N} F(|m|)\mathcal{F}u(m)(\exp(i\langle m, x\rangle), \eta)_{L^2(\mathbb{T}^N)} < \infty.$$

(II) Let $u, v \in D(F(\sqrt{-\Delta}))$, then apply Definition 2.1, and one has

$$(F(\sqrt{-\Delta})u, v)_{L^2(\mathbb{T}^N)} = \left(\sum_{m\in\mathbb{Z}^N} F(|m|)\mathcal{F}u(m) \exp(i\langle m, x\rangle), \sum_{n\in\mathbb{Z}^N} \mathcal{F}v(n) \exp(i\langle n, x\rangle)\right)_{L^2(\mathbb{T}^N)}$$

$$= \sum_{m\in\mathbb{Z}^N} (F(|m|)\mathcal{F}u(m)\exp(i\langle m, x\rangle), \mathcal{F}v(m)\exp(i\langle m, x\rangle))_{L^2(\mathbb{T}^N)}$$
$$= \left(\sum_{m\in\mathbb{Z}^N} \mathcal{F}u(m)\exp(i\langle m, x\rangle), \sum_{n\in\mathbb{Z}^N} F(|n|)\mathcal{F}v(n)\exp(i\langle n, x\rangle)\right)_{L^2(\mathbb{T}^N)}$$
$$= (u, F(\sqrt{-\Delta})v)_{L^2(\mathbb{T}^N)}. \quad \square$$

Remark 2.3. As a matter of fact, the spectrum of the Laplace–Beltrami operator Δ_T on the torus $\mathbb{T}^N$ is $\{0, -1^2, -2^2, -3^2, \ldots\}$. And the associated orthonormal basis for $L^2(\mathbb{T}^N)$ is

$$\left\{\left(\int_{\mathbb{T}^N} 1dx\right)^{-\frac{1}{2}} \exp(i\langle m, x\rangle) : |m| = 0, 1, 2, 3, \ldots\right\}.$$

In contrast, the spectrum of the Laplace–Beltrami operator Δ_S on the unit sphere $\mathbb{S}^N$ is

$$\{\lambda_k = -k(k + N - 1), k = 0, 1, 2, \ldots\}.$$

Applying Definition 2.1 of the pseudodifferential operator $\sqrt{-\partial_x^2}$ on the compact manifold $\mathbb{T}$, we obtain the following lemma.

Lemma 2.4. *For $a \in \mathbb{Z}$, $x \in \mathbb{T}$ and a continuous function $F : \mathbb{R} \to \mathbb{R}$, one has*

$$F\left(\sqrt{-\partial_x^2}\right)\exp(i|a|x) = F(|a|)\exp(i|a|x).$$

Proof. For $u(x) = \exp(i|a|x)$, $a \in \mathbb{Z}$, the Fourier series expansion is

$$u(x) = \sum_{k\in\mathbb{Z}} \hat{u}(k)\exp(ikx),$$

where

$$\hat{u}(k) = (2\pi)^{-1}\int_{\mathbb{T}} u(x)\exp(-ikx)dx = \begin{cases} 1, & k = |a|; \\ 0, & k \neq |a|. \end{cases}$$

Then apply the definition of pseudodifferential operators on the compact manifold $\mathbb{T}$, and we have

$$F\left(\sqrt{-\partial_x^2}\right)\exp(i|a|x) = \sum_{k\in\mathbb{Z}} \hat{u}(k)F(|k|)\exp(ikx) = F(|a|)\exp(i|a|x).$$

The statement is proved. $\square$

3. Proof of Theorem 1.4

First we introduce some auxiliary functions and sequences.

Definition 3.1. For a sufficiently small $\varepsilon > 0$, we define:

$$w_\varepsilon(t) \triangleq \sin t \exp\left(2\varepsilon\int_0^t \psi(\tau)\sin^2\tau d\tau\right),$$
$$a_\varepsilon(t) \triangleq 1 - 4\varepsilon\psi(t)\sin(2t) - 2\varepsilon\psi'(t)\sin^2 t - 4\varepsilon^2\psi^2(t)\sin^4 t,$$

where the real-valued non-negative smooth function ψ is 2π-periodic on $\mathbb{R}$ and identically 0 in a neighborhood of 0. And assume that ψ satisfies

$$\int_0^{2\pi} \psi(\tau)\sin^2(\tau)d\tau = \pi.$$

It is easy to verify the following fact after simple calculation.

Lemma 3.2. *According to Definition 3.1, $a_\varepsilon \in C^\infty(\mathbb{R})$ and $w_\varepsilon \in C^\infty(\mathbb{R})$. Particularly, w_ε is the unique solution of the following ordinary differential equation with initial data*

$$\partial_t^2 w_\varepsilon(t) + a_\varepsilon(t)w_\varepsilon(t) = 0, \qquad w_\varepsilon(0) = 0, \qquad \partial_t w_\varepsilon(0) = 1.$$

Definition 3.3. Define

$$\{\rho_k\}_k \triangleq \left\{2^{-P_2+2}\pi \frac{\Lambda(t_k)}{\lambda(t_k)} \frac{[\nu(t_k)]}{\nu(t_k)}\right\}_k,$$
$$\{h_k\}_k \triangleq \left\{2^{P_2}\nu(t_k)\frac{\lambda(t_k)}{\Lambda(t_k)}\right\}_k,$$
$$\{\delta_k\}_k \triangleq \left\{\lambda(t_k)\right\}_k,$$

where $\{t_k\}_k$ is a zero sequence satisfying

$$\left(2^{P_2}\nu(t_k)/\Lambda(t_k)\right)^{\frac{1}{\sigma}} \in \mathbb{N}_+,$$

for each $k \in \mathbb{N}$. $[a]$ represents the integer part of a.

Definition 3.4. In addition, we introduce two time-sequences:

$$\{t_k'\} \triangleq \{t_k + \rho_k\}_k$$

and

$$\{t_k''\} \triangleq \{t_k - \rho_k\}_k,$$

and accordingly, one defines three time intervals I_k, I_k' and I_k'',

$$I_k \triangleq \left[t_k - \frac{\rho_k}{2}, t_k + \frac{\rho_k}{2}\right],$$
$$I_k' \triangleq \left[t_k' - \frac{\rho_k}{2}, t_k' + \frac{\rho_k}{2}\right],$$
$$I_k'' \triangleq \left[t_k'' - \frac{\rho_k}{2}, t_k'' + \frac{\rho_k}{2}\right].$$

Remark 3.5. $\{I_k\}_k$ is called sequence of oscillation intervals, $\{I_k'\}_k$ is sequence of right buffer intervals, and $\{I_k''\}_k$ is sequence of left buffer intervals.

Remark 3.6. It is easy to see that the sequences $\{t_k\}_k$, $\{\rho_k\}_k$, tend to 0, while the sequence $\{h_k\}_k$ tends to $+\infty$. Such choice of ρ_k guarantees that I_k is contained in $(0, T]$. Furthermore, $h_k\rho_k/(4\pi)$, $(h_k/\delta_k)^{\frac{1}{\sigma}} \in \mathbb{N}_+$.

We divide our proof of Theorem 1.4 into three steps.

Step 1: *Construction of a sequence of oscillating coefficients*

Define a monotonously increasing function $\mu \in C^\infty(\mathbb{R})$ as

$$\mu(x) \triangleq \begin{cases} 0, & x \in \left(-\infty, -\frac{1}{3}\right]; \\ 1, & x \in \left[\frac{1}{3}, +\infty\right). \end{cases}$$

Now we introduce a family of coefficients $\{a_k = a_k(t)\}_k$:

$$a_k(t) \triangleq \begin{cases} \lambda^2(t), & t \in [0, T] \setminus (I_k' \cup I_k \cup I_k''); \\ \delta_k^2 a_\varepsilon(h_k(t - t_k)), & t \in I_k; \\ \delta_k^2\left(1 - \mu\left(\frac{t - t_k'}{\rho_k}\right)\right) + \lambda^2(t)\mu\left(\frac{t - t_k'}{\rho_k}\right), & t \in I_k'; \\ \delta_k^2\mu\left(\frac{t - t_k''}{\rho_k}\right) + \lambda^2(t)\left(1 - \mu\left(\frac{t - t_k''}{\rho_k}\right)\right), & t \in I_k''. \end{cases}$$

Consequently, the oscillating part $\{b_k(t)\}_k$ is

$$b_k^2(t) \triangleq \begin{cases} 1, & t \in [0,T] \setminus (I_k' \cup I_k \cup I_k''); \\ \dfrac{\delta_k^2}{\lambda^2(t)} a_\varepsilon(h_k(t-t_k)), & t \in I_k; \\ \dfrac{\delta_k^2}{\lambda^2(t)} \left(1-\mu\left(\dfrac{t-t_k'}{\rho_k}\right)\right) + \mu\left(\dfrac{t-t_k'}{\rho_k}\right), & t \in I_k'; \\ \dfrac{\delta_k^2}{\lambda^2(t)} \mu\left(\dfrac{t-t_k''}{\rho_k}\right) + \left(1-\mu\left(\dfrac{t-t_k''}{\rho_k}\right)\right), & t \in I_k''. \end{cases}$$

In addition, we suppose

$$d_0^+ \le \inf_k \frac{\lambda(t_k)}{\lambda(t_k+\rho_k)} \le \sup_k \frac{\lambda(t_k)}{\lambda(t_k+\rho_k)} \le d_1^+,$$
$$d_0^- \le \inf_k \frac{\lambda(t_k)}{\lambda(t_k-\rho_k)} \le \sup_k \frac{\lambda(t_k)}{\lambda(t_k-\rho_k)} \le d_1^-,$$

where d_1^+, d_1^-, d_0^+ and d_0^- are all positive constants. Actually, when P_2 is sufficiently large, then t_k is always the dominating part. Now we verify the above assumption for $\lambda(t_k)$, $k = 1, 2, \ldots$ by the typical finitely degenerating coefficient $\lambda(t) = t^\ell$, $\ell > 0$ and infinitely degenerating coefficient $\lambda(t) = \frac{1}{t^2}\exp(-\frac{1}{t})$.

- $\lambda(t) = t^\ell$, $\ell > 0$. In this case, $\Lambda(t) = \frac{1}{\ell+1}t^{\ell+1}$. According to Definition 3.3,

$$\rho_k = 2^{-P_2+2}\pi \frac{\Lambda(t_k)}{\lambda(t_k)} \frac{[\nu(t_k)]}{\nu(t_k)} = \frac{4\pi}{(\ell+1)2^{P_2}} \frac{[\nu(t_k)]}{\nu(t_k)} t_k.$$

  As a matter of fact, when $\nu(t_k) \ge 1$, one has $\frac{1}{10} < \frac{[\nu(t_k)]}{\nu(t_k)} \le 1$, which is independent of k. Now we choose a sufficiently large P_2 to make sure that

$$\frac{1}{10^6} t_k \le \rho_k \le \frac{1}{10^5} t_k, \quad k = 1, 2, \ldots.$$

  Consequently, for $k = 1, 2, \ldots$,

$$\left(\frac{10^5}{10^5+1}\right)^\ell \le \frac{\lambda(t_k)}{\lambda(t_k+\rho_k)} = \frac{t_k^\ell}{(t_k+\rho_k)^\ell} \le \left(\frac{10^6}{10^6+1}\right)^\ell,$$
$$\left(\frac{10^6}{10^6-1}\right)^\ell \le \frac{\lambda(t_k)}{\lambda(t_k-\rho_k)} = \frac{t_k^\ell}{(t_k-\rho_k)^\ell} \le \left(\frac{10^5}{10^5-1}\right)^\ell.$$

- $\lambda(t) = \frac{1}{t^2}\exp(-\frac{1}{t})$. In this case, $\Lambda(t) = \exp(-\frac{1}{t})$. According to Definition 3.3,

$$\rho_k = 2^{-P_2+2}\pi \frac{\Lambda(t_k)}{\lambda(t_k)} \frac{[\nu(t_k)]}{\nu(t_k)} = \frac{4\pi}{2^{P_2}} \frac{[\nu(t_k)]}{\nu(t_k)} t_k^2.$$

  Indeed, when $\nu(t_k) \ge 1$, one has $\frac{1}{10} < \frac{[\nu(t_k)]}{\nu(t_k)} \le 1$, which is independent of k. Now we choose a sufficiently large P_2 to make sure that

$$\frac{1}{10^6} t_k^2 \le \rho_k \le \frac{1}{10^5} t_k^2, \quad k = 1, 2, \ldots.$$

  Since $\lim_{k\to+\infty} t_k = 0$, there exists a positive T such that $t_k \le T$ for $k = 1, 2, \ldots$. It is evident that, for $k = 1, 2, \ldots$,

$$\frac{\lambda(t_k)}{\lambda(t_k+\rho_k)} = \frac{(t_k+\rho_k)^2}{t_k^2} \exp\left(\frac{1}{t_k+\rho_k} - \frac{1}{t_k}\right).$$

  Now we estimate the two parts respectively. On the one hand, for $k = 1, 2, \ldots$,

$$1 < 1 + \frac{1}{10^6} t_k \le \frac{t_k+\rho_k}{t_k} \le 1 + \frac{1}{10^5} t_k \le 1 + \frac{T}{10^5}.$$

  On the other hand,

$$-\frac{1}{10^5} \le -\frac{1}{10^5+t_k} \le \frac{1}{t_k+\rho_k} - \frac{1}{t_k} \le -\frac{1}{10^6+t_k} \le -\frac{1}{10^6+T}.$$

As a result, for $k = 1, 2, \ldots,$

$$\exp\left(-\frac{1}{10^5}\right) \le \frac{\lambda(t_k)}{\lambda(t_k+\rho_k)} \le \left(1+\frac{T}{10^5}\right)^2 \exp\left(-\frac{1}{10^6+T}\right).$$

Similarly, we have

$$\left(1-\frac{T}{10^5}\right)^2 \exp\left(\frac{1}{10^6}\right) \le \frac{\lambda(t_k)}{\lambda(t_k-\rho_k)} \le \exp\left(\frac{1}{10^5-T}\right).$$

Simple calculations lead to

$$0 < b_0 \le \inf_{t\in(0,T]} b_k(t) \le \sup_{t\in(0,T]} b_k(t) \le b_1 < \infty,$$

where b_0 and b_1 are independent of k. Moreover, in the interval $I_k \bigcup I_k' \bigcup I_k''$, one has

$$|b_k'(t)| \le C\frac{\lambda(t)\nu(t)}{\Lambda(t)}; \qquad |b_k''(t)| \le C\left(\frac{\lambda(t)\nu(t)}{\Lambda(t)}\right)^2,$$

where C is independent of k.

Step 2: *Construction of auxiliary functions*

Next we study the family of Cauchy problems in $[t_k-\rho_k/2, t_k+\rho_k/2]\times\mathbb{T}$,

$$\partial_t^2 u^k + \delta_k^2 a_\varepsilon(h_k(t-t_k))(-\partial_x^2)^\sigma u^k = 0, \qquad u^k(t_k,x) = 0, \qquad \partial_t u^k(t_k,x) = u_1^k(x). \tag{12}$$

Let the initial Cauchy data be

$$u_1^k(x) = \exp\left(i\left(\frac{h_k}{\delta_k}\right)^{\frac{1}{\sigma}} x\right)$$

and apply the coordinate transform

$$s = h_k(t-t_k).$$

At the same time, define

$$v^k(s,x) \triangleq u^k(t(s),x),$$

then for $s\in[-h_k\rho_k/2, h_k\rho_k/2]$, one has

$$\partial_s^2 v^k + \delta_k^2 h_k^{-2} a_\varepsilon(s)(-\partial_x^2)^\sigma v^k = 0, \qquad v^k(0,x) = 0, \qquad \partial_s v^k(0,x) = u_1^k(x)/h_k. \tag{13}$$

As a matter of fact, when we take Lemma 3.2. into account, then we have a unique solution for (13) in the form of

$$v^k(s,x) = h_k^{-1} u_1^k(x) w_\varepsilon(s).$$

Transforming back to $u^k(t,x)$, we arrive at

$$u^k(t,x) = h_k^{-1}\exp\left(i\left(\frac{h_k}{\delta_k}\right)^{\frac{1}{\sigma}} x\right) w_\varepsilon(h_k(t-t_k))$$

in I_k. Further calculations lead to

$$\begin{aligned} &u^k\big(t_k-\rho_k/2,x\big) = 0, \quad \partial_t u^k\big(t_k-\rho_k/2,x\big) = \exp\left(i\left(\frac{h_k}{\delta_k}\right)^{\frac{1}{\sigma}} x\right)\exp(-\varepsilon\rho_k h_k/2),\\ &u^k\big(t_k+\rho_k/2,x\big) = 0, \quad \partial_t u^k\big(t_k+\rho_k/2,x\big) = \exp\left(i\left(\frac{h_k}{\delta_k}\right)^{\frac{1}{\sigma}} x\right)\exp(\varepsilon\rho_k h_k/2). \end{aligned} \tag{14}$$

Step 3: *Existence of* ν*-loss of regularity*

Now we introduce an energy-increasing property in the sense of pseudo-differential operators.

Lemma 3.7. *For the Cauchy problem with* $(t,x)\in\mathbb{R}\times\mathbb{T}$,

$$\partial_t^2 u + z^2(t)(-\partial_x^2)^\sigma u = 0, \qquad u(t_0,x) = 0, \qquad \partial_t u(t_0,x) = c\exp(i|a|x), \tag{15}$$

with $\sigma > 0$, $c\in\mathbb{R}$ *and* $a\in\mathbb{Z}$. *If* $z(t)$ *is non-negative and* $z_t(t)\ge 0$, *then the homogeneous energy increases, that is,*

$$\dot{\mathbb{E}}_s(u)(t) \ge \dot{\mathbb{E}}_s(u)(t_0).$$

Proof. In effect, we have the following explicit representation of the unique solution by virtue of separation of variables:

$$u(t,x) = cy(t)\exp(i|a|x),$$

where $y(t)$ satisfies the ordinary differential equation:

$$y_{tt} + |a|^{2\sigma} z^2(t) y = 0, \qquad y(t_0) = 0, \qquad y_t(t_0) = 1.$$

By applying the definition of homogeneous Sobolev spaces $\dot{H}^s(\mathbb{T})$, $s \in \mathbb{R}$, we calculate the homogeneous energy for the solution $u = u(t,x)$. It holds by applying Lemma 2.4.

$$\begin{aligned}
\frac{\partial}{\partial t}\dot{\mathbb{E}}_s(u)(t) &= \frac{\partial}{\partial t}\left(\|u(t,\cdot)\|^2_{\dot{H}^s(\mathbb{T})} + \|\partial_t u(t,\cdot)\|^2_{\dot{H}^{s-\sigma}(\mathbb{T})}\right) \\
&= \frac{\partial}{\partial t}\left(\sum_{k\in\mathbb{Z}} |\hat{u}(t,k)|^2 |k|^{2s} + \sum_{k\in\mathbb{Z}} |\partial_t \hat{u}(t,k)|^2 |k|^{2(s-\sigma)}\right) \\
&= \frac{\partial}{\partial t}\left(c^2 z^2(t) y^2(t) |a|^{2s} + c^2 y_t^2 |a|^{2(s-\sigma)}\right) \\
&= 2c^2\left(z z_t y^2 |a|^{2s} + z^2 y y_t |a|^{2s} + y_t y_{tt} |a|^{2(s-\sigma)}\right) \\
&= 2c^2 z z_t y^2 |a|^{2s} \geq 0. \quad \square
\end{aligned}$$

From the definition of $a_k(t)$, it is easy to verify that $a_k(t)$ is monotonously increasing and differentiable in the intervals except for I_k. Therefore, according to Lemma 3.7 and (14), one has

$$\dot{\mathbb{E}}_\sigma(u^k)(t) \leq \exp(-\varepsilon\rho_k h_k), \quad \text{for } t \in [0, t_k - \rho_k/2]; \tag{16}$$

$$\dot{\mathbb{E}}_\sigma(u^k)(t) \geq \exp(\varepsilon\rho_k h_k), \quad \text{for } t \in [t_k + \rho_k/2, T]. \tag{17}$$

It is clear that (8) can be deduced immediately from (16). While for $t = t_k + \rho_k/2$, we apply Lemma 2.4 and obtain

$$\begin{aligned}
\dot{\mathbb{E}}_\sigma(\exp(-c_1\nu(\mu^{-1}(2^{P_2}/(\sqrt{1-\Delta})^\sigma)))u^k)(t) &= \dot{\mathbb{E}}_\sigma(\exp(-c_1\nu(\mu^{-1}(2^{P_2}\delta_k/h_k)))u^k)(t) \\
&= \exp(-2c_1\nu(\mu^{-1}(2^{P_2}\delta_k/h_k)))\dot{\mathbb{E}}_\sigma(u^k)(t) \\
&= \exp(-2c_1\nu(\mu^{-1}(2^{P_2}\delta_k/h_k)) + \varepsilon\rho_k h_k) \\
&= \exp(-2c_1\nu(t_k) + \varepsilon\rho_k h_k).
\end{aligned}$$

Taking into account the choice of ρ_k, h_k, we can choose a sufficiently small $c_1(\varepsilon)$ independent of k such that (9) holds. This concludes the proof. Similarly, as for the non-homogeneous energy $\mathbb{E}_s(u)(t) \triangleq \lambda^2(t)\|u(t,\cdot)\|^2_{H^s(\mathbb{T})} + \|\partial_t u(t,\cdot)\|^2_{H^{s-\sigma}(\mathbb{T})}$, $s > \sigma$, once we notice the fact (14), it is easy to check that, (8) and (9) also hold for the non-homogeneous energy $\mathbb{E}_\sigma(u)(t)$.

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