Global exact controllability in infinite time of Schrödinger equation: multidimensional case

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Abstract. We prove that the multidimensional Schrödinger equation is exactly controllable in infinite time near any point which is a finite linear combination of eigenfunctions of the Schrödinger operator. We prove that, generically with respect to the potential, the linearized system is controllable in infinite time. Applying the inverse mapping theorem, we prove the controllability of the nonlinear system.

Contents

1 Introduction

This paper is concerned with the problem of controllability for the following Schrödinger equation

$$i\dot{z} = -\Delta z + V(x)z + u(t)Q(x)z, \quad x \in D, \tag{1.1}$$

$$z|_{\partial D} = 0, \tag{1.2}$$

$$z(0,x) = z_0(x), \tag{1.3}$$

where $D \subset \mathbb{R}^d, d \geq 1$ is a rectangle, $V, Q \in C^\infty(\overline{D}, \mathbb{R})$ are given functions, u is the control, and z is the state. We prove that (1.1)-(1.3) is exactly controllable in infinite time near any point which is a finite linear combination of eigenfunctions of the Schrödinger operator, extending the results of [24] to the multidimensional case.

Recall that in the papers [6, 8, 10] it is proved that the 1D Schrödinger equation is exactly controllable in finite time in a neighborhood of any finite linear combination of eigenfunctions of Laplacian. In [13, 26, 19], approximate controllability in L^2 is proved for multidimensional Schrödinger equation, generically with respect to functions V, Q and domain D. In [20, 11, 23, 22, 21], stabilization results and approximate controllability properties are proved. In particular, combination of the results of [23] with the above mentioned local exact controllability properties gives global exact controllability in finite time for 1D case in the spaces $H^{3+\varepsilon}, \varepsilon > 0$. See also papers [28, 29, 3, 2, 1, 9] for controllability of finite-dimensional systems and papers [16, 17, 5, 31, 14, 15] for controllability properties of various Schrödinger systems.

The linearization of (1.1)-(1.3) around the trajectory $e^{-i\lambda_{k,V}t}e_{k,V}$ with $u = 0$ and $z_0 = e_{k,V}$ ($e_{k,V}$ is an eigenfunction of the Schrödinger operator $-\Delta + V$ corresponding to some eigenvalue $\lambda_{k,V}$) is of the form

$$i\dot{z} = -\Delta z + V(x)z + u(t)Q(x)e^{-i\lambda_{k,V}t}e_{k,V}, \quad x \in D, \tag{1.4}$$

$$z|_{\partial D} = 0, \tag{1.5}$$

$$z(0,x) = 0. \tag{1.6}$$

Writing this in the Duhamel form

$$z(T) = -i\int_0^T S(T-s)[u(s)Qe^{-i\lambda_{k,V}s}e_{k,V}]\mathrm{d}s, \tag{1.7}$$

where $S(t) = e^{it(\Delta - V)}$ is the free evolution, we see that (1.4)-(1.6) is equivalent to the following moment problem for $d_{mk} := \frac{ie^{i\lambda_{m,V}T}}{\langle Qe_{m,V}, e_{k,V}\rangle}\langle z(T), e_{m,V}\rangle$

$$d_{mk} = \int_0^T e^{i\omega_{mk}s}u(s)\mathrm{d}s,\ m \geq 1, \quad \omega_{mk} = \lambda_{m,V} - \lambda_{k,V}. \tag{1.8}$$

It is well known that a gap condition for the frequencies ω_{mk} is necessary for the solvability of this moment problem when $T < +\infty$ (e.g., see [30]). The asymptotic formula for the eigenvalues $\lambda_{m,V} \sim C_d m^{\frac{2}{d}}$ implies that there is no gap in the case $d \geq 3$ (when $d = 2$, existence of a domain for which there is a gap between the eigenvalues is an open problem). Moreover, it follows from [4] that there is a linear dependence between the exponentials: there is a non-zero $\{c_m\} \in \ell^2$ such that $\sum_{m=1}^{+\infty} c_m e^{i\omega_{mk} s} = 0$ for $t \in [0, T]$. Hence (1.4)-(1.6) is non-controllable in finite time $T < +\infty$. The situation is different when $T = +\infty$. Indeed, by Lemma 3.10 in [22], the exponentials are independent on $[0, +\infty)$, and moreover, (1.4)-(1.6) is controllable, by Theorem 2.6 in [24]. In [24], we used the controllability of linearized system (1.4)-(1.6) to prove the controllability of nonlinear system only in the case $d = 1$. In the multidimensional case, we were able to prove the controllability of (1.4)-(1.6) in a more regular Sobolev space than the one where nonlinear system (1.1)-(1.3) is well posed. We do not know if this difficulty of loss of regularity can be treated using the Nash–Moser inverse function theorem in the spirit of [6]. More precisely, in the multidimensional case, it is very difficult to prove that the inverse of the linearization satisfies the estimates in the Nash–Moser theorem. In this paper, we find a space $\mathcal{H}$ (see (1.11) for the definition), where the nonlinear problem is well posed and the linearized problem is controllable. Applying the inverse inverse function theorem in the space $\mathcal{H}$, we get controllability for (1.1)-(1.3). Let us notice that $\mathcal{H}$ is a sufficiently large space of functions, it contains the Sobolev space H^{3d}. Thus, in particular, we prove controllability in H^{3d}. The result of this paper is optimal in the sense that it seems that the multidimensional Schrödinger equation (1.1)-(1.3) is not exactly controllable in finite time.

Acknowledgments. The authors would like to thank J.-P Puel for providing them in privat communication [27] some results about regularity questions for the Schrödinger equation. The first author was partially supported by the Agence Nationale de la Recherche, Projets STOSYMAP ANR 2011 BS01 015 01 and Blanc EMAQS ANR 2011 BS01 017 01. The second author was supported by the Grant MTM2011-29306-C02-00 of the MICINN (Spain), project PI2010-04 of the Basque Government, the ERC Advanced Grant FP7-246775 NUMERIWAVES.

Notation

In this paper, we use the following notation. Let us define the Banach spaces

$$
\begin{aligned}
\ell^2 &:= \{\{a_j\} \in \mathbb{C}^\infty : \|\{a_j\}\|_{\ell^2}^2 = \sum_{j=1}^{+\infty} |a_j|^2 < +\infty\},\\
\ell_0^2 &:= \{\{a_j\} \in \ell^2 : a_1 \in \mathbb{R}\},\\
\ell^\infty &:= \{\{a_j\} \in \mathbb{C}^\infty : \|\{a_j\}\|_{\ell^\infty} = \sup_{j\geq 1} |a_j| < +\infty\},\\
\ell_0^\infty &:= \{\{a_j\} \in \ell^\infty : \lim_{j\to+\infty} a_j = 0\},\\
\ell_{01}^\infty &:= \{\{a_j\} \in \ell_0^\infty : a_1 \in \mathbb{R}\}.
\end{aligned}
$$

We denote by $H^s := H^s(D)$ the Sobolev space of order $s \geq 0$. Consider the Schrödinger operator $-\Delta + V$, $V \in C^\infty(\overline{D}, \mathbb{R})$ with $\mathcal{D}(-\Delta + V) := H_0^1 \cap H^2$. Let $\{\lambda_{j,V}\}$ and $\{e_{j,V}\}$ be the sets of eigenvalues and normalized eigenfunctions of this operator. Let $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ be the scalar product and the norm in the space L^2. Define the space $H^s_{(V)} := D((-\Delta + V)^{\frac{s}{2}})$ endowed with the norm $\|\cdot\|_{s,V} = \|(\lambda_{j,V})^{\frac{s}{2}} \langle \cdot, e_{j,V}\rangle\|_{\ell^2}$. When D is the rectangle $(0,1)^d$ and $V(x_1, \ldots, x_d) = V_1(x_1) + \ldots + V_d(x_d)$, $V_k \in C^\infty([0,1], \mathbb{R})$, the eigenvalues and eigenfunctions of $-\Delta + V$ on D are of the form

$$\lambda_{j_1,\ldots,j_d,V} = \lambda_{j_1,V_1} + \ldots + \lambda_{j_d,V_d}, \tag{1.9}$$

$$e_{j_1,\ldots,j_d,V}(x_1, \ldots, x_d) = e_{j_1,V_1}(x_1) \cdot \ldots \cdot e_{j_d,V_d}(x_d), \quad (x_1, \ldots, x_d) \in D, \tag{1.10}$$

where $\{\lambda_{j,V_k}\}$ and $\{e_{j,V_k}\}$ are the eigenvalues and eigenfunctions of operator $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_k$ on $(0,1)$. Define the spaces

$$
\begin{aligned}
\mathcal{H} = \{z \in L^2 : (j_1^3 \cdot \ldots \cdot j_d^3)\langle z, e_{j_1,\ldots,j_d,V}\rangle \in \ell_0^\infty,&\\
\|z\|_{\mathcal{H}} := \|(j_1^3 \cdot \ldots \cdot j_d^3)\langle z, e_{j_1,\ldots,j_d,V}\rangle\|_{\ell^\infty} < +\infty\},& \qquad (1.11)
\end{aligned}
$$

$$\mathcal{V} = \{z \in L^2 : \|z\|_{\mathcal{V}} := \sum_{j_1,\ldots,j_d=1}^{+\infty} (j_1^3 \cdot \ldots \cdot j_d^3) |\langle z, e_{j_1,\ldots,j_d,V}\rangle| < +\infty\}. \tag{1.12}$$

The eigenvalues and eigenfunctions of Dirichlet Laplacian on the interval $(0,1)$ are $\lambda_{k,0} = k^2\pi^2$ and $e_{k,0}(x) = \sqrt{2}\sin(k\pi x)$, $x \in (0,1)$. It is well known that for any $V \in L^2([0,1], \mathbb{R})$

$$\lambda_{k,V} = k^2\pi^2 + \int_0^1 V(x)\mathrm{d}x + r_k, \tag{1.13}$$

$$\|e_{k,V} - e_{k,0}\|_{L^\infty} \leq \frac{C}{k}, \tag{1.14}$$

$$\left\|\frac{\mathrm{d}e_{k,V}}{\mathrm{d}x} - \frac{\mathrm{d}e_{k,0}}{\mathrm{d}x}\right\|_{L^\infty} \leq C, \tag{1.15}$$

where $\sum_{k=1}^{+\infty} r_k^2 < +\infty$ (e.g., see [25]). For a Banach space X, we shall denote by $B_X(a, r)$ the open ball of radius $r > 0$ centered at $a \in X$. The integer part of $x \in \mathbb{R}$ is denoted by $[x]$. We denote by C a constant whose value may change from line to line.

2 Main results

2.1 Well-posedness of Schrödinger equation

We assume that $V(x_1, \dots, x_d) = V_1(x_1) + \dots + V_d(x_d), x_k \in [0,1]$ and $V_k \in C^\infty([0,1], \mathbb{R}), k = 1, \dots, d$. Let us consider the following Schrödinger equation

$$i\dot{z} = -\Delta z + V(x)z + u(t)Q(x)z + v(t)Q(x)y, \tag{2.1}$$

$$z|_{\partial D} = 0, \tag{2.2}$$

$$z(0,x) = z_0(x). \tag{2.3}$$

The following lemma shows the well-posedness of this system in $H^2_{(V)}$.

Lemma 2.1. *For any $z_0 \in H^2_{(V)}$, $u, v \in L^1_{loc}([0,\infty), \mathbb{R})$ and $y \in L^1([0,\infty), H^2_{(V)})$ problem (2.1)-(2.3) has a unique solution $z \in C([0,\infty), H^2_{(V)})$. Furthermore, if $v = 0$, then for all $t \geq 0$ we have*

$$\|z(t)\| = \|z_0\|. \tag{2.4}$$

See [12] for the proof. In [10] it is proved that this problem is well posed in $H^3_{(V)}$ for $d = 1$, and in [27] the well-posedness in $H^3_{(V)}$ is proved for $d \geq 1$.

For any integer $l \geq 3$, let $m = m(l) := [\frac{l-1}{2}]$ and define the space

$$C_0^m := \{u \in C^m([0,\infty), \mathbb{R}) : \frac{\mathrm{d}^k u}{\mathrm{d}t^k}(0) = 0, k \in [0,m]\}$$

endowed with the norm of $C^m([0,\infty), \mathbb{R})$. The following lemma shows that problem (2.1)-(2.3) is well posed in higher Sobolev spaces when u, v and y are more regular.

Lemma 2.2. *For any integer $l \geq 3$, any $z_0 \in H^l_{(V)}$, any $y \in W^{m,1}_{loc}([0,\infty), H^2_{(V)})$ and any $u, v \in C_0^m$ the solution z in Lemma 2.1 belongs to the space $C([0,\infty), H^l) \cap C^1([0,\infty), H^{l-2})$. Moreover, there is a constant $C > 0$ such that*

$$\|z(t)\|_{H^l} + \|z\|_{W^{m,1}([0,t],H^2_{(V)})} \leq C(\|z_0\|_{l,V} + \|v\|_{C_0^m}\|y\|_{W^{m,1}([0,t],H^2_{(V)})}) \times e^{C(\|u\|_{C_0^m}+1)t}. \tag{2.5}$$

See Appendix of [6] for the proof.

Lemma 2.3. *Denote by $\mathcal{U}_t(\cdot,\cdot) : H^2_{(V)} \times L^1_{loc}(\mathbb{R}_+, \mathbb{R}) \to H^2_{(V)}$ the resolving operator of (1.1), (1.2). Then $\mathcal{U}_t(\cdot,\cdot)$ is locally Lipschitz continuous: there is $C > 0$ such that*

$$\|\mathcal{U}_t(z_0, u) - \mathcal{U}_t(z_0', u')\|_{H^l} \leq C(\|z_0 - z_0'\|_{l,V} + \|u - u'\|_{C_0^m}\|z_0'\|_{l,V})e^{C(\|u\|_{C_0^m}+1)t}. \tag{2.6}$$

Proof. Notice that $z(t) := \mathcal{U}_t(z_0, u) - \mathcal{U}_t(z'_0, u')$ is a solution of problem

$$
\begin{aligned}
i\dot{z} &= -\Delta z + V(x)z + u(t)Q(x)z + (u(t) - u'(t))Q(x)\mathcal{U}_t(z'_0, u'),\\
z|_{\partial D} &= 0,\\
z(0, x) &= z_0(x) - z'_0(x).
\end{aligned}
$$

Applying Lemma 2.2, we get

$$
\|z(t)\|_{H^l} \leq C(\|z_0 - z'_0\|_{l,V} + \|u - u'\|_{C_0^m}\|\mathcal{U}_\cdot(z'_0, u')\|_{W^{m,1}([0,t],H^2_{(V)})})e^{C(\|u\|_{C_0^m}+1)t}, \tag{2.7}
$$

$$
\|\mathcal{U}_\cdot(z'_0, u')\|_{W^{m,1}([0,t],H^2_{(V)})} \leq C\|z'_0\|_{l,V}e^{C(\|u\|_{C_0^m}+1)t}. \tag{2.8}
$$

Replacing (2.8) into (2.7), we get (2.6). □

Let us rewrite (1.1)-(1.3) in the Duhamel form

$$
z(t) = S(t)z_0 - i\int_0^t S(t-s)[u(s)Qz(s)]\mathrm{d}s, \tag{2.9}
$$

where $S(t) = e^{it(\Delta - V)}$ is the free evolution. Let us take any $w \in L^1(\mathbb{R}_+, \mathbb{R})$ and estimate the following integral

$$
G_t(z) := \int_0^t S(-s)[w(s)Qz(s)]\mathrm{d}s.
$$

We take controls from the weighted space

$$
\mathcal{G} := \{u \in L^1(\mathbb{R}_+, \mathbb{R}) : u(\cdot)e^{B\cdot} \in L^1(\mathbb{R}_+, \mathbb{R})\}
$$

endowed with the norm $\|u\|_{\mathcal{G}} = \|u(\cdot)e^{B\cdot}\|_{L^1}$, where the constant $B > 0$ will be chosen later. For $B > C + 1$, where C is the constant in Lemma 2.2, we have the following result.

Proposition 2.4. *Let us take any $l \geq 4d$, $z_0 \in H^l_{(V)}, w \in \mathcal{G}$ and $u \in C_0^m$, and let $z(t) := \mathcal{U}_t(z_0, u)$. Then there are constants $\delta, C > 0$ such that for any $u \in B_{C_0^m}(0, \delta)$ and for any $t > s \geq 0$*

$$
\|G_t(z) - G_s(z)\|_{\mathcal{H}} \leq C\int_s^t \|z(\tau)\|_{H^l}|w(\tau)|\mathrm{d}\tau, \tag{2.10}
$$

and the following integral converges in $\mathcal{H}$

$$
G_\infty(z) := \int_0^{+\infty} S(-\tau)[w(\tau)Qz(\tau)]\mathrm{d}\tau. \tag{2.11}
$$

Proof. Using (2.5) with $v = 0$, the definition of $\mathcal{G}$, and choosing $\delta > 0$ sufficiently small, we see that

$$
\int_0^{+\infty} \|z(\tau)\|_{H^l}|w(\tau)|\mathrm{d}\tau < +\infty.
$$

Combining this with (2.10), we prove the convergence of the integral in (2.11). Let us prove (2.10). To simplify the notation, let us suppose that $d = 2$; the proof of the general case is similar. Let $V(x_1, x_2) = V_1(x_1)+V_2(x_2)$. Integration by parts gives

$$\begin{aligned}\langle Qz(s), e_{j_1,V_1}e_{j_2,V_2}\rangle =&\frac{1}{\lambda_{j_1,V_1}}\langle(-\frac{\partial^2}{\partial x_1^2}+V_1)(Qz), e_{j_1,V_1}e_{j_2,V_2}\rangle\\ =&\frac{1}{\lambda_{j_1,V_1}^2}\langle(-\frac{\partial^2}{\partial x_1^2}+V_1)(Qz), (-\frac{\partial^2}{\partial x_1^2}+V_1)e_{j_1,V_1}e_{j_2,V_2}\rangle\\ =&\frac{1}{\lambda_{j_1,V_1}^2}\int_0^1\frac{\partial^2}{\partial x_1^2}(Qz)e_{j_2,V_2}\mathrm{d}x_2\frac{\partial}{\partial x_1}e_{j_1,V_1}\Big|_{x_1=0}^{x_1=1}\\ &+\frac{1}{\lambda_{j_1,V_1}^2}\Big(\langle V_1(-\frac{\partial^2}{\partial x_1^2}+V_1)(Qz), e_{j_1,V_1}e_{j_2,V_2}\rangle\\ &+\langle\frac{\partial}{\partial x_1}(-\frac{\partial^2}{\partial x_1^2}+V_1)(Qz), \frac{\partial}{\partial x_1}e_{j_1,V_1}e_{j_2,V_2}\rangle\Big)\\ =:&\, I_j+J_j.\end{aligned}$$

Let us estimate I_j. Since $\frac{\partial^2}{\partial x_1^2}(Qz(s)) = 0$ for all $x_1 \in [0, 1]$ and for $x_2 = 0$ and $x_2 = 1$, integration by parts in x_2 implies

$$\begin{aligned}I_j =& \frac{1}{\lambda_{j_1,V_1}^2\lambda_{j_2,V_2}}\int_0^1(-\frac{\partial^2}{\partial x_2^2}+V_2)\Big(\frac{\partial^2}{\partial x_1^2}(Qz)\Big)e_{j_2,V_2}\mathrm{d}x_2\frac{\partial}{\partial x_1}e_{j_1,V_1}\Big|_{x_1=0}^{x_1=1}\\ =& \frac{1}{\lambda_{j_1,V_1}^2\lambda_{j_2,V_2}^2}\int_0^1(-\frac{\partial^2}{\partial x_2^2}+V_2)\Big(\frac{\partial^2}{\partial x_1^2}(Qz)\Big)(-\frac{\partial^2}{\partial x_2^2}+V_2)e_{j_2,V_2}\mathrm{d}x_2\frac{\partial}{\partial x_1}e_{j_1,V_1}\Big|_{x_1=0}^{x_1=1}\\ =& \frac{1}{\lambda_{j_1,V_1}^2\lambda_{j_2,V_2}^2}(-\frac{\partial^2}{\partial x_2^2}+V_2)\Big(\frac{\partial^2}{\partial x_1^2}(Qz)\Big)\frac{\partial}{\partial x_2}e_{j_2,V_2}\frac{\partial}{\partial x_1}e_{j_1,V_1}\Big|_{x_2=0}^{x_2=1}\Big|_{x_1=0}^{x_1=1}\\ &+\frac{1}{\lambda_{j_1,V_1}^2\lambda_{j_2,V_2}^2}\int_0^1 V_2(-\frac{\partial^2}{\partial x_2^2}+V_2)\Big(\frac{\partial^2}{\partial x_1^2}(Qz)\Big)e_{j_2,V_2}\mathrm{d}x_2\frac{\partial}{\partial x_1}e_{j_1,V_1}\Big|_{x_1=0}^{x_1=1}\\ &+\frac{1}{\lambda_{j_1,V_1}^2\lambda_{j_2,V_2}^2}\int_0^1\frac{\partial}{\partial x_2}(-\frac{\partial^2}{\partial x_2^2}+V_2)\Big(\frac{\partial^2}{\partial x_1^2}(Qz)\Big)\frac{\partial}{\partial x_2}e_{j_2,V_2}\mathrm{d}x_2\frac{\partial}{\partial x_1}e_{j_1,V_1}\Big|_{x_1=0}^{x_1=1}\\ =:&\, I_{j,1}+I_{j,2}+I_{j,3}.\end{aligned}\tag{2.12}$$

Let us consider the term $I_{j,1}$:

$$\begin{aligned}I_{j,1} =& \Big(\frac{2j_1j_2\pi^2}{\lambda_{j_1,V_1}^2\lambda_{j_2,V_2}^2}(-\frac{\partial^2}{\partial x_2^2}+V_2)\Big(\frac{\partial^2}{\partial x_1^2}(Qz)\Big)\cos(j_1\pi x_1)\cos(j_2\pi x_2)\\ &+\frac{1}{\lambda_{j_1,V_1}^2\lambda_{j_2,V_2}^2}(-\frac{\partial^2}{\partial x_2^2}+V_2)\Big(\frac{\partial^2}{\partial x_1^2}(Qz)\Big)\frac{\partial^2}{\partial x_1\partial x_2}(e_{j_1,V_1}e_{j_2,V_2}-e_{j_1,0}e_{j_2,0})\Big)\Big|_{x_2=0}^{x_2=1}\Big|_{x_1=0}^{x_1=1}.\end{aligned}$$

Using (1.13), (1.15) and the Sobolev embedding $H^s \hookrightarrow L^\infty, s > \frac{d}{2}$, we get

$$\sup_{j_1,j_2\geq 1}\Big|j_1^3j_2^3\int_s^t e^{i(\lambda_{j_1,V_1}+\lambda_{j_2,V_2})\tau}w(\tau)I_{j,1}\mathrm{d}\tau\Big| \leq C\int_s^t \|z(\tau)\|_{H^l}|w(\tau)|\mathrm{d}\tau.$$

The Riemann–Lebesgue theorem and (1.15) imply that

$$j_1^3 j_2^3 \int_s^t e^{i(\lambda_{j_1,V_1}+\lambda_{j_2,V_2})\tau} w(\tau) I_{j,1} \mathrm{d}\tau \rightarrow 0 \quad \text{as } j_1+j_2 \rightarrow +\infty.$$

Thus

$$j_1^3 j_2^3 \int_s^t e^{i(\lambda_{j_1,V_1}+\lambda_{j_2,V_2})\tau} w(\tau) I_{j,1} \mathrm{d}\tau \in \ell_0^\infty.$$

The terms $I_{j,2}, I_{j,3}$ and J_j are treated exactly in the same way. We omit the details. Thus we get that

$$\|G_t(z)-G_s(z)\|_{\mathcal{H}} = \|\int_s^t S(-\tau)[w(\tau)Qz(\tau)]\mathrm{d}\tau\|_{\mathcal{H}} \leq C\int_s^t \|z(\tau)\|_{H^l} |w(\tau)| \mathrm{d}\tau.$$

□

Let $T_n \rightarrow +\infty$ be a sequence such that $e^{-i\lambda_{V,j}T_n} \rightarrow 1$ as $n\rightarrow\infty$ for any $j\geq 1$ (e.g., see Lemma 2.1 in [24]). Then

$$S(T_n)z \rightarrow z \text{ as } n \rightarrow +\infty \text{ in } \mathcal{H} \text{ for any } z\in\mathcal{H} \text{ and } t\geq 0. \tag{2.13}$$

Indeed, since

$$S(t)z = \sum_{j=1}^{+\infty} e^{-i\lambda_{j,V}t} \langle z, e_{j,V}\rangle e_{j,V}, \tag{2.14}$$

we have

$$\begin{aligned}\|S(T_n)z-z\|_{\mathcal{H}} \leq & \sup_{\lambda_{j_1,\ldots,j_d,V}\leq N} (j_1^3\cdot\ldots\cdot j_d^3)|e^{-i\lambda_{j_1,\ldots,j_d,V}T_n}-1||\langle z, e_{j_1,\ldots,j_d,V}\rangle| \\ & +2\sup_{\lambda_{j_1,\ldots,j_d,V}>N} (j_1^3\cdot\ldots\cdot j_d^3)|\langle z, e_{j_1,\ldots,j_d,V}\rangle| \leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon\end{aligned}$$

for sufficiently large integers $N, n\geq 1$.

Let us take $t=T_n$ in (2.9) and pass to the limit $n\rightarrow\infty$. Using Proposition 2.4, the embedding $H^l_{(V)} \hookrightarrow \mathcal{H}$ and (2.13), we obtain the following result.

Lemma 2.5. *Let us take any $l\geq 4d$ and $z_0\in H^l_{(V)}$. There is a constant $\delta>0$ such that for any $u\in B_{C_0^m}(0,\delta)\cap\mathcal{G}$ the following limit exists in $\mathcal{H}$*

$$\lim_{n\rightarrow+\infty} \mathcal{U}_{T_n}(z_0,u) =: \mathcal{U}_\infty(z_0,u). \tag{2.15}$$

2.2 Exact controllability in infinite time

Let $l\geq 4d$ be the integer in Proposition 2.4. Take any integer $s\geq l$ and let

$$H_0^s(\mathbb{R}_+,\mathbb{R}) := \{u\in H^s(\mathbb{R}_+,\mathbb{R}) : u^{(k)}(0)=0, k=0,\ldots,s-1\}.$$

The set of admissible controls is the Banach space

$$\mathcal{F} := \mathcal{G} \cap H_0^s(\mathbb{R}_+, \mathbb{R}) \tag{2.16}$$

endowed with the norm $\|u\|_{\mathcal{F}} := \|u\|_{\mathcal{G}} + \|u\|_{H^s}$. Equality (2.4) implies that it suffices to consider the controllability properties of (1.1), (1.2) on the unit sphere S in L^2.

We prove the controllability of (1.1), (1.2) under below condition.

Condition 2.6. *Suppose that the functions $V, Q \in C^\infty(\overline{D}, \mathbb{R})$ are such that*

(i) $\inf_{p_1,j_1,\dots,p_d,j_d \geq 1} |(p_1 j_1 \cdot \ldots \cdot p_d j_d)^3 Q_{pj}| > 0, Q_{pj} := \langle Q e_{p_1,\dots,p_d,V}, e_{j_1,\dots,j_d,V} \rangle$,

(ii) $\lambda_{i,V} - \lambda_{j,V} \neq \lambda_{p,V} - \lambda_{q,V}$ *for all* $i, j, p, q \geq 1$ *such that* $\{i, j\} \neq \{p, q\}$ *and* $i \neq j$.

See [24] and [26, 23, 18] for the proof of genericity of (i) and (ii), respectively. Let us set

$$\mathcal{E} := \operatorname{span}\{e_{j,V}\}. \tag{2.17}$$

Below theorem is the main result of this paper.

Theorem 2.7. *Under Condition 2.6, for any $\tilde{z} \in S \cap \mathcal{E}$ there is $\sigma > 0$ such that problem (1.1), (1.2) is exactly controllable in infinite time in $S \cap B_{\mathcal{H}}(\tilde{z}, \sigma)$, i.e., for any $z_1 \in S \cap B_{\mathcal{H}}(\tilde{z}, \sigma)$ there is a control $u \in \mathcal{F}$ such that limit (2.15) exists in $\mathcal{H}$ and $z_1 = \mathcal{U}_\infty(\tilde{z}, u)$.*

See Section 3.3 for the proof. Since the space $H^{3d}_{(V)}$ is continuously embedded into $\mathcal{H}$, we obtain

Theorem 2.8. *Under Condition 2.6, for any $\tilde{z} \in S \cap \mathcal{E}$ there is $\sigma > 0$ such that for any $z_1 \in S \cap B_{H^{3d}_{(V)}}(\tilde{z}, \sigma)$ there is a control $u \in \mathcal{F}$ such that limit (2.15) exists in $\mathcal{H}$ and $z_1 = \mathcal{U}_\infty(\tilde{z}, u)$.*

Remark 2.9. As in the case $d = 1$ (see Theorems 3.7 and 3.8 in [24]) here also one can prove controllability in higher Sobolev spaces with more regular controls, and a global controllability property using a compactness argument.

3 Proof of Theorem 2.8

3.1 Controllability of linearized system

In this section, we study the controllability of the linearization of (1.1), (1.2) around the trajectory $\mathcal{U}_t(\tilde{z}, 0), \tilde{z} \in S \cap \mathcal{E}$:

$$i\dot{z} = -\Delta z + V(x)z + u(t)Q(x)\mathcal{U}_t(\tilde{z}, 0), \tag{3.1}$$

$$z|_{\partial D} = 0, \tag{3.2}$$

$$z(0, x) = z_0. \tag{3.3}$$

The controllability in infinite time of this system is proved in [24], Section 2. For the proof of Theorem 2.8 we need to show controllability of (3.1)-(3.3) in $\mathcal{H}$ which is larger than the space considered in [24]. Hence a generalization of the arguments of [24] is needed.

Let S be the unit sphere in L^2. For $y \in S$, let T_y be the tangent space to S at $y \in S$:

$$T_y = \{z \in L^2 : \mathrm{Re}\langle z, y\rangle = 0\}.$$

By Lemma 2.1, for any $z_0 \in H^2_{(V)}$ and $u \in L^1_{loc}(\mathbb{R}_+, \mathbb{R})$, problem (3.1)-(3.3) has a unique solution $z \in C(\mathbb{R}_+, H^2_{(V)})$. Let

$$\begin{aligned} R_t(\cdot,\cdot) : H^2_{(V)} \times L^1([0,t],\mathbb{R}) &\to H^2_{(V)}, \\ (z_0, u) &\to z(t) \end{aligned}$$

be the resolving operator. Then $R_t(z_0, u) \in T_{\mathcal{U}_t(\tilde{z},0)}$ for any $z_0 \in T_{\tilde{z}} \cap H^2_{(V)}$ and $t \geq 0$. Indeed,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Re}\langle R_t, \mathcal{U}_t\rangle &= \mathrm{Re}\langle \dot{R}_t, \mathcal{U}_t\rangle + \mathrm{Re}\langle R_t, \dot{\mathcal{U}}_t\rangle \\ &= \mathrm{Re}\langle i(\Delta - V)R_t - iu(t)Q(x)\mathcal{U}_t, \mathcal{U}_t\rangle + \mathrm{Re}\langle R_t, i(\Delta - V)\mathcal{U}_t\rangle \\ &= \mathrm{Re}\langle i(\Delta - V)R_t, \mathcal{U}_t\rangle + \mathrm{Re}\langle R_t, i(\Delta - V)\mathcal{U}_t\rangle = 0. \end{aligned}$$

Since $\mathrm{Re}\langle R_0, \mathcal{U}_0\rangle = \mathrm{Re}\langle z_0, \tilde{z}\rangle = 0$, we get $R_t(z_0, u) \in T_{\mathcal{U}_t(\tilde{z},0)}$.

As (3.1)-(3.3) is a linear control problem, the controllability of system with $z_0 = 0$ is equivalent to that with any $z_0 \in T_{\tilde{z}}$. Henceforth, we take $z_0 = 0$ in (3.3). Let us rewrite this problem in the Duhamel form

$$z(t) = -i \int_0^t S(t-s)u(s)Q(x)\mathcal{U}_s(\tilde{z}, 0)\mathrm{d}s. \tag{3.4}$$

Let $T_n \to \infty$ be the sequence defined in Section 2.1. For any $u \in \mathcal{F}$ the following limit exists in $\mathcal{H}$

$$R_\infty(0, u) := \lim_{n\to+\infty} z(T_n) = \lim_{n\to+\infty} R_{T_n}(0, u). \tag{3.5}$$

Using (2.14) and (3.4), we obtain

$$\langle z(t), e_{m,V}\rangle = -i \sum_{k=1}^{+\infty} e^{-i\lambda_{m,V}t}\langle \tilde{z}, e_{k,V}\rangle Q_{mk} \int_0^t e^{i\omega_{mk}s}u(s)\mathrm{d}s,\ m \geq 1, \tag{3.6}$$

where $\omega_{mk} = \lambda_{m,V} - \lambda_{k,V}$ and $Q_{mk} := \langle Qe_{m,V}, e_{k,V}\rangle$. Let us take $t = T_n$ in (3.6) and pass to the limit as $n \to +\infty$. The choice of the sequence T_n implies that

$$\langle R_\infty(0, u), e_{m,V}\rangle = -i \sum_{k=1}^{+\infty} \langle \tilde{z}, e_{k,V}\rangle Q_{mk} \int_0^{+\infty} e^{i\omega_{mk}s}u(s)\mathrm{d}s. \tag{3.7}$$

Moreover, $R_\infty(0, u) \in T_{\tilde{z}}$. Indeed, using (3.5) and the convergence $\mathcal{U}_{T_n}(\tilde{z}, 0) \to \tilde{z}$ in $\mathcal{H}$, we get

$$\mathrm{Re}\langle R_\infty(0,u),\tilde z\rangle=\lim_{n\to\infty}\mathrm{Re}\langle R_{T_n}(0,u),\mathcal{U}_{T_n}(\tilde z,0)\rangle=0.$$

Lemma 3.1. *The mapping $R_\infty(0,\cdot)$ is linear continuous from $\mathcal{F}$ to $T_{\tilde z}\cap\mathcal{H}$.*

Proof. By (2.24) in [24], there is a constant $C>0$ such that for any $m_j,k_j\geq 1$, $j=1,\dots,d$ we have

$$\Big|\frac{(m_1\cdot\ldots\cdot m_d)^3}{(k_1\cdot\ldots\cdot k_d)^3}\langle Qe_{k_1,\ldots,k_d,V},e_{m_1,\ldots,m_d,V}\rangle\Big|\leq C. \tag{3.8}$$

Then (3.7), (3.8) and the Schwarz inequality imply that

$$\begin{aligned}
&\|R_\infty(0,u)\|_{\mathcal{H}}=\sup_{m_1,\ldots,m_d\geq1}|(m_1^3\cdot\ldots\cdot m_d^3)\langle R_\infty(0,u),e_{m_1,\ldots,m_d,V}\rangle|\\
&\leq C\sup_{m_1,\ldots,m_d\geq1}\Big|(m_1^3\cdot\ldots\cdot m_d^3)\langle\tilde z,e_{m_1,\ldots,m_d,V}\rangle\langle Qe_{m,V},e_{m,V}\rangle\int_0^{+\infty}u(s)\mathrm{d}s\Big|\\
&\quad+C\|\tilde z\|_{\mathcal{V}}\sup_{m,k\geq1,m\neq k}\Big|\frac{(m_1\cdot\ldots\cdot m_d)^3}{(k_1\cdot\ldots\cdot k_d)^3}\langle Qe_{k_1,\ldots,k_d,V},e_{m_1,\ldots,m_d,V}\rangle\int_0^{+\infty}e^{i\omega_{mk}s}u(s)\mathrm{d}s\Big|\\
&\leq C\|\tilde z\|_{\mathcal{V}}^2\|u\|_{\mathcal{F}}^2<+\infty,
\end{aligned}$$

where $\mathcal{V}$ is defined by (1.12).

□

Let us introduce the set

$$\mathcal{E}_0:=\{z\in S:\exists p,q\geq1,p\neq q,z=c_pe_{p,V}+c_qe_{q,V},\ |c_p|^2\langle Qe_{p,V},e_{p,V}\rangle-|c_q|^2\langle Qe_{q,V},e_{q,V}\rangle=0\}.$$

Theorem 3.2. *Under Condition 2.6, for any $\tilde z\in S\cap\mathcal{E}\setminus\mathcal{E}_0$, the mapping $R_\infty(0,\cdot):\mathcal{F}\to T_{\tilde z}\cap\mathcal{H}$ admits a continuous right inverse, where the space $T_{\tilde z}\cap\mathcal{H}$ is endowed with the norm of $\mathcal{H}$. If $\tilde z\in S\cap\mathcal{E}_0$, then $R_\infty(0,\cdot)$ is not invertible.*

Remark 3.3. The invertibility of the mapping $R_T(0,\cdot)$ with finite $T>0$ and $\tilde z=e_1$ is studied by Beauchard et al. [7]. They prove that for space dimension $d\geq3$ the mapping is not invertible. By Beauchard [6], R_T is invertible in the case $d=1$ and $\tilde z=e_1$. The case $d=2$ is open to our knowledge.

For any $u\in L^1(\mathbb{R}_+,\mathbb{R})$, denote by $\check u$ the inverse Fourier transform of the function obtained by extending u as zero to $\mathbb{R}_-^*$:

$$\check u(\omega):=\int_0^{+\infty}e^{i\omega s}u(s)\mathrm{d}s. \tag{3.9}$$

Proof of Theorem 3.2. Let us take any $\tilde z\in S\cap\mathcal{E}\setminus\mathcal{E}_0$ and $y\in T_{\tilde z}\cap\mathcal{H}$. There is an integer $N\geq1$ such that $\langle\tilde z,e_{k,V}\rangle=0$ for any $k\geq N+1$. Let us define

$$d_{mk}:=\frac{i\langle y,e_{m,V}\rangle\langle e_{k,V},\tilde z\rangle-i\langle e_{k,V},y\rangle\langle\tilde z,e_{m,V}\rangle}{Q_{mk}}+C_{mk},$$

for $k \leq N$, where $C_{mk} \in \mathbb{C}$. Notice that

$$\sup_{m,k\geq 1} \left| \frac{\langle y, e_{m,V}\rangle \langle e_{k,V}, \tilde{z}\rangle}{Q_{mk}} \right| \leq C\|y\|_{\mathcal{H}}\|\tilde{z}\|_{\mathcal{H}} < +\infty.$$

Repeating the arguments of the proof of Theorem 2.6 in [24], one can show that the constants C_{mk} can be chosen such that

$$\sup_{m,k\geq 1} |d_{mk}| < +\infty, d_{mm} = d_{11},\ d_{mk} = \overline{d}_{km} \text{ for all } 1 \leq m, k \leq N,$$

$$d_{mk} \to 0 \text{ as } m \to \infty \text{ for any fixed } k \geq 1,$$

and $y = R_\infty(0, u)$ holds for any solution $u \in \mathcal{F}$ of system

$$d_{mk} = \check{u}(\omega_{mk}) \quad \text{for all } m \geq 1 \text{ and } k \in [1, N].$$

It remains to use the following proposition, which is proved in next subsection.

Proposition 3.4. *If the strictly increasing sequence $\omega_m \in \mathbb{R}, m \geq 1$ is such that $\omega_1 = 0$ and $\omega_m \to +\infty$ as $m \to +\infty$, then there is a linear continuous operator A from ℓ^∞_{01} to $\mathcal{F}$ such that $\{\check{A(d)}(\omega_m)\} = d$ for any $d \in \ell^\infty_{01}$.*

The proof of the non-invertibility of $R_\infty(0, \cdot)$ is a remark by Beauchard and Coron [8] (cf. Step 2 of the proof of Theorem 2.6 in [24]).

□

Remark 3.5. The proof of Theorem 3.2 does not work in the multidimensional case for a general $\tilde{z} \notin \mathcal{E}$. Indeed, assume that $\langle z, e_{k_n,V}\rangle \neq 0$ for some sequence $k_n \to +\infty$. Then the well-known asymptotic formula for eigenvalues $\lambda_{k,V} \sim C_d k^{\frac{2}{d}}$ implies that the frequencies $\omega_{m_n k_n} \to 0$ for some integers $m_n \geq 1$ for space dimension $d \geq 3$. Thus the moment problem $\check{u}(\omega_{mk}) = d_{mk}$ cannot be solved in the space $L^1(\mathbb{R}_+, \mathbb{R})$ for a general d_{mk}. Clearly, this does not imply the non-controllability in infinite time of linearized system.

3.2 Proof of Proposition 3.4

The proof of Proposition 3.4 is close to that of Proposition 2.9 in [24]. Let

$$\tilde{\mathcal{G}} := \{u \in L^1(\mathbb{R}_+, \mathbb{R}) : u^2(\cdot)e^{\tilde{B}\cdot} \in L^1(\mathbb{R}_+, \mathbb{R})\}$$

endowed with the norm $\|u\|_{\tilde{\mathcal{G}}} = \|u^2(\cdot)e^{\tilde{B}\cdot}\|_{L^1}$, where the constant $\tilde{B} > 2B$. Then

$$\tilde{\mathcal{F}} := \tilde{\mathcal{G}} \cap H^s_0(\mathbb{R}_+, \mathbb{R})$$

is a subspace of $\mathcal{F}$ defined by (2.16). Moreover, $\tilde{\mathcal{F}}$ is a Hilbert space. The construction of the operator A is based on the following lemma.

Lemma 3.6. *Under the conditions of Proposition 3.4, for any $d \in \ell^\infty_{01}$ there is $u \in \tilde{\mathcal{F}}$ such that $\{\check{u}(\omega_m)\} = d$.*

Proof of Proposition 3.4. By Lemma 3.6, the mapping $u \to \{\check{u}(\omega_m)\}$ is surjective linear bounded form Hilbert space $\tilde{\mathcal{F}}$ onto Banach space ℓ^∞_{01}. Hence it admits a linear bounded right inverse $A : \ell^\infty_{01} \to \tilde{\mathcal{F}}$. □

Proof of Lemma 3.6. Let us show that there is a constant $M > 0$ such that for any $d \in \ell^\infty_{01}, \|d\|_{\ell^\infty_{01}} \leq 1$ there is $u \in B_{\tilde{\mathcal{F}}}(0, M)$ satisfying $\{\check{u}(\omega_m)\} = d$.

Let us introduce the functional

$$F(u) := \|\{\check{u}(\omega_m)\} - d\|_{\ell^\infty}$$

defined on the space $\tilde{\mathcal{F}}$.

Step 1. First, let us show that for any $M > 0$ there is $u_0 \in \overline{B_{\tilde{\mathcal{F}}}(0, M)}$ such that

$$F(u_0) = \inf_{u \in \overline{B_{\tilde{\mathcal{F}}}(0,M)}} F(u). \tag{3.10}$$

To this end, let $u_n \in \overline{B_{\tilde{\mathcal{F}}}(0, M)}$ be an arbitrary minimizing sequence. Since $\tilde{\mathcal{F}}$ is reflexive, without loss of generality, we can assume that there is $u_0 \in \overline{B_{\tilde{\mathcal{F}}}(0, M)}$ such that $u_n \rightharpoonup u_0$ in $\tilde{\mathcal{F}}$. Using the compactness of the injection $H^1([0, N]) \to C([0, N])$ for any $N > 0$ and a diagonal extraction, we can assume that $u_n(t) \to u_0(t)$ uniformly for $t \in [0, N]$. Again extracting a subsequence, if it is necessary, one gets $\{\check{u}_n(\omega_m)\} \to \{\check{u}_0(\omega_m)\}$ in ℓ^∞ as $n \to +\infty$. Indeed, the tails on $[T, +\infty)$, $T \gg 1$ of the integrals (3.9) are small uniformly in n (this comes from the boundedness of u_n in $\tilde{\mathcal{G}}$), and on the finite interval $[0, T]$ the convergence is uniform. This implies that

$$F(u_0) \leq \inf_{u \in \overline{B_{\tilde{\mathcal{F}}}(0,M)}} F(u).$$

Since $u_0 \in \overline{B_{\tilde{\mathcal{F}}}(0, M)}$, we have (3.10).

Step 2. To complete the proof, we need to show that $F(u_0) = 0$.

Lemma 3.7. *Under the conditions of Proposition 3.4, the set*

$$U := \{\{\check{u}(\omega_m)\} : u \in \tilde{\mathcal{F}}\}$$

is dense in ℓ^∞_{10}.

Combining this with the Baire lemma, we get that for sufficiently large $M > 0$

$$\tilde{U} := \{\{\check{u}(\omega_m)\} : u \in B_{\tilde{\mathcal{F}}}(0, M)\}$$

is dense in $B_{\ell^\infty_{10}}(0, 1)$. Thus $F(u_0) = 0$.

□

Proof of Lemma 3.7. It is well known that the dual of ℓ^∞_0 is ℓ^1. Let us suppose that $h = \{h_m\} \in \ell^1$ is such that

$$\langle h, \{\check{u}(\omega_m)\}\rangle_{\ell^1, \ell^\infty_0} = 0$$

for all $u \in \tilde{\mathcal{F}}$. Then replacing in this equality $\check{u}(\omega_m)$ by its integral representation, we get

$$0 = \sum_{m=1}^{+\infty} \int_0^{+\infty} e^{i\omega_m s} u(s) \mathrm{d}s \overline{h_m} = \int_0^{+\infty} u(s) \Big(\sum_{m=1}^{+\infty} e^{i\omega_m s} \overline{h_m} \Big) \mathrm{d}s.$$

Since $\omega_i \neq \omega_j$ for $i \neq j$, by Lemma 3.10 in [22], we have $h_m = 0$ for any $m \geq 1$. This proves that U is dense.

□

3.3 Application of the inverse mapping theorem

The proof is based on the inverse mapping theorem. We project the system onto the tangent space $T_{\tilde{z}}$ and apply the inverse mapping theorem to the following mapping

$$\tilde{\mathcal{U}}_\infty(\cdot) : \mathcal{F} \to T_{\tilde{z}} \cap \mathcal{H}, \quad u \to P\mathcal{U}_\infty(\tilde{z}, u),$$

where P is the orthogonal projection in L^2 onto $T_{\tilde{z}}$, i.e., $Pz = z - \mathrm{Re}\langle z, \tilde{z}\rangle \tilde{z}, z \in L^2$. Notice that $P^{-1} : B_{T_{\tilde{z}}}(0, \delta) \to S$ is well defined for sufficiently small $\delta > 0$. The following result proves that $\tilde{\mathcal{U}}_\infty$ is C^1.

Proposition 3.8. *For a sufficiently small $\delta > 0$ the mapping*

$$\mathcal{U}_\infty(\tilde{z}, \cdot) : B_{\mathcal{F}}(0, \delta) \to \mathcal{H}, \quad u \to \mathcal{U}_\infty(\tilde{z}, u),$$

is C^1. Moreover, $\mathrm{d}\mathcal{U}_\infty(\tilde{z}, u)v = R_\infty(u, v)$, where

$$R_\infty(u, v) := \lim_{n \to +\infty} R_{T_n}(u, v) \quad \textit{in } \mathcal{H}, \tag{3.11}$$

and R_t is the resolving operator of

$$i\dot{z} = -\Delta z + V(x)z + u(t)Q(x)z + v(t)Q(x)\mathcal{U}_t(\tilde{z}, u), \tag{3.12}$$

$$z|_{\partial D} = 0, \tag{3.13}$$

$$z(0, x) = z_0. \tag{3.14}$$

This proposition implies that $\tilde{\mathcal{U}}_\infty \in C^1(B_{\mathcal{F}}(0, \delta))$. By the definition of T_n, we have $\lim_{n \to +\infty} \mathcal{U}_{T_n}(\tilde{z}, 0) = \tilde{z}$. Hence $\mathcal{U}_\infty(\tilde{z}, 0) = \tilde{z}$ and $\tilde{\mathcal{U}}_\infty(0) = 0$. We have $\mathrm{d}\tilde{\mathcal{U}}_\infty(0)v = R_\infty(0, v)$, which is invertible for $\tilde{z} \notin \mathcal{E}_0$ in view of Theorem 3.2. Thus applying the inverse mapping theorem, we complete the proof of Theorem 2.8 for $\tilde{z} \notin \mathcal{E}_0$.

In the case $\tilde{z} \in \mathcal{E}_0$ the linearized system is not controllable, and R_∞ is not invertible. Controllability near $\tilde{z}$ in finite time and for $d = 1$ is proved by Beauchard and Coron [8]. They show that the linearized system is controllable up to codimension one. This implies that the nonlinear system is also controllable up to codimnsion one. The controllability in the missed directions is proved using the intermediate values theorem. In the case $d \geq 1$ and $T = +\infty$, the proof repeats literally the arguments of [8]. We omit the details.

Proof of Proposition 3.8. See [10] for the proof the fact that $\mathcal{U}_T(\tilde{z}, \cdot)$ is C^1 when T is finite, $d = 1$ and phase space is H^3. Let us show that $\mathcal{U}_\infty(\tilde{z}, \cdot)$ is differentiable at any $u \in B_{\mathcal{F}}(0, \delta)$ for sufficiently small $\delta > 0$. We need to prove that

$$\|\mathcal{U}_\infty(\tilde{z}, u + v) - \mathcal{U}_\infty(\tilde{z}, u) - R_\infty(u, v)\|_{\mathcal{H}} = o(\|v\|_{\mathcal{F}}). \tag{3.15}$$

Notice that $h = \mathcal{U}_t(\tilde{z}, u + v) - \mathcal{U}_t(\tilde{z}, u) - R_t(u, v)$ is a solution of

$$\begin{aligned} i\dot{h} &= -\Delta h + V(x)h + (u(t) + v(t))Q(x)h + v(t)Q(x)R_t(u, v), \\ h|_{\partial D} &= 0, \\ h(0, x) &= 0. \end{aligned}$$

Using Proposition 2.4 and Lemma 2.2, we get

$$\begin{aligned} &\|h(\infty)\|_{\mathcal{H}} \leq C \int_0^{+\infty} (\|h(\tau)\|_{H^l} |u(\tau) + v(\tau)| + \|R_\tau(u, v)\|_{H^l} |v(\tau)|) \mathrm{d}\tau \\ &\leq C \int_0^{+\infty} (\|v\|_{C_0^m} \|R_\cdot(u, v)\|_{W^{m,1}([0,\tau], H^2_{(V)})} |u(\tau) + v(\tau)| e^{C(\|u+v\|_{C_0^m} + 1)\tau} \\ &\quad + \|R_\tau(u, v)\|_{H^l} |v(\tau)|) \mathrm{d}\tau \\ &\leq C \int_0^{+\infty} (\|v\|^2_{C_0^m} \|\mathcal{U}_\cdot(\tilde{z}, u)\|_{W^{m,1}([0,\tau], H^2_{(V)})} |u(\tau) + v(\tau)| e^{C(\|u+v\|_{C_0^m} + \|v\|_{C_0^m} + 2)\tau} \\ &\quad + \|v\|_{C_0^m} \|\mathcal{U}_\cdot(\tilde{z}, u)\|_{W^{m,1}([0,\tau], H^2_{(V)})} |v(\tau)| e^{C(\|v\|_{C_0^m} + 1)\tau}) \mathrm{d}\tau \\ &\leq C \|v\|^2_{\mathcal{F}}, \end{aligned}$$

for any $v \in B_{\mathcal{F}}(0, \varepsilon)$, sufficiently small $\varepsilon > 0$, and for sufficiently large $B > 0$ in the definition of $\mathcal{G}$.

It remains to prove that $R_\infty(u, \cdot)$ is continuous in $B_{\mathcal{F}}(0, \delta)$. For $g := R_t(u_1, v) - R_t(u_2, v)$ we have

$$\begin{aligned} i\dot{g} &= -\Delta g + V(x)g + u_1(t)Q(x)g + (u_1(t) - u_2(t))Q(x)R_t(u_2, v) \\ &\quad + v(t)Q(x)(\mathcal{U}_t(\tilde{z}, u_1) - \mathcal{U}_t(\tilde{z}, u_2)), \\ g|_{\partial D} &= 0, \\ g(0, x) &= 0. \end{aligned}$$

By Proposition 2.4,

$$\begin{aligned} \|g(\infty)\|_{\mathcal{H}} \leq C \int_0^{+\infty} &(\|g(\tau)\|_{H^l} |u_1(\tau)| + \|R_\tau(u_2, v)\|_{H^l} |u_1(\tau) - u_2(\tau)| \\ &+ \|\mathcal{U}_\tau(\tilde{z}, u_1) - \mathcal{U}_\tau(\tilde{z}, u_2))\|_{H^l} |v(\tau)|) \mathrm{d}\tau =: I_1 + I_2 + I_3. \end{aligned}$$

Lemmas 2.2 and 2.3 imply

$$
\begin{aligned}
I_1 \leq C \int_0^{+\infty} & (\|R.(u_2, v)\|_{W^{m,1}([0,\tau], H^2_{(V)})} \|u_1(\tau) - u_2(\tau)\|_{C_0^m} \\
& + \|\mathcal{U}.(\tilde{z}, u_1) - \mathcal{U}.(\tilde{z}, u_2)))\|_{W^{m,1}([0,\tau], H^2_{(V)})} ||v(t)\|_{C_0^m}) |u_1(\tau)| e^{C(\|u_1\|_{C_0^m}+1)\tau} \mathrm{d}\tau \\
\leq & C \|u_1 - u_2\|_{\mathcal{F}}.
\end{aligned}
$$

The terms I_2, I_3 are treated in a similar way. Thus we get the continuity of $R_\infty(u, \cdot)$.

□

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