ON THE EXACT CONTROLLABILITY OF HYPERBOLIC MAGNETIC EQUATIONS

XIAOJUN LU

Basque Center for Applied Mathematics(BCAM),
Alameda de Mazarredo 14, 48009 Bilbao, Bizkaia, Spain

Abstract. In this paper, we address the exact controllability problem for the hyperbolic magnetic equation, which plays an important role in the research of quantum mechanics. Typical techniques, such as Hamiltonian induced Hilbert spaces and pseudodifferential operators are introduced. By choosing an appropriate multiplier, we proved the observability inequality with sharp constants. In particular, a genuine compactness-uniqueness argument is applied to obtain the minimal time. In the final analysis, a suitable boundary control is constructed by the systematic Hilbert Uniqueness Method introduced by J. L. Lions. Compared with the micro-local discussion in [2], we do not require the coefficients belong to C^∞. Actually, C^1 is already sufficient for the vector potential of the hyperbolic electromagnetic equation.

Résumé. Dans cet article, on considère le problème de contrôlabilité exacte pour l'équation magnétique hyperbolique, qui joue un rôle important dans la recherche de la mécanique quantique. Les techniques typiques, tels que les espaces de Hilbert induits de l'opérateur hamiltonien et des opérateurs pseudo-différentiels, sont introduites. En choisissant un multiplicateur approprié, on a démonstré l'inégalité d'observabilité avec des constantes fortes. En particulier, l'argument authentique de compacité-unicité est appliqué pour obtenir le temps minimal. Enfin, un contrôle frontière est construit par la méthode systématique, la méthode hilbertienne de l'unicité introduite par J. L. Lions. Par rapport à la discussion dans [2], il n'est pas nécessaire que les coefficients appartiennent à C^∞. En fait, C^1 est déjà suffisante pour le potentiel vecteur de l'équation électromagnétique hyperbolique.

Key words and phrases:. Hamiltonian operator, pseudodifferential operators, generalized Poincaré's inequality, trace theorem, energy conservation law, multiplier, observability inequality, sidewise energy estimate, Hilbert Uniqueness Method, unique continuation theorem, compactness-uniqueness argument

Mots-clés:. Opérateur hamiltonien, opérateurs pseudo-différentiels, inégalité générale de Poincaré, théorème de trace, la loi de conservation d'énergie, multiplicateur, inégalité d'observabilité, estimation latérale d'énergie, la méthode hilbertienne de l'unicité, théorème de prolongement unique, l'argument de compacité-unicité

Contents

Corresponding author: lvxiaojun1119@hotmail.de.
AMS 2010 subject classification: 35A23, 35J25, 35L20, 35Q40, 35S11, 93B05, 93B07.

1. Introduction to hyperbolic magnetic equations and exact controllability

In quantum mechanics, a magnetic field is produced by electric fields varying in time, spinning of the elementary particles, or moving electric charges, etc. For instance, the earth's magnetic field is a consequence of the movement of convection currents in the outer ferromagnetic liquid of the core. Nowadays, with the fast development of modern technology, electromagnetic theory is widely utilized in medical research of organs' biomagnetism, studying the vortex in the superconductor which carries quantized magnetic flux, and predicting geographical cataclysms, such as earthquakes, volcanic eruptions, geomagnetic reversal, etc.

From the viewpoint of mathematics, the magnetic field $\mathbf{B}$ is a solenoidal vector field whose field line either forms a closed curve or extends to infinity. In contrast, a field line of the electric field $\mathbf{E}$ starts at a positive charge and ends at a negative charge.

Let $\mathbf{A}(x)$ be the vector potential of $\mathbf{B}$, which does not depend on time, that is, $\mathbf{B} = \nabla \times \mathbf{A}$. Evidently, $\nabla \cdot \mathbf{B} = \text{div rot}\mathbf{A} = 0$. From one of the Maxwell's equations(μ is the magnetic permeability)

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{B}}{\partial t} = 0,$$

we deduce that $\mathbf{E} = -\nabla\phi$, where the scalar ϕ represents the electric potential. Next we choose an appropriate Lagrangian for the charged particle in the electromagnetic field(q is the electric charge of the particle, and $\mathbf{v}$ is its velocity, m is mass),

$$\mathscr{L} = \frac{m\mathbf{v}^2}{2} - q\phi + q\mathbf{v}\cdot\mathbf{A}.$$

In particular, the canonical momentum is specified by the equation

$$\mathbf{p} = \nabla_{\mathbf{v}}\mathscr{L} = m\mathbf{v} + q\mathbf{A}.$$

Then we define the classical Hamiltonian by Legendre transform,

$$\mathscr{H} \triangleq \mathbf{p}\cdot\mathbf{v} - \mathscr{L} = m\mathbf{v}^2 + q\mathbf{A}\cdot\mathbf{v} - (\frac{m\mathbf{v}^2}{2} - q\phi + q\mathbf{v}\cdot\mathbf{A}) = \frac{(\mathbf{p} - q\mathbf{A})^2}{2m} + q\phi.$$

In quantum mechanics, we replace $\mathbf{p}$ by $-i\hbar\nabla$,($\hbar$ is the Planck constant)

$$\mathscr{H} = \frac{(i\hbar\nabla + q\mathbf{A})^2}{2m} + q\phi.$$

When we do not consider the influence from the electric field $\mathbf{E}$, then the above Hamiltonian can be simplified as the differential operator $\mathscr{H}_{\mathbf{A}}^2 \triangleq (i\nabla + \mathbf{A})^2 : \mathcal{H} \to \mathcal{H}^*$. $\mathcal{H}$ and $\mathcal{H}^*$ will be explained in Section 2. This Hamiltonian operator phenomenologically describes a number of behaviors discovered in superconductors and quantum electrodynamics(QED). Ginzburg-Landau equations, Schrödinger equations, Dirac equations and the matrix Pauli operator are famous examples in this respect. For more details, please refer to [3][7][8][11][26][28].

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with a time-independent vector potential $\mathbf{A}(x)$. In this work, we mainly address the exact controllability of the hyperbolic magnetic equation in the following form,

$$\begin{cases} u_{tt} + \mathscr{H}_{\mathbf{A}}^2 u = 0 & (t,x) \in (0,T)\times\Omega \\ u = \psi & (t,x) \in (0,T)\times\Gamma \\ u(0,x) = u_0, u_t(0,x) = u_1 & x \in \Omega. \end{cases} \tag{1}$$

The concerned function spaces will be explained in the next section. We are interested in the property, i.e. for every initial data (u_0, u_1) and every target (u_0^T, u_1^T), whether there exists a boundary control ψ such that the solution u of (1) satisfies

$$(u(T,x), u_t(T,x)) = (u_0^T, u_1^T) \ \text{ for a. e. } x \in \Omega. \tag{2}$$

Due to the time-reversibility for hyperbolic operators, we can decompose the linear problem (1) into two parts, u^a and u^b. u^a is solution of the homogeneous Dirichlet problem

$$\begin{cases} u_{tt}^a + \mathscr{H}_{\mathbf{A}}^2 u^a = 0 & (t,x) \in (0,T)\times\Omega \\ u^a = 0 & (t,x) \in (0,T)\times\Gamma \\ u^a(T,x) = u_0^T, u_t^a(T,x) = u_1^T & x \in \Omega. \end{cases}$$

Assume that there exists a function ψ such that the solution u^b of the problem

$$\begin{cases} u^b_{tt} + \mathscr{H}^2_{\mathbf{A}} u^b = 0 & (t,x) \in (0,T) \times \Omega \\ u^b = \psi & (t,x) \in (0,T) \times \Gamma \\ u^b(0,x) = u_0(x) - u^a(0,x), u^b_t(0,x) = u_1(x) - u^a_t(0,x) & x \in \Omega \end{cases}$$

satisfies

$$u^b(T) = u^b_t(T) = 0.$$

It is evident that $u = u^a + u^b$ is the solution of (1) and it satisfies (2). Due to this fact, it is sufficient to consider the null controllability of (1). Henceforth, we shall assume the target $u^T_0 = u^T_1 = 0$. Now we give the main theorem.

Theorem 1.1. *Assume that* $\mathbf{A} \in (C^1(\overline{\Omega}))^N$, *and the boundary* $\Gamma \in C^2$. *When* $T > 2\max_\Omega \|x\|_2$, *then for any initial data* $(u_0, u_1) \in L^2 \times H^{-1}$, *we can find a boundary control* $\psi \in L^2([0,T]; L^2(\Gamma))$ *such that the hyperbolic magnetic problem (1) is exactly controllable.*

In the above theorem, one applies a control on the whole boundary Γ. Next we consider the partial boundary control problem. For fixed $x^0 \in \mathbb{R}^N$, let

$$\Gamma_+ \triangleq \{x \in \Gamma : (x - x^0) \cdot \nu(x) > 0\},$$

$$\Gamma_- \triangleq \{x \in \Gamma : (x - x^0) \cdot \nu(x) \le 0\}.$$

And our control problem is stated in the following form,

$$\begin{cases} u_{tt} + \mathscr{H}^2_{\mathbf{A}} u = 0 & (t,x) \in (0,T) \times \Omega \\ u = \psi & (t,x) \in (0,T) \times \Gamma_+ \\ u = 0 & (t,x) \in (0,T) \times \Gamma_- \\ u(0,x) = u_0, u_t(0,x) = u_1 & x \in \Omega. \end{cases} \tag{3}$$

Apply the same techniques as in Theorem 1.1, and one proves the partial boundary control problem (3).

Theorem 1.2. *Assume that* $\mathbf{A} \in (C^1(\overline{\Omega}))^N$, *and the boundary* $\Gamma \in C^2$. *When* $T > 2\max_\Omega \|x - x^0\|_2$, *then for any initial data* $(u_0, u_1) \in L^2 \times H^{-1}$, *we can find a boundary control* $\psi \in L^2([0,T]; L^2(\Gamma_+))$ *such that the hyperbolic magnetic problem (3) is exactly controllable.*

When we include the influence from the electric field $\mathbf{E}$, e.g. replacing $\mathscr{H}^2_{\mathbf{A}}$ by $(i\nabla + \mathbf{A}(x))^2 + \phi(x)$, actually, by applying the same multipliers and compactness-uniqueness argument, one is able to prove the following fact.

Theorem 1.3. *Assume that* $\mathbf{A} \in (C^1(\overline{\Omega}))^N$, $\phi \in L^\infty(\Omega)$ *is a nonnegative real function, and the boundary* $\Gamma \in C^2$. *When* $T > 2\max_\Omega \|x - x^0\|_2$, *then for any initial data* $(u_0, u_1) \in L^2 \times H^{-1}$, *we can find a boundary control* $\psi \in L^2([0,T]; L^2(\Gamma_+))$ *such that the hyperbolic magnetic problem (3) is exactly controllable.*

Remark 1.4. *Compared with the classical multiplier* $\mathbf{H}(x) \cdot \nabla v$, *the new multiplier* $\mathbf{H}(x) \cdot \mathscr{H}_{\mathbf{A}} v$ *has several advantages. On the one hand, it allows to utilize the special quantum structure of the Hamiltonian, such as the magnetic energy conservation law, the test matrix for magnetic field* $\Xi_{\mathbf{A}}$ *(which will be explained later), Coulomb gauge condition, etc. While* $\mathbf{H}(x) \cdot \nabla v$ *will destroy the particular physical structure. On the other hand, it helps to obtain the optimal minimal control time. In contrast, if we use the multiplier* $\mathbf{H}(x) \cdot \nabla v$, *by the compactness-uniqueness argument, there exist several remainder terms which can only be estimated by uncertain constants, such as Poincaré constant, etc., which keep us from getting the optimal minimal control time* $2\|x - x^0\|_2$. *Moreover, this new multiplier can be successfully applied to the discussion of exact controllability for magnetic Schrödinger equations.*

The rest of the paper is organized as follows. Section 2 is devoted to the description of $\mathscr{H}^2_{\mathbf{A}}$-induced function spaces $\mathcal{H}^1_0$, $\mathcal{H}^{-1}$ and introduction of $\mathscr{H}^2_{\mathbf{A}}$-pseudodifferential operators in the distributional sense. And the usual compactness-uniqueness argument is applied to demonstrate a generalized Poincaré's inequality. In Section 3, we use the idea of HUM(Hilbert Uniqueness Method) and apply suitable multipliers to obtain the hidden regularity inequality and observability inequality of the adjoint problem of (1), respectively. Readers can also refer to [6][14][15][16][19][20][31] for the philosophy of HUM. In order to prove the minimal time for exact controllability, one uses a genuine compactness-uniqueness argument introduced in [2][32]. In addition, a rigorous proof of the unique continuation theorem for the elliptic operator $\mathscr{L} = \mathscr{H}^2_{\mathbf{A}} - \phi$ is given in [23]. Particularly, for the 1-dimensional case, one can also apply the sidewise energy estimate concerned with the observability inequality. In the final analysis, some physical interpretation of the theory and open problems conclude this paper.

2. Prerequisites: basic function spaces and pseudodifferential operators

2.1. $\mathscr{H}_{\mathbf{A}}^2$-induced Hilbert spaces. Assume that $\mathcal{H}(\Omega)$ is a Hilbert space. From the Hamiltonian, we can define the corresponding vector operator

$$\mathscr{H}_{\mathbf{A}} \triangleq i\nabla + \mathbf{A}(x) : \mathcal{H}(\Omega) \to (\mathcal{H}(\Omega))^N,$$

where $\mathbf{A}(x) \in \mathbb{R}^N$ is the real-valued potential vector. Now we give a function space induced by the vector operator $\mathscr{H}_{\mathbf{A}}$.

Definition 2.1. *Let $\mathbf{A} \in (L^\infty(\Omega))^N$, we define a complex function space*

$$\mathcal{H}^1(\Omega) \triangleq \{\omega : \omega \in L^2(\Omega), \mathscr{H}_{\mathbf{A}}\omega \in (L^2(\Omega))^N\},$$

which is equipped with the norm

$$\|\omega\|_{\mathcal{H}^1} \triangleq (\|\omega\|_{L^2}^2 + \|\mathscr{H}_{\mathbf{A}}\omega\|_{(L^2)^N}^2)^{\frac{1}{2}},$$

where

$$\|(\omega_1, \cdots, \omega_N)\|_{(L^2)^N} \triangleq (\sum_{\ell=1}^{N} \|\omega_\ell\|_{L^2}^2)^{\frac{1}{2}}.$$

Correspondingly, one defines $\mathcal{H}_0^1$ as the closure of $\mathscr{D}(\Omega)$ in $\mathcal{H}^1$, and $\mathcal{H}^{-1}$ as the dual of $\mathcal{H}_0^1$.

Lemma 2.2. *Actually, $\mathcal{H}^1$ is an equivalent definition of the Sobolev space H^1. Consequently, $\mathcal{H}_0^1 = H_0^1$, $\mathcal{H}^{-1} = H^{-1}$ and the imbeddings $\mathcal{H}_0^1 \hookrightarrow L^2$ and $L^2 \hookrightarrow \mathcal{H}^{-1}$ are both dense and compact.*

Proof. Indeed, by using the definition of norm in each space, one has

- $\mathcal{H}^1 \hookrightarrow H^1$

$$\begin{aligned}\|\omega\|_{H^1}^2 &= \|\omega\|_{L^2}^2 + \|\nabla\omega\|_{(L^2)^N}^2 \\ &= \|\omega\|_{L^2}^2 + \|\mathscr{H}_{\mathbf{A}}\omega - \mathbf{A}\omega\|_{(L^2)^N}^2 \\ &\leq \|\omega\|_{L^2}^2 + 2\|\mathscr{H}_{\mathbf{A}}\omega\|_{(L^2)^N}^2 + 2\|\mathbf{A}\omega\|_{(L^2)^N}^2 \\ &\leq (1 + 2N\|\mathbf{A}\|_{L^\infty}^2)\|\omega\|_{L^2}^2 + 2\|\mathscr{H}_{\mathbf{A}}\omega\|_{(L^2)^N}^2.\end{aligned}$$

- $\mathcal{H}^1 \hookleftarrow H^1$

$$\begin{aligned}\|\omega\|_{\mathcal{H}^1}^2 &= \|\omega\|_{L^2}^2 + \|\mathscr{H}_{\mathbf{A}}\omega\|_{(L^2)^N}^2 \\ &= \|\omega\|_{L^2}^2 + \|(i\nabla + \mathbf{A}(x))\omega\|_{(L^2)^N}^2 \\ &\leq \|\omega\|_{L^2}^2 + 2\|\nabla\omega\|_{(L^2)^N}^2 + 2\|\mathbf{A}(x)\omega\|_{(L^2)^N}^2 \\ &\leq (1 + 2N\|\mathbf{A}\|_{L^\infty}^2)\|\omega\|_{L^2}^2 + 2\|\nabla\omega\|_{(L^2)^N}^2.\end{aligned}$$

Q. E. D. □

In order to introduce the pseudodifferential operators, we need to make some necessary preparation. First we give a series of generalized Green's formulas for the second order operator $(i\mathscr{H}_{\mathbf{A}})^2$ on H^2.

Lemma 2.3. *For $u, v \in H^2$, $\Gamma \in C^1$, one has*

$$\int_\Omega (i\mathscr{H}_{\mathbf{A}})^2 u\bar{v}dx = \int_\Gamma \frac{\partial u}{\partial \nu_{i\mathscr{H}_{\mathbf{A}}}} \cdot \bar{v}d\Gamma - \int_\Omega i\mathscr{H}_{\mathbf{A}}u \cdot \overline{i\mathscr{H}_{\mathbf{A}}v}dx, \tag{4}$$

$$\int_\Omega (i\mathscr{H}_{\mathbf{A}})^2 u\bar{v}dx - \int_\Omega u\overline{(i\mathscr{H}_{\mathbf{A}})^2 v}dx = \int_\Gamma \frac{\partial u}{\partial \nu_{i\mathscr{H}_{\mathbf{A}}}} \cdot \bar{v}d\Gamma - \int_\Gamma u \cdot \overline{\frac{\partial v}{\partial \nu_{i\mathscr{H}_{\mathbf{A}}}}}d\Gamma, \tag{5}$$

$$\int_\Omega (i\mathscr{H}_{\mathbf{A}})^2 udx = \int_\Gamma \frac{\partial u}{\partial \nu_{i\mathscr{H}_{\mathbf{A}}}}d\Gamma - \int_\Omega \mathbf{A}(x) \cdot \mathscr{H}_{\mathbf{A}}udx, \tag{6}$$

where

$$\frac{\partial}{\partial \nu_{i\mathscr{H}_{\mathbf{A}}}} \triangleq (\nabla - i\mathbf{A}) \cdot \nu = \frac{\partial}{\partial \nu} - i\mathbf{A}(x) \cdot \nu, \tag{7}$$

and ν is the unit outward normal vector.

Proof. Keep in mind the classical trace theory in [1]. If Ω is bounded and $\Gamma \in C^1$, then $\mathscr{D}(\overline{\Omega})$ is dense in H^2. And the trace mapping $v \mapsto \overrightarrow{\gamma} v = (\gamma_0 v, \gamma_1 v) = (v|_\Gamma, \frac{\partial v}{\partial \nu}\Big|_\Gamma)$ from $H^2(\Omega)$ to $H^{\frac{3}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$ is linear and continuous. So we prove these identities on $\mathscr{D}(\overline{\Omega})$. On the one hand,

$$\begin{aligned} & \int_\Omega i\mathscr{H}_{\mathbf{A}} u \cdot \overline{i\mathscr{H}_{\mathbf{A}} v} dx \\ = & \int_\Omega \nabla u \cdot \nabla \bar{v} dx - \int_\Omega i\mathbf{A}(x) u \cdot \nabla \bar{v} dx + \int_\Omega i \nabla u \cdot \mathbf{A}(x) \bar{v} dx + \int_\Omega \mathbf{A}\mathbf{A}^T u \bar{v} dx \\ = & \int_\Omega \nabla u \cdot \nabla \bar{v} dx + \langle u, i\mathbf{A} \cdot \nabla v \rangle_{L^2} + \int_\Gamma i u \bar{v} \cdot (\mathbf{A} \cdot \nu) d\Gamma + \langle u, i \nabla \cdot \mathbf{A} v \rangle_{L^2} \\ & + \int_\Omega \mathbf{A}\mathbf{A}^T u \bar{v} dx. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \int_\Omega (i\mathscr{H}_{\mathbf{A}})^2 u \bar{v} dx \\ = & \int_\Gamma \frac{\partial u}{\partial \nu} \cdot \bar{v} d\Gamma - 2 \int_\Gamma i u \bar{v} \cdot (\mathbf{A}(x) \cdot \nu) d\Gamma \\ & - \int_\Omega \nabla u \cdot \nabla \bar{v} dx - \langle u, i\mathbf{A} \cdot \nabla v \rangle_{L^2} - \langle u, i \nabla \cdot \mathbf{A} v \rangle_{L^2} - \int_\Omega \mathbf{A}\mathbf{A}^T u \bar{v} dx. \end{aligned}$$

Notice the definition (7), and one concludes the proof of the first identity. The second identity follows immediately when we consider the conjugate of the first identity. Finally, the third identity is the special case of $v \equiv 1$ of the first identity. Q. E. D. □

Remark 2.4. *When $\mathbf{A} \equiv 0$, one has the classical Green's formulas for Laplacian Δ.*

Remark 2.5. *From the identity (5), we know that $\mathscr{H}_{\mathbf{A}}^2$ is a self-adjoint differential operator on $H^2 \bigcap H_0^1$. In this case, (7) becomes the usual unit outward normal derivative, i.e.*

$$\frac{\partial}{\partial \nu_{i\mathscr{H}_{\mathbf{A}}}} = \frac{\partial}{\partial \nu}.$$

Let ∇_j denote $\frac{\partial}{\partial x_j}$. Here we introduce an important matrix, i.e. the compatibility matrix $\Xi_{\mathbf{A}}$,

$$\Xi_{\mathbf{A}} \triangleq \begin{pmatrix} \xi_{11} & \xi_{12}, & \cdots & \xi_{1N} \\ \xi_{21} & \xi_{22} & \cdots & \xi_{2N} \\ \vdots & \vdots & \cdots & \vdots \\ \xi_{N1} & \xi_{N2} & \cdots & \xi_{NN} \end{pmatrix} \tag{8}$$

where

$$\xi_{jk} \triangleq \begin{vmatrix} \nabla_j & \nabla_k \\ a_j & a_k \end{vmatrix}. \tag{9}$$

Clearly, $\Xi_{\mathbf{A}}$ is an antisymmetric matrix. In quantum mechanics, $\Xi_{\mathbf{A}} \equiv 0$ stands for the case without magnetic field, i.e.

$$\mathbf{B} = \mathrm{rot}\mathbf{A} = 0.$$

Once the magnetic field exists, then $\Xi_{\mathbf{A}} \neq 0$. Consequently, $\Xi_{\mathbf{A}}$ serves as a test matrix for the magnetic field. In the following, one introduces a considerably significant result, which plays a crucial role in the description of the dual of $\mathcal{H}_0^1$.

Lemma 2.6. *(Generalized Poincaré's inequality) Let $\Gamma \in C^1$. Then for any $\omega \in \mathcal{H}_0^1$, there is a constant $C(\Omega) > 0$ such that*

$$\|\omega\|_{L^2} \leq C(\Omega) \|(i\nabla + \mathbf{A})\omega\|_{(L^2)^N}. \tag{10}$$

Proof. 1) $\mathbf{A}$ is a constant vector $\mathbf{a}$. In quantum mechanics, we say the vector potential $\mathbf{A}$ satisfies the famous Coulomb gauge condition $\nabla \cdot \mathbf{A} = 0$. In this case, one applies the method in Fourier analysis. Assume that

Supp $\omega \subset \mathbb{B}$, $\mathbb{B}$ is bounded. For a fixed $\epsilon > 0$, divide the frequency into higher and lower parts, then apply Hölder's inequality, and one has

$$\begin{aligned}
\|\omega\|_{L^2}^2 &= \int_{|\mathbf{a}-\xi|\le\epsilon} |\hat{\omega}(\xi)|^2 d\xi + \int_{|\mathbf{a}-\xi|\ge\epsilon} |\hat{\omega}(\xi)|^2 d\xi \\
&= \int_{|\mathbf{a}-\xi|\le\epsilon} |\hat{\omega}(\xi)|^2 d\xi + \int_{|\mathbf{a}-\xi|\ge\epsilon} \tfrac{|\mathbf{a}-\xi|^2 |\hat{\omega}(\xi)|^2}{|\mathbf{a}-\xi|^2} d\xi \\
&= \int_{|\mathbf{a}-\xi|\le\epsilon} |\int_{\mathbb{B}} \omega(x)\exp(-ix\cdot\xi)dx|^2 d\xi + \int_{|\mathbf{a}-\xi|\ge\epsilon} \tfrac{|\mathbf{a}-\xi|^2 |\hat{\omega}(\xi)|^2}{|\mathbf{a}-\xi|^2} d\xi \\
&\le \|\omega\|_{L^2}^2 \sigma(1)\mathrm{Vol}(\mathbb{B})\epsilon^N + \epsilon^{-2}\|(i\nabla + \mathbf{a})\omega\|_{L^2}^2,
\end{aligned}$$

where $\sigma(1)$ is the volume of a unit ball. Let us choose

$$\epsilon = \sqrt[N]{\frac{1-\beta}{\sigma(1)\mathrm{Vol}(\mathbb{B})}}, \quad \beta \in (0,1),$$

then we have

$$\|\omega\|_{L^2}^2 \le \frac{1}{\beta}\Big(\frac{\sigma(1)\mathrm{Vol}(\mathbb{B})}{1-\beta}\Big)^{\frac{2}{N}} \|(i\nabla + \mathbf{a})\omega\|_{L^2}^2.$$

2) $\mathbf{A}(x)$ is not a constant vector. Let us define a semi-norm on $\mathcal{H}^1$, i.e.

$$|\omega|_{\dot{\mathcal{H}}^1} \triangleq \|(i\nabla + \mathbf{A})\omega\|_{(L^2)^N} = \Big(\sum_{j=1}^N \|i\frac{\partial\omega}{\partial x_j} + a_j(x)\omega\|_{L^2}^2\Big)^{\frac{1}{2}}.$$

First we prove that the above semi-norm is actually a norm on $\mathcal{H}_0^1$. In fact, let ω be a function from $\mathcal{H}_0^1(\Omega)$ such that $|\omega|_{\dot{\mathcal{H}}^1} = 0$. Then one has a system of ordinary differential equations in Ω,

$$\forall j = 1, \cdots, N, \quad i\frac{\partial\omega}{\partial x_j} + a_j(x)\omega = 0.$$

Notice the fact $\omega \in \mathcal{H}_0^1$, then one has a unique solution $\omega = 0$ in Ω.

Define an equivalent norm in $\mathcal{H}^1$, i.e.

$$\|\omega\|_{\mathcal{H}^1} \triangleq \|\omega\|_{L^2} + |\omega|_{\dot{\mathcal{H}}^1}.$$

We prove the inequality (10) by the method of contradiction. If there does not exist any constant $C(\Omega)$ such that, $\forall \omega \in \mathcal{H}_0^1$,

$$\|\omega\|_{L^2} \le C(\Omega)|\omega|_{\dot{\mathcal{H}}^1},$$

then one can find a sequence $\{\omega_m\}_m$ from $\mathcal{H}_0^1$ such that

$$\frac{1}{m}\|w_m\|_{L^2} > |w_m|_{\dot{\mathcal{H}}^1}.$$

Let

$$v_m \triangleq \frac{w_m}{\|w_m\|_{\mathcal{H}^1}},$$

then one defines a sequence $\{v_m\}_m$ from $\mathcal{H}_0^1$ such that

$$\|v_m\|_{\mathcal{H}^1} = 1, \tag{11}$$

$$|v_m|_{\dot{\mathcal{H}}^1} < \frac{1}{m}. \tag{12}$$

Since Ω is bounded and open in $\mathbb{R}^N$, $\Gamma \in C^1$, then the canonical injection from $\mathcal{H}^1(\Omega)$ to $L^2(\Omega)$ is compact. According to (11), one can extract a subsequence $\{v_\mu\}_\mu$ from the sequence $\{v_m\}_m$ such that

$$v_\mu \to v \quad \text{in} \quad L^2(\Omega).$$

From (12), one has

$$\forall j = 1, \cdots, N, \quad i\frac{\partial v_\mu}{\partial x_j} + a_j(x)v_\mu \to 0 \quad \text{in} \quad L^2(\Omega).$$

Since $\mathcal{H}_0^1$ is complete, then one obtains

$$v_\mu \to v \quad \text{in} \quad \mathcal{H}_0^1,$$

with

$$\forall j = 1, \cdots, N, \quad i\frac{\partial v}{\partial x_j} + a_j(x)v = 0.$$

Accordingly, one has

$$v \equiv 0.$$

This is impossible since

$$\|v\|_{\mathcal{H}^1} = \lim_{\mu\to\infty} \|v_\mu\|_{\mathcal{H}^1} = 1.$$

And we conclude the proof. Q. E. D. □

2.2. **$\mathscr{H}_{\mathbf{A}}^2$-induced pseudodifferential operators.** According to Lemma 2.6, we introduce an equivalent norm in $\mathcal{H}_0^1$,

$$\|u\|_{\mathcal{H}_0^1} \triangleq \|\mathscr{H}_{\mathbf{A}} u\|_{(L^2)^N}, \quad \forall u \in \mathcal{H}_0^1. \tag{13}$$

Lemma 2.3 indicates

$$(\mathscr{H}_{\mathbf{A}}^2 u, v)_{L^2} = (u, v)_{\mathcal{H}_0^1}, \quad \forall u \in \mathcal{H}_0^1 \quad \text{such that } \mathscr{H}_{\mathbf{A}}^2 u \in L^2, \quad \forall v \in \mathcal{H}_0^1. \tag{14}$$

We know the fact, the imbeddings $\mathcal{H}_0^1 \hookrightarrow L^2$ and $L^2 \hookrightarrow \mathcal{H}^{-1}$ are both dense and compact. Consequently, $\mathcal{H}_0^1 \hookrightarrow \mathcal{H}^{-1}$ is dense and compact. As a result, it is reasonable to introduce the duality mapping

$$\mathscr{H}_{\mathbf{A}}^2 : \mathcal{H}_0^1 \to \mathcal{H}^{-1}$$

defined by

$$\langle \mathscr{H}_{\mathbf{A}}^2 u, v\rangle_{\mathcal{H}^{-1},\mathcal{H}_0^1} \triangleq (u, v)_{\mathcal{H}_0^1}, \ \forall u, v \in \mathcal{H}_0^1. \tag{15}$$

By Riesz-Fréchet representation theorem, it holds that $\mathscr{H}_{\mathbf{A}}^2$ is an isometric isomorphism of $\mathcal{H}_0^1$ onto $\mathcal{H}^{-1}$. This indicates, $\mathscr{D}(\Omega)$ is also dense in $\mathcal{H}^{-1}$. Particularly, when $\mathbf{A} \in (C^\infty(\overline{\Omega}))^N$, then $\mathscr{H}_{\mathbf{A}}^2(\mathscr{D}(\Omega)) = \mathscr{D}(\Omega)$. Denoting the compact imbedding as

$$I : \mathcal{H}_0^1 \to \mathcal{H}^{-1}.$$

Then we define a linear and compact mapping

$$S \triangleq (\mathscr{H}_{\mathbf{A}}^2)^{-1} \circ I : \mathcal{H}_0^1 \to \mathcal{H}_0^1.$$

Furthermore, S is positive and self-adjoint. Indeed, for $\forall u, v \in \mathcal{H}_0^1$, according to (14), on the one hand,

$$(Su, v)_{\mathcal{H}_0^1} = ((\mathscr{H}_{\mathbf{A}}^2)^{-1} u, v)_{\mathcal{H}_0^1} = (u, v)_{L^2}.$$

On the other hand,

$$(u, Sv)_{\mathcal{H}_0^1} = (u, (\mathscr{H}_{\mathbf{A}}^2)^{-1} v)_{\mathcal{H}_0^1} = (u, v)_{L^2}.$$

Hence,

$$(Su, v)_{\mathcal{H}_0^1} = (u, Sv)_{\mathcal{H}_0^1}.$$

Applying the spectral theorem in [29][30] to the compact, self-adjoint and positive linear operator S, we conclude that the spectrum for $\mathscr{H}_{\mathbf{A}}^2$ on $\mathcal{H}_0^1$ is discrete, which we denote as $\Lambda_{\mathscr{H}_{\mathbf{A}}^2} \triangleq \{\lambda_k\}_k$. And the point spectrum satisfies

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \to \infty,$$

with finite multiplicity. In addition, there exists an orthogonal system of complex-valued eigenfunctions $\{\phi_\lambda(x)\}_{\lambda\in\Lambda_{\mathscr{H}_{\mathbf{A}}^2}}$ in $\mathcal{H}_0^1$, and for each $\lambda \in \Lambda_{\mathscr{H}_{\mathbf{A}}^2}$,

$$\|\phi_\lambda\|_{L^2} = 1.$$

More importantly, $\{\phi_\lambda(x)\}_{\lambda\in\Lambda_{\mathscr{H}_{\mathbf{A}}^2}}$ is dense in $\mathcal{H}_0^1$. Hereafter, we denote by Z the finite combinations of eigenfunctions ϕ_λ. Obviously, Z is dense in $\mathcal{H}_0^1$.

Remark 2.7. *$\{\phi_\lambda(x)\}_{\lambda\in\Lambda_{\mathscr{H}_{\mathbf{A}}^2}}$ has orthogonality in both L^2 and $\mathcal{H}^{-1}$. Indeed, for $k \neq l$,*

$$0 = (\phi_{\lambda_k}, \phi_{\lambda_l})_{\mathcal{H}_0^1} = \langle \mathscr{H}_{\mathbf{A}}^2 \phi_{\lambda_k}, \phi_{\lambda_l}\rangle_{\mathcal{H}^{-1},\mathcal{H}_0^1} = \lambda_k(\phi_{\lambda_k}, \phi_{\lambda_l})_{L^2}. \tag{16}$$

While by using the isometric property of $\mathscr{H}_{\mathbf{A}}^2$ on $\mathcal{H}_0^1$, one has

$$(\phi_{\lambda_k}, \phi_{\lambda_l})_{\mathcal{H}^{-1}} = ((\mathscr{H}_{\mathbf{A}}^2)^{-1}\phi_{\lambda_k}, (\mathscr{H}_{\mathbf{A}}^2)^{-1}\phi_{\lambda_l})_{\mathcal{H}_0^1} = {\lambda_k}^{-1}{\lambda_l}^{-1}(\phi_{\lambda_k}, \phi_{\lambda_l})_{\mathcal{H}_0^1} = 0. \tag{17}$$

Remark 2.8. *Moreover, $\{\phi_\lambda(x)\}_{\lambda\in\Lambda_{\mathscr{H}_{\mathbf{A}}^2}}$ is also dense in both L^2 and $\mathcal{H}^{-1}$ since the density of the imbeddings*

$$\mathcal{H}_0^1 \hookrightarrow L^2 \hookrightarrow \mathcal{H}^{-1}. \tag{18}$$

Remark 2.9. *Let $\Omega = (0, \pi)$, for the Dirichlet operator*

$$(i\frac{\partial}{\partial x} - 1)^2 : \mathcal{H}_0^1 \to \mathcal{H}^{-1},$$

it is easy to know that $\{1, 2^2, 3^2, \cdots, N^2, \cdots\}$ is the set of eigenvalues which are bounded away from 0. And the associated orthonormal basis (in the sense of L^2-norm) in $\mathcal{H}_0^1$ is

$$\Big\{\sqrt{\frac{2}{\pi}}\sin(x)e^{-ix}, \sqrt{\frac{2}{\pi}}\sin(2x)e^{-ix}, \sqrt{\frac{2}{\pi}}\sin(3x)e^{-ix}, \cdots, \sqrt{\frac{2}{\pi}}\sin(Nx)e^{-ix}, \cdots\Big\}.$$

Similar as the harmonic operator $-\Delta + |x|^2$, if $\mathbf{A}(x)$ is not a constant vector, then the eigenfunctions in $\mathcal{H}_0^1$ will not always take the trigonometric form.

With the above notations, one can define the generalized Fourier transform for $f \in \mathcal{H}^{-1}$ as follows:

$$\mathcal{F}f(\lambda) \triangleq \langle f, \phi_\lambda\rangle_{\mathcal{H}^{-1},\mathcal{H}_0^1}. \tag{19}$$

And the corresponding Fourier series, a unique orthogonal expansion in $\mathcal{H}^{-1}$, is of the form

$$f(x) = \sum_{\lambda\in\Lambda_{\mathscr{H}_{\mathbf{A}}^2}} \mathcal{F}f(\lambda)\phi_\lambda(x), \tag{20}$$

with the RHS(right hand-side) converging in $\mathcal{H}^{-1}$. Indeed, for $\forall f \in \mathcal{H}^{-1}$, there is a unique $u_f \in \mathcal{H}_0^1$ such that $\langle f, v\rangle_{\mathcal{H}^{-1},\mathcal{H}_0^1} = (u_f, v)_{\mathcal{H}_0^1}$ for $\forall v \in \mathcal{H}_0^1$. Then

$$\sum_{\lambda\in\Lambda_{\mathscr{H}_{\mathbf{A}}^2}} |\mathcal{F}f(\lambda)|^2\|\phi_\lambda\|_{\mathcal{H}^{-1}}^2 = \sum_{\lambda\in\Lambda_{\mathscr{H}_{\mathbf{A}}^2}} |(u_f, \phi_\lambda)_{\mathcal{H}_0^1}|^2\|\phi_\lambda\|_{\mathcal{H}^{-1}}^2 = \sum_{\lambda\in\Lambda_{\mathscr{H}_{\mathbf{A}}^2}} \lambda|(u_f, \phi_\lambda)_{L^2}|^2\|\phi_\lambda\|_{L^2}^2 < \infty.$$

Remark 2.10. *Notice Remark 2.7 and 2.8, when $f \in \mathcal{H}_0^1$ (resp. $f \in L^2$), (20) is the unique orthogonal expansion converging in $\mathcal{H}_0^1$ (resp. in L^2). Particularly, when $f \in \mathcal{H}_0^1$, then*

$$\|f\|_{\mathcal{H}_0^1}^2 = \sum_{\lambda\in\Lambda_{\mathscr{H}_{\mathbf{A}}^2}} |\mathcal{F}f(\lambda)|^2(\phi_\lambda, \phi_\lambda)_{\mathcal{H}_0^1} = \sum_{\lambda\in\Lambda_{\mathscr{H}_{\mathbf{A}}^2}} \lambda|\mathcal{F}f(\lambda)|^2 < \infty,$$

$$\|f\|_{L^2}^2 = \sum_{\lambda\in\Lambda_{\mathscr{H}_{\mathbf{A}}^2}} |\mathcal{F}f(\lambda)|^2(\phi_\lambda, \phi_\lambda)_{L^2} = \sum_{\lambda\in\Lambda_{\mathscr{H}_{\mathbf{A}}^2}} |\mathcal{F}f(\lambda)|^2 < \infty,$$

$$\|f\|_{\mathcal{H}^{-1}}^2 = \sum_{\lambda\in\Lambda_{\mathscr{H}_{\mathbf{A}}^2}} |\mathcal{F}f(\lambda)|^2(\phi_\lambda, \phi_\lambda)_{\mathcal{H}^{-1}} = \sum_{\lambda\in\Lambda_{\mathscr{H}_{\mathbf{A}}^2}} \frac{1}{\lambda}|\mathcal{F}f(\lambda)|^2 < \infty.$$

At the moment, we are ready to introduce the pseudodifferential operators induced by $\mathscr{H}_{\mathbf{A}}^2$.

Definition 2.11. *Let $\mathbf{A} \in (C^\infty(\overline{\Omega}))^N$. Assume that the complex-valued functional $F \in C(\mathbb{R}_+)$ is polynomially bounded. One defines a generalized linear pseudodifferential operator as follows:*

$$F\Big(\sqrt{\mathscr{H}_{\mathbf{A}}^2}\Big) : D\Big(F\Big(\sqrt{\mathscr{H}_{\mathbf{A}}^2}\Big)\Big) \subset \mathcal{H}^{-1}(\Omega) \to \mathscr{D}'(\Omega), \tag{21}$$

$$F\Big(\sqrt{\mathscr{H}_{\mathbf{A}}^2}\Big)u(x) \triangleq \sum_{\lambda\in\Lambda_{\mathscr{H}_{\mathbf{A}}^2}} F(\sqrt{\lambda})\mathcal{F}u(\lambda)\phi_\lambda(x). \tag{22}$$

The sequence $\{F(\sqrt{\lambda})\}_{\lambda\in\Lambda_{\mathscr{H}_{\mathbf{A}}^2}}$ is referred to as the symbol of $F\Big(\sqrt{\mathscr{H}_{\mathbf{A}}^2}\Big)$.

$F\Big(\sqrt{\mathscr{H}_{\mathbf{A}}^2}\Big)$ is defined in the distributional sense. Indeed, for $\forall\eta\in\mathscr{D}(\Omega)$, since $\mathscr{H}_{\mathbf{A}}^2(\mathscr{D}(\Omega))=\mathscr{D}(\Omega)$, then there exists a unique $\eta_k\in\mathscr{D}(\Omega)$ such that $\underbrace{\mathscr{H}_{\mathbf{A}}^2\cdots\mathscr{H}_{\mathbf{A}}^2}_{k}\eta=\eta_k$ for each $k\in\mathbb{N}$. As a result,

$$\begin{aligned}
(\phi_\lambda,\eta_k)_{L^2} &= (\phi_\lambda,\underbrace{\mathscr{H}_{\mathbf{A}}^2\cdots\mathscr{H}_{\mathbf{A}}^2}_{k}\eta)_{L^2}\\
&= (\mathscr{H}_{\mathbf{A}}^2\phi_\lambda,\underbrace{\mathscr{H}_{\mathbf{A}}^2\cdots\mathscr{H}_{\mathbf{A}}^2}_{k-1}\eta)_{L^2}\\
&= \lambda(\phi_\lambda,\underbrace{\mathscr{H}_{\mathbf{A}}^2\cdots\mathscr{H}_{\mathbf{A}}^2}_{k-1}\eta)_{L^2}\\
&= \lambda^k(\phi_\lambda,\eta)_{L^2}.
\end{aligned}$$

On the one hand, Hölder's inequality tells that $|(\phi_\lambda,\eta_k)_{L^2}|\le\|\phi_\lambda\|_{L^2}\|\eta_k\|_{L^2}=\|\eta_k\|_{L^2}$. As a result, $\{|(\phi_\lambda,\eta)_{L^2}|\}_\lambda$ is a rapidly decreasing sequence with respect to λ. On the other hand, for $\forall u\in\mathcal{H}^{-1}$, there exists a $u_f\in\mathcal{H}_0^1$ such that

$$\mathcal{F}u(\lambda)=\langle u,\phi_\lambda\rangle_{\mathcal{H}^{-1},\mathcal{H}_0^1}=(u_f,\phi_\lambda)_{\mathcal{H}_0^1}=\overline{\langle\mathscr{H}_{\mathbf{A}}^2\phi_\lambda,u_f\rangle}_{\mathcal{H}^{-1},\mathcal{H}_0^1}=\lambda\overline{(\phi_\lambda,u_f)}_{L^2}.$$

Apply Hölder's inequality and one has

$$|\mathcal{F}u(\lambda)|=|\lambda\overline{(\phi_\lambda,u_f)}_{L^2}|\le\lambda\|\phi_\lambda\|_{L^2}\|u_f\|_{L^2}=\lambda\|u_f\|_{L^2}.$$

This indicates, $\{|\mathcal{F}u(\lambda)|\}_\lambda$ is a polynomially bounded sequence with respect to λ. Since F is also a polynomially bounded functional, consequently, the sum on the RHS converges. i.e.

$$\Big\langle F\Big(\sqrt{\mathscr{H}_{\mathbf{A}}^2}\Big)u(x),\eta\Big\rangle_{\mathscr{D}',\mathscr{D}}=\sum_{\lambda\in\Lambda_{\mathscr{H}_{\mathbf{A}}^2}}F(\sqrt{\lambda})\mathcal{F}u(\lambda)(\phi_\lambda,\eta)_{L^2}<\infty.$$

Remark 2.12. *In particular, when $F\in C(\overline{\mathbb{R}_+})$, and $F\Big(\sqrt{\mathscr{H}_{\mathbf{A}}^2}\Big)$ is defined on $\mathcal{H}_0^1$ (resp. L^2), it is reasonable to substitute $\mathscr{D}'(\Omega)$ by $\mathcal{H}_0^1$ (resp. L^2). In this respect, (20) is a natural example. Recall the fact when we consider the unbounded operator $\mathscr{H}_{\mathbf{A}}^2$ on $\mathcal{H}_0^1$, then $\mathscr{D}'(\Omega)$ is replaced by $\mathcal{H}^{-1}$. More information about pseudodifferential operators on different manifolds please refer to* [22][30].

3. Hilbert Uniqueness Method

The systematic method, i.e. Hilbert Uniqueness Method, as is indicated by the terminology, is based on Hilbert spaces constructed by using uniqueness results. In [15][19][20], the authors considered the boundary control and stabilization of wave equations, Klein-Gordon equations and Petrovsky systems etc. in real Hilbert spaces by applying this method. In [31], J. Vancostenoble and E. Zuazua addressed the exact controllability of the wave and Schrödinger equations perturbed by a singular inverse-square potential. Under suitable geometric conditions, exact boundary controllability is proved in the range of subcritical coefficients of the singular potential by the multiplier method.

How to choose suitable multipliers according to the principal elliptic operator is critical. In this section, we are going to apply the multiplier method to solve the exact controllability of the hyperbolic magnetic equations in complex Hilbert spaces. Mechanically speaking, one is going to exert some external force on the boundary of a bounded magnetic field to make the object reach its exact target at a given time T. This theory plays a very important part in the research of plasm physics and liquid crystal technology.

3.1. Definition of weak solutions.

With the terminology introduced in Section 2, we are ready to treat the exact controllability of problem (1) in the time interval $[L,S]$, i.e.

$$\left\{\begin{array}{ll} u_{tt}+\mathscr{H}_{\mathbf{A}}^2u=0 & (t,x)\in(L,S)\times\Omega\\ u=\psi & (t,x)\in(L,S)\times\Gamma\\ u(L,x)=u(L),u_t(L,x)=u_t(L) & x\in\Omega. \end{array}\right. \tag{23}$$

As is discussed in Section 1, the time-reversibility property indicates the equivalence between exact controllability and null controllability for (23). In order to define a solution in the weak sense, first we introduce the following homogeneous adjoint problem

$$\begin{cases} v_{tt} + \mathscr{H}_{\mathbf{A}}^2 v = 0 & (t,x) \in (L,S) \times \Omega \\ v(t,x) = 0 & (t,x) \in (L,S) \times \Gamma \\ v(L,x) = v(L), v_t(L,x) = v_t(L) & x \in \Omega. \end{cases} \tag{24}$$

Let $u, v \in C^2([L,S]; \mathscr{D}(\overline{\Omega}))$. We multiply (24) with u and integrate by parts.

$$\begin{aligned} 0 &= \int_L^S \int_\Omega u \cdot \overline{(v_{tt} + \mathscr{H}_{\mathbf{A}}^2 v)} dx dt \\ &= \int_\Omega (u\overline{v}_t - u_t \overline{v}) dx \Big|_L^S - \int_L^S \int_\Gamma \Big(u \cdot \overline{\tfrac{\partial v}{\partial \nu_{i\mathscr{H}_{\mathbf{A}}}}} - \tfrac{\partial u}{\partial \nu_{i\mathscr{H}_{\mathbf{A}}}} \cdot \overline{v} \Big) d\Gamma dt + \int_L^S \int_\Omega (u_{tt} + \mathscr{H}_{\mathbf{A}}^2 u) \cdot \overline{v} dx dt \\ &= \int_\Omega \Big(u(S)\overline{v}_t(S) - u_t(S)\overline{v}(S) + u_t(L)\overline{v}(L) - u(L)\overline{v}_t(L) \Big) dx - \int_L^S \int_\Gamma \psi \cdot \overline{\tfrac{\partial v}{\partial \nu_{i\mathscr{H}_{\mathbf{A}}}}} d\Gamma dt. \end{aligned}$$

Let $(u, u_t) \in C([L,S]; L^2 \times \mathcal{H}^{-1})$ and fix $(v(L), v_t(L)) \in \mathcal{H}_0^1 \times L^2$, it is reasonable to define

$$\mathscr{L}_L^S \Big(v(L), v_t(L) \Big) \triangleq -u_t(L) \bigotimes u(L) \Big(v(L), v_t(L) \Big) + \int_L^S \int_\Gamma \psi \cdot \overline{\frac{\partial v}{\partial \nu_{i\mathscr{H}_{\mathbf{A}}}}} d\Gamma dt,$$

where

$$-u_t(L) \bigotimes u(L) \Big(v(L), v_t(L) \Big) \triangleq \Big\langle \Big(-u_t(L), u(L) \Big), \Big(v(L), v_t(L) \Big) \Big\rangle_{\mathcal{H}^{-1} \times L^2, \mathcal{H}_0^1 \times L^2}.$$

Then the identity is rewritten as

$$\mathscr{L}_L^S \Big(v(L), v_t(L) \Big) = -u_t(S) \bigotimes u(S) \Big(v(S), v_t(S) \Big). \tag{25}$$

Now we give the definition of weak solution of (23).

Definition 3.1. *We say that (u, u_t) is a weak solution of the non-homogeneous problem (23) if $(u, u_t) \in C([L,S]; L^2 \times \mathcal{H}^{-1})$ and the identity (25) is satisfied for any $S, L \in \mathbb{R}$ and every $(v(L), v_t(L)) \in \mathcal{H}_0^1 \times L^2$ of the homogeneous problem (24).*

Clearly, the term $\int_L^S \int_\Gamma \psi \cdot \overline{\frac{\partial v}{\partial \nu_{i\mathscr{H}_{\mathbf{A}}}}} d\Gamma dt$ in $\mathscr{L}_L^S$ must make sense. This indicates, we are looking for an appropriate boundary control ψ, which is interacting with the homogeneous problem in a suitable Sobolev space on the boundary manifold $\mathcal{M}(\Gamma)$. Now we investigate the homogeneous problem (24) to gain more insightful information of the outward normal derivative with respect to $i\mathscr{H}_{\mathbf{A}}$.

3.2. **Representation of solutions and energy conservation laws for the adjoint problem.** In the whole space $\mathbb{R}$, it is easy to check that, if $\omega(t,x) = F(x+t) + G(x-t)$ solves the classical wave equation $\omega_{tt} - \omega_{xx} = 0$, then $v(t,x) = \exp(ix)F(x+t) + \exp(-ix)G(x-t)$ is the solution of $v_{tt} + (i\frac{\partial}{\partial x} - 1)^2 v = 0$. One can discover that, the propagation speed is 1. We are interested to know the structure of the solution for (24).

Theorem 3.2. *For any given initial data $(v(L), v_t(L)) \in \mathcal{H}_0^1 \times L^2$ and $S \in \mathbb{R}$, the homogeneous problem (24) has a unique solution such that*

$$v \in C([L,S]; \mathcal{H}_0^1) \bigcap C^1([L,S]; L^2) \bigcap C^2([L,S]; \mathcal{H}^{-1}).$$

Proof. At first, we assume sufficient regularity for the solution. Apply the generalized Fourier transform (19) (20) to the homogeneous equation (24), then one has

$$\sum_{\lambda \in \Lambda_{\mathscr{H}_{\mathbf{A}}^2}} \Big((\mathcal{F}v(\lambda))_{tt} + \lambda \mathcal{F}v(\lambda) \Big) \phi_\lambda(x) = 0.$$

Since $\{\phi_\lambda(x)\}_{\lambda \in \Lambda_{\mathscr{H}_{\mathbf{A}}^2}}$ is a complete orthogonal system of complex-valued eigenfunctions in $\mathcal{H}^{-1}$, then for $\forall \lambda \in \Lambda_{\mathscr{H}_{\mathbf{A}}^2}$,

$$(\mathcal{F}v(\lambda))_{tt} + \lambda \mathcal{F}v(\lambda) = 0.$$

As a result,

$$\mathcal{F}v(\lambda) = \cos(t\sqrt{\lambda})\mathcal{F}v(L)(\lambda) + \frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}} \mathcal{F}v_t(L)(\lambda).$$

Since $(v(L), v_t(L)) \in \mathcal{H}_0^1 \times L^2$, then in $\mathcal{H}_0^1$,

$$v(x) = \sum_{\lambda \in \Lambda_{\mathscr{H}_{\mathbf{A}}^2}} \mathcal{F}v(\lambda)\phi_\lambda(x).$$

Consequently, Definition 2.11 gives the unique converging form by pseudodifferential operators,

$$v(t,x) = \cos\left(t\sqrt{\mathscr{H}_{\mathbf{A}}^2}\right)v(L) + \frac{\sin\left(t\sqrt{\mathscr{H}_{\mathbf{A}}^2}\right)}{\sqrt{\mathscr{H}_{\mathbf{A}}^2}}v_t(L). \tag{26}$$

Notice Remark 2.12, then it can be checked that $v \in C([L,S]; \mathcal{H}_0^1)$. Other regularity follows immediately from (26) through symbol calculus. Indeed, if the initial data are composed of finite combinations of eigenfunctions, then elliptic regualarity theory gives $u \in C^\infty(\mathbb{R}; H^2(\Omega))$. Q. E. D. □

Now we turn to the energy conservation property for (24).

Definition 3.3. *We define the energy of homogeneous problem (24) as*

$$\mathbb{E}(v)(t) \triangleq \|v_t\|_{L^2}^2 + \|v\|_{\mathcal{H}_0^1}^2.$$

Lemma 3.4. *The energy conservation law holds for the homogeneous problem (24).*

Proof. Here we apply the symbol calculus for pseudodifferential operators. All the steps are rigorously deduced by the definition of (21)(22) induced by $\mathscr{H}_{\mathbf{A}}^2$ on $\mathcal{H}_0^1$. From (26), one has

$$\begin{aligned}
\|v_t(t,\cdot)\|_{L^2}^2 &= \|-\sqrt{\mathscr{H}_{\mathbf{A}}^2}\sin\left(t\sqrt{\mathscr{H}_{\mathbf{A}}^2}\right)v(L) + \cos\left(t\sqrt{\mathscr{H}_{\mathbf{A}}^2}\right)v_t(L)\|_{L^2}^2 \\
&= \|\sqrt{\mathscr{H}_{\mathbf{A}}^2}\sin\left(t\sqrt{\mathscr{H}_{\mathbf{A}}^2}\right)v(L)\|_{L^2}^2 + \|\cos\left(t\sqrt{\mathscr{H}_{\mathbf{A}}^2}\right)v_t(L)\|_{L^2}^2 \\
&\quad -2\mathrm{Re}\Big(\sqrt{\mathscr{H}_{\mathbf{A}}^2}\sin\left(t\sqrt{\mathscr{H}_{\mathbf{A}}^2}\right)v(L), \cos\left(t\sqrt{\mathscr{H}_{\mathbf{A}}^2}\right)v_t(L)\Big)_{L^2}, \\
\|v(t,\cdot)\|_{\mathcal{H}_0^1}^2 &= \|\sqrt{\mathscr{H}_{\mathbf{A}}^2}\cos\left(t\sqrt{\mathscr{H}_{\mathbf{A}}^2}\right)v(L)\|_{L^2}^2 + \|\sin\left(t\sqrt{\mathscr{H}_{\mathbf{A}}^2}\right)v_t(L)\|_{L^2}^2, \\
&\quad +2\mathrm{Re}\Big(\sqrt{\mathscr{H}_{\mathbf{A}}^2}\sin\left(t\sqrt{\mathscr{H}_{\mathbf{A}}^2}\right)v(L), \cos\left(t\sqrt{\mathscr{H}_{\mathbf{A}}^2}\right)v_t(L)\Big)_{L^2}.
\end{aligned}$$

As a result,

$$\begin{aligned}
\mathbb{E}(v)(t) &= \|\sqrt{\mathscr{H}_{\mathbf{A}}^2}\sin\left(t\sqrt{\mathscr{H}_{\mathbf{A}}^2}\right)v(L)\|_{L^2}^2 + \|\sqrt{\mathscr{H}_{\mathbf{A}}^2}\cos\left(t\sqrt{\mathscr{H}_{\mathbf{A}}^2}\right)v(L)\|_{L^2}^2 \\
&\quad +\|\sin\left(t\sqrt{\mathscr{H}_{\mathbf{A}}^2}\right)v_t(L)\|_{L^2}^2 + \|\cos\left(t\sqrt{\mathscr{H}_{\mathbf{A}}^2}\right)v_t(L)\|_{L^2}^2 \\
&= (\mathscr{H}_{\mathbf{A}}^2 v(L), v(L))_{L^2} + \|v_t(L)\|_{L^2}^2 = \|v(L)\|_{\mathcal{H}_0^1}^2 + +\|v_t(L)\|_{L^2}^2 \\
&= \mathbb{E}(v)(L).
\end{aligned}$$

Q. E. D. □

3.3. **Hidden regularity inequality.** Recall the definition of $\mathscr{L}_L^S$. In order to know more about the boundary control ψ, one needs to investigate the term $\frac{\partial v}{\partial \nu}$ carefully. By the density method, traditional trace theory tells us that $\frac{\partial v}{\partial \nu}(t) \in H^{\frac{1}{2}}(\Gamma)$. In the following, we will give an important inequality which indicates the continuity of $\frac{\partial v}{\partial \nu}(t)$ with respect to initial data.

Theorem 3.5. *(Hidden regularity inequality) For any given $L, S \in \mathbb{R}$ and initial data $(v(L), v_t(L)) \in \mathcal{H}_0^1 \times L^2$, the outward normal derivative defined in the form of* (7) *satisfies*

$$\frac{\partial v}{\partial \nu} \in L^2(L, S; L^2(\Gamma))$$

and

$$\int_L^S \int_\Omega |\frac{\partial v}{\partial \nu}|^2 d\Gamma dt \le C(1 + (S-L))\mathbb{E}(v)(L).$$

In particular, for the 1-dimensional case, let $\Omega=(-1,1)$, then the above inequality can be rewritten as

$$\int_L^S(|v_x(t,-1)|^2+|v_x(t,1)|^2)dt\le C(1+(S-L))\mathbb{E}(v)(L).$$

3.3.1. *Deduction of the first identity.* In order to prove the above theorem, first we introduce an important identity. In the following, we consider the problem (24) with initial data from $Z\times Z$ due to its density in $\mathcal{H}_0^1\times L^2$. To begin with, we apply the multiplier $\mathbf{H}(x)\cdot\mathscr{H}_{\mathbf{A}}v$ to (24), where the vector field $\mathbf{H}\in C^1(\overline{\Omega};\mathbb{R}^N)$. One decomposes the following integral into two parts, i.e.

$$\begin{aligned}&\textstyle\int_L^S(v_{tt}+\mathscr{H}_{\mathbf{A}}^2v,\mathbf{H}(x)\cdot\mathscr{H}_{\mathbf{A}}v)_{L^2}dt\\ =\;&\underbrace{\int_L^S(v_{tt},\mathbf{H}(x)\cdot\mathscr{H}_{\mathbf{A}}v)_{L^2}dt}_{(I)}+\underbrace{\int_L^S(\mathscr{H}_{\mathbf{A}}^2v,\mathbf{H}(x)\cdot\mathscr{H}_{\mathbf{A}}v)_{L^2}dt}_{(II)}.\end{aligned}$$

Apply the generalized Green's formula in Lemma 2.3, then one has

$$\begin{aligned}(I)&=(v_t,\mathbf{H}(x)\cdot\mathscr{H}_{\mathbf{A}}v)_{L^2}\Big|_L^S-\textstyle\int_L^S(v_t,\mathbf{H}(x)\cdot\mathscr{H}_{\mathbf{A}}v_t)_{L^2}dt\\&=(v_t,\mathbf{H}(x)\cdot\mathscr{H}_{\mathbf{A}}v)_{L^2}\Big|_L^S-\textstyle\int_L^S\int_\Omega v_t\cdot\Big(\mathbf{H}(x)\cdot\overline{\mathscr{H}_{\mathbf{A}}v_t}\Big)dxdt.\\(II)&=\textstyle\int_L^S\int_\Omega\mathscr{H}_{\mathbf{A}}^2v\cdot\Big(\mathbf{H}(x)\cdot\overline{\mathscr{H}_{\mathbf{A}}v}\Big)dxdt\\&=\textstyle\int_L^S\int_\Gamma-\frac{\partial v}{\partial\nu}\cdot\Big(\mathbf{H}(x)\cdot\overline{\mathscr{H}_{\mathbf{A}}v}\Big)d\Gamma dt+\underbrace{\int_L^S\int_\Omega\mathscr{H}_{\mathbf{A}}v\cdot\overline{\mathscr{H}_{\mathbf{A}}\Big\{\mathbf{H}(x)\cdot\mathscr{H}_{\mathbf{A}}v\Big\}}dxdt}_{(III)}.\end{aligned}$$

And we calculate the term (III).

$$\begin{aligned}(III)&=\textstyle\int_L^S\int_\Omega\sum\limits_j\Big(i\nabla_jv+a_j(x)v\Big)\cdot\overline{\Big(i\nabla_j+a_j(x)\Big)\Big\{\sum\limits_k h_k(x)\cdot\Big(i\nabla_kv+a_k(x)v\Big)\Big\}}dxdt\\&=\textstyle\int_L^S\int_\Omega\sum\limits_j\Big(i\nabla_jv+a_j(x)v\Big)\cdot\overline{i\nabla_j\Big\{\sum\limits_k h_k(x)\cdot\Big(i\nabla_kv+a_k(x)v\Big)\Big\}}dxdt\\&\quad+\textstyle\int_L^S\int_\Omega\sum\limits_j\Big(i\nabla_jv+a_j(x)v\Big)\cdot\overline{a_j(x)\Big\{\sum\limits_k h_k(x)\cdot\Big(i\nabla_kv+a_k(x)v\Big)\Big\}}dxdt\\&=\textstyle\int_L^S\int_\Omega\sum\limits_{j,k}\Big(i\nabla_jv+a_j(x)v\Big)\cdot\overline{i\nabla_jh_k(x)\cdot\Big(i\nabla_kv+a_k(x)v\Big)}dxdt\\&\quad+\textstyle\int_L^S\int_\Omega\sum\limits_{j,k}\Big(i\nabla_jv+a_j(x)v\Big)\cdot\overline{h_k(x)\cdot\Big(i\nabla_j+a_j(x)\Big)\Big(i\nabla_kv+a_k(x)v\Big)}dxdt.\end{aligned}$$

Combining (I)(II)(III), one has the identity,

$$\begin{aligned}&\textstyle\int_L^S\int_\Gamma\frac{\partial v}{\partial\nu}\cdot\Big(\mathbf{H}(x)\cdot\overline{\mathscr{H}_{\mathbf{A}}v}\Big)d\Gamma dt\\=\;&(v_t,\mathbf{H}(x)\cdot\mathscr{H}_{\mathbf{A}}v)_{L^2}\Big|_L^S-\textstyle\int_L^S\int_\Omega v_t\cdot\Big(\mathbf{H}(x)\cdot\overline{\mathscr{H}_{\mathbf{A}}v_t}\Big)dxdt\\&+\textstyle\int_L^S\int_\Omega\sum\limits_{j,k}\Big(i\nabla_jv+a_j(x)v\Big)\cdot\overline{i\nabla_jh_k(x)\cdot\Big(i\nabla_kv+a_k(x)v\Big)}dxdt\\&+\textstyle\int_L^S\int_\Omega\sum\limits_{j,k}\Big(i\nabla_jv+a_j(x)v\Big)\cdot\overline{h_k(x)\cdot\Big(i\nabla_j+a_j(x)\Big)\Big(i\nabla_kv+a_k(x)v\Big)}dxdt.\end{aligned}\tag{27}$$

Moreover, for the 1-dimensional case, identity (27) is simplified as

$$(28)\qquad -i\int_L^S |v_x|^2 h(x)dt\Big|_{-1}^1 = 2i\ \mathrm{Im}\int_L^S\int_{-1}^1 v_{tt}\cdot\Big(h(x)\cdot\overline{\mathscr{H}_{\mathbf{A}}v}\Big)dxdt + \int_L^S\int_{-1}^1 \mathscr{H}_{\mathbf{A}}v\cdot\overline{i\nabla h(x)\cdot\mathscr{H}_{\mathbf{A}}v}dxdt.$$

3.3.2. *Proof of Theorem 3.5 in 1-dimensional case.* Choose $g(x)\in C^2([-1,1])$ such that $\nabla g\neq 0$ and define $h(x)\triangleq\frac{\nabla g}{|\nabla g|}$. It is evident that $h(x)\in C^1([-1,1])$ and $h=\nu$ on Γ. From (28), one has,

$$(29)\qquad \begin{aligned} -i\int_L^S |v_x|^2 h(x)dt\Big|_{-1}^1 \;=\;& 2i\ \mathrm{Im}(v_t, h(x)\cdot\mathscr{H}_{\mathbf{A}}v)_{L^2}\Big|_L^S - 2i\ \mathrm{Im}\int_L^S\int_{-1}^1 v_t\cdot\Big(h(x)\cdot\overline{\mathscr{H}_{\mathbf{A}}v_t}\Big)dxdt\\ & -i\int_L^S\int_{-1}^1 \nabla h(x)|\mathscr{H}_{\mathbf{A}}v|^2dxdt.\end{aligned}$$

On the one hand,

$$-i\int_L^S |v_x|^2 h(x)dt\Big|_{-1}^1 = -i\int_L^S(|v_x(t,-1)|^2+|v_x(t,1)|^2)dt.$$

On the other hand,

$$\begin{aligned} &\int_L^S\int_{-1}^1 v_t\cdot\Big(h(x)\cdot\overline{\mathscr{H}_{\mathbf{A}}v_t}\Big)dxdt\\ =\;&\int_L^S v_t\cdot h(x)\cdot\overline{iv_t}dt\Big|_{-1}^1 - \int_L^S\int_{-1}^1\nabla h(x)\cdot v_t\cdot\overline{iv_t}dxdt\\ &-\int_L^S\int_{-1}^1\nabla v_t\cdot\Big(h(x)\cdot\overline{iv_t}\Big)dxdt + \int_L^S\int_{-1}^1 v_t\cdot a(x)\cdot\Big(h(x)\cdot\overline{v}_t\Big)dxdt\\ =\;& i\int_L^S\int_{-1}^1\nabla h(x)|v_t|^2dxdt + \int_L^S\int_{-1}^1\mathscr{H}_{\mathbf{A}}v_t\cdot h(x)\cdot\overline{v}_tdxdt.\end{aligned}$$

Transfer the term $\int_L^S\int_{-1}^1\mathscr{H}_{\mathbf{A}}v_t\cdot h(x)\cdot\overline{v}_tdxdt$ on the RHS to the LHS, then we have

$$2i\ \mathrm{Im}\int_L^S\int_{-1}^1 v_t\cdot\Big(h(x)\cdot\overline{\mathscr{H}_{\mathbf{A}}v_t}\Big)dxdt = i\int_L^S\int_{-1}^1\nabla h(x)|v_t|^2dxdt.$$

And (29) can be rewritten as

$$(30)\qquad \begin{aligned} &\tfrac{1}{2}\int_L^S(|v_x(t,-1)|^2+|v_x(t,1)|^2)dt\\ =\;& -\mathrm{Im}(v_t,h(x)\cdot\mathscr{H}_{\mathbf{A}}v)_{L^2}\Big|_L^S + \tfrac{1}{2}\int_L^S\int_{-1}^1\nabla h(x)(|v_t|^2+|\mathscr{H}_{\mathbf{A}}v|^2)dxdt.\end{aligned}$$

Consider the first term on the RHS,

$$\Big|\mathrm{Im}(v_t,h(x)\cdot\mathscr{H}_{\mathbf{A}}v)_{L^2}\Big|_L^S\Big| \le 2\sup_x|h(x)|\mathbb{E}(v)(L).$$

As for the second term on the RHS, it is clear that

$$\Big|\frac{1}{2}\int_L^S\int_{-1}^1\nabla h(x)(|v_t|^2+|\mathscr{H}_{\mathbf{A}}v|^2)dxdt\Big| \le \sup_x|\nabla h(x)|(S-L)\mathbb{E}(v)(L).$$

In the final analysis,

$$\int_L^S(|v_x(t,-1)|^2+|v_x(t,1)|^2)dt \le C(\mathbf{H})(1+(S-L))\mathbb{E}(v)(L).$$

This concludes the proof for 1-dimensional case. Q. E. D. □

3.3.3. *Deduction of the second identity.* For multidimensional case, (27) is far from enough. It is necessary to develop a new identity to obtain more information. First we decompose (27) into four parts, i.e.

$$
\begin{aligned}
&\textstyle\int_L^S \int_\Gamma \frac{\partial v}{\partial \nu} \cdot \Big(\mathbf{H}(x) \cdot \overline{\mathscr{H}_{\mathbf{A}} v}\Big) d\Gamma dt \\
= \; & \underbrace{(v_t, \mathbf{H}(x) \cdot \mathscr{H}_{\mathbf{A}} v)_{L^2}\Big|_L^S}_{(I)} - \underbrace{\int_0^T \int_\Omega \sum_j v_t \cdot h_j(x) \cdot \overline{\Big(i\nabla_j v_t + a_j(x) v_t\Big)} dxdt}_{(II)} \\
& \underbrace{+ \int_L^S \int_\Omega \sum_{j,k} \Big(i\nabla_j v + a_j(x) v\Big) \cdot \overline{i\nabla_j h_k(x) \cdot \Big(i\nabla_k v + a_k(x) v\Big)} dxdt}_{(III)} \\
& \underbrace{+ \int_L^S \int_\Omega \sum_{j,k} \Big(i\nabla_j v + a_j(x) v\Big) \cdot \overline{h_k(x) \cdot \Big(i\nabla_j + a_j(x)\Big)\Big(i\nabla_k v + a_k(x) v\Big)} dxdt}_{(IV)}.
\end{aligned}
$$

We treat the second term (II).

$$
\begin{aligned}
(II) \; &= \; \textstyle\int_L^S \int_\Omega v_t \cdot \Big(\mathbf{H}(x) \cdot \overline{\mathscr{H}_{\mathbf{A}} v_t}\Big) dxdt \\
&= \; \sum_j i \int_L^S \int_\Omega \nabla_j v_t \cdot h_j(x) \cdot \overline{v}_t dxdt + \sum_j \int_L^S \int_\Omega v_t \cdot h_j(x) \cdot a_j(x) \cdot \overline{v}_t dxdt \\
& \quad + \sum_j i \int_L^S \int_\Omega \nabla_j h_j(x) \cdot |v_t|^2 dxdt \\
&= \; \textstyle\int_L^S \int_\Omega \overline{v}_t \cdot \Big(\mathbf{H}(x) \cdot \mathscr{H}_{\mathbf{A}} v_t\Big) dxdt + i \int_L^S \int_\Omega \Big(\nabla \cdot \mathbf{H}(x)\Big) \cdot |v_t|^2 dxdt.
\end{aligned}
$$

This leads to the following fact,

$$
2 \; \mathrm{Im} \textstyle\int_L^S \int_\Omega v_t \cdot \Big(\mathbf{H}(x) \cdot \overline{\mathscr{H}_{\mathbf{A}} v_t}\Big) dxdt = \int_L^S \int_\Omega \Big(\nabla \cdot \mathbf{H}(x)\Big) \cdot |v_t|^2 dxdt.
$$

Next we turn to the third term (III). Let

$$
\Theta_{\mathbf{H}} \triangleq \begin{pmatrix} \nabla_1 h_1 & \nabla_1 h_2, & \cdots & \nabla_1 h_N \\ \nabla_2 h_1 & \nabla_2 h_2, & \cdots & \nabla_2 h_N \\ \vdots & \vdots & \cdots & \vdots \\ \nabla_N h_1 & \nabla_N h_2, & \cdots & \nabla_N h_N \end{pmatrix}.
$$

Then we represent (III) in the sense of matrix calculus, i.e.

$$
(III) = -i \int_L^S \int_\Omega \mathscr{H}_{\mathbf{A}} v \times \Theta_{\mathbf{H}} \times \overline{\mathscr{H}_{\mathbf{A}}^T v} dxdt,
$$

where the quadratic form is

$$
\mathscr{H}_{\mathbf{A}} v \times \Theta_{\mathbf{H}} \times \overline{\mathscr{H}_{\mathbf{A}}^T v} = (i\nabla_1 v + a_1 v, \cdots) \begin{pmatrix} \nabla_1 h_1 & \nabla_1 h_2, & \cdots & \nabla_1 h_N \\ \nabla_2 h_1 & \nabla_2 h_2, & \cdots & \nabla_2 h_N \\ \vdots & \vdots & \cdots & \vdots \\ \nabla_N h_1 & \nabla_N h_2, & \cdots & \nabla_N h_N \end{pmatrix} \begin{pmatrix} \overline{i\nabla_1 v + a_1 v} \\ \overline{i\nabla_2 v + a_2 v} \\ \vdots \\ \overline{i\nabla_N v + a_N v} \end{pmatrix}.
$$

Then we consider the fourth item (IV), which is the most complicated part. (In the following, $[,]$ is the Lie bracket.)

$$\begin{aligned}(IV) &= \int_L^S \int_\Omega \sum_{j,k} \Big(i\nabla_j v + a_j(x)v\Big) \cdot \overline{h_k(x) \cdot \Big(i\nabla_j + a_j(x)\Big)\Big(i\nabla_k v + a_k(x)v\Big)} dxdt \\ &= \int_L^S \int_\Omega \sum_{j,k} \Big(i\nabla_j v + a_j(x)v\Big) \cdot h_k(x) \cdot \overline{i\nabla_k \Big(i\nabla_j v + a_j(x)v\Big)} dxdt \\ &\quad \underbrace{+\int_L^S \int_\Omega \sum_{j,k} \Big(i\nabla_j v + a_j(x)v\Big) \cdot h_k(x) \cdot a_k(x) \cdot \overline{\Big(i\nabla_j v + a_j(x)v\Big)} dxdt}_{(V)} \\ &\quad + \int_L^S \int_\Omega \sum_{j,k} \Big(i\nabla_j v + a_j(x)v\Big) \cdot h_k(x) \cdot \overline{\Big[i\nabla_j + a_j(x), i\nabla_k + a_k(x)\Big] v} dxdt.\end{aligned}$$

It is worth noticing that the item (V) is purely real. When we consider only the imaginary part of (IV), (V) disappears and one has

$$\begin{aligned}\mathrm{Im}(IV) &= -\tfrac{1}{2}\int_L^S \int_\Gamma \Big(\mathbf{H}(\mathbf{x}) \cdot \nu\Big) \cdot \Big|\mathscr{H}_{\mathbf{A}} v\Big|^2 d\Gamma dt + \tfrac{1}{2}\int_L^S \int_\Omega \Big(\nabla \cdot \mathbf{H}(\mathbf{x})\Big)\Big|\mathscr{H}_{\mathbf{A}} v\Big|^2 dxdt \\ &\quad + \mathrm{Im} \int_L^S \int_\Omega \overline{iv}\mathscr{H}_{\mathbf{A}} v \times \Xi_{\mathbf{A}} \times \mathbf{H}^T dxdt,\end{aligned}$$

where $\Xi_{\mathbf{A}}$ is the test matrix of magnetic field defined in (8), i.e.

$$\mathscr{H}_{\mathbf{A}} v \times \Xi_{\mathbf{A}} \times \mathbf{H}^T = (i\nabla_1 v + a_1 v, \cdots, i\nabla_N v + a_N v) \begin{pmatrix} \xi_{11} & \xi_{12}, & \cdots & \xi_{1N} \\ \xi_{21} & \xi_{22} & \cdots & \xi_{2N} \\ \vdots & \vdots & \cdots & \vdots \\ \xi_{N1} & \xi_{N2} & \cdots & \xi_{NN} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_N \end{pmatrix}.$$

Combining (I)(II)(III)(IV), one has a new identity,

$$\begin{aligned}& -\tfrac{1}{2}\int_L^S \int_\Gamma \Big|\tfrac{\partial v}{\partial \nu}\Big|^2 \cdot \Big(\mathbf{H}(x) \cdot \nu\Big) d\Gamma dt \\ (31) \qquad &= \mathrm{Im}(v_t, \mathbf{H}(x) \cdot \mathscr{H}_{\mathbf{A}} v)_{L^2}\Big|_L^S - \tfrac{1}{2}\int_L^S \int_\Omega \Big(\nabla \cdot \mathbf{H}(\mathbf{x})\Big) \cdot \Big(|v_t|^2 - \Big|\mathscr{H}_{\mathbf{A}} v\Big|^2\Big) dxdt \\ &\quad - \mathrm{Re}\int_L^S \int_\Omega \mathscr{H}_{\mathbf{A}} v \times \Theta_{\mathbf{H}} \times \overline{\mathscr{H}_{\mathbf{A}}^T v} dxdt - \mathrm{Re}\int_L^S \int_\Omega \overline{v}\mathscr{H}_{\mathbf{A}} v \times \Xi_{\mathbf{A}} \times \mathbf{H}^T dxdt.\end{aligned}$$

3.3.4. *Proof of Theorem 3.5 in multidimensional case.* Since $\overline{\Omega}$ is compact, then it can be covered with a finite number of neighborhoods $\mathcal{O}_k \subset \mathbb{R}^N$, $k = 1, 2, \cdots, m$, in which there is a unique $\mathcal{O}_1$ satisfying $\mathcal{O}_1 \cap \Gamma = \varnothing$. Similar as the procedure in 1-dimensional case, we choose $\zeta_1 = 0$ in $\mathcal{O}_1$ and $\zeta_k \in C^1(\overline{\mathcal{O}_k})$ such that $\zeta_k = \nu$ on $\mathcal{O}_k \cap \Gamma$ for $k = 2, \cdots, m$. Let $\theta_k \in C_0^2(\mathcal{O}_k)$, $k = 1, \cdots, m$ be a partition of unity, corresponding to the covering $\{\mathcal{O}_k\}_k$. Then we define

$$\mathbf{H}(x) \triangleq \Big(\sum_k \theta_k \zeta_k\Big)\Big|_{\overline{\Omega}}.$$

It is easy to check that $\mathbf{H} \in C^1(\overline{\Omega})$ and $\mathbf{H} = \nu$ on Γ. Keep in mind the energy conservation law, now we begin to estimate each term on the RHS of (31).

1) For the first term,

$$\begin{aligned}
&\Big|\mathrm{Im}(v_t, \mathbf{H}(x)\cdot \mathscr{H}_{\mathbf{A}} v)_{L^2}\Big|_L^S\Big| \\
\leq\ & \Big|\int_\Omega v_t(S)\cdot \Big(\mathbf{H}(x)\cdot \overline{\mathscr{H}_{\mathbf{A}} v(S)}\Big)dx\Big| + \Big|\int_\Omega v_t(L)\cdot \Big(\mathbf{H}(x)\cdot \overline{\mathscr{H}_{\mathbf{A}} v(L)}\Big)dx\Big| \\
\leq\ & \tfrac{1}{2}(1+\max_x \|\mathbf{H}\|_2)\Big(\int_\Omega |v_t(S)|^2 dx + \int_\Omega |\mathscr{H}_{\mathbf{A}} v(S)|^2 dx\Big) \\
& +\tfrac{1}{2}(1+\max_x \|\mathbf{H}\|_2)\Big(\int_\Omega |v_t(L)|^2 dx + \int_\Omega |\mathscr{H}_{\mathbf{A}} v(L)|^2 dx\Big) \\
=\ & 2(1+\max_x \|\mathbf{H}\|_2)\mathbb{E}(v)(L).
\end{aligned}$$

2) For the second term,

$$\Big|\tfrac{1}{2}\int_L^S\int_\Omega \Big(\nabla\cdot \mathbf{H}(\mathbf{x})\Big)\cdot\Big(|v_t|^2 - \Big|\mathscr{H}_{\mathbf{A}} v\Big|^2\Big)dxdt\Big| \leq \max_x |\nabla\cdot\mathbf{H}|(S-L)\mathbb{E}(v)(L).$$

3) For the third term, by taking into account the inequality for Frobenius norm in matrix analysis, i.e.

$$\|\Theta_{\mathbf{H}}\times \overline{\mathscr{H}^T_{\mathbf{A}} v}\|_2 \leq \|\Theta_{\mathbf{H}}\|_F \|\mathscr{H}_{\mathbf{A}} v\|_2,$$

where

$$\|\Theta_{\mathbf{H}}\|_F \triangleq \Big(\sum_{j,k}|\nabla_j h_k|^2\Big)^{\frac{1}{2}},$$

one has

$$\begin{aligned}
&\Big|\mathrm{Re}\int_L^S\int_\Omega \mathscr{H}_{\mathbf{A}} v\times\Theta_{\mathbf{H}}\times\overline{\mathscr{H}^T_{\mathbf{A}} v}dxdt\Big| \\
\leq\ & \int_L^S\int_\Omega \|\mathscr{H}_{\mathbf{A}} v\|_2\|\Theta_{\mathbf{H}}\|_F\|\mathscr{H}_{\mathbf{A}} v\|_2 dxdt. \\
\leq\ & \max_x \|\Theta_{\mathbf{H}}\|_F\int_L^S\int_\Omega |\mathscr{H}_{\mathbf{A}} v|^2 dxdt \\
\leq\ & 2\max_x \|\Theta_{\mathbf{H}}\|_F (S-L)\mathbb{E}(v)(L).
\end{aligned}$$

4) For the last term, recall the generalized Poincaré's inequality in Lemma 2.6, and one has

$$\begin{aligned}
&\Big|\mathrm{Re}\int_L^S\int_\Omega \overline{v}\mathscr{H}_{\mathbf{A}} v\times\Xi_{\mathbf{A}}\times\mathbf{H}^T dxdt\Big| \\
\leq\ & \int_L^S\int_\Omega |v|\|\mathscr{H}_{\mathbf{A}} v\|_2\|\Xi_{\mathbf{A}}\|_F\|\mathbf{H}\|_2 dxdt \\
\leq\ & \tfrac{1}{2}\max_x(\|\Xi_{\mathbf{A}}\|_F\|\mathbf{H}\|_2)\int_L^S\int_\Omega (|\mathscr{H}_{\mathbf{A}} v|^2 + |v|^2)dxdt \\
\leq\ & 2C_p\max_x(\|\Xi_{\mathbf{A}}\|_F\|\mathbf{H}\|_2)(S-L)\mathbb{E}(v)(L).
\end{aligned} \tag{32}$$

Consequently,

$$\int_S^L\int_\Gamma \Big|\frac{\partial v}{\partial\nu}\Big|^2 d\Gamma dt \leq C(\mathbf{A},\mathbf{H})(1+(S-L))\mathbb{E}(v)(L).$$

Up to now, Theorem 3.5 is completely proved. Compared with the inequality for 1-dimensional case, here the constant depends on both $\mathbf{H}$ and the magnetic potential $\mathbf{A}$ in multidimensional case. Q. E. D. □

3.4. **Well-posedness of the hyperbolic magnetic problem.** Now we turn to our previous question. Recall the term $\int_L^S\int_\Gamma \psi\cdot\overline{\frac{\partial v}{\partial\nu}}d\Gamma dt$ in $\mathscr{L}_L^S$. Theorem 3.5 shows that for any given initial data $(v(L), v_t(L))\in\mathcal{H}_0^1\times L^2$, then $\frac{\partial v}{\partial\nu}\in L^2_{loc}(\mathbb{R}; L^2(\Gamma))$. If $\psi\in L^2_{loc}(\mathbb{R}; L^2(\Gamma))$, then $\mathscr{L}_L^S$ is well defined. We are eager to know, whether problem (23) is well-posed with such a control ψ. Indeed, one has the following positive answer.

Theorem 3.6. *Given any $(u(L), u_t(L))\in L^2\times\mathcal{H}^{-1}$ and any $\psi\in L^2([L,S]; L^2(\Gamma))$, problem (23) has a unique solution satisfying $(u(S), u_t(S))\in L^2\times\mathcal{H}^{-1}$ and the linear mapping $(u(L), u_t(L), \psi)\longmapsto(u(S), u_t(S))$ is continuous from $L^2\times\mathcal{H}^{-1}\times L^2([L,S]; L^2(\Gamma))$ into $L^2\times\mathcal{H}^{-1}$ with respect to these topologies.*

Proof. Recall the identity (25). i.e.

$$\mathscr{L}_L^S\Big(v(L), v_t(L)\Big) = -u_t(S)\bigotimes u(S)\Big(v(S), v_t(S)\Big).$$

From Theorem 3.5, it follows that for any $L, S \in \mathbb{R}$, the linear operator $\mathscr{L}_L^S$ is bounded on $\mathcal{H}_0^1 \times L^2$. Pay attention to Theorem 3.2 and the energy conservation law in Lemma 3.4, due to time-reversibility, then one finds the linear mapping

$$(v(S), v_t(S)) \longmapsto (v(L), v_t(L))$$

is an isometric isomorphism of $\mathcal{H}_0^1 \times L^2$ onto itself. Consequently, the linear form

$$(v(S), v_t(S)) \longmapsto \mathscr{L}_L^S(v(L), v_t(L))$$

is also bounded on $\mathcal{H}_0^1 \times L^2$. According to Riesz-Fréchet representation theorem, there exists a unique pair $(u_t(S), u(S)) \in \mathcal{H}^{-1} \times L^2$ satisfying (25). The existence and uniqueness are proved.

Next we show that $\|(u_t(S), u(S))\|_{\mathcal{H}^{-1}\times L^2}$ is uniformly bounded in each finite time interval I. Indeed, by applying Hölder's inequality, Theorem 3.5 and the energy conservation law in Lemma 3.4, one has

$$\begin{aligned}
& \Big| - u_t(S)\bigotimes u(S)\Big(v(S), v_t(S)\Big)\Big| \\
\le\ & \Big| - u_t(L)\bigotimes u(L)\Big(v(L), v_t(L)\Big)\Big| + \Big|\textstyle\int_L^S\int_\Gamma \psi \cdot \overline{\frac{\partial v}{\partial \nu}} d\Gamma dt\Big| \\
\le\ & \|(-u_t(L), u(L))\|_{\mathcal{H}^{-1}\times L^2}\|(v(L), v_t(L))\|_{\mathcal{H}_0^1\times L^2} + \|\psi\|_{L^2(I,L^2(\Gamma))}\|\tfrac{\partial v}{\partial \nu}\|_{L^2(I,L^2(\Gamma))} \\
\le\ & \Big(\|(-u_t(L), u(L))\|_{\mathcal{H}^{-1}\times L^2} + C(I)\|\psi\|_{L^2(I,L^2(\Gamma))}\Big)\|(v(L), v_t(L))\|_{\mathcal{H}_0^1\times L^2} \\
=\ & \Big(\|(-u_t(L), u(L))\|_{\mathcal{H}^{-1}\times L^2} + C(I)\|\psi\|_{L^2(I,L^2(\Gamma))}\Big)\|(v(S), v_t(S))\|_{\mathcal{H}_0^1\times L^2}.
\end{aligned}$$

The definition of norm for the tensor product operator indicates,

$$\|(u_t(S), u(S))\|_{\mathcal{H}^{-1}\times L^2} \le \|(-u_t(L), u(L))\|_{\mathcal{H}^{-1}\times L^2} + C(I)\|\psi\|_{L^2(I,L^2(\Gamma))}.$$

This demonstrates the continuous dependence of the solution on $(u(L), u_t(L), \psi)$. Q. E. D. □

3.5. **Observability inequality.** At the moment, we begin to search an appropriate $\psi \in L^2_{loc}(\mathbb{R}; L^2(\Gamma))$ such that (23) is exactly controllable. First of all, we need to construct a Hilbert space with a suitable norm concerned with the outward normal derivative of the homogeneous adjoint system (24). We will prove that, under certain conditions, $\|\frac{\partial v}{\partial \nu}\|_{L^2_{loc}(\mathbb{R};L^2(\Gamma))}$ is actually an equivalent norm on $\mathcal{H}_0^1 \times L^2$.

Theorem 3.7. *Let $T^* \triangleq 2\max_\Omega \|x\|_2$. For any given $L, S \in \mathbb{R}$ and initial data $(v(L), v_t(L)) \in \mathcal{H}_0^1 \times L^2$, then when $S - L > T^*$, the outward normal derivative defined by (7) satisfies*

$$((S - L) - T^*)\mathbb{E}(v)(L) \le C(\Omega)\int_L^S\int_\Gamma |\frac{\partial v}{\partial \nu}|^2 d\Gamma dt.$$

In particular, in 1-dimensional case, let $\Omega = (-1, 1)$, the above inequality can be rewritten as

$$((S - L) - T^*)\mathbb{E}(v)(L) \le C(\Omega)\int_L^S (|v_x(t, -1)|^2 + |v_x(t, 1)|^2)dt.$$

3.5.1. *Proof of Theorem 3.7 in 1-dimensional case: multiplier method.* It is sufficient to prove the estimate for $(u(L), u_t(L)) \in Z \times Z$, the general case then follows by a density argument. Similar as the proof of Theorem 3.5, different identities will be applied accordingly. Moreover, a sidewise energy estimate, which is introduced by E. Fernández-Cara and E. Zuazua in [6] can also be applied. Interesting readers can check this fact.

Pay attention to the identity (28) for 1-dimensional case. Let $h(x) \triangleq x$, then (28) is simplified as

$$\begin{aligned}
& -i\textstyle\int_L^S(|v_x(t, -1)|^2 + |v_x(t, 1)|^2)dt \\
=\ & 2i\ \mathrm{Im}(v_t, x \cdot \mathscr{H}_{\mathbf{A}}v)_{L^2}\Big|_L^S - 2i\ \mathrm{Im}\textstyle\int_L^S\int_{-1}^1 v_t \cdot \Big(x \cdot \overline{\mathscr{H}_{\mathbf{A}}v_t}\Big)dxdt - i\int_L^S\int_{-1}^1 |\mathscr{H}_{\mathbf{A}}v|^2 dxdt.
\end{aligned} \tag{33}$$

For the second term on the RHS of (33), integrating by parts, one has

$$\int_L^S\int_{-1}^1 v_t\cdot\Big(x\cdot\overline{\mathscr{H}_{\mathbf{A}}v_t}\Big)dxdt = i\int_L^S\int_{-1}^1|v_t|^2dxdt+\int_L^S\int_{-1}^1\mathscr{H}_{\mathbf{A}}v_t\cdot x\cdot\overline{v}_t dxdt.$$

Consequently,

$$2i\ \mathrm{Im}\int_L^S\int_{-1}^1 v_t\cdot\Big(x\cdot\overline{\mathscr{H}_{\mathbf{A}}v_t}\Big)dxdt = i\int_L^S\int_{-1}^1|v_t|^2dxdt.$$

According to energy conservation law in Lemma 3.4, the identity (33) can be rewritten as

$$\frac{1}{2}\int_L^S(|v_x(t,-1)|^2+|v_x(t,1)|^2)dt = -\mathrm{Im}(v_t, x\cdot\mathscr{H}_{\mathbf{A}}v)_{L^2}\Big|_L^S+(S-L)\mathbb{E}(v)(L). \tag{34}$$

Now we investigate the first term on the RHS of (34). Applying Hölder's inequality, one has

$$\begin{aligned}\left|\mathrm{Im}(v_t, x\cdot\mathscr{H}_{\mathbf{A}}v)_{L^2}\Big|_L^S\right| &\le \left|\mathrm{Im}\int_{-1}^1 v_t(S)\cdot x\cdot\overline{\mathscr{H}_{\mathbf{A}}v(S)}dx\right|+\left|\mathrm{Im}\int_{-1}^1 v_t(L)\cdot x\cdot\overline{\mathscr{H}_{\mathbf{A}}v(L)}dx\right|\\ &\le \mathbb{E}(v)(S)+\mathbb{E}(v)(L)\le 2\mathbb{E}(v)(L).\end{aligned}$$

Consequently, when $S-L>2$, we have the observability inequality,

$$((S-L)-2)\mathbb{E}(v)(L)\le\frac{1}{2}\int_L^S(|v_x(t,-1)|^2+|v_x(t,1)|^2)dt.$$

3.5.2. *Step 1 of Proof of Theorem 3.7 in multidimensional case: multiplier method.* Let $\mathbf{H}(x)\triangleq x$, then $\Theta_{\mathbf{H}}$ is the identity matrix and the identity (31) is simplified as

$$\begin{aligned}-\tfrac{1}{2}\int_L^S\int_\Gamma\left|\tfrac{\partial v}{\partial\nu}\right|^2\cdot\Big(x\cdot\nu\Big)d\Gamma dt &= \mathrm{Im}(v_t, x\cdot\mathscr{H}_{\mathbf{A}}v)_{L^2}\Big|_L^S-\tfrac{N}{2}\int_L^S\int_\Omega\Big(|v_t|^2-\Big|\mathscr{H}_{\mathbf{A}}v\Big|^2\Big)dxdt\\ &\quad -\int_L^S\int_\Omega|\mathscr{H}_{\mathbf{A}}v|^2dxdt-\mathrm{Re}\int_L^S\int_\Omega\overline{v}\mathscr{H}_{\mathbf{A}}v\times\Xi_{\mathbf{A}}\times\mathbf{H}^Tdxdt.\end{aligned} \tag{35}$$

Next we apply the multiplier v to (24). By decomposing the following integral into two parts and applying the generalized Green's formula (4) in Lemma 2.3, we have

$$\int_L^S(v_{tt}+\mathscr{H}_{\mathbf{A}}^2v, iv)_{L^2}dt = -i\int_\Omega v_t\cdot\overline{v}dx\Big|_L^S+i\int_L^S\int_\Omega|v_t|^2dxdt-i\int_L^S\int_\Omega|\mathscr{H}_{\mathbf{A}}v|^2dxdt. \tag{36}$$

Combining (35) and (36), we obtain a new identity,

$$\begin{aligned}&\tfrac{1}{2}\int_L^S\int_\Gamma\left|\tfrac{\partial v}{\partial\nu_{i\mathscr{H}_{\mathbf{A}}}}\right|^2\cdot\Big(x\cdot\nu\Big)d\Gamma dt\\ =\ &\underbrace{-\mathrm{Im}(v_t, x\cdot\mathscr{H}_{\mathbf{A}}v)_{L^2}\Big|_L^S}_{(I)}\underbrace{-\mathrm{Im}(v_t, \frac{N-1}{2}iv)_{L^2}\Big|_L^S}_{(II)}+(S-L)\mathbb{E}(v)(L)\\ &\underbrace{+\mathrm{Re}\int_L^S\int_\Omega\overline{v}\mathscr{H}_{\mathbf{A}}v\times\Xi_{\mathbf{A}}\times\mathbf{H}^Tdxdt}_{(III)}.\end{aligned} \tag{37}$$

Actually, we can have a more refined estimate. Since

$$\int_\Omega x\cdot\mathscr{H}_{\mathbf{A}}v\cdot\overline{v}dx = -iN\|v\|_{L^2}^2+\int_\Omega x\cdot\overline{\mathscr{H}_{\mathbf{A}}v}\cdot vdx,$$

then

$$\begin{aligned}
&(v_t, x\cdot \mathscr{H}_{\mathbf{A}}v + \tfrac{N-1}{2}iv)_{L^2}\\
\le\ & \tfrac{1}{2}\max_\Omega \|x\|_2\|v_t\|_{L^2}^2 + \tfrac{1}{2\max_\Omega \|x\|_2}\|x\cdot \mathscr{H}_{\mathbf{A}}v + \tfrac{N-1}{2}iv\|_{L^2}^2\\
=\ & \tfrac{1}{2}\max_\Omega \|x\|_2\|v_t\|_{L^2}^2 + \tfrac{1}{2\max_\Omega \|x\|_2}\Big(\|x\cdot \mathscr{H}_{\mathbf{A}}v\|_{L^2}^2 + \|\tfrac{N-1}{2}v\|_{L^2}^2 - \tfrac{(N-1)N}{2}\|v\|_{L^2}^2\Big)\\
\le\ & \tfrac{1}{2}\max_\Omega \|x\|_2\|v_t\|_{L^2}^2 + \tfrac{1}{2\max_\Omega \|x\|_2}\Big((\max_\Omega \|x\|_2)^2\|\mathscr{H}_{\mathbf{A}}v\|_{L^2}^2\Big)\\
\le\ & \max_\Omega \|x\|_2\mathbb{E}(v)(L).
\end{aligned}$$

As a result,

$$|(I)+(II)| \le 2\max_\Omega \|x\|_2\mathbb{E}(v)(L).$$

5
As to (III), by applying Schwartz's inequality, we have

$$\begin{aligned}
&\Big|\mathrm{Re}\int_L^S\int_\Omega \overline{v}\mathscr{H}_{\mathbf{A}}v\times \Xi_{\mathbf{A}}\times \mathbf{H}^T dxdt\Big|\\
\le\ & \int_L^S\int_\Omega |v|\|\mathscr{H}_{\mathbf{A}}v\|_2\|\Xi_{\mathbf{A}}\|_F\|\mathbf{H}\|_2 dxdt\\
\le\ & \epsilon\max_x(\|\Xi_{\mathbf{A}}\|_F\|\mathbf{H}\|_2)\int_L^S\int_\Omega |\mathscr{H}_{\mathbf{A}}v|^2dxdt\\
& +\tfrac{1}{4\epsilon}\max_x(\|\Xi_{\mathbf{A}}\|_F\|\mathbf{H}\|_2)\int_L^S\int_\Omega |v|^2dxdt\\
\le\ & \epsilon\max_x(\|\Xi_{\mathbf{A}}\|_F\|\mathbf{H}\|_2)(S-L)\mathbb{E}(v)(L)\\
& +\tfrac{1}{4\epsilon}\max_x(\|\Xi_{\mathbf{A}}\|_F\|\mathbf{H}\|_2)\int_L^S\int_\Omega |v|^2dxdt.
\end{aligned}\tag{38}$$

Let $\eta \triangleq \max_x(\|\Xi_{\mathbf{A}}\|_F\|\mathbf{H}\|_2)$, consequently,

$$\begin{aligned}
&\Big((1-\epsilon\eta)(S-L) - 2\max_\Omega \|x\|_2\Big)\mathbb{E}(v)(L)\\
\le\ & \tfrac{\max_\Omega \|x\|_2}{2}\int_L^S\int_\Gamma \Big|\tfrac{\partial v}{\partial\nu}\Big|^2 d\Gamma dt + \tfrac{\eta}{4\epsilon}\int_L^S\int_\Omega |v|^2dxdt.
\end{aligned}\tag{39}$$

Remark 3.8. *In quantum mechanics, $\|\Xi_{\mathbf{A}}\|_F$ stands for the intensity of the magnetic field $\nabla\times\mathbf{A}$. For 1-dimensional case, $\|\Xi_{\mathbf{A}}\|_F\equiv 0$, so there is no influence from the magnetic field. In fact, many typical vector potentials also satisfy $\|\Xi_{\mathbf{A}}\|_F\equiv 0$, such as $\mathbf{A}=\eta(|x|^2)x$, where $\eta\in C^\infty(\mathbb{R}^3)$, etc.*

3.5.3. *Step 2 of Proof of Theorem 3.7 in multidimensional case: exact controllability in a minimal time.* In this section, one applies the compactness-uniqueness argument introduced in [2][14][32]. With this method, the lower order terms can be absorbed by the boundary integral.

Lemma 3.9. *There exists a positive constant $C(\epsilon,\Omega)$ such that*

$$\int_L^S\int_\Omega |v|^2dxdt \le C(\epsilon,\Omega)\int_L^S\int_\Gamma \Big|\frac{\partial v}{\partial\nu}\Big|^2 d\Gamma dt. \tag{40}$$

Proof. We prove it by the method of contradiction. Assume that (40) does not hold, then there exists a sequence of initial data $(v_0^m, v_1^m)\in\mathcal{H}_0^1\times L^2$ such that the corresponding sequence of solutions $\{v^m\}_m$ of the homogeneous adjoint problem (24) satisfies

$$\int_L^S\int_\Omega |v^m|^2dxdt = 1 \quad \text{for all } m; \tag{41}$$

$$\int_L^S\int_\Gamma \Big|\frac{\partial v^m}{\partial\nu}\Big|^2 d\Gamma dt \le \frac{1}{m}. \tag{42}$$

Recall the norm in $H^1([L,S]\times\Omega)$, e.g.

$$\|\omega\|_{H^1([L,S]\times\Omega)} \triangleq \|\omega\|_{L^2_t(L^2)} + \|\omega_t\|_{L^2_t(L^2)} + \|\mathscr{H}_{\mathbf{A}}\omega\|_{L^2_t(L^2)},$$

Since $\omega \in C([L,S];\mathcal{H}^1_0)$, according to Lemma 2.6, one can define an equivalent norm in $H^1((L,S)\times\Omega)$ as

$$\|\omega\|_{H^1([L,S]\times\Omega)} \triangleq \|\omega_t\|_{L^2_t(L^2)} + \|\mathscr{H}_{\mathbf{A}}\omega\|_{L^2_t(L^2)}.$$

According to (39) (41) and (42), for $(1-\epsilon\eta)S - L > 2\max_\Omega \|x\|_2$, one knows that (v_0^m, v_1^m) is bounded in $\mathcal{H}^1_0 \times L^2$, therefore,

$$(v_0^m, v_1^m) \rightharpoonup (v_0^*, v_1^*) \quad \text{in } \ \mathcal{H}^1_0 \times L^2. \tag{43}$$

Since the canonical injection from $H^1([L,S]\times\Omega)$ to $L^2([L,S]\times\Omega)$ is compact, then one can extract a subsequence, (for convenience's sake, we still use the notation $\{v^m\}_m$), such that

$$v^m \rightharpoonup v^* \quad \text{in } \ H^1([L,S]\times\Omega), \tag{44}$$

$$v^m \to v^* \quad \text{in } \ L^2([L,S]\times\Omega), \tag{45}$$

where v^* is the solution corresponding to the limit initial data (v_0^*, v_1^*). (For rigorous proof please see [4][21].) Furthermore, for any $\phi \in H^1([L,S]\times\Omega)$ with $\phi(L,x) = \phi(S,x) = 0$, we have

$$\begin{aligned} 0 &= \textstyle\int_L^S \int_\Omega ((v_{tt}^m - v_{tt}^*)\overline{\phi} + \mathscr{H}_{\mathbf{A}}^2(v^m - v^*)\overline{\phi})dxdt \\ &= -\textstyle\int_L^S \int_\Omega (v_t^m - v_t^*)\overline{\phi}_t dxdt - \int_L^S \int_\Gamma (\frac{\partial v^m}{\partial \nu} - \frac{\partial v^*}{\partial \nu})\overline{\phi} d\Gamma dt \\ &\quad + \textstyle\int_L^S \int_\Omega (\mathscr{H}_{\mathbf{A}} v^m - \mathscr{H}_{\mathbf{A}} v^*)\overline{\mathscr{H}_{\mathbf{A}}\phi} dxdt. \end{aligned} \tag{46}$$

From (44), we have

$$\lim_m \int_L^S \int_\Gamma (\frac{\partial v^m}{\partial \nu} - \frac{\partial v^*}{\partial \nu})\overline{\phi} d\Gamma dt = 0. \tag{47}$$

This gives

$$\frac{\partial v^m}{\partial \nu} \rightharpoonup \frac{\partial v^*}{\partial \nu} \quad \text{in } \ L^2([L,S]\times\Gamma). \tag{48}$$

According to the lower semicontinuity property for weak convergence, from (42) we deduce that

$$\frac{\partial v^*}{\partial \nu} = 0 \quad \text{in } \ L^2([L,S]\times\Gamma). \tag{49}$$

Definition 3.10. *A weak solution $v \in L^2([L,S];\mathcal{H}^1_0)$ with $v_t \in L^2([L,S];L^2)$ and $v_{tt} \in L^2([L,S];\mathcal{H}^{-1})$ of $v_{tt} + \mathscr{H}_{\mathbf{A}}^2 v = 0$ is called invisible if it satisfies $v|_\Gamma = \frac{\partial v}{\partial \nu}\Big|_\Gamma = 0$. The set of all invisible solutions is denoted as $\mathscr{N}$.*

Remark 3.11. *$\mathscr{N}$ is a finite dimensional subspace of $L^2([L,S]\times\Omega)$. Indeed, when $(1-\epsilon\eta)S - L > 2\max_\Omega \|x\|_2$, for $\forall v \in \mathscr{N}$, it holds,*

$$\int_L^S \int_\Omega \Big(|\mathscr{H}_{\mathbf{A}} v|^2 + |v_t|^2\Big) dxdt \le C(\eta,\epsilon,L,S) \int_L^S \int_\Omega |v|^2 dxdt. \tag{50}$$

Since the canonical injection from $H^1([L,S]\times\Omega)$ to $L^2([L,S]\times\Omega)$ is compact, according to Riesz Theorem, (50) indicates that dim($\mathscr{N}$) is finite. Since

$$\Big\{\sqrt{\frac{1}{S-L}}\exp^{it\sqrt{\lambda}}\phi_\lambda, \sqrt{\frac{1}{S-L}}\exp^{-it\sqrt{\lambda}}\phi_\lambda\Big\}_{\lambda\in\Lambda_{\mathscr{H}_{\mathbf{A}}^2}}$$

forms a basis satisfying (24) in $L^2([L,S]\times\Omega)$, so $\mathscr{N}$ has a basis of the form

$$\Big\{\sqrt{\frac{1}{S-L}}\exp^{it\sqrt{\lambda_i}}\phi_{\lambda_i}, \sqrt{\frac{1}{S-L}}\exp^{-it\sqrt{\lambda_i}}\phi_{\lambda_i}\Big\}_{i=1}^n.$$

Consequently, due to the elliptic regularity theory, for $\forall v \in \mathscr{N}$, $v \in C^\infty([L,S];H^2)$.

In the following, we show that $\mathscr{N}$ is indeed $\{0\}$. First, we introduce an important unique continuation theorem concerned with the magnetic operator $\mathscr{H}_{\mathbf{A}}^2$. The specific proof based on the multiplier method is given in [23].

Theorem A.*(Unique continuation theorem) For $N \geq 2$, let $\omega \in H^2(\mathbb{B}_1)$ be a solution of the elliptic problem*

$$\mathscr{H}_{\mathbf{A}}^2 \omega = \phi(x)\omega \quad in\ \mathbb{B}_1,$$

where $\mathbb{B}_1$ is a unit ball and the complex function $\phi \in L^\infty(\mathbb{R}^N)$. If ω vanishes in a neighborhood of $x_0 \in \mathbb{B}_1$, then $\omega \equiv 0$ in $\mathbb{B}_1$.

From the above theorem, we can deduce the following result.

Corollary 3.12. *Let Ω be a bounded open domain in $\mathbb{R}^N$ with the boundary $\Gamma \in C^2$. Let $\omega \in H^2$ be a solution of*

$$\mathscr{H}_{\mathbf{A}}^2 \omega = \phi\omega \qquad in \quad \Omega$$

$$\omega = \tfrac{\partial \omega}{\partial_{i\mathscr{H}_{\mathbf{A}}}} = 0 \quad on \quad \Gamma.$$

Then $\omega \equiv 0$ in Ω.

Proof. Let $\mathbb{B}$ be an arbitrarily small open ball such that

$$\Gamma \cap \mathbb{B} \neq \varnothing.$$

Set

$$\Omega^1 \triangleq \Omega \cup \mathbb{B},$$

and define

$$\omega^1 \triangleq \begin{cases} \omega & \text{in} \quad \Omega; \\ \\ 0 & \text{in} \quad \mathbb{B}\backslash\Omega. \end{cases}$$

It is sufficient to verify that $\omega^1 \in H^2$. Denote by ω_j^1, ω_{jk}^1 the extension by zero to Ω^1 of the derivatives $\nabla_j \omega$, $\nabla_j \nabla_k \omega$, $j, k = 1, \cdots, N$. Then ω_j, $\omega_{jk} \in L^2(\Omega^1)$ and it is necessary to demonstrate that, for $\forall \zeta \in \mathscr{D}(\Omega^1)$,

$$\int_{\Omega^1} \omega^1 \nabla_j \overline{\zeta} dx = -\int_{\Omega^1} \omega_j^1 \overline{\zeta} dx,$$

and

$$\int_{\Omega^1} \omega_j^1 \nabla_k \overline{\zeta} dx = -\int_{\Omega^1} \omega_{jk}^1 \overline{\zeta} dx.$$

Indeed, since $\omega_j^1 = \omega_{jk}^1 \equiv 0$ outside of Ω, $\zeta \equiv 0$ on $\Gamma\backslash(\Gamma \cap \mathbb{B})$ and $\omega = \frac{\partial \omega}{\partial_{i\mathscr{H}_{\mathbf{A}}}} \equiv 0$ on $\Gamma \cap \mathbb{B}$, one has

$$\begin{aligned} \textstyle\int_{\Omega^1} \omega^1 \nabla_j \overline{\zeta} dx &= \textstyle\int_\Omega \omega \nabla_j \overline{\zeta} dx = \int_\Gamma \omega \overline{\zeta} \nu_j d\Gamma - \int_\Omega (\nabla_j \omega) \overline{\zeta} dx \\ &= \textstyle\int_{\Gamma\cap\mathbb{B}} \omega \overline{\zeta} \nu_j d\Gamma - \int_\Omega (\nabla_j \omega) \overline{\zeta} dx = -\int_\Omega (\nabla_j \omega) \overline{\zeta} dx = -\int_{\Omega^1} \omega_j^1 \overline{\zeta} dx, \end{aligned}$$

and

$$\begin{aligned} \textstyle\int_{\Omega^1} \omega_j^1 \nabla_k \overline{\zeta} dx &= \textstyle\int_\Omega \nabla_j \omega \nabla_k \overline{\zeta} dx = \int_\Gamma \nabla_j \omega \overline{\zeta} \nu_k d\Gamma - \int_\Omega (\nabla_k \nabla_j \omega) \overline{\zeta} dx \\ &= \textstyle\int_{\Gamma\cap\mathbb{B}} \nabla_j \omega \overline{\zeta} \nu_k d\Gamma - \int_\Omega (\nabla_k \nabla_j \omega) \overline{\zeta} dx = -\int_\Omega (\nabla_k \nabla_j \omega) \overline{\zeta} dx = -\int_{\Omega^1} \omega_{jk}^1 \overline{\zeta} dx. \end{aligned}$$

Q. E. D. □

If $\mathscr{N} \neq \{0\}$, at least there should exist a nonzero ϕ_{λ_i} such that

$$\mathscr{H}_{\mathbf{A}}^2 \phi_{\lambda_i} = \lambda_i \phi_{\lambda_i}, \quad \phi_{\lambda_i}|_\Gamma = \frac{\partial \phi_{\lambda_i}}{\partial \nu}\Big|_\Gamma = 0.$$

The above unique continuation property implies that $\phi_{\lambda_i} \equiv 0$. Consequently, $\mathscr{N} \equiv \{0\}$. This is in direct contradiction to the fact that $\int_L^S \int_\Omega |v^*|^2 dxdt = 1$. Q. E. D.

□

Theorem 3.5 and Theorem 3.7 indicate, for $T > 2\max_\Omega \|x\|_2$,

$$\int_0^T \int_\Gamma |\frac{\partial v}{\partial \nu}|^2 d\Gamma dt$$

defines an equivalent norm in $\mathcal{H}_0^1 \times L^2$ with respect to Definition 2.1. While Theorem 3.2 and Theorem 3.5 demonstrate that, for any given $T \in \mathbb{R}$ and initial data $(v(0), v_t(0)) \in \mathcal{H}_0^1 \times L^2$, there exists a unique solution

$$v \in C([0,T];\mathcal{H}_0^1)\bigcap C^1([0,T];L^2)\bigcap C^2([0,T];\mathcal{H}^{-1})$$

for the homogeneous problem (24) and the outward normal derivative defined by (7) satisfies

$$\frac{\partial v}{\partial \nu} \in L^2([0,T];L^2(\Gamma)),$$

which is continuous with respect to the initial data. If we choose

$$\psi \triangleq \frac{\partial v}{\partial \nu} \in L^2([0,T];L^2(\Gamma))$$

and consider the problem (23) with the initial data $(u(T), u_t(T)) = 0$, then Theorem 3.6 shows that problem (23) has a unique solution satisfying

$$(u_0, u_1) \triangleq (u(0), u_t(0)) \in L^2 \times \mathcal{H}^{-1}.$$

And the linear mapping

$$(u(T), u_t(T), \psi) \longmapsto (u(0), u_t(0))$$

is continuous from $L^2 \times \mathcal{H}^{-1} \times L^2([L,S];L^2(\Gamma))$ into $L^2 \times \mathcal{H}^{-1}$ with respect to these topologies. Let (v_0, v_1) be the initial data of (24). Thus, in a unique fashion, one can define a linear and bounded mapping

$$\mathscr{S} : \mathcal{H}_0^1 \times L^2(\Omega) \longrightarrow \mathcal{H}^{-1} \times L^2(\Omega), \quad (v_0, v_1) \mapsto (u_1, -u_0). \tag{51}$$

It is evident that, if $\mathscr{S}$ is surjective, then $\psi \triangleq \frac{\partial v}{\partial \nu}$ is an appropriate boundary control which drives $(u_0, u_1) \in L^2 \times \mathcal{H}^{-1}$ to rest. In the following lemma we will show this fact.

Lemma 3.13. *Let $T > 2\max_\Omega \|x\|_2$, then $\mathscr{S}$ is an isomorphism from $\mathcal{H}_0^1 \times L^2$ onto $\mathcal{H}^{-1} \times L^2$.*

Proof. Recall the Lions-Lax-Milgram lemma in [21].

Lemma 3.14. *Let $\mathscr{V}$ be a real or complex Hilbert space, and $\mathscr{V}^*$ be its dual. $\mathscr{V}$ is continuously imbedded into $\mathscr{V}^*$. If there exists a $\kappa > 0$ such that $\mathscr{A} \in \mathscr{L}(\mathscr{V}, \mathscr{V}^*)$ satisfies the so-called $\mathscr{V}$-elliptic condition, i.e.*

$$\langle \mathscr{A}\omega, \omega\rangle_{\mathscr{V}^*,\mathscr{V}} \geq \kappa\|\omega\|_{\mathscr{V}}^2, \quad \forall \omega \in \mathscr{V},$$

then $\mathscr{A}$ is an isomorphism from $\mathscr{V}$ onto $\mathscr{V}^*$.

It is sufficient to check the $\mathscr{V}$-elliptic condition for $\mathscr{S}$. Taking into account of the data $u(T) = u_t(T) = 0$, we multiply the magnetic equation (23) by the solution v of the adjoint homogeneous problem (24). By applying the generalized Green's formula, we have

$$\begin{aligned} 0 &= \int_0^T \int_\Omega v \cdot \overline{(u_{tt} + \mathscr{H}_{\mathbf{A}}^2 u)} dxdt \\ &= \int_\Omega (v\overline{u}_t - v_t\overline{u})dx\Big|_0^T - \int_0^T\int_\Gamma \left(v \cdot \overline{\frac{\partial u}{\partial \nu_{i\mathscr{H}_{\mathbf{A}}}}} - \frac{\partial v}{\partial \nu_{i\mathscr{H}_{\mathbf{A}}}} \cdot \overline{u}\right) d\Gamma dt \\ &\quad + \int_0^T \int_\Omega (v_{tt} + \mathscr{H}_{\mathbf{A}}^2 v) \cdot \overline{u} dxdt \\ &= \int_\Omega (v(T)\overline{u}_t(T) - v_t(T)\overline{u}(T) + v_1\overline{u}(0) - v_0\overline{u}_t(0))dx \\ &\quad + \int_0^T \int_\Gamma |\tfrac{\partial v}{\partial \nu}|^2 d\Gamma dt. \end{aligned}$$

Hence,

$$\langle \mathscr{S}(v_0, v_1), (v_0, v_1)\rangle_{\mathcal{H}^{-1}\times L^2, \mathcal{H}_0^1 \times L^2} = \int_0^T \int_\Gamma |\frac{\partial v}{\partial \nu}|^2 d\Gamma dt.$$

For $T > 2\max_\Omega \|x\|_2$, the observability result in Theorem 3.7 gives the $\mathscr{V}$-elliptic condition. Consequently, $\mathscr{S}$ is an isomorphism from $\mathcal{H}_0^1 \times L^2$ onto $\mathcal{H}^{-1} \times L^2$. Q. E. D. □

Remark 3.15. *Theorem 1.2 and Theorem 1.3 can be proved similarly by using the same multipliers as in Theorem 1.1. Here we do not go to details of the proofs. Actually, by applying the transmutation method introduced in* [24][25]*, one can check the null controllability results for magnetic heat equations and magnetic Schrödinger equations.*

Remark 3.16. *During the proof, we find that, in a magnetic field with large $\|\Xi_{\mathbf{A}}\|_F$, it is difficult to impose some exterior force on the boundary to influence the interior activity. Astronomically speaking, when the solar wind with coronal mass ejections encounters Earth's magnetosphere, most of the radioactive particles are deflected around the earth instead of impacting the atmosphere or the earth's surface, although some leakage occurs, resulting in auroras and Van Allen belts.*

Remark 3.17. *It is really challenging to investigate the control theory in the field of quantum mechanics. Many interesting problems, such as the exact controllability of Maxwell's equations, nonlinear Ginzburg-Landau equations, etc., are to be addressed. More references are to be found in* [17][18].

Acknowledgement

This project was suggested by Prof. Enrique Zuazua during the author's post-doctoral research in BCAM from 2010-2011. The author thanks his advisor Prof. Zuazua for the helpful guidance into this challenging field and many fruitful discussions during the research. And the author thanks Qi Lü for the beneficial discussion about the philosophy of control theory. Furthermore, the author thanks BCAM for providing the nice research environment. This project is partially supported by Grant MTM2008-03541 and MTM2011-29306 of the MICINN(Spain), the ERC Advanced Grant FP7-246775 NUMERIWAVES, the ESF Research Networking Programme OPTPDE and the Grant PI2010-04 of the Basque Government.

References

[1] R. Adams, Sobolev Spaces, Academic Press, New York, 1975.
[2] C. Bardos, G. Lebeau and J. Rauch, Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary, SIAM J. Control and Optimization, 30(1992), 1024-1065.
[3] T. Cazanave, Semilinear Schrödinger Equations (Courant Lecture Notes), AMS, 2003.
[4] J. C. Díaz, J. C. Calvo, F. Maestre and J. D. M. Gómez, Homogenization and corrector for the wave equation with discontinuous coefficients in time, J. Math. Anal. Appl. 379(2011), 664-681.
[5] L. C. Evans, Partial Differential Equations, Graduate Studies in Mathematics(19), AMS, 1997.
[6] E. Fernández-Cara, E. Zuazua, On the null controllability of the one-dimensional heat equation with non-smooth coefficients, preprint, 2000.
[7] L. Fanelli, Electromagnetic Schrödinger flow: multiplier methods for dispersion, Proc. Journées EDP, Port D'Albret, GDR 2434(CNRS), 2010.
[8] M. Goldberg, Strichartz estimates for Schrödinger operators with a non-smooth magnetic potential, Discrete and Continuous Dynmaical Systems, 31(2011), 109-118.
[9] N. Garofalo and F. H. Lin, Monotonicity properties of variational integrals, A_p weights and unique continuation, Indiana Univ. Math. J., 35(1986), 245-268.
[10] N. Garofalo and F. H. Lin, Unique continuation for elliptic operators: A geometric variational approach, Comm. Pure Appl. Math., 40(1987), 347-366.
[11] H. Kovarik, Large time behavior of the heat kernel of two-dimensional magnetic Schrödinger operators, preprint, 2010.
[12] K. Kurata, A unique continuation theorem for the Schrödinger equation with singular magnetic field, Proc. AMS, 125(1997), 853-860.
[13] K. Kurata, Local boundedness and continuity for weak solutions of $-(\nabla - ib)^2u + Vu = 0$, Math. Z., 224(1997), 641-653.
[14] V. Komornik, A new method of exact controllability in short time and applications, Annales Faculté des Sciences de Toulouse, Série 5, 10(1989), 415-464.
[15] V. Komornik, Exact Controllability and Stabilization, The Multiplier Method, Masson, Paris, 1994.
[16] V. Komornik, Contrôlabilité exacte en un temps minimal, C. R. Acad. Sci. Paris, Série I, 304(1987), 223-225.
[17] V. Komornik, Boundary stabilization, observation and control of Maxwell's equations, Panamerican Mathematical Journal 4(1994), 47-61.
[18] V. Komornik, Rapid boundary stabilization of Maxwell's equations, Équations aux Dérivées Partielles et Applications. Articles dédiés à Jacques-Louis Lions, Gauthier-Villars, 1998, 611-622.
[19] J. L. Lions, Exact controllability, stabilization and perturbations for distributed systems, SIAM Review, 30(1988), 1-68.
[20] J. L. Lions, Contrôlabilité Exacte et Stabilisation de Systèmes Distribués, Vol. 1, Masson, Paris, 1988.
[21] J. L. Lions and E. Magenes, Problèmes aux limites non homogènes et applications I-III, Dunod, Paris, 1968-1970.
[22] X. Lu, A note on fractional order Poincaré's inequalities, 2012.
[23] X. Lu, A unique continuation result for the magnetic Schrödinger operator, 2012.
[24] L. Miller, Controllability cost of conservative systems: resolvent condition and transmutation, Journal of Functional Analysis 218, 2(2005), 425-444.
[25] L. Miller, Geometric bounds on the growth rate of null-controllability cost for the heat equation in small time, Journal of Differential Equations 204, 1(2004), 202-226.
[26] M. E. Peskin and D. V. Schröder, An Introduction to Quantum Field Theory, Westview Press, 1995.
[27] P. A. Raviart and J. M. Thomas, Introduction à L'analyse Numérique des Équations aux Dérivées Partielles, Dunod, Paris, 1998.
[28] M. Reed and B. Simon, Methods of Modern Mathematical Physics, I-IV, Academic Press, London, 1978.
[29] Michael E. Taylor, Partial Differential Equations I, Basic Theory, Springer Verlag, 1999.
[30] Michael E. Taylor, Partial Differential Equations II, Qualitative Studies of Linear Equations, Springer Verlag, 1999.
[31] J. Vancostenoble and E. Zuazua, Hardy inequalities, observability, and control for the wave and Schrödinger equations with singular potentials, SIAM J. Math. Anal., 41(2009), 1508-1532.
[32] E. Zuazua, Controllability of the linear system of thermoelasticity, J. Math. pures et appl.,74(1995), 291-315.