A UNIQUE CONTINUATION RESULT FOR THE MAGNETIC SCHRÖDINGER OPERATOR

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Abstract. In this paper, we mainly address the unique continuation property for the magnetic Schrödinger operator, which plays an important role in the research of quantum mechanics.

1. Introduction

Let $\mathbf{A}(x)$ be the vector potential of $\mathbf{B}$, that is, $\mathbf{B} = \nabla \times \mathbf{A}$, which does not depend on time. Clearly, $\nabla \cdot \mathbf{B} = \text{div rot}\mathbf{A} = 0$. From one of the well-known Maxwell's equations (μ is the magnetic permeability)

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{B}}{\partial t} = 0,$$

we deduce that $\mathbf{E} = -\nabla \phi$, where the scalar ϕ represents the electric potential. We choose an appropriate Lagrangian for the charged particle in the electromagnetic field (q is the electric charge of the particle, and $\mathbf{v}$ is its velocity, m is mass),

$$\mathscr{L} = \frac{m\mathbf{v}^2}{2} - q\phi + q\mathbf{v} \cdot \mathbf{A}.$$

Particularly, the canonical momentum is specified by the equation

$$\mathbf{p} = \nabla_{\mathbf{v}} \mathscr{L} = m\mathbf{v} + q\mathbf{A}.$$

Next we define the classical Hamiltonian by Legendre transform,

$$\mathscr{H} \triangleq \mathbf{p} \cdot \mathbf{v} - \mathscr{L} = m\mathbf{v}^2 + q\mathbf{A} \cdot \mathbf{v} - (\frac{m\mathbf{v}^2}{2} - q\phi + q\mathbf{v} \cdot \mathbf{A}) = \frac{(\mathbf{p} - q\mathbf{A})^2}{2m} + q\phi.$$

In quantum mechanics, we replace $\mathbf{p}$ by $-i\hbar\nabla$,($\hbar$ is the Planck constant), then we have

$$\mathscr{H} = \frac{(i\hbar\nabla + q\mathbf{A})^2}{2m} + q\phi.$$

There is lots of literature concerned with this operator[2][3]. In [4][5], N. Garofalo and F. H. Lin considered the unique continuation problem for the Laplacian operator by applying the multiplier method. In [6][7], K. Kurata used the same multipliers to prove the unique continuation property for the magnetic operator with $\mathbf{A}\mathbf{A}^T \in \mathscr{K}_N^{loc}(\Omega)$, where $\mathscr{K}_N^{loc}(\Omega)$ denotes the Kato class. We say $f \in L^1_{loc}(\Omega)$ belongs to the Kato class $\mathscr{K}_N^{loc}$ if

$$\lim_{r \to 0} \eta(r; f\chi_{\Omega_0}) = 0$$

for every compact subdomain Ω_0 of Ω. Here

$$\eta(r; f\chi_{\Omega_0}) \triangleq \sup_{x \in \mathbb{R}^N} \int_{\{|x-y|<r\} \cap \Omega_0} \frac{|f(y)|}{|x-y|^{N-2}} dy.$$

For the Hamiltonian with $\mathbf{A} \in (L^\infty(\Omega))^N$, in effect, it does not belong to the Kato class. Denote $\mathscr{H}^2_{\mathbf{A}} \triangleq (i\nabla + \mathbf{A})^2 : \mathcal{H} \to \mathcal{H}^*$. Assume that $\mathcal{H}(\Omega)$ is a Hilbert space. From the Hamiltonian, we can define the corresponding vector operator

$$\mathscr{H}_{\mathbf{A}} \triangleq i\nabla + \mathbf{A}(x) : \mathcal{H}(\Omega) \to (\mathcal{H}(\Omega))^N,$$

where $\mathbf{A}(x) \in C^1(\overline{\Omega})$ is the real-valued potential vector. By developing new multipliers, one proves the unique continuation result for the magnetic Schrödinger operator.

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Theorem 1.1. *For $N \geq 2$, let $\omega \in H^2(\mathbb{B}_1)$ be a solution of the elliptic problem*

$$\mathscr{H}_{\mathbf{A}}^2 \omega = \phi(x)\omega \quad \text{in } \mathbb{B}_1,$$

where $\mathbb{B}_1$ is a unit ball and the complex function $\phi \in L^\infty(\mathbb{R}^N)$. If ω vanishes in a neighborhood of $x_0 \in \mathbb{B}_1$, then $\omega \equiv 0$ in $\mathbb{B}_1$.

2. Proof of the main theorem: multiplier method

Let ϕ^R denote the real part of ϕ. For every $r \in (0,1)$, we define the following two quantities

$$\Phi(r) \triangleq \int_{\partial\mathbb{B}_r} |\omega|^2 dS_x,$$

$$\Psi(r) \triangleq \int_{\mathbb{B}_r} (|\mathscr{H}_{\mathbf{A}}\omega|^2 - \phi^R|\omega|^2) dV_x.$$

On the one hand,

$$\begin{aligned}
\int_{\mathbb{B}_r} \mathscr{H}_{\mathbf{A}}^2 |\omega|^2 dV_x &= -\int_{\partial\mathbb{B}_r} \Big(\nabla|\omega|^2 - i\mathbf{A}\cdot|\omega|^2\Big)\cdot \tfrac{x}{r} dS_x + \int_{\mathbb{B}_r} \mathbf{A}\cdot \mathscr{H}_{\mathbf{A}}|\omega|^2 dV_x \\
&= -\int_{\partial\mathbb{B}_r} \Big(\nabla|\omega|^2 - i\mathbf{A}|\omega|^2\Big)\cdot \tfrac{x}{r} dS_x \qquad (1)\\
&\quad + \int_{\mathbb{B}_r} \mathbf{A}\cdot(i\overline{\omega}\nabla\omega + i\omega\nabla\overline{\omega} + \mathbf{A}|\omega|^2) dV_x.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_{\mathbb{B}_r} \mathscr{H}_{\mathbf{A}}^2 |\omega|^2 dV_x &= \int_{\mathbb{B}_r} \Big(-\overline{\omega}\Delta\omega - \omega\Delta\overline{\omega} - 2|\nabla\omega|^2 + i\nabla\cdot\mathbf{A}|\omega|^2\Big) dV_x \qquad (2)\\
&\quad + \int_{\mathbb{B}_r} \Big(i\overline{\omega}\mathbf{A}\cdot\nabla\omega + i\omega\mathbf{A}\cdot\nabla\overline{\omega} + \mathbf{A}\mathbf{A}^T|\omega|^2\Big) dV_x.
\end{aligned}$$

Since

$$\nabla\cdot\mathbf{A}|\omega|^2 = \overline{\omega}\nabla\cdot\mathbf{A}\omega + \omega\mathbf{A}\cdot\nabla\overline{\omega} = \overline{\overline{\omega}\nabla\cdot\mathbf{A}\omega + \omega\mathbf{A}\cdot\nabla\overline{\omega}}, \qquad (3)$$

then by combining (1) and (2), one has

$$\begin{aligned}
&-\mathrm{Re}\int_{\partial\mathbb{B}_r} \Big(\nabla|\omega|^2 - i\mathbf{A}|\omega|^2\Big)\cdot \tfrac{x}{r} dS_x \\
&= \int_{\mathbb{B}_r} \Big(-\overline{\omega}\Delta\omega - \omega\Delta\overline{\omega} - 2|\nabla\omega|^2\Big) dV_x \\
&= \int_{\mathbb{B}_r} \Big(-2|\nabla\omega|^2 + 2i\omega\mathbf{A}\cdot\nabla\overline{\omega} - 2i\overline{\omega}\mathbf{A}\cdot\nabla\omega - 2\mathbf{A}\mathbf{A}^T|\omega|^2\Big) dV_x \qquad (4)\\
&\quad + \int_{\mathbb{B}_r} \Big(-\overline{\omega}\Delta\omega + i\overline{\omega}\mathbf{A}\cdot\nabla\omega + i\overline{\omega}\nabla\cdot\mathbf{A}\omega + \mathbf{A}\mathbf{A}^T|\omega|^2\Big) dV_x \\
&\quad + \int_{\mathbb{B}_r} \Big(-\omega\Delta\overline{\omega} - i\omega\mathbf{A}\cdot\nabla\overline{\omega} - i\omega\nabla\cdot\mathbf{A}\overline{\omega} + \mathbf{A}\mathbf{A}^T|\omega|^2\Big) dV_x \\
&= \int_{\mathbb{B}_r} \Big(-2|\mathscr{H}_{\mathbf{A}}\omega|^2 + 2\phi^R|\omega|^2\Big) dV_x.
\end{aligned}$$

Next we calculate the derivative of $\Phi(r)$ with respect to r.

$$\begin{aligned}
\Phi'(r) &= \int_{\partial\mathbb{B}_1} (|\omega(ry)|^2 r^{N-1})'_r dS_y \\
&= \int_{\partial\mathbb{B}_1} \Big((\nabla\omega\cdot y)\overline{\omega} + \omega(\nabla\overline{\omega}\cdot y) r^{N-1} + |\omega|^2(N-1)r^{N-2}\Big) dS_y \\
&= \int_{\partial\mathbb{B}_r} \Big((\nabla\omega\cdot \tfrac{x}{r})\overline{\omega} + \omega(\nabla\overline{\omega}\cdot \tfrac{x}{r})\Big) dS_x + \tfrac{N-1}{r}\int_{\partial\mathbb{B}_r} |\omega|^2 dS_x \qquad (5)\\
&= \tfrac{N-1}{r}\Phi(r) + \mathrm{Re}\int_{\partial\mathbb{B}_r} \Big(\nabla|\omega|^2 - i\mathbf{A}|\omega|^2\Big)\cdot \tfrac{x}{r} dS_x \\
&= \tfrac{N-1}{r}\Phi(r) + 2\Psi(r).
\end{aligned}$$

As for the derivative of $\Psi(r)$ with respect to r, one has

$$
\begin{aligned}
\Psi'(r) &= \int_{\partial\mathbb{B}_r} |\mathscr{H}_{\mathbf{A}}\omega|^2 dS_x - \int_{\partial\mathbb{B}_r} \phi^R |\omega|^2 dS_x \\
&= \tfrac{1}{r}\int_{\partial\mathbb{B}_r} |\mathscr{H}_{\mathbf{A}}\omega|^2 x\cdot \tfrac{x}{r} dS_x - \int_{\partial\mathbb{B}_r} \phi^R |\omega|^2 dS_x \\
&= \tfrac{1}{r}\int_{\mathbb{B}_r} \operatorname{div}(|\mathscr{H}_{\mathbf{A}}\omega|^2 x) dV_x - \int_{\partial\mathbb{B}_r} \phi^R |\omega|^2 dS_x \\
&= \tfrac{N}{r}\int_{\mathbb{B}_r} |\mathscr{H}_{\mathbf{A}}\omega|^2 dV_x + \underbrace{\frac{1}{r}\int_{\mathbb{B}_r} x\cdot\nabla|\mathscr{H}_{\mathbf{A}}\omega|^2 dV_x}_{(I)} - \int_{\partial\mathbb{B}_r} \phi^R |\omega|^2 dS_x.
\end{aligned}
\tag{6}
$$

Now we treat the term (I) carefully.

$$
\begin{aligned}
(I) &= \sum_{j,k}\frac{1}{r}\int_{\mathbb{B}_r} x_j\nabla_j\Big((i\nabla_k\omega + a_k\omega)\overline{(i\nabla_k\omega + a_k\omega)}\Big)dV_x \\
&= \sum_{j,k}\frac{2}{r}\operatorname{Re}\int_{\mathbb{B}_r} x_j\Big(\nabla_j(i\nabla_k\omega + a_k\omega)\overline{(i\nabla_k\omega + a_k\omega)}\Big)dV_x \\
&= \sum_{j,k}\frac{2}{r}\operatorname{Re}\int_{\mathbb{B}_r} ix_j\nabla_j\nabla_k\omega\overline{(i\nabla_k\omega + a_k\omega)}dV_x + \sum_{j,k}\frac{2}{r}\operatorname{Re}\int_{\mathbb{B}_r} x_j\nabla_j(a_k\omega))\overline{(i\nabla_k\omega + a_k\omega)}dV_x \\
&= \sum_{j,k}\frac{2}{r}\operatorname{Re}\int_{\partial\mathbb{B}_r} ix_j\nabla_j\omega\overline{(i\nabla_k\omega + a_k\omega)}\nu_k dS_x - \sum_{j,k}\frac{2}{r}\operatorname{Re}\int_{\mathbb{B}_r} i\nabla_k x_j\nabla_j\omega\overline{(i\nabla_k\omega + a_k\omega)}dV_x \\
&\quad -\sum_{j,k}\frac{2}{r}\operatorname{Re}\int_{\mathbb{B}_r} ix_j\nabla_j\omega\overline{\nabla_k(i\nabla_k\omega + a_k\omega)}dV_x + \sum_{j,k}\frac{2}{r}\operatorname{Re}\int_{\mathbb{B}_r} x_j\omega\nabla_j a_k\overline{(i\nabla_k\omega + a_k\omega)}dV_x \\
&\quad +\sum_{j,k}\frac{2}{r}\operatorname{Re}\int_{\mathbb{B}_r} a_k x_j\nabla_j\omega\overline{(i\nabla_k\omega + a_k\omega)}dV_x \\
&= \sum_{j,k}\frac{2}{r}\operatorname{Re}\int_{\partial\mathbb{B}_r} x_j(i\nabla_j\omega + a_j\omega)\overline{(i\nabla_k\omega + a_k\omega)}\nu_k dS_x \\
&\quad -\sum_{j,k}\frac{2}{r}\operatorname{Re}\int_{\partial\mathbb{B}_r} x_j a_j\omega\overline{(i\nabla_k\omega + a_k\omega)}\nu_k dS_x - \sum_{j,k}\frac{2}{r}\operatorname{Re}\int_{\mathbb{B}_r} i\nabla_k x_j\nabla_j\omega\overline{(i\nabla_k\omega + a_k\omega)}dV_x \\
&\quad -\sum_{j,k}\frac{2}{r}\operatorname{Re}\int_{\mathbb{B}_r} ix_j\nabla_j\omega\overline{\nabla_k(i\nabla_k\omega + a_k\omega)}dV_x + \sum_{j,k}\frac{2}{r}\operatorname{Re}\int_{\mathbb{B}_r} x_j\omega\nabla_j a_k\overline{(i\nabla_k\omega + a_k\omega)}dV_x \\
&\quad +\sum_{j,k}\frac{2}{r}\operatorname{Re}\int_{\mathbb{B}_r} a_k x_j\nabla_j\omega\overline{(i\nabla_k\omega + a_k\omega)}dV_x
\end{aligned}
$$

$$
\begin{aligned}
&= 2\int_{\partial\mathbb{B}_r}|\nu\cdot\mathscr{H}_{\mathbf{A}}\omega|^2dS_x - 2\ \mathrm{Re}\int_{\partial\mathbb{B}_r}(\mathbf{A}\omega\cdot\nu)(\overline{\mathscr{H}_{\mathbf{A}}\omega\cdot\nu})dS_x\\
&\quad -\tfrac{2}{r}\int_{\mathbb{B}_r}|\mathscr{H}_{\mathbf{A}}\omega|^2dV_x + \tfrac{2}{r}\ \mathrm{Re}\int_{\mathbb{B}_r}\mathbf{A}\omega\cdot\overline{\mathscr{H}_{\mathbf{A}}\omega}dV_x + \tfrac{2}{r}\ \mathrm{Re}\int_{\mathbb{B}_r}(x\cdot\nabla\omega)\cdot\overline{\mathscr{H}^2_{\mathbf{A}}\omega}dV_x\\
&\quad +\tfrac{2}{r}\ \mathrm{Re}\int_{\mathbb{B}_r}\omega x\times\Theta_{\mathbf{A}}\times\overline{\mathscr{H}_{\mathbf{A}}\omega}^TdV_x\\
&= 2\int_{\partial\mathbb{B}_r}|\nu\cdot\mathscr{H}_{\mathbf{A}}\omega|^2dS_x - 2\ \mathrm{Re}\int_{\partial\mathbb{B}_r}(\mathbf{A}\omega\cdot\nu)(\overline{\mathscr{H}_{\mathbf{A}}\omega\cdot\nu})dS_x\\
&\quad -\tfrac{2}{r}\int_{\mathbb{B}_r}|\mathscr{H}_{\mathbf{A}}\omega|^2dV_x + \tfrac{2}{r}\ \mathrm{Re}\int_{\mathbb{B}_r}\mathbf{A}\omega\cdot\overline{\mathscr{H}_{\mathbf{A}}\omega}dV_x + \tfrac{2}{r}\ \mathrm{Re}\int_{\mathbb{B}_r}(x\cdot\nabla\omega)\cdot\overline{\phi\omega}dV_x\\
&\quad +\tfrac{2}{r}\ \mathrm{Re}\int_{\mathbb{B}_r}\omega x\times\Theta_{\mathbf{A}}\times\overline{\mathscr{H}_{\mathbf{A}}\omega}^TdV_x,
\end{aligned}
$$

where

$$
\Theta_{\mathbf{A}} \triangleq \begin{pmatrix} \nabla_1 a_1 & \nabla_1 a_2, & \cdots & \nabla_1 a_N \\ \nabla_2 a_1 & \nabla_2 a_2 & \cdots & \nabla_2 a_N \\ \vdots & \vdots & \cdots & \vdots \\ \nabla_N a_1 & \nabla_N a_2 & \cdots & \nabla_N a_N \end{pmatrix}.
$$

Consequently, one has the following identity,

$$
\begin{aligned}
\Psi'(r) &= \tfrac{N-2}{r}\Psi(r) + \tfrac{(N-2)}{r}\int_{\mathbb{B}_r}\phi^R|\omega|^2dV_x + \tfrac{2}{r}\ \mathrm{Re}\int_{\mathbb{B}_r}\mathbf{A}\omega\cdot\overline{\mathscr{H}_{\mathbf{A}}\omega}dV_x\\
&\quad +\tfrac{2}{r}\ \mathrm{Re}\int_{\mathbb{B}_r}(x\cdot\nabla\omega)\cdot\overline{\phi\omega}dV_x + \tfrac{2}{r}\ \mathrm{Re}\int_{\mathbb{B}_r}\omega x\times\Theta_{\mathbf{A}}\times\overline{\mathscr{H}_{\mathbf{A}}\omega}^TdV_x\\
&\quad +2\int_{\partial\mathbb{B}_r}|\nu\cdot(i\nabla\omega+\mathbf{A}\omega)|^2dS_x - 2\ \mathrm{Re}\int_{\partial\mathbb{B}_r}(\mathbf{A}\omega\cdot\nu)(\overline{\mathscr{H}_{\mathbf{A}}\omega\cdot\nu})dS_x\\
&\quad -\int_{\partial\mathbb{B}_r}\phi^R|\omega|^2dS_x.
\end{aligned}
$$

Now we show an important comparison lemma.

Lemma 2.1. *There exists an $r_0\in(0,1)$ such that for every $r\in(0,r_0)$, one has*

$$
\int_{\mathbb{B}_r}|\omega|^2dV_x \le r\int_{\partial\mathbb{B}_r}|\omega|^2dS_x.
$$

Proof. On the one hand,

$$
\begin{aligned}
&\int_{\mathbb{B}_r}\mathscr{H}^2_{\mathbf{A}}|\omega|^2\cdot(r^2-|x|^2)dV_x\\
&= \int_{\mathbb{B}_r}|\omega|^2\cdot\overline{\mathscr{H}^2_{\mathbf{A}}(r^2-|x|^2)}dV_x + \int_{\partial\mathbb{B}_r}|\omega|^2\cdot\overline{\tfrac{\partial(r^2-|x|^2)}{\partial\nu_{i\mathscr{H}_{\mathbf{A}}}}}dS_x\\
&= \int_{\mathbb{B}_r}|\omega|^2\Big(2N+2i\mathbf{A}\cdot x - i\nabla\cdot\mathbf{A}(r^2-|x|^2)+\mathbf{A}\mathbf{A}^T(r^2-|x|^2)\Big)dV_x\\
&\quad -2r\int_{\partial\mathbb{B}_r}|\omega|^2dS_x.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\int_{\mathbb{B}_r}\mathscr{H}^2_{\mathbf{A}}|\omega|^2\cdot(r^2-|x|^2)dV_x\\
&= \int_{\mathbb{B}_r}\Big(-\overline{\omega}\Delta\omega-\omega\Delta\overline{\omega}-2|\nabla\omega|^2\Big)(r^2-|x|^2)dV_x + \int_{\mathbb{B}_r}\mathbf{A}\mathbf{A}^T|\omega|^2(r^2-|x|^2)dV_x\\
&\quad +\int_{\mathbb{B}_r}\Big(i\nabla\cdot\mathbf{A}|\omega|^2+i\overline{\omega}\mathbf{A}\cdot\nabla\omega+i\omega\mathbf{A}\cdot\nabla\overline{\omega}\Big)(r^2-|x|^2)dV_x\\
&= \int_{\mathbb{B}_r}\Big(-2|\mathscr{H}_{\mathbf{A}}\omega|^2+2\phi^R|\omega|^2\Big)(r^2-|x|^2)dV_x + \int_{\mathbb{B}_r}\mathbf{A}\mathbf{A}^T|\omega|^2(r^2-|x|^2)dV_x\\
&\quad +\int_{\mathbb{B}_r}\Big(i\nabla\cdot\mathbf{A}|\omega|^2+i\overline{\omega}\mathbf{A}\cdot\nabla\omega+i\omega\mathbf{A}\cdot\nabla\overline{\omega}\Big)(r^2-|x|^2)dV_x.
\end{aligned}
$$

As a result,

$$\int_{\mathbb{B}_r}\Big(2N|\omega|^2+2|\mathscr{H}_{\mathbf{A}}\omega|^2(r^2-|x|^2)-2\phi^R|\omega|^2(r^2-|x|^2)\Big)dV_x=2r\int_{\partial\mathbb{B}_r}|\omega|^2dS_x.$$

When $\|\phi^R\|_{L^\infty}>0$, then we choose $r_0\in(0,\frac{1}{2})$ such that

$$r_0^2\le\frac{N-1}{\|\phi^R\|_{L^\infty}}.$$

It follows immediately that

$$\int_{\mathbb{B}_r}|\omega|^2dV_x\le\int_{\mathbb{B}_r}\Big(N|\omega|^2-\phi^R|\omega|^2(r^2-|x|^2)\Big)dV_x\le r\int_{\partial\mathbb{B}_r}|\omega|^2dS_x.$$

When $\|\phi^R\|_{L^\infty}=0$, then

$$\int_{\mathbb{B}_r}|\omega|^2dV_x\le\frac{r}{N}\int_{\partial\mathbb{B}_r}|\omega|^2dS_x.$$

Q. E. D. □

Assume that there exists a small $r_1\in(0,1)$ such that

$$\Phi(r)\ne 0\quad\text{for}\quad\forall\ r\in(0,r_1).$$

Define

$$\digamma(r)\triangleq\frac{r\Psi(r)}{\Phi(r)},\qquad r\in(0,r_1).$$

Let $r^*\triangleq\min\{r_0,r_1\}$, and we set

$$\daleth_{r^*}\triangleq\Big\{r\in(0,r^*):\digamma(r)>1\Big\}.$$

Under this assumption, one has

$$\int_{\mathbb{B}_r}|\mathscr{H}_{\mathbf{A}}\omega|^2dV_x>(\frac{1}{r^2}-\|\phi^R\|_{L^\infty})\int_{\mathbb{B}_r}|\omega|^2dV_x.$$

Indeed,

$$\begin{aligned}\textstyle\int_{\mathbb{B}_r}|\mathscr{H}_{\mathbf{A}}\omega|^2dV_x &= \textstyle\int_{\mathbb{B}_r}(|\mathscr{H}_{\mathbf{A}}\omega|^2-\phi^R|\omega|^2)dV_x+\int_{\mathbb{B}_r}\phi^R|\omega|^2dV_x\\ &> \textstyle\frac{1}{r}\int_{\partial\mathbb{B}_r}|\omega|^2dS_x+\int_{\mathbb{B}_r}\phi^R|\omega|^2dV_x\\ &\ge \textstyle\frac{1}{r^2}\int_{\mathbb{B}_r}|\omega|^2dV_x+\int_{\mathbb{B}_r}\phi^R|\omega|^2dV_x\\ &\ge \textstyle(\frac{1}{r^2}-\|\phi^R\|_{L^\infty})\int_{\mathbb{B}_r}|\omega|^2dV_x.\end{aligned}$$

This indicates the integral $\int_{\mathbb{B}_r}|\mathscr{H}_{\mathbf{A}}\omega|^2dV_x$ is the dominating part in $\Psi(r)$. By calculating $\digamma'(r)$ with respect to r, one has the following identity.

$$\begin{aligned}
\digamma'(r) &= \digamma(r)\Big(\Psi'(r)/\Psi(r)+1/r-\Phi'(r)/\Phi(r)\Big)\\
&= \digamma(r)\underbrace{\Big(\frac{2\int_{\partial\mathbb{B}_r}|\nu\cdot(i\nabla\omega+\mathbf{A}\omega)|^2dS_x}{\mathrm{Re}\int_{\partial\mathbb{B}_r}(\frac{x}{r}\cdot\nabla\omega)\overline{\omega}dS_x}-\frac{2\ \mathrm{Re}\int_{\partial\mathbb{B}_r}(\frac{x}{r}\cdot\nabla\omega)\overline{\omega}dS_x}{\int_{\partial\mathbb{B}_r}|\omega|^2dS_x}\Big)}_{(II)}\\
&\quad+\digamma(r)\Big\{\underbrace{\frac{(N-2)}{r}\int_{\mathbb{B}_r}\phi^R|\omega|^2dV_x}_{(III)}+\underbrace{\frac{2}{r}\ \mathrm{Re}\int_{\mathbb{B}_r}\mathbf{A}\omega\cdot\overline{\mathscr{H}_{\mathbf{A}}\omega}dV_x}_{(IV)}\\
&\quad+\underbrace{\frac{2}{r}\ \mathrm{Re}\int_{\mathbb{B}_r}(x\cdot\nabla\omega)\cdot\overline{\phi\omega}dV_x}_{(V)}+\underbrace{\frac{2}{r}\ \mathrm{Re}\int_{\mathbb{B}_r}\omega x\times\Theta_{\mathbf{A}}\times\overline{\mathscr{H}_{\mathbf{A}}\omega}^TdV_x}_{(VI)}\\
&\quad\underbrace{-2\ \mathrm{Re}\int_{\partial\mathbb{B}_r}(\mathbf{A}\omega\cdot\nu)(\overline{\mathscr{H}_{\mathbf{A}}\omega\cdot\nu})dS_x}_{(VII)}\\
&\quad\underbrace{-\int_{\partial\mathbb{B}_r}\phi^R|\omega|^2dS_x}_{(VIII)}\Big\}\Big/\Big\{\tfrac{1}{2}\int_{\partial\mathbb{B}_r}\tfrac{x}{r}\cdot\nabla|\omega|^2dS_x\Big\}.
\end{aligned}$$

We estimate each term respectively. For (II), one applies Hölder's inequality,

$$\begin{aligned}
(II) &= 2\int_{\partial\mathbb{B}_r}|\nu\cdot(i\nabla\omega+\mathbf{A}\omega)|^2dS_x/\mathrm{Re}\int_{\partial\mathbb{B}_r}(\tfrac{x}{r}\cdot\nabla\omega)\overline{\omega}dS_x\\
&\quad-2\ \mathrm{Re}\int_{\partial\mathbb{B}_r}(\tfrac{x}{r}\cdot\nabla\omega)\overline{\omega}dS_x/\int_{\partial\mathbb{B}_r}|\omega|^2dS_x\\
&= 2\int_{\partial\mathbb{B}_r}|\nu\cdot(i\nabla\omega+\mathbf{A}\omega)|^2dS_x/\mathrm{Re}\int_{\partial\mathbb{B}_r}(\tfrac{x}{r}\cdot(\nabla\omega-i\mathbf{A}\omega))\overline{\omega}dS_x\\
&\quad-2\ \mathrm{Re}\int_{\partial\mathbb{B}_r}(\tfrac{x}{r}\cdot(\nabla\omega-i\mathbf{A}\omega))\overline{\omega}dS_x/\int_{\partial\mathbb{B}_r}|\omega|^2dS_x\\
&\geq 2\int_{\partial\mathbb{B}_r}|\nu\cdot(i\nabla\omega+\mathbf{A}\omega)|^2dS_x/\Big(\Big\{\int_{\partial\mathbb{B}_r}|(\tfrac{x}{r}\cdot(\nabla\omega-i\mathbf{A}\omega))|^2dS_x\Big\}^{\frac{1}{2}}\Big\{\int_{\partial\mathbb{B}_r}|\omega|^2dS_x\Big\}^{\frac{1}{2}}\Big)\\
&\quad-2\Big\{\int_{\partial\mathbb{B}_r}|(\tfrac{x}{r}\cdot(\nabla\omega-i\mathbf{A}\omega))|^2dS_x\Big\}^{\frac{1}{2}}\Big\{\int_{\partial\mathbb{B}_r}|\omega|^2dS_x\Big\}^{\frac{1}{2}}/\int_{\partial\mathbb{B}_r}|\omega|^2dS_x\\
&\geq 0.
\end{aligned}$$

Moreover, one has

Lemma 2.2. *There exists a constant $C^*(\phi)$ independent of r such that*

$$\frac{\int_{\partial\mathbb{B}_r}|\nu\cdot(i\nabla\omega+\mathbf{A}\omega)|^2dS_x}{\Psi(r)}\leq\frac{C^*(\phi)}{r}.$$

Proof. Indeed, by multiplying $\mathbf{H}(x)\cdot\mathscr{H}_{\mathbf{A}}\omega$ to $\mathscr{H}^2_{\mathbf{A}}\omega=\phi\omega$ and integrating by parts, one has the following identity,

$$\begin{aligned}
&-\tfrac{1}{2}\int_{\partial\mathbb{B}_r}\Big|\frac{\partial\omega}{\partial\nu_{i\mathscr{H}_{\mathbf{A}}}}\Big|^2\cdot\Big(\mathbf{H}(x)\cdot\nu\Big)dS_x\\
&= \tfrac{1}{2}\int_{\mathbb{B}_r}\Big(\nabla\cdot\mathbf{H}(\mathbf{x})\Big)\cdot\Big|\mathscr{H}_{\mathbf{A}}\omega\Big|^2dV_x-\mathrm{Im}\int_{\mathbb{B}_r}\phi\omega\cdot\Big(\mathbf{H}(x)\cdot\overline{\mathscr{H}_{\mathbf{A}}\omega}\Big)dV_x\\
&\quad-\mathrm{Re}\int_{\mathbb{B}_r}\mathscr{H}_{\mathbf{A}}\omega\times\Theta_{\mathbf{H}}\times\overline{\mathscr{H}^T_{\mathbf{A}}\omega}dV_x-\mathrm{Re}\int_{\mathbb{B}_r}\overline{\omega}\mathscr{H}_{\mathbf{A}}\omega\times\Xi_{\mathbf{A}}\times\mathbf{H}^TdV_x.
\end{aligned}$$

Since

$$\int_{\mathbb{B}_r} |\mathscr{H}_{\mathbf{A}}\omega|^2 dV_x \geq (\frac{1}{r^2} - \|\phi^R\|_{L^\infty}) \int_{\mathbb{B}_r} |\omega|^2 dV_x,$$

by choosing

$$\mathbf{H}(x) \triangleq \frac{x}{r},$$

one has the following estimate,

$$\begin{aligned}
& \tfrac{1}{2} \int_{\partial\mathbb{B}_r} \left|\frac{\partial \omega}{\partial \nu_{i\mathscr{H}_{\mathbf{A}}}}\right|^2 dS_x \\
\leq\ & \tfrac{N}{2r} \int_{\mathbb{B}_r} |\mathscr{H}_{\mathbf{A}}\omega|^2 dV_x + \tfrac{\|\phi\|_{L^\infty}}{2} \int_{\mathbb{B}_r} (|\omega|^2 + |\mathscr{H}_{\mathbf{A}}\omega|^2) dV_x \\
& + \tfrac{1}{r} \int_{\mathbb{B}_r} |\mathscr{H}_{\mathbf{A}}\omega|^2 dV_x + \tfrac{1}{2} \max \|\Xi_{\mathbf{A}}\|_F \int_{\mathbb{B}_r} (|\omega|^2 + |\mathscr{H}_{\mathbf{A}}\omega|^2) dV_x \\
=\ & \underbrace{\Big(\frac{N+2}{2r} + \frac{\|\phi\|_{L^\infty} + \max \|\Xi_{\mathbf{A}}\|_F}{2}\Big)}_{\alpha} \int_{\mathbb{B}_r} |\mathscr{H}_{\mathbf{A}}\omega|^2 dV_x + \underbrace{\frac{\|\phi\|_{L^\infty} + \max \|\Xi_{\mathbf{A}}\|_F}{2}}_{\beta} \int_{\mathbb{B}_r} |\omega|^2 dV_x \\
\leq\ & \Big(\alpha + \tfrac{r^2}{1 - r^2\|\phi^R\|_{L^\infty}} \beta\Big) \int_{\mathbb{B}_r} |\mathscr{H}_{\mathbf{A}}\omega|^2 dV_x.
\end{aligned}$$

Since

$$\Psi(r) = \int_{\mathbb{B}_r} (|\mathscr{H}_{\mathbf{A}}\omega|^2 - \phi^R|\omega|^2) dV_x \geq \frac{1 - 2r^2\|\phi^R\|_{L^\infty}}{1 - r^2\|\phi^R\|_{L^\infty}} \int_{\mathbb{B}_r} |\mathscr{H}_{\mathbf{A}}\omega|^2 dV_x,$$

therefore,

$$\frac{\int_{\partial\mathbb{B}_r} |\nu \cdot (i\nabla\omega + \mathbf{A}\omega)|^2 dS_x}{\Psi(r)} \leq \frac{1 - r^2\|\phi^R\|_{L^\infty}}{1 - 2r^2\|\phi^R\|_{L^\infty}} \Big(2\alpha + \frac{2r^2}{1 - r^2\|\phi^R\|_{L^\infty}} \beta\Big).$$

Q. E. D. □

Keeping Lemma 2.2 into account and noticing the fact $\mathcal{F}(r) > 1$, for the term (III), one has

$$(III)/\Psi(r) = \frac{(N-2)\int_{\mathbb{B}_r} \phi^R |\omega|^2 dV_x}{r\Psi(r)} \leq \frac{(N-2)\|\phi^R\|_{L^\infty}\Phi(r)}{\Psi(r)} \leq r(N-2)\|\phi^R\|_{L^\infty}.$$

For the term (IV),

$$\begin{aligned}
|(IV)/\Psi(r)| &= |2 \text{ Re} \int_{\mathbb{B}_r} \mathbf{A}\omega \cdot \overline{\mathscr{H}_{\mathbf{A}}\omega} dV_x| / (r\Psi(r)) \\
& \leq 2 \int_{\mathbb{B}_r} |\mathbf{A}\omega| \cdot |\overline{\mathscr{H}_{\mathbf{A}}\omega}| dV_x / (r\Psi(r)) \\
&\leq \Big(\tfrac{1}{2\epsilon r} \int_{\mathbb{B}_r} |\mathbf{A}\omega|^2 dV_x + \tfrac{2\epsilon}{r} \int_{\mathbb{B}_r} |\mathscr{H}_{\mathbf{A}}\omega|^2 dV_x\Big) / \Psi(r) \\
&= \Big(\tfrac{1}{2\epsilon r} \int_{\mathbb{B}_r} \mathbf{A}\mathbf{A}^T |\omega|^2 dV_x + \tfrac{2\epsilon}{r} \int_{\mathbb{B}_r} (|\mathscr{H}_{\mathbf{A}}\omega|^2 - \phi^R|\omega|^2) dV_x + \tfrac{2\epsilon}{r} \int_{\mathbb{B}_r} \phi^R |\omega|^2 dV_x\Big) / \Psi(r) \\
&\leq \|\mathbf{A}\mathbf{A}\|_{L^\infty} \Phi(r) / (2\epsilon\Psi(r)) + 2\epsilon/r + 2\epsilon\|\phi^R\|_{L^\infty} \Phi(r)/\Psi(r).
\end{aligned}$$

Let $\epsilon = \frac{r}{2}$, since $\mathcal{F}(r) > 1$, then

$$|(IV)/\Psi(r)| \leq \|\mathbf{A}\mathbf{A}\|_{L^\infty} + 1 + r^2\|\phi^R\|_{L^\infty}.$$

For the term (V),

$$\begin{aligned}
|(V)/\Psi(r)| &= |2\ \mathrm{Re}\int_{\mathbb{B}_r}(x\cdot\nabla\omega)\cdot\overline{\phi\omega}dV_x|/(r\Psi(r))\\
&= |2\ \mathrm{Re}\int_{\mathbb{B}_r}(x\cdot(\nabla\omega-i\mathbf{A}\omega))\cdot\overline{\phi\omega}dV_x|/(r\Psi(r))\\
&\leq \Big(\tfrac{\|\phi\|_{L^\infty}}{2\epsilon r}\int_{\mathbb{B}_r}|\omega|^2dV_x+\tfrac{2\epsilon\|\phi\|_{L^\infty}}{r}\int_{\mathbb{B}_r}|x\cdot\mathscr{H}_{\mathbf{A}}\omega|^2dV_x\Big)/\Psi(r)\\
&\leq \Big(\tfrac{\|\phi\|_{L^\infty}}{2\epsilon r}\int_{\mathbb{B}_r}|\omega|^2dV_x+2r\epsilon\|\phi\|_{L^\infty}\int_{\mathbb{B}_r}|\mathscr{H}_{\mathbf{A}}\omega|^2dV_x\Big)/\Psi(r)\\
&= \Big(\tfrac{\|\phi\|_{L^\infty}}{2\epsilon r}\int_{\mathbb{B}_r}|\omega|^2dV_x+2r\epsilon\|\phi\|_{L^\infty}\int_{\mathbb{B}_r}(|\mathscr{H}_{\mathbf{A}}\omega|^2-\phi^R|\omega|^2)dV_x\\
&\quad +2r\epsilon\|\phi\|_{L^\infty}\int_{\mathbb{B}_r}\phi^R|\omega|^2dV_x\Big)/\Psi(r)\\
&\leq r\|\phi\|_{L^\infty}/(2\epsilon)+2r\epsilon\|\phi\|_{L^\infty}+2r^3\epsilon\|\phi\|^2_{L^\infty}.
\end{aligned}$$

Let $\epsilon=\frac{r}{2}$, then

$$|(V)/\Psi(r)|\leq\|\phi\|_{L^\infty}(1+r^2+r^4\|\phi\|_{L^\infty}).$$

For the term (VI),

$$\begin{aligned}
|(VI)/\Psi(r)| &= |2\ \mathrm{Re}\int_{\mathbb{B}_r}\omega x\times\Theta_{\mathbf{A}}\times\overline{\mathscr{H}_{\mathbf{A}}\omega}^T dV_x|/(r\Psi(r))\\
&\leq 2\int_{\mathbb{B}_r}\|\omega x\|_2\|\Theta_{\mathbf{A}}\times\overline{\mathscr{H}_{\mathbf{A}}\omega}^T\|_2dV_x/(r\Psi(r))\\
&\leq 2\int_{\mathbb{B}_r}\|\omega x\|_2\|\Theta_{\mathbf{A}}\|_F\|\mathscr{H}_{\mathbf{A}}\omega\|_2dV_x/(r\Psi(r))\\
&\leq 2\max\|\Theta_{\mathbf{A}}\|_F\int_{\mathbb{B}_r}|\omega||\mathscr{H}_{\mathbf{A}}\omega|dV_x/\Psi(r)\\
&= \Big(\tfrac{\max\|\Theta_{\mathbf{A}}\|_F}{2\epsilon}\int_{\mathbb{B}_r}|\omega|^2dV_x+2\epsilon\max\|\Theta_{\mathbf{A}}\|_F\{\int_{\mathbb{B}_r}(|\mathscr{H}_{\mathbf{A}}\omega|^2-\phi^R|\omega|^2)dV_x\\
&\quad +\int_{\mathbb{B}_r}\phi^R|\omega|^2dV_x\}\Big)/\Psi(r)\\
&\leq \max\|\Theta_{\mathbf{A}}\|_F(r^2/(2\epsilon)+2\epsilon+2\epsilon r^2\|\phi^R\|_{L^\infty}).
\end{aligned}$$

Let $\epsilon=\frac{r}{2}$, then

$$|(VI)/\Psi(r)|\leq\max\|\Theta_{\mathbf{A}}\|_F(2r+r^3\|\phi^R\|_{L^\infty}).$$

For the term (VII) with $\epsilon=\frac{r}{2}$,

$$\begin{aligned}
|(VII)/\Psi(r)| &= |-2\ \mathrm{Re}\int_{\partial\mathbb{B}_r}(\mathbf{A}\omega\cdot\nu)(\overline{\mathscr{H}_{\mathbf{A}}\omega\cdot\nu})dS_x|/\Psi(r)\\
&\leq 2\int_{\partial\mathbb{B}_r}|\mathbf{A}\omega\cdot\nu||\overline{\mathscr{H}_{\mathbf{A}}\omega\cdot\nu}|dS_x/\Psi(r)\\
&\leq \Big(\tfrac{1}{2\epsilon}\int_{\partial\mathbb{B}_r}|\mathbf{A}\omega|^2dS_x+2\epsilon\int_{\partial\mathbb{B}_r}|\mathscr{H}_{\mathbf{A}}\omega\cdot\nu|^2dS_x\Big)/\Psi(r)\\
&\leq \|\mathbf{A}\mathbf{A}^T\|_{L^\infty}\int_{\partial\mathbb{B}_r}|\omega|^2dS_x/(2\epsilon\Psi(r))+2\epsilon\int_{\partial\mathbb{B}_r}|\mathscr{H}_{\mathbf{A}}\omega\cdot\nu|^2dS_x/\Psi(r)\\
&\leq r\|\mathbf{A}\mathbf{A}^T\|_{L^\infty}/(2\epsilon)+2\epsilon\int_{\partial\mathbb{B}_r}|\mathscr{H}_{\mathbf{A}}\omega\cdot\nu|^2dS_x/\Psi(r)\\
&\leq \|\mathbf{A}\mathbf{A}^T\|_{L^\infty}+C^*(\phi),
\end{aligned}$$

where $C^*(\phi)$ is from Lemma 2.2. For the last term (VIII),

$$|(VIII)/\Psi(r)|=\int_{\partial\mathbb{B}_r}\phi^R|\omega|^2dS_x/\Psi(r)\leq r\|\phi^R\|_{L^\infty}.$$

From the above estimates, we conclude that there exists a positive constant $\tau = \tau(N, \phi)$ which is independent of r such that

$$\mathcal{F}'(r) \geq -\mathcal{F}(r)\tau.$$

It follows that $\exp(\tau r)\mathcal{F}(r)$ is monotonously increasing on $(0, r^*)$, that is to say,

$$\exp(\tau r)\mathcal{F}(r) \leq \exp(\tau r^*)\mathcal{F}(r^*).$$

Thus, $\mathcal{F}(r)$ is bounded on $(0, r^*)$. Since

$$\Phi'(r) = \frac{N-1}{r}\Phi(r) + 2\Psi(r),$$

then

$$\Big(\log \frac{\Phi(r)}{r^{N-1}}\Big)' = \frac{2\Psi(r)}{\Phi(r)} = \frac{2\mathcal{F}(r)}{r} \leq \frac{C(\tau)}{r}.$$

We integrate from γ to 2γ, then

$$\log \frac{2^{1-N}\Phi(2\gamma)}{\Phi(\gamma)} \leq C(\tau)\log 2.$$

It follows that

$$\Phi(2\gamma) \leq 2^{C(\tau)+N-1}\Phi(\gamma).$$

Finally, integrating with respect to γ gives

$$\int_{\mathbb{B}_{2\gamma}} |\omega|^2 dV_x \leq 2^{C(\tau)+N-1}\int_{\mathbb{B}_\gamma} |\omega|^2 dV_x.$$

Since $\mathbb{B}_1$ is connected, then our conclusion follows immediately.

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