An Arithmetic Poisson Formula for the multi–variate resultant

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Abstract

In these pages we compute the expectation of several functions of multi-variate complex polynomials. The common thread of all our outcomes is the basic technique used in their proofs. The used techniques combine essentially the unitary invariance of Bombieri-Weyl’s Hermitian product and some elementary Integral Geometry. Using different combinations of these techniques we compute the expectation of the logarithm of the absolute value of an affine polynomial and we compute the expected value of Akatsuka Zeta Mahler’s measure. As main consequences of these results and techniques, we show a probabilistic answer to question (d) in [4], concerning the complexity of one point homotopy, and an Arithmetic Poisson Formula for the multi-variate resultant. These two last statements and bounds are related to the complexity of algorithms for polynomial equation solving.

Keywords: Akatsuka’z Zeta Mahler Function, Arithmetic Poisson Formula, Multi-variate resultant.

1. Introduction and Statement of the Main Results

In these pages we compute the expectation of several functions of multi-variate complex polynomials. The common thread of all our outcomes is the basic technique used in their proofs. The used techniques combine essentially the unitary invariance of Bombieri-Weyl’s Hermitian product and some elementary Integral Geometry. Using different combinations of these techniques we compute the expectation of the logarithm of the absolute value of an affine polynomial (Theorem 1.1) and we compute the expected value of Akatsuka Zeta Mahler’s measure (Theorem 1.2). As main consequences of these results and techniques, we show a probabilistic answer to question (d) in [4] (cf. Corollary 1.3), concerning the complexity...
of one point homotopy, and an Arithmetic Poisson Formula for the multi-variate resultant (cf. Theorem 1.4). These two last statements and bounds are related to the complexity of algorithms for polynomial equation solving.

We begin by introducing some notations to state the main outcomes. As in [13], given two positive integers \( d, n \in \mathbb{N} \), we denote by \( H_d^{(n+1)} \) the complex vector space of all homogeneous polynomials \( f \in \mathbb{C}[X_0, \ldots, X_n] \) of degree \( d \). For every degree list \( (d) := (d_0, \ldots, d_n) \), we denote by \( H_d := H_{d_0}^{(n+1)} \times \cdots \times H_{d_n}^{(n+1)} \) the Cartesian product of the spaces \( H_{d_0}^{(n+1)}, \ldots, H_{d_n}^{(n+1)} \). This space may be interpreted as the complex vector space of over-determined systems of homogeneous equations of degrees respectively equal to the degrees in the list \( (d) \). For every polynomial \( f \in H_d^{(n+1)} \) we denote by \( af \) its affine trace (i.e. \( af := f(1, X_1, \ldots, X_n) \in \mathbb{C}[X_1, \ldots, X_n] \)).

We equip \( H_d^{(n+1)} \) with a unitarily invariant Hermitian structure usually called Bombieri-Weyl’s Hermitian product (cf. [43] or [4] and its references, for instance). A definition of this metric is given in Subsection 2.2 below. Its invariance under the action of the Unitary group \( U(n+1) \) has been shown to be a useful property in studies of the complexity of polynomial equation solving. For instance, it has been used by M. Shub and S. Smale in the series of works developed between 1981 and 1995 (the “Bézout series”, which go from [37] to [40]). It became a central notion to define probability distributions in spaces of systems of complex equations which led to a Las Vegas solution of Smale’s 17th Problem (cf. [8], [9], [15], [41] for further information). In the forthcoming book [16], the authors explain in much more detail the interest of using Bombieri-Weyl’s Hermitian product as natural norm to define probability distributions in the space of polynomial systems. These works inspired our techniques here.

For two positive integers \( n, d \in \mathbb{N} \), we define the quantity:

\[
\mathcal{E}(d, n) := \frac{1}{2} (dH_n - H_{Nd}),
\]

where \( N_d := \binom{d+n}{n} - 1 \) is the complex dimension of the complex projective space \( \mathbb{P}(H_d^{(n+1)}) \) and \( H_r \) denotes the \( r \)-th harmonic number. In the defective case \( (n = 0) \), we define \( \mathcal{E}(d, 0) := -\frac{H_d}{2} \).

**Theorem 1.1.** Let \( n \in \mathbb{N} \) be a positive integer. Let \( S(H_d^{(n+1)}) \) be the unit sphere in \( H_d^{(n+1)} \) with respect to the Bombieri-Weyl’s Hermitian product and let \( \varphi_0 : \mathbb{C}^n \to \mathbb{P}_n(\mathbb{C}) \) the standard embedding of the affine space into its projective closure. Then, the following equality holds:

\[
E_{f \in S(H_d^{(n+1)})}[E_{\mathbb{P}_n(\mathbb{C})}(|\log |af \circ \varphi_0^{-1}||)] = \mathcal{E}(d, n).
\]  

(1)

where \( E_{f \in S(H_d^{(n+1)})} \) denotes expectation with respect to the probability distribution induced on \( S(H_d^{(n+1)}) \) by its natural structure of complex Riemannian manifold of finite volume. Accordingly, \( E_{\mathbb{P}_n(\mathbb{C})} \) is the expectation with respect to the probability distribution induced on \( \mathbb{P}_n(\mathbb{C}) \) by its Fubini-Study Riemannian structure of finite volume.

This expectation satisfies (cf. Subsection 2.3):

\[
2\mathcal{E}(d, n) \leq (d - 1)\gamma + (d + n + 1/2) \log \left( \frac{n}{(d + n)} \right) + (d + 1/2) \log d + c \left( \frac{d}{n} \right) + k_1,
\]

(2)
where $\gamma$ is Euler-Mascheroni number and $c > 0$, $k_1 > 0$ are some universal constants. In the case of linear forms, we have $E(1,n) = 0$. Asymptotically when $n \to \infty$, we have

$$E(d,n) \approx \frac{1}{2} \left((d-1)\gamma + \log(\Gamma(d+1))\right),$$

where $\Gamma$ is the Gamma function.

The manuscript [4] deals with the complexity of numerical methods for solving systems of multi-variate polynomial equations, based on “homotopy at one point”. The main outcome of [4] is Theorem 1.1 which exhibits an upper bound $E$ of the average complexity of these algorithmic procedures. This quantity $E$ is the expectation of a sophisticated function involving condition numbers and a function $\Theta(h, \eta)$ which is also an expectation on a basin $B(h, \eta)$. As claimed by the authors of [4]: “Essentially nothing is known about the integrals”. In view of the preliminary difficulties to determine the value of $E$, the authors of [4] stated four open questions (questions (a) to (d)) and claimed “Even for $n = 1$ these are interesting questions....”. Question (d) asks for estimates of a much simpler function that we call $I$ here (cf. also Subsection 4.3 for its definition). In these pages, our contribution consists in the exact computation of the expectation $E[I]$ of this function $I$, with respect to a normal probability distribution based Bombieri-Weyl’s Hermitian product. This just gives a probabilistic estimate of $I$.

With similar techniques to those used to prove the previous Theorem 1.1, we first obtain the expected value of Akatsuka’s Zeta Mahler measure (introduced in [1]) and, then, we conclude the value $E[I]$, with respect to a normal probability distribution based Bombieri-Weyl’s Hermitian product. This just gives a probabilistic estimate of $I$.

Let $m \in \mathbb{N}$ be a positive integer and let $(d) := (d_1, \ldots, d_m)$ be a list of $m$ positive integers. We denote by $H^{(m)}(d)$ the Cartesian product of the complex vector spaces $H_{d_1}^{(n+1)}, \ldots, H_{d_m}^{(n+1)}$. Namely,

$$H^{(m)}(d) := H_{d_1}^{(n+1)} \times \cdots \times H_{d_m}^{(n+1)}.$$

The complex vector space $H^{(m)}(d)$ is the set of all lists $f = (f_1, \ldots, f_m)$ of $m$ homogeneous polynomials $f_i \in \mathbb{C}[X_0, \ldots, X_n]$ of respective degree $d_i \in \mathbb{N}$. We denote by $N(d)$ (or simply by $N$) the complex dimension of the complex projective space $\mathbb{P}(H^{(m)}(d))$.

**Definition 1.1 (Akatsuka’s Zeta Mahler measure function).** Let $(d) := (d_1, \ldots, d_m) \in \mathbb{N}^m$ be a degree list of positive integers. For every list of polynomials $f := (f_1, \ldots, f_m) \in H^{(m)}(d)$, and every $s \in \mathbb{C}$, with positive real part $\text{Re}(s) > 0$, we define the Akatsuka Zeta Mahler measure function $Z^{(m)}(s, f)$ by the following identity:

$$Z^{(m)}(s, f) := \frac{1}{\nu_S(S^{2n+1})} \int_{S^{2n+1}} \|f(z)\|^s d\nu_S(z),$$

where $f(z) \in \mathbb{C}^m$ and the norm is the standard Hermitian norm in $\mathbb{C}^m$ and $\nu_S(S^{2n+1})$ is the volume of the complex hyper-sphere $S^{2n+1} \subseteq \mathbb{C}^{n+1}$.

While this manuscript was in the editing process, Arrmentano and Shub have found an upper bound for the average complexity of their homotopy at one point that is polynomial in the input dimension and the Bézout number (cf. the second version of their manuscript, published the 7th of March 2013, arXiv:1204.0036v2)
In the case of a single equation (i.e. \( m = 1 \)) \( Z(s, f) := Z^{(1)}(s, f) \) is Akatsuka’s zeta Mahler measure function as defined in [1], whereas \( Z(2, f) \) is the square of the Bombieri-Weyl norm of \( f \). This function is (as Mahler’s measure and Bombieri-Weyl’s norm) invariant under the action of the unitary group \( \mathcal{U}(n + 1) \) on \( \mathcal{H}^{(m)}_{(d)} \). The function is also related with the moments of the logarithm of the norm of a systems of polynomial equations \( f = (f_1, \ldots, f_m) \) through the following equality:

\[
Z^{(m)}(s, f) := \sum_{k=0}^{\infty} \frac{s^k m_k^{(m)}(f)}{k!},
\]

where

\[
m_k^{(m)}(f) := \frac{1}{\nu_S(S^{2n+1})} \int_{S^{2n+1}} \log^k ||f(z)|| d\nu_S(z).
\]

With the same techniques as in Theorem 1.1, we also prove the following statement.

**Definition 1.2 (Expected Akatsuka Zeta Mahler measure function).** With the same notations as above, for every complex number \( s \in \mathbb{C} \), with positive real part \( \text{Re}(s) > 0 \), let us define the expected Akatsuka Zeta Mahler measure function \( Z^{(m)}_E(s) \) as the expected value of \( Z^{(m)}(s, f) \), where \( f \) is randomly chosen in the unit sphere \( S(H^{(m)}_{(d)}) \) with respect to probability distribution induced by Bombieri-Weyl’s norm. Namely, we define:

\[
Z^{(m)}_E(s) := E_{f \in S(H^{(m)}_{(d)})}[Z^{(m)}(s, f)].
\]

**Theorem 1.2.** With these notations, we have:

\[
Z^{(m)}_E(s) := \frac{B(m + \frac{s}{2}, N(d) - m + 1)}{B(N(d) - m + 1, m)} = \frac{\Gamma(N(d) + 1)}{\Gamma(N(d) + s/2 + 1)} \frac{\Gamma(m + s/2)}{\Gamma(m)},
\]

where \( N(d) \) is the complex dimension of the complex projective space \( \mathbb{P}(H^{(m)}_{(d)}) \) and \( B \) is the Beta function.

In the case \( m = 1 \) the expected value \( Z_E(s) \) is given by:

\[
Z_E(s) := Z^{(1)}_E(s) = \frac{\Gamma(N_d + 1)\Gamma(s/2 + 1)}{\Gamma(N_d + s/2 + 1)} = N_d B(N_d, s/2 + 1),
\]

where \( N_d \) is the complex dimension of the complex projective space \( \mathbb{P}(H^{(n+1)}_{(d)}) \).

The following consequence probabilistically answers question \((d)\) of [4]:

**Corollary 1.3.** Let \( I : \mathcal{H}_{(d)} \rightarrow \mathbb{R} \) be the function introduced in question \((d)\) of [4](see Subsection 4.2 below for a precise definition). Then, the expected value of \( I(h) \), where \( h \) belongs to the Bombieri-Weyl unit sphere \( S(H^{(n)}_{(d)}) \) satisfies:

\[
E_{h \in S(H^{(n)}_{(d)})}[I(h)] = \frac{\Gamma(N(d) + 1)}{4\pi^{n+2} n} \sum_{k=0}^{\infty} \frac{1}{2^k k!} \frac{\Gamma(k + 1/2)}{\Gamma(N(d) + k - n + 3/2)},
\]

where \( N(d) \) is the complex dimension of the complex projective space \( \mathbb{P}(H^{(n)}_{(d)}) \).
Our techniques and Theorem 1.1 yield an Arithmetic Poisson Formula for the multivariate resultant. An Arithmetic Poisson Formula could also be derived from the computations of the height of the multi-variate resultant done in the nineties in works as [14] or combining the techniques of [33] with those in [34] and [35] (see additional information of Philippon’s school in the references of these last works).

Our proof techniques here are completely different from theirs. Here, we just point out that the unitary invariance of Bombieri-Weyl’s norm plus some elementary Integral Geometry (no more than the Co-area Formula) suffice to prove an Arithmetic Poisson Formula for the multi-variate resultant and, hence, a complete calculation of the logarithmic Mahler’s measure of the central polynomial in Elimination Theory.

For a degree list \((d)\) the multi-variate resultant \(\text{Res}(d)\) is a multi-homogeneous and irreducible diophantine polynomial whose variables are the coefficients of lists \(f := (f_0, \ldots, f_n) \in \mathcal{H}(d)\).

We denote by \(\text{Res}(d) (f_0, \ldots, f_n)\) or \(\text{Res}(d) (f)\) the value of the resultant \(\text{Res}(d)\) at the coefficients of the polynomials in the list \(f\). The multi-variate resultant is defined as the unique irreducible diophantine polynomial such that the following property holds:

For every degree list \((d) := (d_0, \ldots, d_n)\), given \(f := (f_0, \ldots, f_n) \in \mathcal{H}(d)\) the over-determined system of equations \(f\) has a common projective zero in \(\mathbb{P}_n(\mathbb{C})\) if and only if \(\text{Res}(d) (f) = 0\).

Namely,

\[ \exists z \in \mathbb{P}_n(\mathbb{C}), f_0(z) = \cdots = f_n(z) = 0 \Leftrightarrow \text{Res}(d) (f_0, \ldots, f_n) = 0. \]

For further properties of the multi-variate resultant, the reader may follow [18] or [25] and references therein (cf. also Subsection 3.1 below). As pointed out in [22], the multi-variate resultant is the central polynomial to understand Cook’s Conjecture \((\mathbb{P} \neq \mathbb{NP})?\). The reason is that any \(\mathbb{NP}\)-complete problem can be rephrased as a particular instance of a Homogeneous Nullstellensatz question (i.e. a question of the kind \(\exists z \in \mathbb{P}_n(\mathbb{C}), f_0(z) = \cdots = f_n(z) = 0?\)). Hence, knowledge of the complexity of the multi-variate resultant is a central matter in Complexity Theory (cf. [29] or [5] and references therein, for instance).

Here, we modestly try to calculate the logarithmic Mahler’s measure of the multivariate resultant which, roughly speaking, measures the bit length of the coefficients in \(\mathbb{Z}\) of \(\text{Res}(d)\).

Other estimates with different techniques and approaches may be found in [42], [2], [26], many of them using the Arithmetic Bézout Inequality (as that in [30], [31], [33], [14], [28]) and applications in [3].

According to P. Philippon, the logarithmic height of a multivariate polynomial is defined as the sum of its local heights. In the case of an irreducible diophantine polynomial, non-archimedean local heights are trivial and, hence, only the archimedean height matters. Following [31], the archimedean logarithmic height of an irreducible diophantine polynomial is defined (up to some normalizing constants) as its logarithmic Mahler measure. In the case of the multi-variate resultants this leads to the following definition.

**Definition 1.3.** The logarithmic Mahler’s measure of the multivariate resultant \(m_{\mathcal{S}((n+1))} (\text{Res}(d))\) is the expectation of the logarithm of the absolute value of \(\text{Res}(d)\) (i.e. the expectation of \(\log |\text{Res}(d)|\) along the product of spheres

\[ \mathcal{S}((n+1)) := \prod_{i=0}^{n} S(H_{d_i}^{(n+1)}) = S(H_{d_0}^{(n+1)}) \times \cdots \times S(H_{d_n}^{(n+1)}), \]
where $S(H_{d_i}^{(n+1)})$ is the unit sphere in $H_{d_i}^{(n+1)}$ with respect to Bombieri-Weyl’s norm.

For technical reasons we introduce the following quantity. For a list of degrees $(d) := (d_0, \ldots, d_n)$, we define:

$$\mathcal{R}(d) := \frac{m_{S(n+1)}(\text{Res}(d))}{D(d)},$$

where $D(d) := \prod_{i=0}^n d_i$ is the Bézout number of the over-determined systems of equations in $\mathcal{H}(d)$. As a consequence theorem 1.1 we conclude the following arithmetic statement:

**Theorem 1.4 (Arithmetic Poisson Formula).** With the previous notations, for every degree list $(d) = (d_0, \ldots, d_n) \in \mathbb{N}^{n+1}$ of positive integers, the following properties hold:

For $n = 1$, we have

$$\mathcal{R}(d_0, d_1) = \frac{1}{2} \left( 1 - \left( \frac{H_{d_1}}{d_1} + \frac{H_{d_0}}{d_0} \right) \right),$$

and, hence, the logarithmic Mahler measure of the bi-variate resultant ($n = 1$) satisfies:

$$m_{S(2)}(\text{Res}(d_0, d_1)) = d_0 \mathcal{E}(d_1, 1) + d_1 \mathcal{E}(d_0, 0).$$

(3)

As for $n \geq 2$, the following equality holds:

$$\mathcal{R}(d) = (\mathcal{R}(d') - \mathcal{I}(d')) + \frac{\mathcal{E}(d_0, n)}{d_0},$$

(4)

where $(d') := (d_1, \ldots, d_n)$, $D(d') := \prod_{i=1}^n d_i$ and where:

$$\mathcal{I}(d') := \left( \sum_{i=1}^n \frac{1}{2d_i}(H_{M_i} - H_{L_i}) \right),$$

and, for $1 \leq i \leq n$,

$$M_i := \dim_{\mathbb{C}}(H_{d_i}^{(n+1)}) - 1 = \left( \frac{d_i + n}{n} \right) - 1, \quad L_i := \dim_{\mathbb{C}}(H_{d_i}^{(n)}) - 1 = \left( \frac{d_i + n - 1}{n - 1} \right) - 1.$$

From this Arithmetic Poisson Formula, an elementary induction argument yields the value of the logarithmic Mahler’s measure of the multi-variate resultant. Formulae for the height of the multi-variate resultant are known since [14] or as a combination of the techniques in [33] and those in [34], [35]. In our case this formula is simply a consequence of Theorem 1.4 plus an elementary inductive argument.

**Corollary 1.5.** With the same notations as in previous statements, for $n \geq 1$ and $(d) = (d_0, \ldots, d_n)$ the following equality holds:

$$m_{S(n+1)}(\text{Res}(d)) = \sum_{i=0}^n \left( \prod_{j \neq i} d_j \right) (\kappa(i, n) - H_{M_i}),$$

where $M_i$ is the constant introduced in Theorem 1.4 above, and

$$\kappa(i, n) := \begin{cases} d_i H_{n-i}, & \text{if } 0 \leq i \leq n - 1, \\ 0, & \text{otherwise.} \end{cases}$$
1.1. Some more comments on Applications.

One of the referees suggested that we add a few more comments on the applications of our results. From our point of view, this manuscript is not intended to have applications, but only to exhibit some quantitative results about some central objects (polynomials and functions) related to algorithmic complexity. Potential applications of our quantitative estimates will be the subject of future research. Trying to combine the two views, we just exhibit some hints on potential applications of our results.

As we already observed, Corollary 1.3 shows the exact value of the expectation $E[I]$ of the function $I$ introduced in question (d) of [4], thus giving a probabilistic answer to this question. As we already observed, our outcome is just an elementary result concerning a very simplified version of the integral required to complete the study of [4]. From our techniques and outcomes we cannot yet conclude anything sharp about the value of the expectation $E$ and the average complexity of algorithmic methods based on “homotopy at one point”.

On the other hand, Corollary 1.5 exhibits the exact computation of the logarithmic Mahler’s measure of the multi-variate resultant $m_{q(n+1)}(\text{Res}_{d})$. Once again, this is just a quantitative result about a central object in Elimination Theory. In terms of applications, it is also related to the complexity of some algorithmic problems. For instance, we may consider the question of deciding whether a list of $n+1$ cubic diophantine homogeneous polynomials $f_0, \ldots, f_n \in \mathbb{Z}[X_0, \ldots, X_n]$ share a common projective zero $\zeta \in \mathbb{P}_n(\mathbb{C})$. This problem is NP-hard (and lies inside the polynomial hierarchy \text{PH}) and a possible treatment could be the following algorithm:

**Test whether** $\text{Res}_{(3)}(f_0, \ldots, f_n) = 0$ **or not,**

where $(3) = (3, 3, \ldots, 3)$ is the degree list for cubic equations. The complexity of this algorithm depends on two substantial quantities. First, this complexity depends on the complexity of evaluating the multi-variate resultant $\text{Res}_{(3)}$. In the present state of our knowledge, no one knows how to evaluate $\text{Res}_{(3)}$ with less than an exponential number of arithmetic operations (see [17] for further reading on lower complexity bounds for the evaluation of elimination polynomials). The second relevant quantity involved in the complexity is the bit length of the integers occurring in intermediate calculations (when evaluating $\text{Res}_{(3)}$). The intermediate results (in $\mathbb{Z}$) may have a doubly exponential absolute value in the number of arithmetic operations, then yielding a time complexity which is exponential in the evaluation complexity of $\text{Res}_{(3)}$ and, hence, it is doubly exponential in the number of variables. An alternative to this drawback is the use of modular zero tests as those introduced in [24]. From [24], we may construct probabilistic algorithms based on the following strategy:

**Guess at random, some positive integer numbers** $m_1, \ldots, m_s \in \{1, \ldots, 2^t\}$

**Test whether** $\text{Res}_{(3)}(f_0, \ldots, f_n) = 0 \mod m_i$, $1 \leq i \leq s$, **or not,**

The advantage is that all calculations are now modular and the largest integer occurring is of bit length smaller than $2t$. Hence, the complexity of this strategy is polynomial in the complexity of evaluating $\text{Res}_{(3)}$, the number $s$ of integers and the bound $t$ of the logarithm of the integer numbers $m_i$. In order to achieve a low error probability, both the number $s$ of integers and the bound $t$ of the logarithm of the integers $m_i$ may be chosen as a low degree polynomial in the double logarithm of the absolute value of $\text{Res}_{(3)}(f_0, \ldots, f_n)$ (this is the main idea in [24]). This last quantity is essentially bounded by the logarithm of the
logarithmic height of the input polynomials $f_0, \ldots, f_n$ and the logarithm of the logarithmic Mahler’s measure of $\text{Res}_{(3)}$ as polynomial. Thus, our computation of $m_{\mathbb{C}^{(n+1)}}(\text{Res}_{(3)})$ in Corollary 1.5 may be used to find sharp bounds for the number $s$ of integers and the bound $t$ of the logarithm of the absolute values of the integers in the list $m_1, \ldots, m_s$, chosen in such a way that the above probabilistic algorithm has a low error probability. Nevertheless, without a sharp result on the complexity of the evaluation of $\text{Res}_{(3)}$ nothing meaningful can be said about the complexity of this probabilistic approach. As the reader may obviously conclude, our quantitative estimates are related but are not sufficient to produce sharp estimates for the complexity of several central algorithms dealing with multi-variate polynomial equations. Those are, in future research advances, a couple of potential applications in Complexity Theory of our estimates.

1.2. Structure of the manuscript.

The manuscript is structured as follows. Section 2 is devoted to recall some known aspects of Bombieri-Weyl’s Hermitian product in the complex space of lists of homogeneous polynomial equations. Section 3 is devoted to recall Shub-Smale “Solution Variety”, some basic facts of the multi-variety resultant (cf. Subsection 3.2), some basic integration formulae (as Federer’s Co-area Formula, cf. Subsection 3.3). This Section is also devoted to introduce a central technical tool in our results (the Double Fibration technique, cf. Subsection 3.4). In this Section we also recall some basic facts about Mahler’s measure and we prove some technical results as Lemma 3.11 (cf. Subsection 3.5). Section 4 is devoted to compute some expectations in the probability distribution induced by Bombieri-Weyl’s Hermitian product: Theorem 1.1 is proved in Subsection 4.1 and Theorem 1.2 is proved in Subsection 4.2. Also in this Section, we exhibit some probabilistic estimates for the quantity established in question “(d)” of [4] (cf. Subsection 4.3). Finally, Section 5 is devoted to prove the arithmetic consequences about the multi-variate resultant: Theorem 1.4 is proved in Subsection 5.1 and Corollary 1.5 is proved in Subsection 5.2.

2. Some basic notions and notations

2.1. Lists of polynomials

We follow most of the notations in [13]. Let $n, d \in \mathbb{N}$ be two positive integers. We denote by $H_{d}^{(n+1)}$ the complex vector space of all homogeneous polynomials in $\mathbb{C}[X_0, \ldots, X_n]$ of degree $d$. Similarly, we denote by $P_{d}^{(n+1)}$ the complex vector space of all polynomials of degree at most $d$ in $\mathbb{C}[X_1, \ldots, X_n]$. Both vector spaces are obviously isomorphic of complex dimension $M(d, n) := \binom{d+n}{n}$ and the isomorphism is given by the mapping $a : H_{d}^{(n+1)} \rightarrow P_{d}^{(n+1)}$, which associates to every homogeneous polynomial $f \in H_{d}^{(n+1)}$ its affine trace $a f := f(1, X_1, \ldots, X_n) \in P_{d}^{(n+1)}$.

As for denoting lists of polynomial equations, let $m \in \mathbb{N}$ be another positive integer and let $(d) := (d_1, \ldots, d_m) \in \mathbb{N}^m$ be a list of degrees. We denote by $H_{(d)}^{(m)}$ the complex vector space of all lists $f := (f_1, \ldots, f_m)$ of homogeneous polynomials $f_i \in \mathbb{C}[X_0, \ldots, X_n]$ of respective degree $d_i$. Namely,

$$H_{(d)}^{(m)} := H_{d_1}^{(n+1)} \times \cdots \times H_{d_m}^{(n+1)}.$$
Similarly, we introduce the complex vector space $P^{(m)}_{(d)} := \prod_{i=1}^{m} P^{(n+1)}_{d_i}$ of all lists $f := (f_1, \ldots, f_m)$ of affine polynomials of respective degrees bounded by the list $(d) = (d_1, \ldots, d_m)$. In the case $m := n + 1$, we respectively write $H_{(d)}$ and $P_{(d)}$ instead of $H_{(n+1)}$ and $P_{(n+1)}$.

The affine trace obviously defines an isomorphism between $P^{(m)}_{(d)}$ and $H^{(m)}_{(d)}$. The complex dimension of both vector spaces satisfies:

$$\dim_{\mathbb{C}}(H^{(m)}_{(d)}) = \dim_{\mathbb{C}}(P^{(m)}_{(d)}) := \sum_{i=1}^{m} \left( d_i + n \right).$$

We sometimes consider the complex projective space defined by any of these spaces. We denote by $\mathbb{P}(H^{(m)}_{(d)})$ this complex vector space and we denote by $N_{(d)}$ (or simply by $N$) the complex dimension of this complex projective space.

For every list $f = (f_1, \ldots, f_m) \in H^{(m)}_{(d)}$, let $V_{\mathbb{P}}(f) \subseteq \mathbb{P}_n(\mathbb{C})$ be the complex projective variety of their common zeros:

$$V_{\mathbb{P}}(f) = \{ z \in \mathbb{P}_n(\mathbb{C}) : f_i(z) = 0, 1 \leq i \leq m \} \subseteq \mathbb{P}_n(\mathbb{C}),$$

Similarly, for every list $g = (g_1, \ldots, g_m) \in P^{(m)}_{(d)}$, we define the affine algebraic variety $V_A(g) \subseteq \mathbb{C}^n$ of its common affine zeros:

$$V_A(g) = \{ x \in \mathbb{C}^n : g_i(x) = 0, 1 \leq i \leq m \} \subseteq \mathbb{C}^n.$$

Let $\varphi_0$ be the standard embedding of $\mathbb{C}^n$ into $\mathbb{P}_n(\mathbb{C})$,

$$\varphi_0 : \mathbb{C}^n \longrightarrow \mathbb{P}_n(\mathbb{C}) \setminus \{ X_0 = 0 \}, (x_1, \ldots, x_n) \longmapsto (1 : x_1 : \ldots : x_n). \quad (5)$$

Observe that $\varphi_0$ identifies $V_A(\cdot)$ with $V_{\mathbb{P}}(f) \cap (\mathbb{P}_n(\mathbb{C}) \setminus \{ X_0 = 0 \})$. Namely, $V_A(\cdot) = \varphi_0^{-1}(V_{\mathbb{P}}(f))$, for every $f \in H^{(m)}_{(d)}$.

In what follows, we denote by $d := \max\{d_i : 1 \leq i \leq m\}$ the maximum of the degrees in the list $(d)$, and by $D_{(d)} := \prod_{i=1}^{m} d_i$ we denote the Bézout number associated with the degree list $(d)$.

2.2. Bombieri-Weyl Hermitian Product.

As in [37] or [13] (Sec. 12.1) we may equip $H^{(m)}_{(d)}$ with the unitarily invariant Bombieri-Weyl’s Hermitian product,

$$\langle \cdot, \cdot \rangle_\Delta : H^{(m)}_{(d)} \times H^{(m)}_{(d)} \longrightarrow \mathbb{C}.$$

This Hermitian product may be introduced as follows. Let $f, g \in H^{(n+1)}_{d}$ be two homogeneous complex polynomials of degree $d$ in $n + 1$ variables and assume that the following are their respective monomial expansions:

$$f = \sum_{\mu \in \mathbb{N}^{n+1}} a_{\mu} X_0^{\mu_0} \cdots X_n^{\mu_n}, \quad g = \sum_{\mu \in \mathbb{N}^{n+1}} b_{\mu} X_0^{\mu_0} \cdots X_n^{\mu_n},$$

$$9$$
where \( \mu = (\mu_0, \ldots, \mu_n) \in \mathbb{N}^{n+1} \) and \( |\mu| = \mu_0 + \ldots + \mu_n, \forall \mu \in \mathbb{N}^{n+1} \). We define the Bombieri-Weyl Hermitian product \( \langle f, g \rangle_d \) by the following identity:

\[
\langle f, g \rangle_d := \sum_{\mu \in \mathbb{N}^{n+1}} \left( \begin{array}{c} d \\ \mu \end{array} \right)^{-1} a_\mu \overline{b}_\mu,
\]

\( \left( \begin{array}{c} d \\ \mu \end{array} \right) = \frac{d!}{\mu_0! \cdots \mu_n!}, \)

is the multi-nomial coefficient and \( \overline{\cdot} \) denotes complex conjugation. For every polynomial \( f \in H_d^{(n+1)} \) we denote by \( ||f||_d := \sqrt{\langle f, f \rangle_d} \) the Bombieri-Weyl’s norm of \( f \).

**Remark 2.1.** We may compare Bombieri-Weyl’s Hermitian product with the canonical one in the following terms. Let \( M(d,n) \) be the set of all monomials in the set of variables \( \{X_0, \ldots, X_n\} \) of degree \( d \). Assume a monomial order in \( M(d,n) \). With this fixed order, \( M(d,n) \) becomes an ordered basis of the complex vector space \( H_d^{(n+1)} \). Bombieri-Weyl’s Hermitian product determines a square diagonal matrix \( \Delta_d \) with \( \left( \begin{array}{c} d+n \\ n \end{array} \right) \) rows associated to this monomial order such that the following equality holds for every two polynomials \( f, g \in H_d^{(n+1)} \):

\[
\langle f, g \rangle_d := \langle \Delta_d f, \Delta_d g \rangle,
\]

where \( \langle \cdot, \cdot \rangle : H_d^{(n+1)} \times H_d^{(n+1)} \rightarrow \mathbb{C} \) is the canonical Hermitian product in \( H_d^{(n+1)} \) and \( \Delta_d f \) and \( \Delta_d g \) are the polynomials in \( H_d^{(n+1)} \) obtained after multiplying the coefficients of \( f \) and \( g \) by the respective terms in the diagonal of \( \Delta_d \). Obviously this matrix is given by the following rule:

\[
\Delta_d := \text{Diag} \left( \sqrt{\left( \begin{array}{c} d \\ \mu \end{array} \right)} \right),
\]

where Diag means diagonal matrix and \( \left( \begin{array}{c} d \\ \mu \end{array} \right) \) is the multi-nomial coefficient used in Equation (6) above.

For every degree list \( (d) := (d_1, \ldots, d_m) \) we extend Bombieri-Weyl Hermitian product to lists of polynomials in the obvious way. Namely, if \( f = (f_1, \ldots, f_m) \in \mathcal{H}^{(m)}_d \) and \( g = (g_1, \ldots, g_m) \in \mathcal{H}^{(m)}_d \), then we define

\[
\langle f, g \rangle_\Delta := \sum_{i=1}^m \langle f_i, g_i \rangle_{d_i}.
\]

We denote by \( ||\cdot||_\Delta := \sqrt{\langle \cdot, \cdot \rangle_\Delta} \) the corresponding norm. We will denote by \( \mathcal{S}(\mathcal{H}^{(m)}_d) \) the sphere of radius one in \( \mathcal{H}^{(m)}_d \) with respect to this metric. Namely,

\[
\mathcal{S}(\mathcal{H}^{(m)}_d) := \{ f \in \mathcal{H}^{(m)}_d : ||f||^2_\Delta = 1 \}.
\]
Similarly, for every degree list \((d) := (d_1, \ldots, d_m)\) we denote by \(\mathcal{S}^{(m)}_{(d)}\) the product of spheres (with Bombieri-Weyl metric) given by the following identity:

\[
\mathcal{S}^{(m)}_{(d)} := \prod_{i=1}^{m} \mathcal{S}(H^{(n+1)}_{d_i}) = \mathcal{S}(H^{(n+1)}_{d_1}) \times \cdots \times \mathcal{S}(H^{(n+1)}_{d_m}),
\]

where

\[
\mathcal{S}(H^{(n+1)}_{d_i}) := \{ f \in H^{(n+1)}_{d_i} : ||f||_{d_i} = 1 \}.
\]

**Notation 2.1.** For every degree list \((d) := (d_1, \ldots, d_m)\) we will denote by \(\Delta_{(d)}\) (or simply by \(\Delta\) when no confusion may arise) the diagonal matrix given as the diagonal concatenation of the matrices \(\Delta_{d_1}, \ldots, \Delta_{d_m}\), where \(\Delta_{d_i}\) is the matrix introduced in Remark 2.1 above. Namely,

\[
\Delta_{(d)} := \Delta_{d_1} \oplus \cdots \oplus \Delta_{d_m}.
\]

With this notations, the following statement is straightforward from equality (7) above:

**Proposition 2.2.** Let \((d) := (d_1, \ldots, d_m)\) be a degree list and let \(N_i := (d_i+n) - 1\) be the complex dimension of \(\mathbb{P}(H^{(n+1)}_{d_i})\). Let us denote by \(S^{2N_i+1} \subseteq H^{(n+1)}_{d_i}\) the unit sphere in \(H^{(n+1)}_{d_i}\) with respect to the canonical Hermitian norm in this space. Then, the following mapping is an isometry of Riemannian manifolds:

\[
\Delta : \mathcal{S}^{(m)}_{(d)} \rightarrow \prod_{i=1}^{m} S^{2N_i+1}, \quad f := (f_1, \ldots, f_m) \mapsto \Delta_{(d)} f := (\Delta_{d_1} f_1, \ldots, \Delta_{d_m} f_m).
\]

According to Proposition 1, III-1, of [37], for every homogeneous polynomial \(f \in H^{(n+1)}_{d} \subseteq \mathbb{C}[X_0, \ldots, X_n]\) and for every \(z \in \mathbb{C}^{n+1}\), we have

\[
|f(z)| \leq ||f||_{d} ||z||^d.
\]

The following well-known statement shows that Bombieri-Weyl’s norm is simply an expectation.

**Proposition 2.3 (cf. [19], for instance).** For every homogeneous polynomial \(f \in H^{(n+1)}_{d}\), its Bombieri’s norm satisfies:

\[
||f||_{d}^2 := \left(\frac{d+n}{n}\right) \frac{1}{\nu_\mathcal{S}[S^{2n+1}]} \int_{S^{2n+1}} |f(z)|^2 d\nu_\mathcal{S}(z),
\]

where \(d\nu_\mathcal{S}\) is the canonical volume form in \(S^{2n+1}\) associated to its Riemannian structure and

\[
\nu_\mathcal{S}[S^{2n+1}] := \frac{2\pi^{(n+1)}}{\Gamma(n + 1)},
\]

is the volume of the sphere \(S^{2n+1} := \{ z \in \mathbb{C}^{n+1} : ||z||^2 = 1 \} \).
Moreover, this Proposition yields another proof of the unitary invariance of Bombieri-Weyl’s Hermitian Product.

**Proposition 2.4 ([13], for instance).** Let $U(n+1)$ be the unitary group of $(n+1) \times (n+1)$ complex matrices. Let us consider the following action of $U(n+1)$ on $\mathcal{H}^{(m)}_{(d)}$:

$$U(n+1) \times \mathcal{H}^{(m)}_{(d)} \rightarrow \mathcal{H}^{(m)}_{(d)}$$

$$(U,(f_1,\ldots,f_m)) \mapsto (f_1 \circ U^*,\ldots,f_m \circ U^*),$$

where $f \circ U^*$ denotes composition. This action is isometric with respect to Bombieri-Weyl’s Hermitian product. Namely, for every $f,g \in \mathcal{H}^{(m)}_{(d)}$ and for every $U \in U(n+1)$ the following equality holds:

$$\langle f, g \rangle_\Delta = \langle f \circ U^*, g \circ U^* \rangle_\Delta.$$

Another famous property of Bombieri-Weyl’s Hermitian product is the following statement:

**Proposition 2.5 (Bombieri’s inequality).** For homogeneous polynomials $f$ and $g$ of respective degrees $d$ and $t$ we have

$$\left( \frac{d+t}{d} \right)^{-1} \|f\|_d \|g\|_t \leq \|fg\|_{d+t} \leq \|f\|_d \|g\|_t.$$

2.3. Some Elementary Facts about Harmonic Numbers.

In what follows we shall use the harmonic series and harmonic numbers to express some of our statements. Thus, let us recall here some basic facts about harmonic numbers. We denote by $H_r$ the $r$-th harmonic number. Namely,

$$H_r := \sum_{k=1}^{r} \frac{1}{k}.$$

Recall that the $r$-th Harmonic number $H_r$ satisfies:

$$(H_r - \log(r)) = \gamma + \sum_{k=2}^{\infty} \frac{\zeta(k,r+1)}{r},$$

where $\gamma$ is the Euler-Mascheroni number and $\zeta(k,r+1)$ is Hurwitz zeta function (cf. [10] and references therein). Recall also that (cf. [21]) using the expansion in terms of Bernoulli numbers, we may write:

$$(H_r - \log(r)) = \gamma + \frac{1}{2r} - \frac{1}{12r^2} + \frac{1}{120r^4} - \varepsilon_r,$$

for

$$0 < \varepsilon_r < \frac{1}{252r^6}.$$
Although there may be sharper bounds, the following inequality may be easily concluded from Stirling formula:

\[
\begin{align*}
\frac{dH_r}{r} - H_{N_d} &\geq (d-1)\gamma + (d+r+1/2)\log \left(\frac{r}{(d+r)}\right) + (d+1/2)\log d + b_2 \left(\frac{d}{2}\right) - k_2, \\
\frac{dH_r}{r} - H_{N_d} &\leq (d-1)\gamma + (d+r+1/2)\log \left(\frac{d}{(d+r)}\right) + (d+1/2)\log d + b_1 \left(\frac{d}{2}\right) + k_1,
\end{align*}
\]

where \( N_d := \left(\frac{d+r}{r}\right) - 1 \), \( \gamma \) is Euler-Mascheroni number and \( b_1, k_1, b_2, k_2 \) are some universal constants.

On the other hand, Euler-Weierstrass infinite product definition of the Gamma function yields the following approximation of the binomial coefficient when \( r \to \infty \):

\[
\binom{d+r}{r} \approx e^{d(H_r - \gamma)} \frac{\Gamma(d+1)}{\Gamma(d+r)}.
\]

Thus, we may also conclude that the following also holds asymptotically when \( n \to \infty \):

\[
\mathcal{E}(d,n) := \frac{1}{2} \left( \frac{dH_n}{n} - H_{N_d} \right) \approx \frac{1}{2} \left( (d-1)\gamma + \log(\Gamma(d+1)) \right).
\]

Using Boros and Moll analytic expansion of \( \log \Gamma(z) \) (cf. [12]), we may also conclude the following asymptotic approximation of \( \mathcal{E}(d,n) \), when \( n \to \infty \):

\[
\mathcal{E}(d,n) \approx -\gamma + \frac{1}{2} \left( -\log(d+1) + \sum_{k=1}^{\infty} \left[ \frac{d+1}{k} - \log \left( \frac{d+k+1}{k} \right) \right] \right).
\]

### 3. Solution Variety and Some Basic Integration Formulae

#### 3.1. The Solution Variety

Some of the main advances in [37, 38, 39, 40] are due to the smart exploration of a geometric structure related to the polynomial system solving: the solution variety. Same can be said about [9]. We follow the notations used in these manuscripts.

Let us fix the set \( \{X_0, \ldots, X_n\} \) of homogeneous variables and let \( m \in \mathbb{N} \) be a positive integer.

Let \( (d) = (d_1, \ldots, d_m) \) be a list of positive degrees. We define the projective solution variety \( V^{(m)}_{(d)} \subseteq \mathbb{P}(\mathcal{H}^{(m)}_{(d)}) \times \mathbb{P}_n(\mathbb{C}) \) by the following equality:

\[
V^{(m)}_{(d)} := \{ (f,z) \in \mathbb{P}(\mathcal{H}^{(m)}_{(d)}) \times \mathbb{P}_n(\mathbb{C}) : f_i(z) = 0, 1 \leq i \leq m \}.
\]

This subset \( V^{(m)}_{(d)} \subseteq \mathbb{P}(\mathcal{H}^{(m)}_{(d)}) \times \mathbb{P}_n(\mathbb{C}) \) is a complex smooth multi-homogeneous algebraic variety of co-dimension \( m \). Thus, its complex dimension is \( N_{(d)} + n - m \). At every point \( (f,z) \in V^{(m)}_{(d)} \) the tangent space \( T_{(f,z)}V^{(m)}_{(d)} \) is given by the following equality:

\[
T_{(f,z)}V^{(m)}_{(d)} := \{ (\dot{f}, \dot{z}) \in T_f \mathbb{P}(\mathcal{H}^{(m)}_{(d)}) \times T_z \mathbb{P}_n(\mathbb{C}) : \dot{f}(z) + T_z f(\dot{w}) = 0 \},
\]
where \( T_z f : T_z \mathbb{P}_n(\mathbb{C}) \to \mathbb{C}^m \) is the restriction of the jacobian matrix \( Df(z) \) to \( T_z \mathbb{P}_n(\mathbb{C}) = z^\perp \), which is the orthogonal complement of \( z \) in \( \mathbb{C}^{n+1} \) with respect to the canonical Hermitian product in \( \mathbb{C}^{n+1} \). Namely, we have:

\[
T_z f := Df(z)|_{z^\perp}.
\]

We have two canonical projections defined in the solution variety:

\[
\pi_1 : V^{(m)}_{(d)} \to \mathbb{P}(H^{(m)}_{(d)}), \quad \pi_2 : V^{(m)}_{(d)} \to \mathbb{P}_n(\mathbb{C}).
\]

The following proposition resumes the main properties of these two canonical projections.

**Proposition 3.1 (cf. [13] and [7]).** With the same notations as above, assume \( 1 \leq m \leq n \). Then, the following properties hold:

1. The mapping \( \pi_2 : V^{(m)}_{(d)} \to \mathbb{P}_n(\mathbb{C}) \) is an onto mapping, a submersion at every point \((f, z) \in V^{(m)}_{(d)} \) and for every \( z \in \mathbb{P}_n(\mathbb{C}) \) the fiber \( \pi_2^{-1}\{ \{z\} \} \) can be identified with a complex projective linear submanifold of \( \mathbb{P}(H^{(m)}_{(d)}) \) of co-dimension \( m \) given by the following equality:

\[
V_z := \pi_2^{-1}\{ \{z\} \} = \{ f \in \mathbb{P}(H^{(m)}_{(d)}): f_i(z) = 0, 1 \leq i \leq m \}\.
\]

2. The mapping \( \pi_1 : V^{(m)}_{(d)} \to \mathbb{P}(H^{(m)}_{(d)}) \) is an onto mapping and for every \( f \in \mathbb{P}(H^{(m)}_{(d)}) \) the following equality holds:

\[
V_f := \pi_1^{-1}\{ \{f\} \}.
\]

3. The set of critical values \( \Sigma_{(d)} \subseteq \mathbb{P}(H^{(m)}_{(d)}) \) of \( \pi_2 \) is a proper non-empty complex subvariety of \( \mathbb{P}(H^{(m)}_{(d)}) \). It is called the discriminant variety.

4. For every system \( f \in \mathbb{P}(H^{(m)}_{(d)}) \setminus \Sigma_{(d)} \), outside the discriminant variety, the fiber \( V_f := \pi_1^{-1}\{ \{f\} \} \) is a smooth complete intersection complex projective subvariety of co-dimension \( m \).

### 3.2. The over-determined case: Multi-variate resultant

Things are somehow different if we consider the over-determined case: we just consider the case \( m = n + 1 \) and we simply denote by \( V_{(d)} \) the solution variety \( V^{(n+1)}_{(d)} \) determined by a degree list \( (d) := (d_0, \ldots, d_n) \). The main properties of this solution variety can be resumed in the following Proposition.

**Proposition 3.2 (cf. [18] and its references, for instance).** With the same notations as in the previous subsection, assume \( m = n + 1 \). Then, the following properties hold:

1. The variety \( V_{(d)} \) is a smooth Riemannian manifold of co-dimension \( n + 1 \) in \( \mathbb{P}(H_{(d)}) \times \mathbb{P}_n(\mathbb{C}) \).

2. The image \( R_{(d)} := \pi_1(V_{(d)}) \subseteq \mathbb{P}(H_{(d)}) \) is a projective variety of co-dimension one (and, hence, a complex hyper-surface).
3. The variety \( R_{(d)} \) is defined as the set of common zeros of an irreducible (and hence primitive) polynomial

\[
\text{Res}_{(d)} \in \mathbb{Z}\left[\bigcup_{i=0}^{n} \{ A^{(i)}_\mu : \mu \in \mathbb{N}^{n+1}, |\mu| = d_i \}\right],
\]

which is multi-homogeneous on each group of variables \( \{ A^{(i)}_\mu : |\mu| = d_i \} \) representing the coefficients of the polynomials in \( H_{di}^{(n+1)} \).

4. Moreover, the degree on \( \text{Res}_{(d)} \) on each group of variables \( \{ A^{(i)}_\mu : |\mu| = d_i \} \) equals

\[ \prod_{j \neq i} d_j. \]

5. The total degree of \( \text{Res}_{(d)} \) is

\[ \sum_{i=1}^{n} \prod_{j \neq i} d_j. \]

The polynomial \( \text{Res}_{(d)} \) is called the multi-variate resultant and the variety \( R_{(d)} \) is called the (multi-variate) resultant variety.

3.3. Normal Jacobians and the Co-area Formula

The Co-area Formula is a classic integral formula which generalizes Fubini’s Theorem. The most general version we know is Federer’s Co-area Formula (cf. [20]), but for our purposes a smooth version, as used in [13] or [23], suffices.

**Definition 3.1.** Let \( X \) and \( Y \) be Riemannian manifolds, and let \( F : X \rightarrow Y \) be a \( C^1 \) surjective map. Let \( k = \dim(Y) \) be the real dimension of \( Y \). For every point \( x \in X \) such that the differential \( DF(x) \) is surjective, let \( v^1_x, \ldots, v^k_x \) be an orthonormal basis of \( \text{Ker}(DF(x))^\perp \). Then, we define the normal Jacobian of \( F \) at \( x \), \( NJ_x F \), as the volume in the tangent space \( T_{F(x)} Y \) of the parallelepiped spanned by \( DF(x)(v^1_x), \ldots, DF(x)(v^k_x) \). In the case that \( DF(x) \) is not surjective, we define \( NJ_x F = 0 \).

The following Proposition is easy to prove from this Definition.

**Proposition 3.3.** Let \( X,Y \) be two Riemannian manifolds, and let \( F : X \rightarrow Y \) be a \( C^1 \) map. Let \( x_1,x_2 \in X \) be two points. Assume that there exist isometries \( \varphi_X : X \rightarrow X \) and \( \varphi_Y : Y \rightarrow Y \) such that \( \varphi_X(x_1) = x_2 \), and

\[ F \circ \varphi_X = \varphi_Y \circ F. \]

Then, the following equality holds:

\[ NJ_{x_1} F = NJ_{x_2} F. \]

Moreover, if there exists an inverse \( G : Y \rightarrow X \), then

\[ NJ_x F = \frac{1}{NJ_{F(x)} G}. \]

A relevant tool to be used in the forthcoming pages in the following classical statement of Integral Geometry:
Theorem 3.4 (Co-area Formula). Consider a differentiable map \( F : X \to Y \), where \( X, Y \) are Riemannian manifolds of real dimensions \( n_1 \geq n_2 \). Consider a measurable function \( f : X \to \mathbb{R} \), such that \( f \) is integrable. Then, for every \( y \in Y \) except a zero-measure set, \( F^{-1}(y) \) is empty or a real submanifold of \( X \) of real dimension \( n_1 - n_2 \). Moreover, the following equality holds (and the integrals appearing on it are well defined):

\[
\int_X fNJ_x F \, dX = \int_{y \in Y} \int_{x \in F^{-1}(y)} f(x) \, dF^{-1}(y) \, dY.
\]

The following statement is Lemma 21 of [6].

Lemma 3.5. Let \( \varphi_0 : \mathbb{C}^n \to \mathbb{P}_n(\mathbb{C}) \) be the standard embedding introduced in Equation (5) in Subsection 2.1 above. Then, the following equality holds for every \( z \in \mathbb{C}^n \):

\[
NJ_z \varphi_0 = \frac{1}{(1 + ||z||^2)^{n+1}}.
\]

In particular, for every \( f \in \mathbb{C}[X_1, \ldots, X_n] \), the following equality holds:

\[
\int_{z \in \mathbb{C}^n} \log |f(z_1, \ldots, z_n)| \frac{1}{(1 + ||z||^2)^{n+1}} \, dz = \int_{x \in \mathbb{P}_n(\mathbb{C})} \log |f(\varphi_0^{-1}(x))| \, d\nu_x(x),
\]

where \( d\nu_x \) is the canonical form associated with the Fubini-Study metric in \( \mathbb{P}_n(\mathbb{C}) \).

3.4. Double fibration and some integral formulae

As in [7], the following statement will be extensively used in the forthcoming pages. Let \( (d) := (d_1, \ldots, d_n) \) be a degree list. We may introduce a variation of the solution variety \( V^{(n)}_{(d)} \) introduced in previous pages. We define the incidence variety \( V_{(d)} := V_{(d)}^{(n)} \subseteq \mathcal{G}_{(d)}^{(n)} \times \mathbb{P}_n(\mathbb{C}) \) in the following terms:

\[
V_{(d)} = V_{(d)}^{(n)} := \{ (f, z) \in \mathcal{G}_{(d)}^{(n)} \times \mathbb{P}_n(\mathbb{C}) : f_i(z) = 0, 1 \leq i \leq n, f = (f_1, \ldots, f_n) \}.
\]

As in previous pages we may also consider two canonical projections:

- \( \pi_1 : V_{(d)} \to \mathcal{G}_{(d)}^{(n)} \), \( \pi_1(f, z) := f \), \( \forall (f, z) \in V_{(d)} \),
- \( \pi_2 : V_{(d)} \to \mathbb{P}_n(\mathbb{C}) \), \( \pi_2(f, z) := z \), \( \forall (f, z) \in V_{(d)} \).

From Proposition 2.4 there is an isometric action of the unitary group \( \mathcal{U}(n+1) \) on \( V_{(d)} \) given in the following terms:

\[
\mathcal{U}(n+1) \times V_{(d)} \ni (U, (f, z)) \mapsto U( (f, z) ) \mapsto (f \circ U^*, Uz).
\]

The following double fibration argument is a well-known statement and technique, used in different forms in former manuscripts (cf. [8], [11] and their references, for instance). We just include it for sake of completeness and the calculation of the involved constants.
Proposition 3.6. Let \( g : \mathbb{P}_n(\mathbb{C}) \rightarrow \mathbb{R}_+ \) be an integrable function. Then, the following equality holds:

\[
\int_{f \in \Theta_{(d)}(n)} \left( \sum_{z \in V(f)} g(z) \right) d\nu_\Theta(f) = \frac{D_{(d)}\nu_\Theta[\Theta_{(d)}(n)]}{\nu_\mathbb{P}[\mathbb{P}_n(\mathbb{C})]} \int_{\mathbb{P}_n(\mathbb{C})} g(z) d\nu_\mathbb{P}(z),
\]

where \( d\nu_\mathbb{P} \) is the differential form associated to the canonical Fubini-Study metric in \( \mathbb{P}_n(\mathbb{C}) \).

Proof.– Note that applying the Co-area Formula with respect to \( \pi_1 \) we have:

\[
I(g) := \int_{f \in \Theta_{(d)}(n)} \left( \sum_{z \in V(f)} g(z) \right) d\nu_\Theta(f) = \int_{(f,z) \in V(d)} g(z) NJ(f,z) \pi_1 d\nu_{(d)}(f,z).
\]

Now, we apply the Co-area Formula with respect to the projection \( \pi_2 \) to conclude:

\[
I(g) := \int_{z \in \mathbb{P}_n(\mathbb{C})} \left( \int_{\pi_2^{-1}(z)} g(z) \frac{NJ(f,z) \pi_1}{NJ(f,z) \pi_2} d\pi_2^{-1}(z)(f,z) \right) d\nu_\mathbb{P}(z).
\]

Now, due to the isometric action of \( \mathcal{U}(n+1) \) on \( V(d) \), we conclude that the following quantity:

\[
T := T_z := \left( \int_{\pi_2^{-1}(z)} \frac{NJ(f,z) \pi_1}{NJ(f,z) \pi_2} d\pi_2^{-1}(z)(f,z) \right),
\]

is constant and independent of \( z \). Moreover, we have

\[
I(g) := T \left( \int_{z \in \mathbb{P}_n(\mathbb{C})} g(z) d\nu_\mathbb{P}(z) \right).
\]

Taking \( g \equiv 1 \) in this identity and noting that \( \sharp(\pi_1^{-1}(f)) = D_{(d)} = \prod_{i=1}^n d_i \) holds almost everywhere in \( \mathbb{P}(\mathcal{H}_{(d)}(n)) \), we conclude that \( I(1) = D_{(d)}\nu_\Theta[\Theta_{(d)}(n)] \) and, hence,

\[
D_{(d)}\nu_\Theta[\Theta_{(d)}(n)] = \nu_\mathbb{P}[\mathbb{P}_n(\mathbb{C})] T.
\]

We thus conclude:

\[
T := \frac{D_{(d)}\nu_\Theta[\Theta_{(d)}(n)]}{\nu_\mathbb{P}[\mathbb{P}_n(\mathbb{C})]},
\]

and hence

\[
I(g) = \frac{D_{(d)}\nu_\Theta[\Theta_{(d)}(n)]}{\nu_\mathbb{P}[\mathbb{P}_n(\mathbb{C})]} \int_{\mathbb{P}_n(\mathbb{C})} g(z) d\nu_\mathbb{P}(z),
\]

and the claim follows. \( \square \)

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Corollary 3.7. With the previous notations and assumptions, this Proposition may be rephrased as follows:

\[ E_{\mathbb{S}(d)} \left[ \sum_{z \in V(f)} g(z) \right] = D(d) E_{\mathbb{P}(\mathbb{C})} [g]. \]

Same arguments as those used in the previous Proposition, using unitary invariance, imply the following statements for the natural probability distributions in \( \mathbb{S}(\mathbb{H}(m)) \) and \( \mathbb{P}(\mathbb{H}(m)) \).

Corollary 3.8. Let \( g : \mathbb{P}_n(\mathbb{C}) \to \mathbb{R} \) be an integrable function, \( 1 \leq m \leq a \) positive integer and \( (d) = (d_1, \ldots, d_m) \) a list of degrees. Then, the following equalities hold:

\[ E_{f \in \mathbb{S}(\mathbb{H}(m))} \left[ \int_{V(f)} g(z) dV(f) \right] = D(d) E_{z \in \mathbb{P}_n(\mathbb{C})} [g(z)] = E_{f \in \mathbb{P}(\mathbb{H}(m))} \left[ \int_{V(f)} g(z) dV(f) \right], \]

where \( D(d) = \prod_{i=1}^{m} d_i \) is the Bézout number associated to the degree list \( (d) = (d_1, \ldots, d_m) \).

3.5. Mahler’s measure

Mahler’s measure has been used as an effective measure of the size of polynomials. It has been considered better suited than the canonical norm \( \| \cdot \|_2 \) because of its good behaviour with respect to the product of polynomials (namely \( M(f \cdot g) = M(f) \cdot M(g) \)). Bombieri-Weyl’s norm is the only serious competitor since it satisfies Bombieri’s inequality (cf. Proposition 2.5). Nevertheless, Mahler’s measure remains as the main archimedean measure of polynomials with coefficients in a number field (cf. [31], for instance).

The usual logarithmic Mahler’s measure is, in fact, an expectation of the logarithm of the polynomial along a product of unit complex circles \( \prod_{n=1}^{r} S^1 \subseteq \mathbb{C}^n \) (compare with Propositions 2.3 and 2.4).

In the early nineties, P. Philippon (cf. [31], [33]) introduced some variations of the logarithmic Mahler’s measure, replacing the product of circles by products of spheres or even the sphere \( S^{2n-1} \subseteq \mathbb{C}^n \). Its main outcome in [31] was Theorem 3.9 below just comparing the values of the logarithmic Mahler’s measures when we choose either the product \( \prod_{n=1}^{r} S^1 \) or the sphere \( S^{2n-1} \).

In this Section we recall the main notations about the Mahler’s measure according to Philippon and we show some technical results concerning Mahler’s measure of multivariate polynomials. Let \( (m) := (m_1, \ldots, m_r) \) be a list of positive integer numbers. Suppose \( (m_1 + 1) + \cdots + (m_r + 1) = n + 1 \). Let us denote by \( S^{(m)} \) the product of spheres:

\[ S^{(m)} := \prod_{i=1}^{r} S^{2m_i + 1} \subseteq \mathbb{C}^{m+1}, \]

where \( S^{2m_i + 1} := \{ z \in \mathbb{C}^{m_i + 1} : \| z \|_2 = 1 \} \) is the sphere of radius one in \( \mathbb{C}^{m_i + 1} \) centered at the origin, and \( \| \cdot \|_2 \) is the canonical Hermitian norm in \( \mathbb{C}^{m+1} \).
Definition 3.2 (Logarithmic Mahler’s measure, [31]). Let \( f \in \mathbb{C}[X_0, X_1, \ldots, X_n] \) be a complex polynomial (not necessarily homogeneous). We define the logarithmic Mahler’s measure of \( f \) with respect to \( S^{(m)} \) as:

\[
m_{S^{(m)}}(f) := \frac{1}{\prod_{i=1}^{n} \nu_S[S^{2m_i+1}]} \int_{S^{(m)}} \log |f(z)|d\nu_S(z),
\]

where \( d\nu_S \) is the volume form associated to the standard Fubini-Study metric in \( S^{(m)} \). As in [31], we define the absolute Mahler measure of \( f \) with respect to \( S^{(m)} \) as the exponential of the logarithmic Mahler measure, namely

\[
M_{S^{(m)}}(f) := \exp(m_{S^{(m)}}(f)).
\]

Mahler’s measure of a polynomial is usually defined as \( m(p) := m_S(p) \), where \( S := \prod_{i=0}^{n} S^1 \).

Theorem 3.9 ([31], [27]). For every polynomial \( f \in \mathbb{C}[X_0, X_1, \ldots, X_n] \), the following inequalities hold:

\[
m_{S^{2n+1}}(f) \leq M_{S^{(m)}}(f) \leq m_{S^{2n+1}}(f) + 4 \deg(f) \log(n+1),
\]

\[
m_{S^{2n+1}}(f) \leq M_{S^{(m)}}(f) \leq m_{S^{2n+1}}(f) + \frac{\deg(f)H_n}{2}.
\]

The following elementary and well-known properties hold for Mahler’s measure:

Proposition 3.10. With these notations we have:

1. Absolute Mahler’s measure is multiplicative \( (M_S(fg) = M_S(f)M_S(g)) \) and invariant under the action of the unitary group \( U(n+1) \) acting on \( S^{2n+1} \).
2. According to Proposition 2.3 and Jensen’s Inequality, for every \( f \in H_d^{n+1} \), we have:

\[
m_{S^{2n+1}}(f) + \frac{1}{2} \log \left( \frac{d+n}{n} \right) \leq \log \|f\|_d.
\]

where \( \|f\|_d \) is the Bombieri-Weyl’s norm of \( f \) and \( H_n \) is the \( n \)-th harmonic number.
3. Conversely, for every \( f \in H_d^{n+1} \) we also have:

\[
\log \|f\|_d \leq \frac{1}{2} \log \left( \frac{d+n}{n} \right) + (n+1)d + \frac{dH_n}{2} + m_{S^{2n+1}}(f).
\]

Proof.– Most of these claims are either known or elementary consequences of the definitions. □

Let us give more precise relations among some of these Mahler’s measures in the multi-homogeneous case. Let \( \mathcal{X} := \{X_0, \ldots, X_n\} \) be a set of variables and assume it decomposes as a disjoint union of sets of variables \( \mathcal{X} := \mathcal{X}^{(1)} \cup \cdots \cup \mathcal{X}^{(r)} \). After renaming the variables, assume that

\[
\mathcal{X}^{(i)} := \{Y_0^{(i)}, \ldots, Y_{m_i}^{(i)}\}.
\]
Let us denote by \((m)\) the list \((m_1, \ldots, m_r)\). Let us consider now a list \((\ell) := (\ell_1, \ldots, \ell_r)\), where \(\ell_i \leq m_i\) and sets of variables \(Y(\ell) := \{Y_0^{(i)}, \ldots, Y_{\ell_i}^{(i)}\} \subseteq \mathcal{X}^{(i)}\).

Let \(f \in \mathbb{C}[X_0, \ldots, X_n]\) be a multi-homogeneous polynomial with respect to the groups \(\mathcal{X}^{(1)}, \ldots, \mathcal{X}^{(r)}\). Let \(d_i\) be the degree of \(f\) with respect to the variables in the group \(\mathcal{X}^{(i)}\). Assume that \(f\) depends only on the variables in \(Y^{(1)} \cup \cdots \cup Y^{(r)}\) (i.e. \(f \in \mathbb{C}[Y^{(1)} \cup \cdots \cup Y^{(r)}]\)). Thus, we have two possible Mahler’s measures for \(f\) depending on the probability space considered \(\mathcal{S}^{(m)}\) or \(\mathcal{S}^{(\ell)} = \prod_{i=1}^r S^{2\ell_i+1}\). The following statements shows the existing relation between the logarithmic Mahler’s measures \(m_{\mathcal{S}^{(m)}}(f)\) and \(m_{\mathcal{S}^{(\ell)}}(f)\).

**Lemma 3.11.** With these notations, assuming \(1 \leq \ell_i \leq m_i\), for \(1 \leq i \leq r\), the following equality holds:

\[
m_{\mathcal{S}^{(m)}}(f) = m_{\mathcal{S}^{(\ell)}}(f) - \sum_{i=1}^r \frac{d_i}{2} (H_{m_i} - H_{\ell_i}),
\]

**Proof.** We proceed inductively. To this purpose, we introduce the list \((m') := (m_1 - 1, m_2, \ldots, m_r)\) and the product of spheres \(\mathcal{S}^{(m')} := S^{2m_1-1} \times \prod_{i=2}^r S^{2m_i+1}\).

We consider now the projection that forgets the coordinate \(y_{m_1}^{(1)}:\)

\[
\pi := (\pi_1, Id_2, \ldots, Id_r) : S^{2m_1+1} \times \prod_{i=2}^r S^{2m_i+1} \rightarrow B_{\mathbb{C}^{m_1}}(0,1) \times \prod_{i=2}^r S^{2m_i+1},
\]

where \(B_{\mathbb{C}^{m_1}}(0,1) \subseteq \mathbb{C}^{m_1}\) is the closed ball of radius one centered at the origin, \(\pi_1(z) := (y_0^{(1)}, \ldots, y_{m_1-1}^{(1)}) \in \mathbb{C}^{m_1}\), being

\[
z := (y_0^{(1)}, \ldots, y_{m_1}^{(1)}, y_0^{(2)}, \ldots, y_{m_2}^{(2)}, \ldots, y_0^{(r)}, \ldots, y_{m_r}^{(r)}).
\]

and, for every \(i\), \(Id_i : S^{2m_i+1} \rightarrow S^{2m_i+1}\) is the identity mapping. According to Lemma 2, p. 206, of [13] the normal Jacobian of \(\pi\) is known and satifies:

\[
NJ_{z\pi} := (1 - ||\pi_1(z)||^2)^{1/2}.
\]

Now we apply the Co-area Formula (cf. Theorem 3.4 above) and we have:

\[
m_{\mathcal{S}^{(m)}}(f) = \frac{1}{\nu_{\mathcal{S}^{(m')}}} \int_{x \in B_{\mathbb{C}^{m_1}}(0,1) \times \prod_{i=2}^r S^{2m_i+1}} \int_{z \in \pi^{-1}(x)} \frac{\log |f(z)|}{(1 - ||\pi_1(z)||^2)^{1/2}} d\pi^{-1}(x) d\mu_{S^{2m_1+1}} \prod_{i=2}^r d\mu_{S^{2m_i+1}}.
\]

As both \(f(z)\) and \(\pi(z)\) do not depend on the variable \(y_{m_1}^{(1)}\), \(f(z) = f(x)\) and \(\pi_1(z)\) depends only on \(x\). Thus, this equality may also be written as:

\[
m_{\mathcal{S}^{(m)}}(f) = \frac{1}{\nu_{\mathcal{S}^{(m')}}} \int_{x \in B_{\mathbb{C}^{m_1}}(0,1) \times \prod_{i=2}^r S^{2m_i+1}} \frac{\log |f(x)|}{(1 - ||\pi_1(x)||^2)^{1/2}} \left(\int_{\pi^{-1}(x)} d\pi^{-1}(x)\right) d\mu_{B_{\mathbb{C}^{m_1}}} \prod_{i=2}^r d\mu_{S^{2m_i+1}}.
\]
Now, observe that $\pi^{-1}(x)$ can be identified with the circle in $\mathbb{C}$ of radius $(1-||\pi_1(z)||^2)^{1/2}$ and its total volume is $2\pi(1-||\pi_1(z)||^2)^{1/2}$. This yields the following equality:

$$m_{\mathcal{S}(m)}(f) = \frac{2\pi}{\nu_{\mathcal{S}}[\mathcal{S}(m)]} \int_{x \in B_{\mathbb{C}^m}(0,1) \times \prod_{i=m}^{2m_1+1} \mathbb{S}} \log |f(x)| dB_{\mathbb{C}^m} \prod_{i=2}^{r} d\nu_{\mathbb{S}^{2m_1+1}}.$$

Now we fix $y := (y_0,\ldots,y_{m_2},\ldots;y_0^{(r)},\ldots,y_{m_r}) \in \prod_{i=2}^{r} \mathbb{S}^{2m_1+1}$. As $f$ is a non-zero multihomogeneous polynomial, $f(-,y) := f(y_0^{(1)},\ldots,Y_{m_1},y)$ is a non-zero polynomial of degree $d_1$ for almost all $y \in \prod_{i=2}^{r} \mathbb{S}^{2m_1+1}$. As $f(-,y)$ does not depend on the variable $Y_{m_1}^{(1)}$, for fixed $y$, $f(-,y) \in \mathbb{C}[Y_0^{(1)},\ldots,Y_{m_1}]$ is homogeneous of degree $d_1$. Proposition 1 and its Nota Bene in [31] claims:

$$\frac{1}{\text{vol} \mathcal{B}_{\mathbb{C}^m}(0,1)} \int_{x \in \mathcal{B}_{\mathbb{C}^m}(0,1)} \log |f(z,y)| dB_{\mathbb{C}^m}(z) = m_{\mathcal{S}^{2m_1-1}}(f(-,y)) - \deg(f(-,y)) \frac{d_1}{2m_1}.$$

Thus, replacing this identity in our equality above, we conclude

$$m_{\mathcal{S}(m)}(f) = \frac{2\pi \text{vol} \mathcal{B}_{\mathbb{C}^m}(0,1)}{\nu_{\mathcal{S}}[\mathcal{S}(2m_1+1)]} \int_{y \in \prod_{i=2}^{r} \mathbb{S}^{2m_1+1}} m_{\mathcal{S}^{2m_1-1}}(f(-,y)) \prod_{i=2}^{r} d\nu_{\mathbb{S}^{2m_1+1}}(y) - \frac{d_1}{2m_1}.$$

As

$$\frac{2\pi \text{vol} \mathcal{B}_{\mathbb{C}^m}(0,1)}{\nu_{\mathcal{S}}[\mathcal{S}(2m_1+1)]} = 1,$$

we conclude

$$m_{\mathcal{S}(m)}(f) = \prod_{i=2}^{r} \nu_{\mathcal{S}}[\mathcal{S}(2m_1+1)] \int_{y \in \prod_{i=2}^{r} \mathbb{S}^{2m_1+1}} m_{\mathcal{S}^{2m_1-1}}(f(-,y)) \prod_{i=2}^{r} d\nu_{\mathbb{S}^{2m_1+1}}(y) - \frac{d_1}{2m_1}.$$

Namely, we have shown:

$$m_{\mathcal{S}(m)}(f) = m_{\mathcal{S}(m')} (f) - \frac{d_1}{2m_1}.$$

Proceeding inductively, we conclude:

$$m_{\mathcal{S}(m)}(f) = m_{\mathcal{S}(1,m_2,\ldots,m_r)} (f) - \frac{d_1}{2} (H_{m_1} - H_{i_1}).$$

And repeating the induction for $i = 2$ to $r$, we conclude

$$m_{\mathcal{S}(m)}(f) = m_{\mathcal{S}(m)} (f) - \sum_{i=1}^{r} \frac{d_1}{2} (H_{m_i} - H_{i_1}),$$

as wanted. \qed
4. Expectations in the Bombieri-Weyl’s unit sphere.

In this Subsection we state several results concerning exact values of expectations of several functions on the space of polynomials and systems of equations when the chosen probability distribution is the uniform one in the Bombieri-Weyl’s unit sphere. This includes the Proof of Theorem 1.1, the expectation in Bombieri-Weyl’s unit sphere of the Akatsuka’s Zeta Mahler measure (Theorem 1.2 of the Introduction) and a probabilistic answer to question “(d)” in [4].

4.1. Proof of Theorem 1.1

As in previous sections, let $H_{d}^{(n+1)}$ be the complex vector space of all homogeneous complex polynomials in $\mathbb{C}[X_0, \ldots, X_n]$ of degree $d$, $S(H_{d}^{(n+1)})$ is the unit sphere in $H_{d}^{(n+1)}$ with respect to Bombieri-Weyl’s metric and $N_d$ is the complex dimension of $H_{d}^{(n+1)}$. We also follow the notations of the statement of Theorem 1.1 as stated at the Introduction.

Let us denote by $E(d,n)$ the following expectation:

$$E(d,n) = \mathbb{E}_{f \in S(H_{d}^{(n+1)})} \left[ \mathbb{E}_{P_n(\mathbb{C})} \left[ \log |f(1, z)| \right] \right].$$

As in Lemma 4.6 of [8], we apply Lemma 21 of [6] to conclude:

$$E(d,n) = \frac{1}{\nu_{P_n(\mathbb{C})}} \nu_{S(H_{d}^{(n+1)})} \int_{f \in S(H_{d}^{(n+1)})} \left( \int_{\mathbb{C}^n} \frac{\log |f(1, z)|}{(1 + \|z\|^2)^{n+1}} \, dz \right) \, d\nu_{S}(f).$$

For every $z \in \mathbb{C}^n$, let us define $E(z)$ the following quantity:

$$E(z) := \frac{1}{\nu_{S(H_{d}^{(n+1)})}} \left( \int_{f \in S(H_{d}^{(n+1)})} \log |f(1, z)| \, d\nu_{S}(f) \right).$$

Observe that the following holds

$$E(d,n) = \frac{1}{\nu_{P_n(\mathbb{C})}} \int_{\mathbb{C}^n} \left( 1 + \|z\|^2 \right)^{-n+1} E(z) \, dz.$$

Now, for fixed $z \in \mathbb{C}^n$, there is an unitary matrix $U \in U(n + 1)$ such that $U(1, z) = ((1 + \|z\|^2)^{1/2}, 0, \ldots, 0) \in \mathbb{C}^{n+1}$. Moreover, due to the unitary invariance of the Bombieri-Weyl’s norm, the following is an isometry:

$$U^* : S(H_{d}^{(n+1)}) \rightarrow S(H_{d}^{(n+1)}), \quad f \mapsto f \circ U^*.$$

As $f$ is a homogeneous polynomial of degree $d$, we also have:

$$|f(1, z)| = |(f \circ U^*)((1 + \|z\|^2)^{1/2}, 0, \ldots, 0)| = (1 + \|z\|^2)^{d/2}|(f \circ U^*)(1, 0, \ldots, 0)|.$$
Using this equality plus the isometry defined by $U^*$, we have the following equality for every $z \in \mathbb{C}^n$:

$$E(z) = \frac{d}{2} \log(1 + \|z\|^2) + \frac{1}{\nu_S[H_d^{(n+1)}]} \left( \int_{f \in S(H_d^{(n+1)})} \log |f \circ U^*(1,0,\ldots,0)| \, d\nu_S(f) \right).$$

Or, equivalently,

$$E(z) = \frac{d}{2} \log(1 + \|z\|^2) + \frac{1}{\nu_S[H_d^{(n+1)}]} \left( \int_{f \in S(H_d^{(n+1)})} \log |f(1,0,\ldots,0)| \, d\nu_S(f) \right).$$

Thus, we conclude:

$$E(d,n) := J_1 + J_2, \quad J_2 := J_3J_4, \quad (11)$$

where

$$J_1 := \frac{d}{2\nu_P[\mathbb{P}_n(\mathbb{C})]} \int_{\mathbb{C}^n} \log(1 + \|z\|^2) \, dz \geq 0,$$

and

$$J_3 := \frac{1}{\nu_P[\mathbb{P}_n(\mathbb{C})]} \int_{\mathbb{C}^n} \frac{1}{(1 + \|z\|^2)^{n+1}} \, dz,$$

$$J_4 := \frac{1}{\nu_S[H_d^{(n+1)}]} \left( \int_{f \in S(H_d^{(n+1)})} \log |f(1,0,\ldots,0)| \, d\nu_S(f) \right).$$

Integrating in polar coordinates we have:

$$J_1 := \frac{d}{2\nu_P[\mathbb{P}_n(\mathbb{C})]} \int_0^\infty \int_{S^{2n-1}} \frac{r^{2n-1} \log(1 + r^2)}{(1 + r^2)^{n+1}} \, d\nu_S dr,$$

$$J_3 := \frac{1}{\nu_P[\mathbb{P}_n(\mathbb{C})]} \int_0^\infty \int_{S^{2n-1}} \frac{r^{2n-1}}{(1 + r^2)^{n+1}} \, d\nu_S dr,$$

As $\frac{\nu_S[S^{2n-1}]}{\nu_P[\mathbb{P}_n(\mathbb{C})]} = 2n$, we conclude:

$$J_3 = \frac{\nu_S[S^{2n-1}]}{\nu_P[\mathbb{P}_n(\mathbb{C})]} \frac{1}{2} B(n,1) = \frac{\nu_S[S^{2n-1}]}{\nu_P[\mathbb{P}_n(\mathbb{C})]} \frac{1}{2n} = 1,$$

and

$$J_1 = \frac{d\nu_S[S^{2n-1}]}{2\nu_P[\mathbb{P}_n(\mathbb{C})]} H = dnH,$$

where

$$H := \left( \int_0^\infty \frac{r^{2n-1} \log(1 + r^2)}{(1 + r^2)^{n+1}} \, dr \right).$$

Now, observe that:

$$H = \frac{1}{2} \int_0^\infty \frac{r^{2(n-1)} \log(1 + r^2)}{(1 + r^2)^{n+1}} \, 2r \, dr = \frac{1}{2} \int_0^\infty \frac{(n-1) \log(1 + t)}{(1 + t)^{n+1}} \, dt.$$
And
\[ H = \frac{1}{2} \int_{1}^{\infty} \frac{(x - 1)^{n-1} \log(x)}{x^{n+1}} \, dx = \frac{1}{2} \left( \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \int_{1}^{\infty} \frac{\log(x)}{x^{k+2}} \, dx \right). \]

Thus,
\[ H = \frac{1}{2} \left( \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \frac{1}{(k+1)^2} \right) = \frac{1}{2n} \left( \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} \frac{1}{(k+1)} \right) = \frac{H_n}{2n}, \]
in terms of the \( n \)-th harmonic number \( H_n \). We conclude
\[ J_1 := \frac{1}{2} \frac{H_n}{2n} = \frac{d}{H_n} \frac{1}{2}. \]

As for computing \( J_4 \), let us introduce for \( \alpha \in \{0, 1\} \) the following quantity:
\[ J_4(\alpha) := \frac{1}{\nu_{\mathbb{S}}(H^{(n+1)}_d)} \left( \int_{f \in \mathbb{S}(H^{(n+1)}_d)} \log^\alpha |f(1,0,\ldots,0)| d\nu_{\mathbb{S}}(f) \right). \]

Note that \( f(1,0,\ldots,0) = a_{(d,0,\ldots,0)} \) is the coefficient of the monomial \( X_0^d \) in the monomial expansion of \( f \). Note that \( J_4(0) = 1 \). Evaluating at \( (1,0,\ldots,0) \) is a projection that we may introduce in the following form:
\[ \pi_{(0)} : S(H^{(n+1)}_{d_0}) \rightarrow B_{C}(0,1), \]
where \( B_{C}(0,1) \) is the closed ball in \( \mathbb{C} \) with center 0 and radius 1. As observed in [13], Lemma 2, page 206, the normal Jacobian of \( \pi_{(0)} \) is known and it is given by \( NJ_f \pi_{(0)} = (1 - |\pi_{(0)}(f)|^2)^{1/2} \).

Thus, using the co-area formula, we may rewrite \( J_4(\alpha) \) in the following terms:
\[ J_4(\alpha) := \frac{1}{\nu_{\mathbb{S}}(H^{(n+1)}_d)} \int_{z \in B_{C}(0,1)} \left( \int_{f \in \pi_{(0)}^{-1}(z)} \log^\alpha |z|(1 - |z|^2)^{-1/2} \, df \right) \, dz. \]

Namely,
\[ J_4(\alpha) := \frac{1}{\nu_{\mathbb{S}}(H^{(n+1)}_d)} \int_{z \in B_{C}(0,1)} (1 - |z|^2)^{-1/2} \log^\alpha |z| \, \text{vol}[\pi_{(0)}^{-1}(z)] \, dz. \]

As \( \pi_{(0)} \) is an onto projection, the inverse image \( \pi_{(0)}^{-1}(z) \) can be viewed as the sphere of radius \( (1 - |z|^2)^{1/2} \) in \( V_{e_0} := V_{(1,0,\ldots,0)} := \{ f \in H^{(n+1)}_{d_0} : f(1,0,\ldots,0) = 0 \} \). Now, let \( N_d = \dim_{\mathbb{C}}(H^{(n+1)}_d) - 1 \) be (as in the statement of Theorem 1.1) the complex dimension of the vector space \( V_{e_0} \), and we have:
\[ J_4(\alpha) = \frac{\nu_{\mathbb{S}}(V_{e_0})}{\nu_{\mathbb{S}}(H^{(n+1)}_d)} \int_{z \in B_{C}(0,1)} (1 - |z|^2)^{N_d-1} \log^\alpha |z| \, dz, \]

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where $S(V_{e_0})$ is the sphere of radius one centered at the origin in $V_{e_0}$. Now, we integrate in polar coordinates in $B_C(0, 1)$ to conclude:

$$J_4(\alpha) = \frac{\nu_S[S(V_{e_0})]}{\nu_S[S(H_d^{n+1})]} \int_0^1 \int_{u \in S^1} (1 - |ru|^2)^{N_d-1} |ru| \log^\alpha |ru| \, d\nu_S(u) \, dr.$$  

Namely,

$$J_4(\alpha) = \frac{\nu_S[S(V_{e_0})]\nu_S[S^1]}{\nu_S[S(H_d^{n+1})]} \int_0^1 (1 - r^2)^{N_d-1} r \log^\alpha(r) \, dr.$$  

Taking $\alpha = 0$ we have:

$$1 = J_4(0) = \frac{\nu_S[S(V_{e_0})]\nu_S[S^1]}{\nu_S[S(H_d^{n+1})]} \int_0^1 (1 - r^2)^{N_d-1} r \, dr = \frac{\nu_S[S(V_{e_0})]\nu_S[S^1]}{\nu_S[S(H_d^{n+1})]} \frac{B(N_d, 1)}{2},$$

and, hence,

$$\frac{\nu_S[S(V_{e_0})]\nu_S[S^1]}{\nu_S[S(H_d^{n+1})]} = \frac{2}{B(N_d, 1)} = 2N_d.$$  

Thus, we conclude:

$$J_4 = J_4(1) = 2N_d \int_0^1 (1 - r^2)^{N_d-1} r \log(r) \, dr = \frac{1}{2} \int_0^1 N_d (1 - t)^{N_d-1} \log(t) \, dt = -\frac{H_{N_d}}{2}.$$  

The equality $E(d, n) = \mathfrak{C}(d, n)$ thus follows from Equation (11).  

**4.2. Proof of Theorem 1.2.**

We follow the notations used in the Introduction. For every $z \in S^{2n+1}$, we introduce the value:

$$J_z(z) := \int_{S(H_d^{m})} \|f(z)\|^s \, d\nu_S(f).$$  

Due to the unitary invariance of Bombieri-Weyl's norm, we easily see that $J(z)$ is a constant independent of $z$. Namely, we have:

$$J_z(z) = J_z(e_0) = \int_{S(H_d^{m})} \|f(e_0)\|^s \, d\nu_S(f),$$  

where $e_0 := (1, 0, \ldots, 0) \in S^{2n+1}$. From Fubini’s Theorem, we obviously conclude that:

$$Z_E^{(m)}(s) = \frac{1}{\nu_S[S^{2n+1}]} \int_{S^{2n+1}} \frac{J_z(z)}{\nu_S[S(H_d^{m})]} \, d\nu_S(z) = \frac{J_z(e_0)}{\nu_S[S(H_d^{m})]}.$$  

Now we consider the following mapping:

$$\pi : S(H_d^{m}) \ni f \mapsto B_C^{m}(0, 1) \ni f(e_0),$$  

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where \( B_{\mathbb{C}^m}(0,1) \) is the closed ball in \( \mathbb{C}^m \) of radius one centered at the origin. This mapping \( \pi \) is an onto projection and, according to Lemma 2, p. 206, of [13], its normal Jacobian satisfies:

\[
NJ_f \pi := (1 - \|f(e_0)\|^2)^{1/2}.
\]

Moreover, for every \( z \in B_{\mathbb{C}^m}(0,1) \), \( \pi^{-1}(z) \) may be identified with a sphere of real dimension \( 2(N - m) + 1 \) and radius \( (1 - \|z\|^2)^{1/2} \), where \( N := N(d) \) is the complex dimension of the complex projective space \( \mathbb{P}(H_d^{(m)}) \). We denote \( \pi^{-1}(z) := S^{2(N - m) + 1}((1 - \|z\|^2)^{1/2} \). Thus, Federer’s Co-area yields the following equality:

\[
Z_E^{(m)}(s) = \frac{1}{\nu_S[S(H_d^{(m)})]} \int_{B_{\mathbb{C}^m}(0,1)} \int_{S^{2(N - m) + 1}((1 - \|z\|^2)^{1/2}} \frac{|z|^s}{(1 - \|z\|^2)^{1/2}} d^2\nu_S.
\]

Namely, this yields:

\[
Z_E(s) = \frac{\nu_S[S^{2(N - m) + 1}]}{\nu_S[S(H_d^{(m)})]} \int_{B_{\mathbb{C}^m}(0,1)} \int_{S^{2(N - m) + 1}} (1 - \|z\|^2)^{N - m}|z|^s dz.
\]

Integrating in spherical coordinates, we have:

\[
Z_E^{(m)}(s) = \frac{\nu_S[S^{2(N - m) + 1}]}{\nu_S[S(H_d^{(m)})]} \int_{S^{2m - 1}} \int_0^1 t^{s + 2m - 1} (1 - t^2)^{N - m} dtd\nu_S.
\]

In other terms:

\[
Z_E^{(m)}(s) = \frac{\nu_S[S^{2(N - m) + 1}]}{\nu_S[S(H_d^{(m)})]} \int_0^1 t^{s + 2m - 1} (1 - t^2)^{N - m} dt.
\]

And, finally, we have:

\[
Z_E^{(m)}(s) = \frac{B(m + \frac{s}{2}, N - m + 1)}{B(N - m + 1, m)} = \frac{B(m + \frac{s}{2}, N(d) - m + 1)}{B(N(d) - m + 1, m)} ,
\]

as wanted. As for the the case of a single equation, we just need to replace \( m \) by 1 to obtain the following equality:

\[
Z_E(s) := Z_E^{(1)}(s) = \frac{\Gamma(N_d + 1)\Gamma(s/2 + 1)}{\Gamma(N_d + s/2 + 1)} = N_dB(N_d, s/2 + 1),
\]

where \( N_d \) is the complex dimension of \( \mathbb{P}(H_d^{(n+1)}) \). Similarly, we may also compute the moments of the logarithm of the norm, although they lead to less aesthetic formulations.

**Proposition 4.1.** With the same notations as above, the following equality holds:

\[
E_{f \in \mathcal{H}_d^{(m)}}[m_k^{(m)}(f)] = 2N_d \left( \frac{N(d) - 1}{m - 1} \right) \sum_{i=0}^{N(d) - m} (-1)^{i+k} \binom{N(d) - m}{i} \frac{k!}{(2(m + i))^{k+1}},
\]

where \( m_k^{(m)}(f) \) is the \( k \)-th moment of the logarithm of the norm of \( f \).
where \( N_d \) is the complex dimension of \( \mathbb{P}(H_d^{(n+1)}) \). In the case of a single equation (i.e. \( m = 1 \)), this means:

\[
E_{f \in \mathbb{S}(H_d^{(n+1)})} [m_k(f)] = 2N_d \sum_{i=0}^{N_d - 1} (-1)^i \binom{N_d - 1}{i} \frac{k!}{(2(i + 1))^{k+1}},
\]

where \( N_d \) is the complex dimension of \( \mathbb{P}(H_d^{(n+1)}) \).

**Proof.** Following the same steps as in the previous Proposition we conclude:

\[
E_{\mathbb{S}(H_d^{(m)})} [m_k^{(m)}] = \frac{\nu_S[S^{2(N_d-m)+1}]\nu_S[S^{2m-1}]}{\nu_S[S(\mathbb{H}_d^{(m))}]}} \int_0^1 t^{2m-1}(1-t^2)^{N_d-m}\log^k(t)dt.
\]

This yields

\[
E_{\mathbb{S}(H_d^{(m)})} [m_k^{(m)}] = \frac{\nu_S[S^{2(N_d-m)+1}]\nu_S[S^{2m-1}]}{\nu_S[S(\mathbb{H}_d^{(m))}]}} \sum_{i=0}^{N_d - m} \binom{N_d - m}{i} (-1)^i \int_0^1 t^{2(m+i)-1}\log^k(t)dt.
\]

Thus

\[
E_{\mathbb{S}(H_d^{(m)})} [m_k^{(m)}] = \frac{\nu_S[S^{2(N_d-m)+1}]\nu_S[S^{2m-1}]}{\nu_S[S(\mathbb{H}_d^{(m))}]}} \sum_{i=0}^{N_d - m} \binom{N_d - m}{i} (-1)^i \frac{(-1)^ki!}{(2(m+i))^{k+1}}.
\]

Which yields:

\[
E_{\mathbb{S}(H_d^{(m)})} [m_k^{(m)}] = 2N_d \binom{N_d - 1}{m - 1} \sum_{i=0}^{N_d - m} (-1)^i \binom{N_d - m}{i} \frac{k!}{(2(m+i))^{k+1}}.
\]

\( \square \)

4.3. A probabilistic estimate for question “(d)” in [4]

In [4] the authors asked several questions about the complexity of homotopy methods at one point. Among them, there is question “(d)” that we reproduce now:

“...Evaluate or estimate:

\[
I(h) := \int_{z \in \mathbb{P}_n(\mathbb{C})} e^{\frac{\|\Delta(\|z\|-d_h(z))\|^2}{z}} \frac{1}{\|\Delta(\|z\|-d_h(z))\|^2} |z|^{2m-1}dz,
\]

where \( \Delta(\|z\|-d_h) \) is the diagonal matrix whose terms in the diagonal are respectively \( \|z\|-d_1 \), \( \ldots \), \( \|z\|-d_n \); \( h := (h_1, \ldots, h_n) \in \mathbb{H}_d^{(n)} \) is a list of homogeneous polynomial equations of respective degrees determined by the list \( d = (d_1, \ldots, d_n) \) and \( \|z\| \) is the norm of any representant in \( \mathbb{C}_{n+1} \) of the projective point \( z \in \mathbb{P}_n(\mathbb{C}) \).”
The canonical projection \( \pi : S^{2n+1} \to \mathbb{P}_n(\mathbb{C}) \), the fact that the polynomials \( h_1, \ldots, h_n \) are homogeneous of respective degrees \( d_1, \ldots, d_n \) and the Co-area Formula imply that

\[
I(h) := \frac{1}{2\pi} AS(h),
\]

where

\[
AS(h) := \int_{z \in S^{2n+1}} e^{\frac{||h(z)||^2}{2}} \nu_S(z).
\]

Now, using the expansion of the exponential function, we conclude that the following equality holds for \( h \in \mathbb{S}(\mathbb{H}_d^{(n)}) \):

\[
AS(h) = \frac{\Gamma(n+1)}{2\pi^{n+1}} \sum_{k=0}^{\infty} \frac{Z^{(n)}(2k-n+1, h)}{2^k k!},
\]

where \( Z^{(n)}(s, h) \) is Akatsuka Zeta Mahler function as defined above. Thus, using Theorem 1.2 above, we conclude:

**Corollary 4.2.** With these notations, the expected value of \( I(h) \) in the Bombieri-Weyl sphere \( \mathbb{S}(\mathbb{H}_d^{(n)}) \) is given by the following identity:

\[
E_{h \in \mathbb{S}(\mathbb{H}_d^{(n)})}[I(h)] = \frac{\Gamma(N(d)+1)}{4\pi^{n+2n}} \sum_{k=0}^{\infty} \frac{1}{2^k k!} \frac{\Gamma(k+1/2)}{\Gamma(N(d)+k-n+3/2)},
\]

where \( N(d) \) is the complex dimension of the complex projective space \( \mathbb{P}(\mathbb{H}_d^{(n)}) \).

**Proof.**— According to our notations, we have:

\[
E_{h \in \mathbb{S}(\mathbb{H}_d^{(n)})}[I] = \frac{1}{2\pi} E_{h \in \mathbb{S}(\mathbb{H}_d^{(n)})}[AS] = \frac{1}{2\pi} \frac{\Gamma(n+1)}{2\pi^{n+1}} \sum_{k=0}^{\infty} \frac{Z^{(n)}(2k-n+1, h) 2^k k!}{\Gamma(N(d)+k-n+3/2)}.
\]

Namely, using Theorem 1.2 and making some elementary calculations, we conclude

\[
E_{h \in \mathbb{S}(\mathbb{H}_d^{(n)})}[I] = \frac{\Gamma(N(d)+1)}{4\pi^{n+2n}} \sum_{k=0}^{\infty} \frac{1}{2^k k!} \frac{\Gamma(k+1/2)}{\Gamma(N(d)+k-n+3/2)},
\]

as wanted. \( \square \)

5. An Arithmetic Poisson Formula

5.1. Proof of Theorem 1.4

The multi-variate resultant \( \text{Res}(d) \) satisfies the Homogeneous Poisson Formula (cf. Proposition 1.6.2 of [18], for instance) which claims that for every list \( f := (f_0, \ldots, f_n) \in \mathbb{H}_d \), the following equality holds:

\[
\text{Res}(d)(f_0, \ldots, f_n) = \text{Res}(d')(f_1, d_1, \ldots, f_n, d_n)^{d_0} \prod_{z \in V_s(f_1, \ldots, f_n)} a f_0(z)^{m_z},
\]

where:
that Res all z
From Theorem 1.1 we conclude:
where Note that up to a subset of zero measure in H
For fixed f
According to the double fibration formula (Proposition 3.6 above), we conclude:
and
(d') = (d_1, \ldots, d_n) \in \mathbb{N}^n.

Note that up to a subset of zero measure in \( \mathcal{H}_{(d')}^{(n)} \), we may always assume \( m_z = 1 \), for all \( z \in V_h(f_1, \ldots, f_n) \). Moreover, up to a set of zero measure in \( \mathcal{H}_{(d)}^{(n)} \) we may assume that \( \text{Res}_{(d')}(f_1, d_1, \ldots, f_n, d_n) \neq 0 \) and \( V_{E}(f_1, \ldots, f_n) \) has no point at the infinity hyper–plane \( X_0 = 0 \).

In the case \( n = 1 \), the Poisson Formula is as follows:
\[
\text{Res}_{(d_0, d_1)}(f_0, f_1) := (f_1(0, 1))^{d_0} \prod_{z \in V_h(f_1)} (a f_0)(z)^{m_z}.
\]

We may write \( \text{Res}_{d_1}(f) := f(0, 1) \). Thus, we have:
\[
m_{\mathbb{E}(2)}(\text{Res}_{(d_0, d_1)}) = I_1 + I_2,
\]
where
\[
I_1 := \frac{1}{\nu_{\mathbb{S}[\mathcal{H}_{d_0}^{(2)}]] \nu_{\mathbb{S}[\mathcal{H}_{d_1}^{(2)}]]}} \int_{(f_0, f_1) \in \mathbb{S}(H_{d_0}^{(2)})) \times \mathbb{S}(H_{d_1}^{(2)}))} d_0 \log |f_1(0, 1)| d\nu_{\mathbb{S}}(f_0, f_1),
\]
and
\[
I_2 := \frac{1}{\nu_{\mathbb{S}[\mathcal{H}_{d_0}^{(2)}]] \nu_{\mathbb{S}[\mathcal{H}_{d_1}^{(2)}]]}} \int_{(f_0, f_1) \in \mathbb{S}(H_{d_0}^{(2)})) \times \mathbb{S}(H_{d_1}^{(2)}))} \sum_{z \in V_h(f_1)} \log |a f_0(z)| d\nu_{\mathbb{S}}(f_0, f_1).
\]

For fixed \( f_0 \), we consider the following quantity:
\[
I_2(f_0) := \frac{1}{\nu_{\mathbb{S}[\mathcal{H}_{d_0}^{(2)}]] \nu_{\mathbb{S}[\mathcal{H}_{d_1}^{(2)}]]}} \int_{f_1 \in \mathbb{S}(H_{d_1}^{(2)}))} \sum_{z \in V_h(f_1)} \log |a f_0(z)| d\nu_{\mathbb{S}}(f_1).
\]

According to the double fibration formula (Proposition 3.6 above), we conclude:
\[
I_2(f_0) := d_1 E_{\mathbb{P}(\mathbb{C})}[\log |a f_0 \circ \varphi_0^{-1}|].
\]

From Theorem 1.1 we conclude:
\[
I_2 := E_{\mathbb{S}(H_{d_0}^{(2)})}(I_2(f_0)) = d_1 E(d_0, 1) = \frac{d_1}{2} (d_0 H_1 - H_{d_0}) = \frac{d_1}{2} (d_0 - H_{d_0}).
\]
As for the computation of $I_1$, observe that:

$$I_1 = \frac{d_0}{\nu_{\mathbb{S}(H_{d_1}^{(2)})}} \int_{f_1 \in \mathbb{S}(H_{d_1}^{(2)})} \log |f_1(0,1)| d\nu_{\mathbb{S}}(f_1),$$

We thus repeat the same strategy as the computation of $J_4$ in the Proof of Theorem 1.1 (cf. Subsection 4.1). We conclude:

$$I_1 = -\frac{d_0 H_{d_1}}{2},$$

and

$$m_{\mathbb{S}(a)}(\text{Res}(d_0,d_1)) = -\frac{d_0 H_{d_1}}{2} + \frac{d_1}{2} (d_0 - H_{d_0}) = \frac{1}{2} (d_0 d_1 - (d_0 H_{d_1} + d_1 H_{d_0})), $$

as wanted.

As for the case $n \geq 2$, let us denote by $R_{n+1}(f_0)$ the following integral for fixed $f_0 \in \mathbb{S}(H_{d_0}^{(n+1)})$:

$$R_{n+1}(f_0) := \int_{\mathbb{S}(H_{d_0}^{(n+1)})} \log |\text{Res}(d_0,f_0,\ldots,f_n)| d\mathcal{S}(n,n,d_0),$$

where $\mathcal{S}(n,n,d_0) := \prod_{i=1}^n \mathbb{S}(H_{d_i}^{(n+1)})$ and $(d') = (d_1,\ldots,d_n)$. Namely, a product of $n$ spheres representing lists of coefficients of $n$ polynomials in $n + 1$ variables of respective degrees $d_1,\ldots,d_n$. Along the proof we also consider $\mathcal{S}(n,n,d_0) := \prod_{i=1}^n \mathbb{S}(H_{d_i}^{(n)})$ which are spheres of coefficients of $n$ polynomials in $n$ variables of respective degrees $d_1,\ldots,d_n$. Observe that

$$m_{\mathbb{S}(a)}(\text{Res}(d')) = m_{\mathbb{S}(d')}(\text{Res}(d')),$$

and

$$\mathcal{R}(d) = \frac{1}{D(d) \text{vol}[\mathcal{S}(n,n,d_0) \times \mathbb{S}(H_{d_0}^{(n+1)})]} \int_{\mathbb{S}(H_{d_0}^{(n+1)})} R_{n+1}(f_0) d\mathcal{S}(H_{d_0}^{(n+1)})(f_0). \quad (12)$$

Next, we use the Homogeneous Poisson Formula to conclude

$$R_{n+1}(f_0) = d_0 \int_{\mathcal{S}(n,n,d_0)} \log |\text{Res}(d')(f_1,d_1,\ldots,f_n,d_n)| d\mathcal{S}(n,n,d_0) + \sum_{z \in \mathcal{V}(f_0,\ldots,f_n)} \log |f_0(\phi_0^{-1}(z))| d\mathcal{S}(n,n,d_0). \quad (13)$$

According to Lemma 3.11 above, we have

$$m_{\mathbb{S}(a)}(\text{Res}(d')) = m_{\mathbb{S}(d')}(\text{Res}(d')) - \left(\sum_{i=1}^n \prod_{j \neq i}^n \frac{d_j}{2} (H_{M_i} - H_{L_i})\right),$$

where

$$M_i := \dim_{\mathbb{C}}(H_{d_i}^{(n+1)}) - 1 = \binom{d_i + n}{n} - 1, \quad L_i := \dim_{\mathbb{C}}(H_{d_i}^{(n)}) - 1 = \binom{d_i + n - 1}{n - 1} - 1.$$
Namely, this equality becomes:

\[ m_{\mathcal{E}(d')}^{(n,n+1)}(\text{Res}(d')) = m_{\mathcal{E}(d')}^{(n)}(\text{Res}(d')) - \left( \sum_{i=1}^{n} \frac{\prod_{j \neq i,j=1}^{n} d_j}{2} (H_{M_i} - H_{L_i}) \right), \]  

(14)

Thus, Equation (13) becomes the following one:

\[ R_{n+1}(f_0) = d_0 \nu_{\mathcal{E}[\mathcal{E}(d')]} \left( m_{\mathcal{E}(d')}^{(n)}(\text{Res}(d')) - \left( \sum_{i=1}^{n} \frac{\prod_{j \neq i,j=1}^{n} d_j}{2} (H_{M_i} - H_{L_i}) \right) \right) + \int_{\mathcal{E}(d')}^{(n,n+1)} \sum_{z \in V_d(f_1, \ldots, f_n)} \log |f_0(\varphi_{0}^{-1}(z))| d\mathcal{E}(d'). \]  

(15)

As \( m_{\mathcal{E}(d)}(\text{Res}(d')) = D(d') \mathcal{R}(d') \), we have proved

\[ R_{n+1}(f_0) = d_0 D(d') \nu_{\mathcal{E}[\mathcal{E}(d')]} \left( \mathcal{R}(d') - \mathcal{J}(d') \right) + J(f_0), \]  

(16)

where:

\[ \mathcal{J}(d') := \frac{1}{D(d')} \left( \sum_{i=1}^{n} \frac{\prod_{j \neq i,j=1}^{n} d_j}{2} (H_{M_i} - H_{L_i}) \right) = \left( \sum_{i=1}^{n} \frac{1}{2d_i} (H_{M_i} - H_{L_i}) \right), \]

and

\[ J(f_0) := \int_{\mathcal{E}(d')}^{(n,n+1)} \sum_{z \in V_d(f_1, \ldots, f_n)} \log |f_0(\varphi_{0}^{-1}(z))| d\mathcal{E}(d'). \]

As \( d_0 D(d') = D(d) \), from Equation (12) we conclude:

\[ \mathcal{R}(d) = \frac{1}{D(d')} \nu_{\mathcal{E}[\mathcal{E}(d')]} \left( \mathcal{S}(d') \times \text{S}(H_{d_0}^{(n+1)}) \right) \int_{\text{S}(H_{d_0}^{(n+1)})} R_{n+1}(f_0) d\text{S}(H_{d_0}^{(n+1)})(f_0) = \left( \mathcal{R}(d') + \mathcal{J}(d') \right) + \frac{1}{D(d')} \nu_{\mathcal{E}[\mathcal{E}(d')]} \left( \mathcal{S}(d') \times \text{S}(H_{d_0}^{(n+1)}) \right) \int_{\text{S}(H_{d_0}^{(n+1)})} J(f_0) d\text{S}(H_{d_0}^{(n+1)})(f_0) \]  

(17)

where

\[ E_{\text{S}(H_{d_0}^{(n+1)})}[J(f_0)] = \frac{1}{\text{S}(H_{d_0}^{(n+1)})} \int_{\text{S}(H_{d_0}^{(n+1)})} J(f_0) d\text{S}(H_{d_0}^{(n+1)})(f_0). \]

We may use the double fibration formula (Proposition 3.6 above) to conclude:

\[ J(f_0) = D(d') \nu_{\mathcal{E}[\mathcal{E}(d')]} \left[ E_{\mathbb{P}_n(\mathbb{C})}[\log |f_0 \circ \varphi_{0}^{-1}|] \right], \]  

(18)

where

\[ E_{\mathbb{P}_n(\mathbb{C})}[\log |f_0 \circ \varphi_{0}^{-1}|] = \frac{1}{\nu_n[\mathbb{P}_n(\mathbb{C})]} \int_{\mathbb{P}_n(\mathbb{C})} \log |f_0(\varphi_{0}^{-1}(z))| d\mathbb{P}_n(\mathbb{C})(z). \]
Applying the inductive hypothesis, we have:

\[
E_{S(H_{d_0}^{(n+1)})} [J(f_0)] = \mathcal{D}(d') \nu_{\mathbb{E}} [\mathcal{E}(d')^{(n,n+1)}] E_{S(H_{d_0}^{(n+1)})} [E_{\mathbb{F}_n(C)} [\log |f_0 \circ \varphi_{-1}^0|]] = \\
= \mathcal{D}(d') \nu_{\mathbb{E}} [\mathcal{E}(d')^{(n,n+1)}] \mathcal{E}(d_0,n),
\]
where \( \mathcal{E}(d_0,n) \) is the expectation computed in Theorem 1.1. Hence, we conclude:

\[
E_{S(H_{d_0}^{(n+1)})} [J(f_0)] = \mathcal{D}(d) \nu_{\mathbb{E}} [\mathcal{E}(d)^{(n,n+1)}] \mathcal{E}(d_0,n),
\]
because \( \mathcal{D}(d) = d_0 \mathcal{D}(d') \). Replacing this equality in the last row of Equation (17) we have:

\[
\mathfrak{R}(d) = (\mathfrak{R}(d') - \mathfrak{I}(d')) + \frac{\mathcal{E}(d_0,n)}{d_0}, \quad (19)
\]
as wanted.

\[5.2. \text{Proof of Corollary 1.5}\]
We proceed by induction on \( n \). For \( n = 1 \), Theorem 1.4 implies:

\[
m_{\mathfrak{E}(2)}(\text{Res}(d_0,d_1)) = d_0 \mathcal{E}(d_1,1) + d_1 \mathcal{E}(d_0,0) = \frac{1}{2} (d_1 (d_0 - H_{d_0})) + \frac{1}{2} (d_0 (-H_{d_1})),
\]
and the formula holds. As for \( n \geq 2 \), we apply Theorem 1.4 to conclude:

\[
m_{(d)} = d_0 \left( m_{(d')} - \left( \sum_{i=1}^n \frac{\prod_{j \neq i,j \neq 0} d_j}{2} (H_{M_i} - H_{L_i}) \right) \right) + \frac{\prod_{i=1}^n d_i}{2} (d_0 H_n - H_{M_0}), \quad (20)
\]
where \( (d') := (d_1, \ldots, d_n) \), \( m_{(d)} = m_{\mathfrak{E}(n+1)}(\text{Res}(d)) \), \( m_{(d')} = m_{\mathfrak{E}(n)}(\text{Res}(d')) \) and, for \( 0 \leq i \leq n \),

\[
M_i := \dim_{\mathbb{C}}(H_{d_i}^{(n+1)}) - 1 = \left( \frac{d_i + n}{n} \right) - 1, \quad L_i := \dim_{\mathbb{C}}(H_{d_i}^{(n)}) - 1 = \left( \frac{d_i + n - 1}{n - 1} \right) - 1.
\]

Applying the inductive hypothesis, we have:

\[
m_{(d')} = \left( \sum_{i=1}^{n-1} \frac{\prod_{j \neq i,j \neq 0} d_j}{2} (d_i H_{(n-1)-(i-1)}) \right) - \left( \sum_{i=1}^n \frac{\prod_{j \neq i,j \neq 0} d_j}{2} H_{L_i} \right).
\]
Replacing this identity in Equation (20), we conclude:

\[
m_{(d)} = d_0 \left( \sum_{i=1}^n \frac{\prod_{j \neq i,j \neq 0} d_j}{2} (\kappa(i,n) - H_{M_i}) \right) + \frac{\prod_{i=1}^n d_i}{2} (d_0 H_n - H_{M_0}),
\]
and we get the wanted identity. \qed
References


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