Periodic bifurcation from families of periodic solutions for semilinear differential equations with Lipschitzian perturbations in Banach spaces

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Abstract.
Let $A : D(A) \to E$, $D(A) \subset E$, be an infinitesimal generator either of an analytic compact semigroup or of a contractive $C_0$-semigroup of linear operators acting in a Banach space $E$. In this paper we give both necessary and sufficient conditions for bifurcation of $T$-periodic solutions for the equation $\dot{x} = Ax + f(t, x) + \varepsilon g(t, x, \varepsilon)$ from a $k$-parameterized family of $T$-periodic solutions of the unperturbed equation corresponding to $\varepsilon = 0$. We show that by means of a suitable modification of the classical Mel’nikov approach we can construct a bifurcation function and to formulate the conditions for the existence of bifurcation in terms of the topological index of the bifurcation function. To do this, since the perturbation term $g$ is only Lipschitzian we need to extend the classical Lyapunov-Schmidt reduction to the present nonsmooth case.

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1 Introduction

The aim of this paper is to give both necessary and sufficient conditions for the bifurcation of $T$-periodic solutions of the semi-linear differential equation

$$\dot{x} = Ax + f(t, x) + \varepsilon g(t, x, \varepsilon)$$

(1.1)

from a $k$-parameterized family of $T$-periodic solutions of the unperturbed system, obtained from (1.1) by letting $\varepsilon = 0$. Here $A : D(A) \to E$, $D(A) \subset E$, is an infinitesimal generator either of an analytic compact semigroup or of a contractive $C_0$-semigroup of linear operators acting in the Banach space $E$, the nonlinear operators $f \in C^1(\mathbb{R} \times E, E)$ and $g \in C^0(\mathbb{R} \times E \times [0, 1], E)$ are $T$-periodic in the first variable.

In the case when the unperturbed system is autonomous the problem was studied by Henry in ([7], Ch. 8), where it is assumed that $g$ is differentiable in the second variable. The author provided sufficient conditions for bifurcation of $T$-periodic solutions from a $T$-periodic cycle $x_0$, the main tool employed in that paper is the classical Lyapunov-Schmidt reduction, see for instance Chow and Hale ([4], Ch. 2, § 4). These conditions are formulated in terms of the existence of nondegenerate zeros of an analogue of the Malkin’s bifurcation function [12] for an infinite dimensional Banach space.

In the finite dimensional case, using topological degree arguments, Felmer and Manásevich in [5] replaced the assumption of the existence of nondegenerate zeros of the bifurcation function by the request that the topological degree of the bifurcation function is different from zero with respect to a suitable set. Starting from [5] there has been a great amount of work for developing bifurcation results by using the topological degree theory, see e.g. Henrard and Zanolin [6] for bifurcation from a cycle of a Hamiltonian system and Kamenskii, Makarenkov and Nistri [8] for bifurcation from a cycle of a self-oscillating system. In the present paper we follow this topological approach, namely by using the topological degree theory we formulate suitable assumptions on the considered bifurcation function to obtain for (1.1) results similar to those of ([7], Ch. 8). To this end we prove an extension of the classical Lyapunov-Schmidt reduction as presented in ([4], Ch. 2, § 4) to the case when the perturbation $g$ is Lipschitzian. We mention in the sequel some problems involving partial differential equations which reduce to the situation considered in this paper. In Chow and Hale [4, Ch. 8, § 6] and Schaeffer and Golubitsky [14] the problem of the dependance of the steady states in chemical reaction models on the relative diffusion coefficients leads to the consideration of perturbed equations in Banach spaces with the property that the corresponding unperturbed equations have a family of solutions.

Another example of such a situation is presented in Berti and Bolle [2], where the problem of finding periodic solutions of a nonlinear wave equation by variational methods gives rise to an unperturbed equation with a family of periodic solutions.

The paper is organized as follows. A modified Lyapunov-Schmidt reduction for Lipschitzian perturbations of an operator of the form $(P - I)$, with $P \in C^1(E, E)$, is obtained in Section 2. In order to apply the results of Section 2 some relevant
properties of the Poincaré map for system (1.1) are established in Section 3. Both necessary and sufficient conditions for bifurcation of periodic solutions to (1.1) are obtained in Section 4. Finally, in the appendix of Section 5 we give a proof of a technical result needed in Section 3.

2 Lyapunov-Schmidt reduction

Let $E$ be a Banach space and consider the function $F : E \times [0, 1] \rightarrow E$ given by

$$F(\xi, \varepsilon) = P(\xi) - \xi + \varepsilon Q(\xi, \varepsilon),$$

where $P : E \rightarrow E$ and $Q : E \times [0, 1] \rightarrow E$. Assume that

(A1) there exist $h_0 \in \mathbb{R}^k$, $r_0 > 0$ and a function $S \in C^1(B_{\mathbb{R}^k}(h_0, r_0), E)$ such that $P(\xi) = \xi$ for any $\xi \in Z = \{S(h) : h \in B_{\mathbb{R}^k}(h_0, r_0)\}$.

Here and in what follows $B_X(c, r)$ denotes the ball in the normed space $X$ centered at $c$ with radius $r > 0$. It is well known that, under the assumption (A1) with $P \in C^1(E, E)$ and $Q \in C^1(E \times [0, 1], E)$, the Lyapunov-Schmidt reduction ([4], Ch. 2, § 4) allows to solve the equation

$$F(\xi, \varepsilon) = 0,$$

for $\varepsilon > 0$ sufficiently small. Next theorem extends this result to the case when $Q$ satisfies the following Lipschitz condition:

(L) For any $R > 0$ there exists $L(R) > 0$ such that

$$||Q(\xi_1, \varepsilon) - Q(\xi_2, \varepsilon)|| \leq L(R) ||\xi_1 - \xi_2||$$

whenever $\xi_1, \xi_2 \in B_E(0, R)$ and $\varepsilon \in [0, 1]$.

Theorem 2.1 Let $P \in C^1(E, E)$, $Q \in C^0(E \times [0, 1], E)$, where $E$ is a Banach space. Assume that $Q$ satisfies (L). Moreover, assume (A1) and

(A2) $\dim S'(h_0)\mathbb{R}^k = k$.

Let $E_1, h = S'(h_0)\mathbb{R}^k$. Let $E_2, h$ be any subspace of $E$ such that $E = E_1, h \bigoplus E_2, h$ and assume that

(A3) there exists $r_0 > 0$ such that both the projectors $\pi_{1, h}$ of $E$ onto $E_{1, h}$ along $E_2, h$ and $\pi_{2, h}$ of $E$ onto $E_{2, h}$ along $E_{1, h}$ are continuous in $h \in B_{\mathbb{R}^k}(h_0, r_0)$,

(A4) for $\xi_0 = S(h_0)$ we have $\pi_{2, h_0}(P'(\xi_0) - I)\pi_{2, h_0}$ is invertible on $E_{2, h_0}$.

(2.2)
Then there exist \(0 < r_2 < r_1 < r_0\) and functions \(H : B_E(\xi_0, r_1) \to \mathbb{R}^k\), with \(H(\xi) \to h_0\) as \(\xi \to \xi_0\) and \(\beta : B_R^k(h_0, r_1) \times [0, r_1] \to E\), \(\beta(\cdot, \varepsilon) \in C^0(B_R^k(h_0, r_1), E)\), \(\|\beta(h, \varepsilon)\| \leq M\varepsilon\) for some \(M > 0\), any \(h \in B_R^k(h_0, r_1)\) and any \(\varepsilon \in [0, r_1]\) with

\[
\beta(h, \varepsilon) \in E_{2,h},
\]

and \(\beta(h, \varepsilon)/\varepsilon \to (\pi_{2,h}(P^r(S(h)) - I)\pi_{2,h})^{-1}\pi_{2,h}Q(S(h), 0)\) as \(\varepsilon \to 0\), uniformly in \(h \in B_{2,R}(h_0, r_1)\) such that the following properties hold:

1) if \((\xi, \varepsilon) \in B_E(\xi_0, r_2) \times [0, r_2]\) is a solution to equation (2.1) then \((h, \varepsilon)\), where \(h = H(\xi)\), is a solution to

\[
(S^r(h))^{-1}\pi_{1,h}[P(\beta(h, \varepsilon) + S(h)) - (\beta(h, \varepsilon) + S(h)) + \varepsilon Q(\beta(h, \varepsilon) + S(h), \varepsilon)] = 0.
\]  

2) if \((h, \varepsilon) \in B_R^k(h_0, r_1) \times [0, r_1]\) solves (2.5) then \((\xi, \varepsilon)\) solves (2.1), with

\[
\xi = \beta(h, \varepsilon) + S(h).
\]

Note, that the existence of \((S^r(h))^{-1}\) on \(E_{1,h}\) for \(h \in \mathbb{R}^k\) sufficiently close to \(h_0\) is guaranteed by (A2) and (A3). To prove Theorem 2.1 we need the following version of the implicit function theorem.

**Lemma 2.1** Let \(E\) be a Banach space and \(V \subset \mathbb{R}^k\) be an open bounded set. Consider a family of projectors \(\{\pi_h\}_{h \in \mathcal{V}}\) on \(E\) continuous in \(h\) and let \(E_h = \pi_h E\) for any \(h \in V\). Let \(\Phi_{h,\varepsilon} : E_h \to E_h\) be defined by

\[
\Phi_{h,\varepsilon}(\beta) = \tilde{P}(h, \beta) + \varepsilon \tilde{Q}(h, \beta, \varepsilon),
\]

where \(\tilde{P} \in C^0(\mathbb{R}^k \times E, E)\), \(\tilde{Q} \in C^0(\mathbb{R}^k \times E \times [0, 1], E)\), \(\tilde{P}(h, \cdot), \tilde{Q}(h, \cdot, \cdot) : E_h \to E_h\) for any \(h \in V, \varepsilon \in [0, 1]\). Assume that

1. the continuity of \(\tilde{P}\) in the first variable is uniform on any bounded subset of \(V \times E\),
2. \(\tilde{P}\) is differentiable with respect to the second variable and the derivative is continuous in \(V \times E\),
3. \(\tilde{Q}\) is Lipschitzian in the second variable uniformly on any bounded subset of \(V \times E \times [0, 1]\).
4. \(\tilde{P}(h, 0) = 0\) for any \(h \in V\),
5. \(\pi_h \tilde{P}_h^0(h, 0) : E_h \to E_h\) is invertible for any \(h \in V\) and \((\pi_h \tilde{P}_h^0(h, 0))^{-1}\pi_h\) is continuous in \(h \in V\).
Then there exist \( r > 0 \), \( M > 0 \) and a function \( \beta : V \times [0, r] \to E \), \( \beta(\cdot, \varepsilon) \in C^0(V, E) \) such that

a) \( \beta(h, \varepsilon) \in E_h \) for any \( h \in V, \varepsilon \in [0, r] \),

b) \( \Phi_{h,\varepsilon}(\beta(h, \varepsilon)) = 0 \) for any \( h \in V, \varepsilon \in [0, r] \),

c) \( \beta(h, \varepsilon) \) is the only zero of \( \Phi_{h,\varepsilon} \) in \( B_{E_h}(0, r) \) for any \( h \in V, \varepsilon \in [0, r] \),

d) \( \|\beta(h, \varepsilon)\| \leq M \varepsilon \) for any \( h \in V, \varepsilon \in [0, r] \).

Although Lemma 2.1 looks well-known, the authors were unable to find a proof of it in the literature, thus for the reader convenience we provide a proof of Lemma 2.1 in the Appendix of Section 5.

**Proof of Theorem 2.1.** In order to define the function \( \beta \) we consider the following auxiliary function \( \Phi_{h,\varepsilon} \in C^0(E_{2h}, E_{2h}) \) given by

\[
\Phi_{h,\varepsilon}(\beta) = \pi_{2,h}[P(\pi_{2,h}\beta + S(h)) - (\pi_{2,h}\beta + S(h)) + \varepsilon Q(\beta + S(h), \varepsilon)].
\]

Since \( P \in C^1(E, E) \) and \( S \in C^1(B_{R_k}(h_0, r_0), E) \) then assumptions 1 and 2 of Lemma 2.1 are satisfied.

By our assumptions we have that the application \( (h, \beta, \varepsilon) \to \Phi_{h,\varepsilon}(\beta) \) is Lipschitz in \( \beta \) uniformly on any bounded subset of \( B_{R_k}(h_0, r_0) \times E \times [0, 1] \) and taking into account \( (A_1) \) we have

1) \( \Phi_{h,0}(0) = 0 \) for any \( h \in B_{R_k}(h_0, r_0) \).

By assumptions \( (A_3)-(A_4) \) \( r_0 > 0 \) can be diminished in such a way that

2) \( (\Phi_{h,0})'(0) = \pi_{2,h}(P'(S(h)) - I)\pi_{2,h} \) is an invertible operator from \( E_{2h} \) to \( E_{2h} \) for \( h \in B_{R_k}(h_0, r_0) \).

Therefore, Lemma 2.1 applies with

\[
\tilde{P}(h, \beta) = \pi_{2,h}[P(\pi_{2,h}\beta + S(h)) - (\pi_{2,h}\beta + S(h))],
\]

\[
\tilde{Q}(h, \beta, \varepsilon) = \pi_{2,h}Q(\beta + S(h), \varepsilon) \quad \text{and} \quad V = B_{R_k}(h_0, r_0)?.
\]

Thus there exist \( r_1 \in [0, r_0], M > 0 \) and a function \( \beta(\cdot, \varepsilon) \in C^0(B_{R_k}(h_0, r_1), E) \) satisfying Properties a), b), c) and d) of Lemma 2.1. In particular, from Property b) we have

\[
\pi_{2,h}[P(\beta(h, \varepsilon) + S(h)) - (\beta(h, \varepsilon) + S(h))] - (P(S(h)) - S(h)) + \varepsilon Q(\beta(h, \varepsilon) + S(h), \varepsilon)] = 0
\]
or equivalently

\[
\pi_{2,h}[(P'(S(h)) - I)\pi_{2,h}(\beta(h, \varepsilon) + o(\beta(h, \varepsilon))) + \varepsilon Q(\beta(h, \varepsilon) + S(h), \varepsilon)] = 0,
\]

for any \( h \in B_{R_k}(h_0, r_1) \).
Therefore
\[ \beta(h, \varepsilon) = - (\pi_2, \rho(P(S(h)) - I)\pi_2) (\pi_2, \rho(\beta(h, \varepsilon)) + \pi_2, \rho Q(\beta(h, \varepsilon) + S(h), \varepsilon)). \]

Due to Property d) the last equation implies (2.4).

We now proceed to define the function \( H \). For this? by (A2) we have that \( r_1 > 0 \) can be taken sufficiently small such that \( S'(h) : \mathbb{R}^k \rightarrow E'_{1, \mathbb{R}} \) is invertible. Thus we can define the function \( \Phi : \mathbb{R}^k \rightarrow \mathbb{R}^k, \xi \in E \), as follows
\[ \Phi(h) = (S'(h))^{-1} \pi_{1, \mathbb{R}}(\xi - S(h)), \quad h \in B_{\mathbb{R}}(h_0, r_1). \]

We have the following properties for \( \Phi \).

1) \( \Phi_{\xi_0} \) is differentiable at \( h_0 \).

2) \( (\Phi_{\xi_0})'(h_0) = (S'(h_0))^{-1} \pi_{1, h_0} (-S'(h_0)) = -I \), namely \( (\Phi_{\xi_0})'(h_0) \) is an invertible \( k \times k \)-matrix.

Observe that property 1) is a direct consequence of the fact that \( \xi_0 - S(h_0) = 0 \) and the continuity of the function \( h \rightarrow S^{-1}(h) \pi_h \), therefore the differentiability of \( \pi_{1, h} \) at \( h = h_0 \) is not necessary for the validity of 1).

Let \( \delta > 0 \) be such that \( h_0 \) is the only zero of \( \Phi_{\xi_0} \) in \( B_{\mathbb{R}}(h_0, \delta) \). By ([10], Theorem 6.3) we can consider \( \delta > 0 \) sufficiently small such that \( d(\Phi_{\xi_0}, B_{\mathbb{R}}(h_0, \delta)) = (-1)^k \). By the continuity property of the topological degree \( r_1 > 0 \) can be diminished, such necessary, in such way that \( d(\Phi_{\xi_0}, B_{\mathbb{R}}(h_0, \delta)) = (-1)^k \) for any \( \xi \in B_{E}(\xi_0, r_1) \). Therefore, for any \( \xi \in B_{E}(\xi_0, r_1) \) there exists \( H(\xi) \in B_{\mathbb{R}}(h_0, \delta) \) such that \( \Phi_\xi(H(\xi)) = 0 \). Let us show that \( H(\xi) \rightarrow h_0 \) as \( \xi \rightarrow \xi_0 \). Indeed, arguing by contradiction we would have a sequence \( \{\xi_n\}_{n \in \mathbb{N}} \subseteq B_{E}(\xi_0, r_1), \quad h_\ast \in B_{\mathbb{R}}(h_0, \delta) \) such that \( H(\xi_n) \rightarrow h_\ast \neq h_0 \) as \( n \rightarrow \infty \) and thus \( \Phi_{\xi_0}(h_\ast) = 0 \) contradicting the choice of \( \delta > 0 \). Therefore
\[ \pi_{1, H(\xi)}(\xi - S(H(\xi))) = 0, \quad \xi \in B_{E}(\xi_0, r_1). \quad (2.8) \]

Moreover, we consider \( r_2 \in (0, r_1] \) sufficiently small to have
\[ \| \xi - S(H(\xi)) \| \leq r_1, \quad \xi \in B_{E}(\xi_0, r_2). \quad (2.9) \]

We are now in the position to complete the proof. For this let \( (\xi, \varepsilon) \in B_{E}(\xi_0, r_2) \times [0, r_2 \] satisfying (2.1). Then \( (\xi, \varepsilon) \) also satisfies
\[
\begin{align*}
\pi_{1, H(\xi)}[P(\xi - S(H(\xi))) + S(H(\xi))] & - (\xi - S(H(\xi))) + \varepsilon Q(\xi - S(H(\xi)) + S(H(\xi)), \varepsilon) = 0, \\
\pi_{2, H(\xi)}[P(\xi - S(H(\xi))) + S(H(\xi))] & - (\xi - S(H(\xi))) + \varepsilon Q(\xi - S(H(\xi)) + S(H(\xi)), \varepsilon) = 0.
\end{align*}
\]

From (2.8), (2.9) and Property c) of Lemma 2.1 we have
\[
\begin{align*}
\pi_{1, H(\xi)}[P(\xi - S(H(\xi))) + S(H(\xi))] & - (\xi - S(H(\xi))) + \varepsilon Q(\xi - S(H(\xi)) + S(H(\xi)), \varepsilon) = 0, \\
\beta(H(\xi), \varepsilon) = \xi - S(H(\xi)). \quad (2.10)
\end{align*}
\]
Therefore,
\[ \pi_{1,h} [P(\beta(h, \varepsilon) + S(h)) - (\beta(h, \varepsilon) + S(h)) + \varepsilon Q(\beta(h, \varepsilon), \varepsilon)] = 0 \] (2.11)
has a solution \( h = H(\xi) \). Since \( r_1 > 0 \) has been chosen in such a way that \( S'(h) \) is invertible on \( E_{1,h} \) for \( h \in B_{2k}(h_0, r_1) \) then (2.11) can be rewritten as (2.5). Assume now that (2.5) is satisfied with some \((h_\ast, \varepsilon_\ast) \in B_{2k}(h_0, r_1) \times [0, r_1]. Define \( \xi \in E \) as
\[ \tilde{\xi} = \beta(h_\ast, \varepsilon_\ast) + S(h_\ast). \] (2.12)
Since \((S'(h_\ast))^{-1}\) is invertible then \( \pi_{1,h_\ast}[P(\tilde{\xi})? - \tilde{\xi} + \varepsilon Q(\tilde{\xi}, \varepsilon_\ast)] = 0 \). On the other hand from (2.12) we have
\[ \pi_{2,h_\ast}[P(\pi_{2,h_\ast}\beta(h_\ast, \varepsilon_\ast) + S(h_\ast)) - (\pi_{2,h_\ast}\beta(h_\ast, \varepsilon_\ast) + S(h_\ast)) + \varepsilon Q(\beta(h_\ast, \varepsilon_\ast) + S(h_\ast), \varepsilon)] = \pi_{2,h_\ast}[P(\xi) - \xi + \varepsilon Q(\tilde{\xi}, \varepsilon)]. \]
Thus \((\xi, \varepsilon_\ast)\) solves (2.1) and so the proof is complete.

The following two results are consequences of Theorem 2.1 and they provide, respectively, a necessary and a sufficient condition for the existence of solutions to (2.1) near \( \xi_0 \) when \( \varepsilon > 0 \) is sufficiently small. These conditions are expressed in terms of the following bifurcation function
\[ M(h) = (S'(h))^{-1} \pi_{1,h}[Q(S(h_0), 0) - (P'(S(h)) - I)(\pi_{2,h}(P'(S(h)) - I))^{-1} \pi_{2,h}Q(S(h), 0)], \]
where \( h \) varies in a sufficiently small neighborhood of \( h_0 \in \mathbb{R}^k \).

We can prove the following.

**Theorem 2.2** Let all the assumptions of Theorem 2.1 be satisfied. Assume that there exist sequences \( \varepsilon_n \to 0 \) and \( \xi_n \to \xi_0 \) as \( n \to \infty \) such that \((\xi_n, \varepsilon_n)\) solves (2.1). Then
\[ M(h_0) = 0. \] (2.13)

**Proof.** By Theorem 2.1, for \( n \geq n_0 \), with \( n_0 \in \mathbb{N} \) sufficiently large, we have that
\[ (S'(h_0))^{-1} \pi_{1,h_0}[P(\beta(h_n, \varepsilon_n) + S(h_n)) - (\beta(h_n, \varepsilon_n) + S(h_n)) + \varepsilon_n Q(\beta(h_n, \varepsilon_n) + S(h_n), \varepsilon)] = 0 \] (2.14)
where \( h_n = H(\xi_n) \). On the other hand \( n_0 \) can be chosen sufficiently large in such a way that
\[ P(S(h_n)) - S(h_n) = 0 \quad \text{for} \quad n \geq n_0 \]
thus, for \( n \geq n_0 \), (2.14) can be rewritten as
\[ (S'(h_n))^{-1} \pi_{1,h_n}[(P'(S(h_n)) - I) \frac{\beta(h_n, \varepsilon_n)}{\varepsilon_n} + \frac{\alpha(\beta(h_n, \varepsilon_n))}{\varepsilon_n} + Q(\beta(h_n, \varepsilon_n) + S(h_n), \varepsilon)] = 0. \] (2.15)
By means of property (2.4) we can pass to the limit as \( n \to \infty \) in (2.15) to obtain (2.13).

**Theorem 2.3** Let all the assumptions of Theorem 2.1 be satisfied. Assume that
\[ h_0 \text{ is an isolated zero of } M \]  \tag{2.16}
and
\[ \text{ind}(h_0, M) \neq 0. \]  \tag{2.17}
Then, for any \( \varepsilon > 0 \) sufficiently small there exists \( \xi_\varepsilon \in E \) such that
\[ F(\xi_\varepsilon, \varepsilon) = 0 \]
and
\[ \xi_\varepsilon \to \xi_0 \text{ as } \varepsilon \to 0. \]  \tag{2.18}

**Proof.** Let \( r_1 > 0 \) be as given by Theorem 2.1. Since
\[ P(S(h)) = S(h) \text{ for any } h \in B_{R_k}(h_0, r_1) \]  \tag{2.19}
then the zeros of the function
\[
\Phi(h, \varepsilon) = (S'(h))^{-1} \pi_{1,h}[P(\beta(h, \varepsilon) + S(h)) - \frac{\beta(h, \varepsilon)}{\varepsilon} + o(\beta(h, \varepsilon)) + Q(\beta(h, \varepsilon) + S(h), \varepsilon)]
\]
coincide with the zeros of the function
\[
M_\varepsilon(h) = (S'(h))^{-1} \pi_{1,h}[(P'(S(h)) - I) \frac{\beta(h, \varepsilon)}{\varepsilon} + \frac{o(\beta(h, \varepsilon))}{\varepsilon} + Q(\beta(h, \varepsilon) + S(h), \varepsilon)].
\]
In order to apply Theorem 2.1 we show now that \( r \in (0, r_1] \) can be chosen in such a way that the function \( M_\varepsilon \) has zeros in \( B_{R_k}(h_0, r) \) for any \( \varepsilon > 0 \) sufficiently small.

By condition (2.16) \( r > 0 \) can be chosen sufficiently small in such a way that
\[ \text{the only zero of } M \text{ in } B_{R_k}(h_0, r) \text{ is } h_0. \]  \tag{2.20}
Therefore, by condition (2.17) we have
\[ d(M, B_{R_k}(h_0, r)) = \text{ind}(h_0, M) \neq 0. \]
On the other hand from property (2.4) we have that
\[ M_\varepsilon(h) \to M(h) \text{ as } \varepsilon \to 0 \]  \tag{2.21}
uniformly with respect to \( h \in B_{R_k}(h_0, r) \). Thus we conclude that
\[ d(M_\varepsilon, B_r(h_0)) \neq 0 \]
for $\varepsilon \in (0, \varepsilon_0]$, where $\varepsilon_0 > 0$ is sufficiently small. Thus for any $\varepsilon \in (0, \varepsilon_0]$ there exists $h_\varepsilon$ such that $M_\varepsilon(h_\varepsilon) = 0$. Moreover, we have that

$$h_\varepsilon \to h_0 \quad \text{as} \ \varepsilon \to 0$$

otherwise $M$ would have zeros in $B_{\mathbb{R}k}(h_0, r)$ different from $h_0$, contradicting (2.20). Finally, (2.18) follows from (2.6).

In finite dimensional spaces results similar to previous Theorems 2.2 and 2.3 have been recently obtained by Buica, Llibre and Makarenkov [3], where the uniqueness of the bifurcating periodic solutions is also proved.

3 The Poincaré map

Since the definition of the Poincaré map for system (1.1) on the time interval $[0, T]$ depends on the assumptions on the linear unbounded operator $A$, we precise in (C1) and (C2) below the two cases that we consider for $A$ in the paper.

(C1) The operator $A$ is a generator of an analytic compact semigroup $e^{At}$ in $E$. The operators $f, g$ are subordinated to some $A^{-\alpha}$, $0 < \alpha < 1$ (see e.g. [11]), the operator $f(\cdot, A^{-\alpha} \cdot)$ is differentiable in the second variable and the operators $f'(\cdot, A^{-\alpha} \cdot), g(\cdot, A^{-\alpha} \cdot, \cdot)$ are continuous from $\mathbb{R} \times E \to E$ and they satisfy a Lipschitz condition in the second variable uniformly with respect to the others.

(C2) The operator $A$ is a generator of a $C_0$-semigroup $e^{At}$. The semigroup $e^{At}$ is contractive, namely

$$\|e^{At}\| \leq e^{-\gamma t},$$

where $\gamma > 0$. The operators $f$ and $g$ are continuous from $\mathbb{R} \times E \to E$ and verify the inequality

$$\chi(f(t, \Omega)) \leq k\chi(\Omega), \ \chi(g(t, \Omega, \varepsilon)) \leq k\chi(\Omega),$$

where $\chi$ is the Hausdorff measure of noncompactness in the space $E$, $k \geq 0$ and $q = k/\gamma < 1$. The operator $f$ is differentiable in the second variable and the operators $f'(\cdot, A^{-\alpha} \cdot)$ and $g(\cdot, A^{-\alpha} \cdot, \cdot)$ are continuous from $\mathbb{R} \times E$ to $E$ and they satisfy a Lipschitz condition in the second variable uniformly with respect to the others.

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1 We recall (see [1]) that for a bounded set $\Omega \subset E$ the Hausdorff measure of noncompactness is defined by the formula

$$\chi(\Omega) = \inf \{r > 0 : \text{there exists } (y_1, ..., y_m) \text{ such that } \Omega \subset \cup_{i=1}^m B(y_i, r)\},$$

where $m \in \mathbb{N}$.

The continuous operator $F : E \to E$ is called $(q, \chi)$-condensing if

$$\chi(F(t, \Omega)) \leq q\chi(\Omega)$$

for any bounded $\Omega \in E$.  

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It is a classical result (see e.g. [11]) that (C1) and (C2) ensures respectively that the integral equations

$$x(t) = e^{At} \xi + \int_0^t A^\alpha e^{A(t-s)} \left[ f(s, A^{-\alpha} x(s)) + \varepsilon g(s, A^{-\alpha} x(s), \varepsilon) \right] ds,$$

(3.1)

$$x(t) = e^{At} \xi + \int_0^t A^\alpha e^{A(t-s)} \left[ f(s, x(s)) + \varepsilon g(s, x(s), \varepsilon) \right] ds$$

(3.2)

have a unique solution $x(\cdot)$ defined on some interval $[0, d], d > 0$. By means of this function $x$ we can define the shift operator as follows.

**Definition 3.1** Let $x : [0, d] \times E \times [0, 1] \to E$ be defined at $(t, \xi, \varepsilon)$ as $x(t, \xi, \varepsilon) = x(t)$ for all $t \in [0, d]$. If for some $\xi \in E$ and $\varepsilon \in [0, 1]$ we have that $x(\cdot, \xi, \varepsilon)$ is defined on the whole time interval $[0, T]$ then for these values $\xi$ and $\varepsilon$ we define the Poincaré map for system (1.1) as

$$\mathcal{P}_x(\xi) = x(T, \xi, \varepsilon).$$

A crucial role in what follows is played by the following technical lemma.

**Lemma 3.1** Assume that either (C1) or (C2) is satisfied. Assume that for some $\xi_0 \in E$ the shift operator $(t, \xi, \varepsilon) \to x(t, \xi, \varepsilon)$ is well defined for $t = T, \xi = \xi_0$ and $\varepsilon = 0$. Then there exists $r > 0$ such that this operator is well defined for $t = T$, any $\xi \in B_E(\xi_0, r)$, any $\varepsilon \in [0, r]$ and the function

$$u(t, \xi, \varepsilon) = \frac{x(t, \xi, \varepsilon) - x(t, \xi_0, 0)}{\varepsilon}$$

is Lipschitz in the second variable uniformly in $[0, T] \times B_E(\xi_0, r) \times (0, r]$, namely there exists $L > 0$ such that

$$\|u(t, \xi_1, \varepsilon) - u(t, \xi_2, \varepsilon)\| \leq L \|\xi_1 - \xi_2\|$$

for any $t \in [0, T], \xi_1, \xi_2 \in B_E(\xi_0, r)$ and $\varepsilon \in (0, r]$.

**Proof.** The fact that the assumptions of the Lemma imply the existence of $r > 0$ such that the operator $(t, \xi, \varepsilon) \mapsto x(t, \xi, \varepsilon)$ is well defined, bounded and continuous on $[0, T] \times B_E(\xi_0, r) \times [0, r]$ is well known, see, for instance, ([9], Theorem 5.2.5). In the sequel we have $\alpha \neq 0$ if (C1) holds, while $\alpha = 0$ if we assume (C2). Since $A^{-\alpha}$ is either a compact operator or the identity then the operator $(t, \xi, \varepsilon) \mapsto A^{-\alpha} x(t, \xi, \varepsilon)$ is well defined, bounded and continuous on $[0, T] \times B_E(\xi_0, r) \times [0, r]$. Therefore, taking into account that $f_x'$ satisfies Lipschitz condition, there exists $\tilde{M} > 0$ such that

$$\|f_x'(s, A^{-\alpha} \{\theta x(s, \xi_1, \varepsilon) + (1 - \theta) x(s, \xi_2, \varepsilon)\})\| \leq \tilde{M}$$

for any $s \in [0, T], \theta \in [0, 1], \xi_1, \xi_2 \in B_E(\xi_0, r)$. 

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From the continuous differentiability of \( f \) and the Lipschitz condition on \( g \) assumed in (C1) and (C2) we deduce the existence of \( M > 0 \) such that

\[
\|f(t, A^{-\alpha}\xi)\| + \|g(t, A^{-\alpha}\xi, \varepsilon)\| \leq M
\]

for any \( t \in [0, T] \), \( \xi \in x([0, T], B_{E}(\xi_{0}, r), [0, r]) \) and \( \varepsilon \in [0, r] \).

Therefore, \( A^{-\alpha}x([0, T], B_{E}(\xi_{0}, r), [0, r]) \) is bounded and by using the Lipschitz condition on \( g \) we obtain the existence of \( L > 0 \) such that

\[
\|g(s, A^{-\alpha}\xi_{1}, \varepsilon) - g(s, A^{-\alpha}\xi_{2}, \varepsilon)\| \leq L\|\xi_{1} - \xi_{2}\|
\]

for any \( s \in [0, T] \), \( \xi_{1}, \xi_{2} \in x([0, T], B_{E}(\xi_{0}, r), [0, r]) \) and \( \varepsilon \in [0, r] \).

Furthermore, by [13, Theorem 6.13] there exists \( c > 0 \) such that \( \sup_{t \in [0, T]} \|e^{At}\| < c \) and \( \|A^{\alpha}e^{At}\| < c/t^\alpha \), where either \( \alpha = 0 \) or \( \alpha > 0 \).

Now given an arbitrary \( \phi \in B_{E^*}(0, 1) \), where \( E^* \) denotes the dual space of \( E \), we evaluate \( \langle \phi, x(t, \xi_{1}, \varepsilon) - x(t, \xi_{2}, \varepsilon) \rangle \) as follows

\[
\langle \phi, x(t, \xi_{1}, \varepsilon) - x(t, \xi_{2}, \varepsilon) \rangle = \\
= \langle \phi, e^{At}(\xi_{1} - \xi_{2}) \rangle + \int_{0}^{t} \langle \phi, A^{\alpha}e^{A(t-s)}f_{x}^{\alpha}(s, A^{-\alpha}\theta(s, \xi_{1}, \xi_{2}, \varepsilon)x(s, \xi_{1}, \varepsilon) + \end{array}

\begin{array}{l}
+(1 - \theta(s, \xi_{1}, \xi_{2}, \varepsilon)x(s, \xi_{2}, \varepsilon)) \rangle A^{-\alpha}(x(s, \xi_{1}, \varepsilon) - x(s, \xi_{2}, \varepsilon)) \rangle ds + \\
+\varepsilon \int_{0}^{t} \langle \phi, A^{\alpha}e^{A(t-s)} \{ g(s, A^{-\alpha}x(s, \xi_{2}, \varepsilon), \varepsilon) - g(s, A^{-\alpha}x(s, \xi_{1}, \varepsilon), \varepsilon) \} \rangle ds \leq \\
\leq c\|\xi_{1} - \xi_{2}\| + \int_{0}^{t} \frac{cM}{(t-s)^{\alpha}} \|x(s, \xi_{1}, \varepsilon) - x(s, \xi_{2}, \varepsilon)\| ds + \\
+\varepsilon \int_{0}^{t} \frac{cL}{(t-s)^{\alpha}} \|x(s, \xi_{1}, \varepsilon) - x(s, \xi_{2}, \varepsilon)\| ds.
\end{array}
\]

(3.3)

Since \( \phi \) is arbitrary we have

\[
\|x(t, \xi_{1}, \varepsilon) - x(t, \xi_{2}, \varepsilon)\| \leq c\|\xi_{1} - \xi_{2}\| + \int_{0}^{t} \frac{cM}{(t-s)^{\alpha}} \|x(s, \xi_{1}, \varepsilon) - x(s, \xi_{2}, \varepsilon)\| ds + \\
+\varepsilon \int_{0}^{t} \frac{cL}{(t-s)^{\alpha}} \|x(s, \xi_{1}, \varepsilon) - x(s, \xi_{2}, \varepsilon)\| ds.
\]

(3.4)
Dividing the last inequality by $\|\xi_1 - \xi_2\|$ one obtains that
\[
\frac{\|x(t, \xi_1, \epsilon) - x(t, \xi_2, \epsilon)\|}{\|\xi_1 - \xi_2\|} \leq c + \int_0^t c\bar{M} + c\bar{L} \frac{\|x(s, \xi_1, \epsilon) - x(s, \xi_2, \epsilon)\|}{\|\xi_1 - \xi_2\|} ds.
\]

Using the generalized Gronwall–Bellman lemma, see ([7], Lemma 7.1.1), from the last inequality we obtain that there exists $M_v > 0$ such that
\[
\frac{\|x(t, \xi_1, \epsilon) - x(t, \xi_2, \epsilon)\|}{\|\xi_1 - \xi_2\|} \leq M_v
\]
for any $(t, \xi_1, \xi_2, \epsilon) \in [0, T] \times B_E(\xi_0, r) \times [0, r]$.

For the function $u(t, \xi, \epsilon)$ we have the following inequality
\[
\langle \phi, u(t, \xi, \epsilon) \rangle = \langle \phi, \frac{1}{\epsilon} \int_0^t A^s e^{A(t-s)} \left[ f(s, A^{-\alpha} x(s, \xi, \epsilon)) - f(s, A^{-\alpha} x(s, \xi, 0)) \right] ds + \int_0^t A^s e^{A(t-s)} g(s, A^{-\alpha} x(s, \xi, \epsilon)) ds \rangle \leq \int_0^t \frac{c\bar{M}}{(t-s)^{\alpha}} \|u(s, \xi, \epsilon)\| ds + \int_0^t \frac{c\bar{M}}{(t-s)^{\alpha}} ds.
\]

Using again the generalized Gronwall–Bellman lemma from the last inequality we obtain that there exists $M_u > 0$ such that
\[
\|u(t, \xi, \epsilon)\| \leq M_u \quad \text{for any } (t, \xi, \epsilon) \in [0, T] \times B_E(\xi_0, r) \times [0, r]. \tag{3.6}
\]

Observe that if $\Psi$ function $\Psi : E \rightarrow E$ is differentiable and there exists $L > 0$ such that $\|\Psi'(\xi) - \Psi'(\zeta)\| \leq L \|\xi - \zeta\|$ for any $\xi, \zeta \in E$ then
\[
\|\Psi(\xi_2) - \Psi(\xi_1) - \Psi(\xi_2 + \theta(\xi_2 - \xi_1))\| \leq \sup_{0 \leq \theta \leq 1} \|\Psi'(\xi_2 + \theta(\xi_2 - \xi_1))\| \|\xi_2 - \xi_1 - \xi_2 + \xi_1\| + L \max\{\|\xi_2 - \xi_1\|, \|\xi_2 - \xi_1\|\} \|\xi_1 - \xi_1\|. \tag{3.7}
\]

To prove this it is sufficient to consider the real function $\gamma : [0, 1] \rightarrow \mathbb{R}$ given by
\[
\gamma(\tau) = \langle \phi, \Psi(\xi_2 + \tau(\xi_2 - \xi_1)) - \Psi(\xi_1 + \tau(\xi_1 - \xi_1)) \rangle, \quad \tau \in [0, 1].
\]

By Lagrange theorem there exists $\theta \in [0, 1]$ such that
\[
\gamma(1) - \gamma(0) = \gamma'(\theta)
\]
and then
\[
| \langle \phi, \Psi(\xi_2) - \Psi(\xi_1) - \Psi(\xi_2) + \Psi(\xi_1) \rangle | = \gamma(1) - \gamma(0) = \gamma'(\theta) =
\]

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Consider now for any $\xi_1, \xi_2 \in \mathbb{R}^n$, $t, \xi \in \mathbb{R}^n$.

By the Lipschitz assumption on $f'$ there exists $\tilde{L} > 0$ such that

$$
\|f'_x(s, A^{-1}\xi_1) - f'_x(s, A^{-1}\xi_2)\| \leq \tilde{L} \|\xi_1 - \xi_2\|
$$

for any $s \in [0, T], \xi_1, \xi_2 \in x([0, T], B_E(x_0, r), [0, r])$.

Consider now

$$
\frac{u(t, \xi_1, \varepsilon) - u(t, \xi_2, \varepsilon)}{\|\xi_1 - \xi_2\|} = \frac{x(t, \xi_1, \varepsilon) - x(t, \xi_1, 0) - x(t, \xi_2, \varepsilon) + x(t, \xi_2, 0)}{\varepsilon \|\xi_1 - \xi_2\|} = \frac{1}{\varepsilon \|\xi_1 - \xi_2\|} \int_0^t A^\alpha e^{A(t-s)} (f(s, A^{-\alpha}x(s, \xi_1, \varepsilon)) - f(s, A^{-\alpha}x(s, \xi_1, 0)) - f(s, A^{-\alpha}x(s, \xi_2, \varepsilon)) + f(s, A^{-\alpha}x(s, \xi_2, 0)))ds +
$$

$$
+ \frac{1}{\|\xi_1 - \xi_2\|} \int_0^t A^\alpha e^{A(t-s)} (g(s, A^{-\alpha}x(s, \xi_1, \varepsilon)) - g(s, A^{-\alpha}x(s, \xi_2, \varepsilon)))ds \leq \sup_{s \in [0, T], \theta \in [0, 1]} \|f'_x(s, A^{-\alpha}(x(s, \xi_2, 0) + \theta(x(s, \xi_1, 0) - x(s, \xi_2, 0))))A^{-\alpha}\|
$$

$$
\leq \int_0^t \frac{c}{(t-s)^{\alpha}} \|x(s, \xi_1, \varepsilon) - x(s, \xi_2, \varepsilon)\| ds + \sup_{s \in [0, T]} \max_{\varepsilon} \left\{ \frac{\|x(s, \xi_1, 0) - x(s, \xi_1, \varepsilon)\|}{\varepsilon \|\xi_1 - \xi_2\|} \right\} \cdot \frac{cL}{(t-s)^{\alpha}} \|x(s, \xi_1, \varepsilon) - x(s, \xi_2, \varepsilon)\| ds +
$$

$$
+ \int_0^t \frac{cL}{(t-s)^{\alpha}} \|x(s, \xi_1, \varepsilon) - x(s, \xi_2, \varepsilon)\| ds.
$$
By (3.6) and (3.5) there exists $M > 0$ such that the last inequality can be rewritten as

$$\frac{u(t, \xi_1, \varepsilon) - u(t, \xi_2, \varepsilon)}{\|\xi_1 - \xi_2\|} \leq \int_0^t \frac{c \hat{M}}{(t-s)^\alpha} \frac{\|u(s, \xi_1, \varepsilon) - u(s, \xi_2, \varepsilon)\|}{\|\xi_1 - \xi_2\|} ds + M$$

and the assertion follows from the generalized Gronwall–Bellman lemma, see ([7], Lemma 7.1.1).

4 Existence of periodic solutions

In this section we assume that either (C1) or (C2) is satisfied, moreover we assume the following condition:

$$(\tilde{A}_0)$$ the solution $x$ of (1.1) with $\varepsilon = 0$ satisfying $x(0) = \xi_0$ is defined on $[0, T]$, namely the Poincaré map $P_0$ is defined at $\xi_0$.

Therefore, from Lemma 3.1 we have that there exists $r > 0$ such that the Poincaré map $P_{\varepsilon}$ for system (1.1) is defined on $B_{E}(\xi_0, r)$ for any $\varepsilon \in [0, r]$ and it has the form

$$P_{\varepsilon}(\xi) = P_0(\xi) + \varepsilon Q(\xi, \varepsilon),$$

where $P_0$ is differentiable and $Q$ satisfies a Lipschitz condition in the first variable $\xi$ uniformly on $B_{E}(\xi_0, r) \times [0, r]$.

Letting $F(\xi, \varepsilon) = P_{\varepsilon}(\xi)$ assumptions (A1)-(A4) of Theorem 2.1 can be rewritten as

$$(\tilde{A}_1)$$ there exists a function $S \in C^1(V, E)$ defined on some open neighborhood $V \subset \mathbb{R}^k$ of $h_0$ such that $S(h_0) = \xi_0$ and $P_0(\xi) = \xi$ for any $\xi \in \mathbb{Z} = \bigcup_{h \in V} S(h),$

$$(\tilde{A}_2)$$ dim$S'(h_0)\mathbb{R}^k = k.$

Let $E_{1,h} = S'(h)\mathbb{R}^k$ and let $E_{2,h}$ be any subspace of $E$ such that $E = E_{1,h} \bigoplus E_{2,h}$ and

$$(\tilde{A}_3)$$ both the projectors $\pi_{1,h}$ of $E$ onto $E_{1,h}$ along $E_{2,h}$ and $\pi_{2,h}$ of $E$ onto $E_{2,h}$ along $E_{1,h}$ are continuous in $h \in V,$

$$(\tilde{A}_4)$$ for $\xi_0 = S(h_0)$ we have

$$\pi_{2,h_0}((P_0)'(\xi_0) - I)\pi_{2,h_0}$$

is invertible on $E_{2,h_0}.$

(4.1)

Furthermore, it can be observed that $Q(\xi, 0)$ is the value of the solution of the problem

$$\dot{y} = Ay + f'_x(t, x(t, \xi, 0))y + g(t, x(t, \xi, 0), 0),$$

$$y(0) = 0$$

at time $t = T.$

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To see this, observe that the function $u$ of Lemma 3.1 satisfies the following integral equation
\[ u(t, \xi, \varepsilon) = \int_0^t A^\alpha e^{\Lambda(t-s)} f(s, A^{-\alpha} x(s, \xi, 0)) u(s, \xi, \varepsilon) ds + \int_0^t A^\alpha e^{\Lambda(t-s)} \frac{\partial( x(s, \xi, \varepsilon) - x(s, \xi, 0))}{\varepsilon} ds + \int_0^t A^\alpha e^{\Lambda(t-s)} g(s, A^{-\alpha} x(s, \xi, \varepsilon), \varepsilon) ds \]
and so $u(T, \xi, 0) = Q(\xi, 0)$. Therefore, we can give an equivalent definition of the bifurcation function $M$ introduced in Section 2, that is $M \in C^0(\mathbb{R}^k, \mathbb{R}^k)$ can be defined as follows
\[ M(h) = (\pi_{1,h})^{-1} \pi_{1,h} \eta(S(h_0)) - \pi_{2,h} ((\pi_{1,h})^{-1} \pi_{2,h} \eta(S(h))) \]
where $h \in B_{\mathbb{R}^k}(h_0, r)$, and $\eta$ is the value of the solution of (4.2) at time $t = T$.

From Theorem 2.2 we have the following necessary condition for the existence of $T$-periodic solutions to (1.1).

**Theorem 4.1** Assume that (C1) or (C2) is satisfied. Assume $(\tilde{A}_0)$-$(\tilde{A}_4)$. Assume that there exists a sequence $\varepsilon_n \to 0$ as $n \to \infty$ and a sequence of $T$-periodic functions $x_n \in C^0([0, T], E)$, $x_n \to x(\cdot, \xi_0, 0)$ as $n \to \infty$ such that $(x_n, \varepsilon_n)$ solves (1.1). Then
\[ M(h_0) = 0. \]

Analogously from Theorem 2.3 we derive the following sufficient condition for the existence of $T$-periodic solutions to (1.1).

**Theorem 4.2** Assume that (C1) or (C2) is satisfied. Assume $(\tilde{A}_0)$-$(\tilde{A}_4)$ and that $h_0$ is an isolated zero of $M$ with
\[ \text{ind}(h_0, M) \neq 0. \]
Then, for any $\varepsilon > 0$ sufficiently small, system (1.1) has a $T$-periodic solution $x_\varepsilon \in C^0([0, T], E)$ and
\[ x_\varepsilon(0) \to \xi_0 \quad \text{as} \quad \varepsilon \to 0. \]

5 **Appendix**

**Proof of Lemma 2.1.** Let $\Phi_{h,\varepsilon} : E \to E$ be defined by
\[ \Phi_{h,\varepsilon}(\xi) = \Phi_{h,\varepsilon}(\pi_h \xi) + (I - \pi_h) \xi. \] (5.1)
Observe, that if there exists \( r > 0, M > 0 \) and \( \xi : \mathbb{R}^k \times [0, r] \to E \) satisfying
\[
\xi(\cdot, \varepsilon) \in C^0(V, E), \quad \xi(h, \varepsilon) \to \xi(h, 0) \text{ as } \varepsilon \to 0 \text{ uniformly in } h \in V,
\] (5.2)
such that
b') \( \overline{\Phi}_{h, \varepsilon}(\xi(h, \varepsilon)) = 0 \) for any \( h \in V, \varepsilon \in [0, r] \),
c') \( \xi(h, \varepsilon) \) is the only zero of \( \overline{\Phi}_{h, \varepsilon} \) in \( B_E(0, r) \),
d') \( \|\xi(h, \varepsilon)\| \leq M \varepsilon \) for any \( h \in V, \varepsilon \in [0, r] \),
then \( \beta(h, \varepsilon) = \pi_h \xi(h, \varepsilon) \) satisfies a), b), c) and d).

To prove this assertion? from assumption 4 we have
\[
\overline{\Phi}_{h, 0}(0) = \Phi_{h, 0}(0) = \overline{P}(h, 0) = 0.
\]

For the derivative \( (\overline{\Phi}_{h, 0})'(\cdot) \) taking into account that \( \overline{P}(h, \cdot) \) acts on \( E_h \) we have
\[
(\overline{\Phi}_{h, 0})'(0) = \pi_h \overline{P}_0(h, 0) \pi_h + (I - \pi_h).
\]
Let us show that \( (\overline{\Phi}_{h, 0})'(0) \) is invertible on \( E \) for \( h \in V \); to do this we show that given \( b \in E \) there exists a unique \( a_b \in E \) such that
\[
(\overline{\Phi}_{h, 0})'(0)a_b = b. \quad (5.3)
\]
Indeed, applying \( I - \pi_h \) to (5.3) we have \( (I - \pi_h)a_b = (I - \pi_h)b \). On the other hand, by assumption 5 \( \pi_h \overline{P}_0(h, 0) \) is invertible and thus applying \( \left( \pi_h \overline{P}_0(h, 0) \right)^{-1} \pi_h \) to (5.3) we obtain \( \pi_h a_b = \left( \pi_h \overline{P}_0(h, 0) \right)^{-1} \pi_h b \). Therefore the unique solution \( a_b \) of (5.3) is given by \( a_b = \left( \pi_h \overline{P}_0(h, 0) \right)^{-1} \pi_h b + (1 - \pi_h)b \). This means that \( \left( (\overline{\Phi}_{h, 0})'(0) \right)^{-1} \pi_h \) is continuous in \( h \). Now, introducing \( \overline{P}(h, \xi) = \overline{P}(h, \pi_h \xi) + (I - \pi_h)\xi \) we have that
\[
1') \quad \overline{\Phi}_{h, \varepsilon}(\xi) = \overline{P}(h, \xi) + \varepsilon \tilde{Q}(h, \xi, \varepsilon),
2') \quad \overline{P}(h, 0) = 0,
3') \quad \overline{P}_0(h, 0) \text{ is invertible and } \left( \overline{P}_0(h, 0) \right)^{-1} \text{ is continuous in } h.
\]

Let \( \tilde{\Phi}_{h, \varepsilon}(\xi) = \left( \overline{P}_0(h, 0) \right)^{-1} \overline{\Phi}_{h, \varepsilon}(\xi) \). Since \( \overline{\Phi}_{h, \varepsilon}(\xi) = 0 \) if and only if \( \tilde{\Phi}_{h, \varepsilon}(\xi) = 0 \) we aim now at finding a solution \( \xi(h, \varepsilon) \) to \( \tilde{\Phi}_{h, \varepsilon}(\xi) = 0 \) satisfying properties b'), c') and d'). By assumption 2 for any \( h \in V \) there exists \( r(h) > 0 \) such that
\[
\|I - (\tilde{\Phi}_{h, 0})'(\xi)\| \leq 1/4
\]
for any \( \|\xi\| \leq r(h) \) and any \( \tilde{h} \in B_{R^k}(h, r(h)) \cap V \).
Since the family $\bigcup_{h \in V} B_{\alpha_k}(h, r(h))$ covers the set $V$ we can extract from it a finite subfamily covering $V$. This implies the existence of $r > 0$ such that
\[
\|I - (\hat{\Phi}_h, 0)\| \leq 1/4
\]
for any $\|\xi\| \leq r$ and any $h \in V$.

By assumption 3 there is $L > 0$ such that $\| (\bar{\mathcal{T}}_\xi(h, 0))^{-1} (\varepsilon \tilde{Q}(h, \xi_1, \varepsilon) - \varepsilon \tilde{Q}(h, \xi_2, \varepsilon)) \| \leq \varepsilon L$ for any $h \in V$, $\xi_1, \xi_2 \in B_E(0, 1)$, $\varepsilon \in [0, 1]$.

Therefore, $r > 0$ can be considered sufficiently small to have
\[
\|\xi_1 - \hat{\Phi}_{h, \varepsilon}(\xi_1) - \xi_2 + \hat{\Phi}_{h, \varepsilon}(\xi_2)\| \leq (1/2)\|\xi_1 - \xi_2\| \quad (5.4)
\]
for any $h \in V$, $\varepsilon \in [0, r]$, $\|\xi_1\| \leq r$, $\|\xi_2\| \leq r$. Therefore, there exists $\xi : V \times [0, r] \to E$ satisfying $b')$ and $c')$. It remains to show that $\xi$ satisfies also (5.2) and $d')$. Indeed, by using $b')$ and (5.4) for any $h_1, h_2 \in V$ and $\varepsilon \in [0, r]$ we have
\[
\|\xi(h_1, \varepsilon_1) - \xi(h_2, \varepsilon_2)\| \leq \\
\leq \|\xi(h_1, \varepsilon_1) - \hat{\Phi}_{h_2, \varepsilon_2}(\xi(h_1, \varepsilon_1)) - \xi(h_2, \varepsilon_2) + \hat{\Phi}_{h_2, \varepsilon_2}(\xi(h_2, \varepsilon_2))\| + \\
+ \|\hat{\Phi}_{h_2, \varepsilon_2}(\xi(h_1, \varepsilon_1)) - \hat{\Phi}_{h_1, \varepsilon_2}(\xi(h_1, \varepsilon_1))\| + \\
+ \|\hat{\Phi}_{h_1, \varepsilon_2}(\xi(h_1, \varepsilon_1)) - \hat{\Phi}_{h_1, \varepsilon_1}(\xi(h_1, \varepsilon_1))\| \leq \\
\leq (1/2)\|\xi(h_1, \varepsilon_1) - \xi(h_2, \varepsilon_2)\| + \|\hat{\Phi}_{h_2, \varepsilon_2}(\xi(h_1, \varepsilon_1)) - \hat{\Phi}_{h_1, \varepsilon_2}(\xi(h_1, \varepsilon_1))\| + \\
+ |\varepsilon_1 - \varepsilon_2| \left\| (\bar{\mathcal{T}}_\xi(h_1, 0))^{-1} (\tilde{Q}(h_1, \xi(h_1, \varepsilon_1), \varepsilon_2) - \tilde{Q}(h_1, \xi(h_1, \varepsilon_1), \varepsilon_1)) \right\|.
\]

Finally the continuity assumptions 1, 2 and 3 imply that $\xi$ satisfies (5.2) and $d')$. \qed

**References**


