Dispersive schemes for the critical Korteweg de Vries equation

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Abstract

In this paper we study semi-discrete finite difference schemes for the critical Korteweg de Vries equation. We prove that the solutions of the equation discretized through a two grid algorithm satisfy dispersive estimates uniformly with respect to the discretization parameter. This implies convergence in a weak sense of the discrete solutions to the solution of the Cauchy problem even for rough $L^2(\mathbb{R})$ initial data. We also prove a scattering result for the discrete equation, and show that the discrete scattering function converges to the continuous one. Finally rates of convergence are studied for initial data that belong to $H^s(\mathbb{R})$, $s > 0$; we do not obtain rates for the approximations of the cKdV equation, but such results are proved for a simpler semi-linear equation. Our method relies essentially on the discrete Fourier transform and standard harmonic analysis on the real line.

Introduction

The well-posedness of the generalized Korteweg de Vries equation

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x u^5/5 = 0, \ (x,t) \in \mathbb{R} \times [0,T], \\ u|_{t=0} = u_0, \ x \in \mathbb{R}, \end{cases} \tag{cKdV}$$

for $L^2(\mathbb{R})$ initial data was obtained by Kenig Ponce and Vega [12] in the beginning of the 90's. Scattering for small initial data (meaning that solutions behave asymptotically as solutions of the linear solutions) was also proved as a direct consequence of the existence theorem. These results were optimal in several ways: any other gKdV equation (nonlinearity of power $p \neq 5$) admits traveling wave solutions of L^2 norm arbitrarily small, and L^2 is the critical space for the scaling of the equation. Actually existence of solutions for the KdV equation was even obtained in negative regularity Sobolev spaces [11], but our paper will only be focused on the cKdV equation. The results aforementionned rely on the existence of several sharp *dispersive estimates*. Namely for $V(t)$ the semigroup associated to the skew symmetric operator ∂_x^3, Kenig Ponce and Vega proved

$$\|\partial_x V(t) u_0\|_{L^\infty(\mathbb{R}_x; L^2(\mathbb{R}_t)} \lesssim \|u_0\|_{L^2(\mathbb{R}_x)}, \tag{D1}$$

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$$\|V(t)u_0\|_{L^5(\mathbb{R}_x;L^{10}(\mathbb{R}_t))} \lesssim \|u_0\|_{L^2(\mathbb{R}_x)}. \tag{D2}$$

If one is not allowed to use dispersive estimates, well-posedness may only be proved by semigroup technics in $H^s,\ s > 3/2$, as was done by Kato [8] in 1979. It is thus essential for numerical dispersive schemes that they mimick correctly the fine properties of dispersive equations if one wishes to compute solutions for rough initial data. Though travelling waves are problably the most well known solutions of gKdV equations and are very smooth, the approximation of very rough solutions can be important in some contexts. For example stochastic versions of the KdV equation appear in modellisation of plasma fluids, it has been studied theoretically [16] and numerically [2, 3] by Debussche and Printems. The typical space of existence for the solutions is L^2.

As was pointed out by Ignat and Zuazua [7] in their work on the nonlinear Schrödinger equation, numerical schemes usually fail to reproduce dispersive properties of solutions. The deep reason is that the "symbol" of the discretized differentiation operators do not behave in the same way as the continuous one: in particular they display a lack of convexity (cancellation of the second order derivative of the symbol) and lack of "slope" (cancellation of the first derivative), which are obstructions to the proof of (D1, D2). To clarify the importance of this let us outline the proof of the estimate (D1) as is done in [10]. By Fourier transform, we have

$$\partial_x u = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}e^{i\xi^3 t}\widehat{u_0}(\xi)d\xi = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\eta^{1/3}}e^{i\eta t}\widehat{u_0}(\eta^{1/3})\frac{d\eta}{3\eta^{1/3}}$$

now using Parseval's identity in the time variable and reversing the change of variables

$$\|\partial_x u(x,\cdot)\|_{L^2_t}^2 = \frac{1}{18\pi}\int_{\mathbb{R}} |\widehat{u_0}|^2\frac{d\eta}{\eta^{2/3}} = \frac{1}{18\pi}\int_{\mathbb{R}} |\widehat{u_0}|^2 3\xi^2\frac{d\xi}{\xi^2} = \frac{1}{3}\|u_0\|_{L^2}^2.$$

Looking carefully at the method of proof, it appear that a crucial point is $|(\xi^3)'| \gtrsim \xi^2$, or naively put *the first derivative of the symbol is large for ξ large.* This is certainly not true for a discrete differentiation operator for its symbol (obtained by discrete Fourier transform) is a periodic function in ξ. Similarly it can be seen (though much less directly) that the gain of integrability (D2) relies on the uniform *curvature* of the symbol, another feature not satisfied for discrete schemes, since the symbol can not remain convex nor concave.

At this point, it is worth mentionning that this kind of difficulty has been overcome by Nixon [14] in the case of fully implicit schemes. Indeed for implicit schemes the numerical dissipation is particularly strong at the frequencies where the symbol of the discretized operator fails to behave like the continuous one. Our aim here is to deal rather with non dissipative schemes - discrete in space and continuous in time- for which dispersive estimates are inherent to the space discretization. We also point out that no convergence of the scheme nor study of scattering was provided in Nixon's work.

Our approach follows the one developped by Ignat and Zuazua for the Schrödinger equation [7], where they used a "two grid filtering" in order to eliminate the bad frequencies. The technics rely on basic harmonic and functional analysis, and we tried to keep a reasonnable balance between self-containedness and length of calculus : when proofs are slight modifications of already existing arguments we often point to a reference instead of repeating them, and if several quantities are bounded by the same method the precise argument is detailed only once. Our core result is that there exists $N \in \mathbb{N}$ and an interpolation operator $\Pi : \ l^2(Nh\mathbb{Z}) \to l^2(h\mathbb{Z})$ (actually constructed for $N = 6$) such that the semi-discrete finite difference problem

$$\begin{cases} \partial_t u_n + \partial_h^3 u_n + (\partial_h \Pi E u)_n^5/5 = 0, \ (n,t) \in \mathbb{Z} \times \mathbb{R} \\ u|_{t=0} = \Pi u_{0,h}, \end{cases}$$

is globally well-posed for $\varphi \in l^2(Nh\mathbb{Z})$ sufficiently small, and the solution satisfies analogous dispersive estimates analogous to (D1, D2) *uniformly in h*. The notations E, Π, ∂_h and ∂_h^3 are defined in the first and second sections. The convergence of the discrete solution to the continuous one is proved by standard weak convergence/compactness arguments. Of course no rate of convergence in h can be obtained if we work at the critical level of regularity $u_0 \in L^2$, but we expect that rates of convergence in $h^{\alpha s}$ should be true for $u_0 \in H^s$ (most likely for $\alpha = 2/5$). Though we did not manage to obtain such results for the quasi-linear critical KdV equation, we investigate this question for a simpler (semi-linear) equation in the last section of the article, and give detailed argument that should apply to the cKdV equation, provided some technical obstructions were overcome.
Finally, let us remind that scattering for the continuous problem means that there exists a function $w \in L_x^2$ such that $\lim_{t \to +\infty} \|u(t) - V(t)w\|_{L^2} \to 0$, where $V(t)$ is the group generated by $-\partial_x^3$. In other words the solution of the quasilinear problem behaves asymptotically as a solution of the linear one. We prove here a discrete version of this result, that is

$$\exists\, w_h \in l^2(h\mathbb{Z}) : \ \lim_t \|u_h(t) - V_h(t)w_h\|_{l^2} \to 0,$$

where $V_h(t)$ is generated by $-\partial_h^3$. We prove also that w_h converges to w in a weak sense.
The paper is organized as follows:

- In section 1 we set up notations.
- Section 2 and 3 are focused on the derivation of *linear* dispersive estimates for the semi-discrete problem, more precisely local smoothing is obtained in section 2 while global gain of integrability is tackled in section 3.
- The global well-posedness of the (semi-)discrete problem for small initial data is proved in section 4.
- We prove in section 5 that, provided the discretized initial data converge to u_0, then the solution of the discrete problem converges strongly in $L^2_{loc}(\mathbb{R}_x \times \mathbb{R}_t)$ (and weakly in stronger spaces) to the solution of the Cauchy problem.
- The study of scattering is done in section 6.

- Finally, section 7 deals with rates of convergence when the initial data is more regular than L^2. Linear estimates with rates are obtained similarly to section 2 and 3. Althouth they are not sufficient for the cKdV equation, they are applied to a semi-linear problem for which well-posedness is preliminary justified.
- The appendix describes discrete versions of some well-known results of Fourier and harmonic analysis, and contains a dispersive estimate with rates in the spirit of section 7 that we did not manage to use, but that may be useful in further studies.

1 Notations and basic properties

The set of sequences defined on $h\mathbb{Z}$ is denoted $\mathcal{S}(h\mathbb{Z})$. As usual, the Lebesgue spaces on the real line are denoted $L^p(\mathbb{R})$ while the discrete analogous spaces are $l^p(h\mathbb{Z})$, or in a shortened notation l^p_h, or l^p when there is no ambiguity. Their norms are respectively denoted $\|\cdot\|_{L^p}$, $\|\cdot\|_{l^p}$. The norm of a sequence $u\in l^p(h\mathbb{Z})$ is defined as

$$\|u\|_{l^p}^p = h\sum |u_j|^p.$$

Similarly a sequence of $\mathcal{S}(h\mathbb{Z})$ will be indexed as u_{jh} or u_j indifferently. We write $\mathbb{R}_t$, $\mathbb{R}_x$ in order to avoid confusion between time and space, and write for conciseness $L^p_xL^q_t := L^p(\mathbb{R}_x;L^q(\mathbb{R}_t))$, and similarly $l^pL^q_t$, $L^p_tL^q_x$, $L^p_tl^q$. When working on bounded time intervals, and if there is no risk of confusion we may also write L^p_T for $L^p([-T,T])$ or $L^p([0,T])$.
When we write for two quantities a, b depending on a number of parameters $a\asymp b$, we mean that there exists two constants $\alpha,\ \beta>0$ independant of the parameters such that

$$\alpha a\leq b\leq \beta a. \tag{1.1}$$

The notation $a\lesssim b$ means that there exists $C>0$ such that $a\leq Cb$, C independant of the parameters.
The Fourier transform is the application $u\mapsto \mathcal{F}(u)=\widehat{u}=\int_{\mathbb{R}}e^{-ix\xi}u(x)dx$, the discrete Fourier transform is the application

$$(u_n)\mapsto \widehat{u}=h\sum e^{-ij\xi}u_j.$$

Up to some multiplicative constant it is an isometry $l^2(h\mathbb{Z})\to L^2([-\pi/h,\pi/h])$ (this is the Parseval identity), and the sequence (u_j) can be deduced from its Fourier transform thanks to the Fourier inversion formula

$$u_j=\frac{1}{2\pi}\int_{-\pi/h}^{\pi/h}e^{ijh\xi}\widehat{u}(\xi)d\xi := (\mathcal{F}^{-1}\widehat{u})_j. \tag{1.2}$$

A Fourier multiplier $M(D)$ of symbol $m(\xi)$ acting on $l^2(h\mathbb{Z})$ is defined by

$$\widehat{Mu}(\xi)|_{[-\pi/h,\pi/h]}=m(\xi)\widehat{u}(\xi)_{[-\pi/h,\pi/h]}.$$

A similar definition stands for multiplier acting on $L^2(\mathbb{R})$: $\forall \xi \in \mathbb{R}$, $\widehat{Mu}(\xi) = m(\xi)\widehat{u}(\xi)$. In particular, we may define the fractional derivative operators

$$|D|^s u := \mathcal{F}^{-1}(|\xi|^s \widehat{u}), \tag{1.3}$$

and emphasize that this definition holds for sequences as well as for functions. It is clear, but important that Fourier multipliers commute.
This leads to the definition of discrete and continuous Sobolev spaces for $s \geq 0$

$$H^s(h\mathbb{Z}) = \{(u_n) \in l^2(h\mathbb{Z}) : \ \|u\|_{H^s} := \|(1+|D|^s)u\|_{l^2} < \infty\},$$
$$H^s(\mathbb{R}) = \{u \in L^2(\mathbb{R}) : \ \|u\|_{H^s} := \|(1+|D|^s)u\|_{L^2} < \infty\}.$$

Generically, discrete differentiation operators are Fourier multipliers. For example, the operator ∂_h defined by

$$(\partial_h u)_{jh} = \frac{u_{(j+1)h} - u_{jh}}{h},$$

has for symbol $(e^{ih\xi} - 1)/h$. In section 5, we will also use the action of ∂_h on functions, defined by $(\partial_h u)(x) = (u(x+h) - u(x))/h$. We use in all the paper the standard discretization of the third order derivative operator

$$(u_j) \to (\partial_h^3 u_h)_j := \frac{u_{h(j+2)} - 2u_{h(j+1)} + 2u_{h(j-1)} - u_{h(j-2)}}{2h^3}.$$

Note that (despite the notation) $\partial_h^3 \neq \partial_h \circ \partial_h \circ \partial_h$. This is a stable (semi-)discretization for the Airy equation $\partial_t + \partial_x^3 u = 0$ indeed it is easily seen that

$$\partial_t \widehat{u} - i\frac{4}{h^3}\sin^2(\xi h/2)\sin(\xi h)\widehat{u} := \partial_t \widehat{u} - ip_h(\xi)\widehat{u} = 0 \Rightarrow \widehat{u} = e^{ip_h(\xi)t}\widehat{u_0}, \tag{1.4}$$

implying the conservation of the l_h^2 norm by the Parseval equality. The first order derivative of p_h is

$$p_h' = 4\sin(\xi h/2)\sin(3\xi h/2)/h^2, \tag{1.5}$$

its second order derivative is

$$p_h'' = 2(2\sin(2\xi h) - \sin(\xi h))/h. \tag{1.6}$$

The group corresponding to $-\partial_h^3$ will be denoted $V_h(t) : u \to \mathcal{F}^{-1}(e^{itp_h}\widehat{u})$, we also remind that in the introduction we denoted $V(t)$ the semigroup corresponding to $-\partial_x^3$, it is the Fourier multiplier of symbol $e^{it\xi^3}$.

2 Discrete dispersive smoothing

The lack of smoothing effect for the operator $V_h(t)$ and arbitrary initial data could be proved as was done in [7, 6] for the Schrödinger equation. This is due to a lack of slope of the derivative of the symbol $4\sin(h\xi/2)\sin(3h\xi/2)/h^2$, which cancels at $\xi_\pm = \pm 2\pi/3h$.
The method that we will follow to tackle this issue consists in filtering the initial data and the nonlinearity in a way such that their spectrum is localized away from $\xi_\pm$. Namely if (u_n) is a sequence of $3h\mathbb{Z}$, we define Πu_j defined on $h\mathbb{Z}$ as follows

$$\begin{cases} \Pi u_{3j} = u_{3j}, \\ \Pi u_{3j+1} = 2/3 u_{3j} + 1/3 u_{3j+3}, \\ \Pi u_{3j+2} = 1/3 u_{3j} + 2/3 u_{3j+3}. \end{cases} \tag{2.1}$$

A word of caution on notations: The operator Π as defined is of course linked to h, though it is clear that their definition poorly depends on it. In order to lighten the notations we have chosen not to display this dependance. The careful reader may want to remember this, especially for the upcoming remark 3 where dependance on h *does* matter.

We have the formula

$$\widehat{\Pi u} = \frac{1}{9}(1+2\cos(h\xi))^2 \widehat{E_3 u}. \tag{2.2}$$

Indeed

$$\begin{aligned} \widehat{\Pi u} = h\sum \Pi u_j e^{-ijh\xi} &= 3h/3 \sum u_{3j} e^{-3ijh\xi}\big(1 + 2/3 e^{ih\xi} + 2/3 e^{-ih\xi} + 1/3 e^{2ih\xi} \\ &\quad + 1/3 e^{-2ih\xi}\big) \\ &= \frac{1}{3}\widehat{E_3 u}(\xi)\big(1 + 4/3\cos h\xi + 2/3\cos(2h\xi)\big) \\ &= \frac{1}{9}\widehat{E_3 u}(\xi)\big(1 + 4\cos h\xi + 4\cos^2 h\xi\big) \\ &= \frac{1}{9}\widehat{E_3 u}(\xi)(1+2\cos h\xi)^2. \end{aligned}$$

Note that it is clear that $\|\Pi E_3 u\|_{l^p(h\mathbb{Z})} \asymp \|E_3 u\|_{l^p(3h\mathbb{Z})} \leq 3^{1/p}\|u\|_{l^p(h\mathbb{Z})}$, this fact will be used during the rest of the article without further notice.

Proposition 1. *Let M be a Fourier multiplier on $l^2(h\mathbb{Z})$ whose symbol m satisfies*

$$\forall\, \xi \in [-\pi/h, \pi/h],\ |m(\xi)| \lesssim |\xi|.$$

Any $u \in l^2_{3h}$ satisfies the following dispersive smoothing:

$$\sup_j \int_{\mathbb{R}} |(MV_h(t)\Pi u)_j|^2 dt \lesssim \|u\|^2_{l^2_{3h}}, \tag{2.3}$$

and the inhomogeneous analogous stands for any $(g_{3j}(t))$

$$\| \int_0^t M^2 V(t-t')\Pi f(t')dt' \|_{l^\infty L^2} \lesssim \|g\|_{l^1_{3h} L^2_t} \tag{2.4}$$

Proof. By scaling, it is sufficient to prove (2.3) for $h=1$. We have

$$V(t)\Pi u_j = \int_{-\pi}^{\pi} e^{ix\xi} e^{itp(\xi)} \frac{(1+2\cos\xi)^2}{9} m(\xi)\widehat{u} d\xi = \int_{-\pi}^{\pi} e^{ix\xi} e^{4it\sin^2(\xi/2)\sin\xi} \frac{(1+2\cos\xi)^2}{9} m\widehat{\Pi u} d\xi$$

On $]-2\pi/3, 2\pi/3[$, $(\sin^2(\xi/2)\sin\xi)' = \sin(\xi/2)\sin(3\xi/2) > 0$, thus p is a diffeormorphism and we may use the change of variables $\eta = 4\sin^2(\xi/2)\sin\xi$. To lighten the notations we write $\xi = \xi(\eta)$ and $d\xi = f(\eta)d\eta$ and keep $\widehat{u}, m$ for $\widehat{u}(\xi(\eta))$, $m(\xi(\eta))$. This gives

$$\begin{aligned}\int_{-2\pi/3}^{2\pi/3} e^{ix\xi} e^{4it\sin^2(\xi/2)\sin\xi} \frac{(1+2\cos\xi)^2}{9} m\widehat{u} d\xi &= \int_{-3\sqrt{3}/8}^{3\sqrt{3}/8} e^{ix\xi(\eta)+it\eta} \frac{(1+2\cos\xi)^2}{9} m\widehat{u} \\ &\qquad f(\eta)d\eta \\ &= \int_{\mathbb{R}} 1_{[-3\sqrt{3}/8, 3\sqrt{3}/8]} e^{ix\xi(\eta)+it\eta} \frac{(1+2\cos\xi)^2}{9} m\widehat{u} f(\eta)d\eta,\end{aligned}$$

which is an inverse Fourier transform in time. Plancherel's formula thus implies

$$\| \int_{-3\sqrt{3}/8}^{3\sqrt{3}/8} e^{ix\xi(\eta)} e^{it\eta} \frac{(1+2\cos\xi)^2}{9} m\widehat{u} f(\eta)d\eta \|_{L^2_t}^2 = \int_{-3\sqrt{3}/8}^{3\sqrt{3}/8} \frac{(1+2\cos\xi)^4}{81} |m|^2 |\widehat{u} f(\eta)|^2 d\eta,$$

then reversing the change of variable $\eta = \eta(\xi)$ we get

$$\int_{-3\sqrt{3}/8}^{3\sqrt{3}/8} (1+2\cos\xi)^4 |m|^2 |\widehat{u} f(\eta)|^2 d\eta = \int_{-2\pi/3}^{2\pi/3} (1+2\cos\xi)^4 |m|^2 |\widehat{u}|^2 \frac{d\xi}{4\sin(\xi/2)\sin(3\xi/2)}.$$

The divisor $\sin(\xi/2)\sin(3\xi/2)$ cancels at the first order in $\xi_\pm = \pm 2\pi/3$ and at the second order in 0, but since it is easily seen that

$$\frac{(1+2\cos\xi)|m(\xi)|^2}{\sin(\xi/2)\sin(3\xi/2)} \text{ is uniformly bounded on } [-2\pi/3, 2\pi/3],$$

this implies

$$\begin{aligned}\int_{-2\pi/3}^{2\pi/3} (1+2\cos\xi)^4 |m|^2 |\widehat{u}|^2 \frac{d\xi}{\sin(\xi/2)\sin(3\xi/2)} &\lesssim \int_{-2\pi/3}^{2\pi/3} (1+2\cos\xi)^3 |\widehat{u}|^2 d\xi \\ &\lesssim \|\widehat{u}\|_{L^2}^2 \\ &\lesssim \|u\|_{l^2_{3h}}^2\end{aligned}$$

The same argument can be applied without further difficulties in the areas $[2\pi/3, \pi]$ and $[-\pi, -2\pi/3]$ where p is also monotone, this gives finally (2.3).
The estimate (2.4) is slightly more technical and is based on the formula

$$M^2 \int_0^t V_1(t-t')\Pi g(t')dt' = \int_{-\pi}^{\pi}\int_{\mathbb{R}} e^{ix\xi}\xi^2 \frac{e^{i\tau t}-e^{ip(\xi)t}}{i(\tau-p(\xi))}\frac{(1+2\cos(\xi))^2}{9}\widehat{f}(\xi,\tau)d\xi d\tau \qquad (2.5)$$

where $\widehat{f}$ is the Fourier transform of f with respect to both x and t. For this we refer to the argument in [12], theorem 3.5.

□

Remark 2.

- It is clear from the proof that the boundedness of $\frac{(1+2\cos\xi)^4|m|^2|}{\sin(\xi/2)\sin(3\xi/2)}$ is the main ingredient. It is implied by the first order cancellation of $(1+2\cos\xi)^4$ at $\pm 2\pi/3$, and any other Fourier multiplier whose symbol cancels in such a way would have also worked. This fact is important since in the next section we will see that different interpolators must be used to gain integrability.
- The Proposition is stated for a general multiplier M but we have in mind essentially two cases: the operator $|D|$ and the first order discrete differentiation ∂_h.

Corollary 1. *The following "dual" estimates also hold:*

$$\forall\, u \in l^2_h, \qquad \|\Pi^* M V(t)u\|_{l^\infty_{3h}L^2_t} \lesssim \|u\|_{l^2}, \qquad (2.6)$$

$$\forall\, g \in l^1_{3h}L^2_t, \qquad \sup_t \|M\int_{-\infty}^{\infty} V(t-t')\Pi g(t')dt'\|_{l^2} \lesssim \|\Pi g\|_{l^1L^2}, \qquad (2.7)$$

$$\|M\int_0^t V(t-t')\Pi g(t')dt'\|_{l^2} \lesssim \|\Pi g\|_{l^1L^2([0,t])}. \qquad (2.8)$$

Proof. We first note that M^* is the Fourier multiplier of symbol $\overline{m}$, thus Prop. 1 is also true for M^*. The inequality (2.7) is a direct duality consequence of (2.6), indeed

$$\begin{aligned}
\|M^*\int_{-\infty}^{\infty} V(t-t')\Pi g(t')dt'\|_{l^2} &= \sup_{\|u\|_{l^2}=1} \sum_j M^* \int_{-\infty}^{\infty} V(t-t')(\Pi g)_j(t')dt' u_j \\
&= \sup_{\|u\|_{l^2}=1} \sum_j \int_{-\infty}^{\infty} g_{3j}(t')(\Pi^* V(t'-t)Mu)_{3j}dt' \\
&\leq \|g\|_{l^1L^2_t}\|\Pi^* V(t'-t)Mu\|_{l^\infty L^2_{t'}} \\
&\leq \|g\|_{l^1L^2_t}\|u\|_{l^2},
\end{aligned}$$

while (2.8) is deduced from (2.7) by replacing g by $1_{[0,t]}g$.
It remains to prove (2.6). We remind that $\psi = (1+2\cos(h\xi))^2/9$ is the symbol of Π. From

the duality formula

$$\begin{aligned}\forall\,(u,\,v)\in l^2(3h\mathbb{Z})\times l^2(h\mathbb{Z}),\ 2\pi\sum_j \Pi u_j v_j &= \int_{-\pi/h}^{\pi/h}\psi\widehat{u}\widehat{v}\\ &= \int_{-\pi/3h}^{\pi/3h}\widehat{u}(\xi)\big(\psi\widehat{v}(\xi-2\pi/3)+\psi\widehat{v}(\xi)\\ &\quad +\psi\widehat{v}(\xi+2\pi/3)\big)d\xi\end{aligned}$$

we see that $2\pi\widehat{\Pi^* v}=\psi\widehat{v}(\xi-2\pi/3)+\psi\widehat{v}(\xi)+\psi\widehat{v}(\xi+2\pi/3)$ so that

$$\Pi^*\widehat{|D|V(t)}u=\psi|\xi-2\pi/3|e^{itp}\widehat{u}(\xi-2\pi/3)+\psi|\xi|e^{itp}\widehat{v}(\xi)+\psi|\xi+2\pi/3|e^{itp}\widehat{v}(\xi+2\pi/3),$$

and the proof of Proposition 1 can be repeated identically because ψ cancels at the appropriate points. □

3 Global dispersive estimates

A key estimate on the solutions of the linear evolution equation $\partial_t u+\partial_x^3 u=0$ is

$$\|u\|_{L^4_x L^\infty_t}\lesssim \||D|^{1/4}u_0\|_{L^2}, \tag{3.1}$$

it relies on the fact that the second derivative of the symbol ξ^3 does not cancel on $\mathbb{R}$ outside 0, which is obviously not the case for the discrete symbol on $[-\pi,\pi]$. This estimate is sharp in the sense that it is scale invariant:

$$\|u(\lambda x,\lambda^3 t)\|_{L^4_x L^{\infty} t}=\|u(x,t)\|_{L^4_x L^{\infty} t}/\lambda^{1/4},\ \||D|^{1/4}u_0(\lambda x)\|_{L^2_x}=\|u_0\|_{L^2}/\lambda^{1/4},$$

and the same scale invariancy holds for sequences of $\mathcal{S}(\mathbb{Z})$ dilated in sequences of $\mathcal{S}(h\mathbb{Z})$. This will be used in order to reduce our proofs to the case $h=1$.
According to (1.6), the second derivative of the discrete symbol p cancels at the points where

$$2\sin(2\xi)=\sin\xi.$$

Except the obvious points $0,\pm\pi$, there are only two solutions (ξ_1,ξ_0) in $]-\pi,\pi[$, and up to reindexing we can assume that $\xi_1\in]-\pi/2,-\pi/4[$ and $\xi_0\in]\pi/4,\pi/2[$. Similarly to the previous section, we will see that the derivation of global dispersive estimates only requires to use an interpolation operator whose symbol cancels at (ξ_0,ξ_1). Unfortunately they are not rational multiple of π, thus no 'barycentric' interpolator may be used to filter those frequencies as in previous section. We chose instead to use an interpolation operator more tailored to this case. According to the previous section, we need to find an interpolator Π such that

$$\widehat{\Pi u}(\xi_0)=\widehat{\Pi u}(\xi_1)=\widehat{\Pi u}(\pm 2\pi/3)=\widehat{\Pi u}(\pm\pi)=0.$$

Given u_{6j} defined on the coarse grid $6h\mathbb{Z}$, we set

$$\Pi_\alpha u_{6j+k} = \alpha_k u_{6j} + (1-\alpha_k) u_{6(j+1)}, \tag{3.2}$$

the discrete Fourier transform of $\Pi_\alpha u_j$ is then

$$\begin{aligned}
\widehat{\Pi_\alpha u} = \sum e^{-ij\xi} u_j &= \sum u_{6j} e^{-6ij\xi} + \sum_{k=1}^{5} e^{-i(6j+k)\xi}\big(\alpha_k u_{6j} + (1-\alpha_k) u_{6(j+1)}\big) \\
&= \sum e^{-i6j\xi} u_{6j} \Big(\sum_{k=0}^{5} e^{-ik\xi} + \alpha_k e^{-ik\xi} - \alpha_k e^{i(6-k)\xi} \Big) \\
&= \frac{1}{6} \widehat{u}(\xi) \Big(\frac{1-e^{i6\xi}}{1-e^{i\xi}} + \sum_{1}^{5} \alpha_k e^{-ik\xi} (1-e^{i6\xi}) \Big) \\
&= \frac{1-e^{6i\xi}}{6} \widehat{u}(\xi) \big(\frac{1}{1-e^{i\xi}} + \sum_{1}^{5} \alpha_k e^{-ik\xi} \big) \\
&= m(\xi) \widehat{u}(\xi).
\end{aligned}$$

It is clear that, without restriction on α_k, we have $m(0)=1$, $m(k\pi/3)=0$, $k \not\equiv 0 \pmod 6$. The system

$$\begin{cases} 1 \le k \le 5, \ m(2k\pi/6) = 0, \\ m(\xi_0) = 0, \\ m(\xi_1) = 0, \end{cases} \tag{3.3}$$

is thus underdetermined and we may arbitrarily choose a solution (α_k). The optimal choice of (α_k) (that minimizes the norm of Π_α) is a question that we will not study.

Remark 3. The construction was performed here for $h=1$, however it is clear that the same construction of an operator $\Pi_\alpha : \mathcal{S}(6h\mathbb{Z}) \to \mathcal{S}(h\mathbb{Z})$ leads to an interpolator whose symbol is simply $m(h\xi)$, and thus cancels automatically at $k\pi/3h$, $k \not\equiv 0 \pmod 6$.
More generally, any interpolator $\mathcal{S}(Nh\mathbb{Z}) \to \mathcal{S}(h\mathbb{Z})$ constructed by this technic has the generic symbol

$$m_h(\xi) = \frac{1-e^{iNh\xi}}{N} \big(\frac{1}{1-e^{ih\xi}} + \sum_{1}^{N-1} \alpha_k e^{-ikh\xi} \big),$$

and thus always cancels at $2k\pi/Nh$, $k \not\equiv 0 \pmod N$. This fact will be important for the sections dealing with convergence to the exact solution.

We simply assume in the rest of this section that an operator $\Pi : \mathcal{S}(Nh\mathbb{Z}) \to \mathcal{S}(h\mathbb{Z})$ is given whose symbol has a first order cancellation at ξ_0, ξ_1, $\pm\pi$.

Proposition 4. *Let $\Pi : Nh\mathbb{Z} \to h\mathbb{Z}$ be a discrete Fourier multiplier, ie an operator such that there exists ψ satisfying $\widehat{\Pi f} = \psi \widehat{f}$. We allow here N to be different of* 1 *in order to handle the*

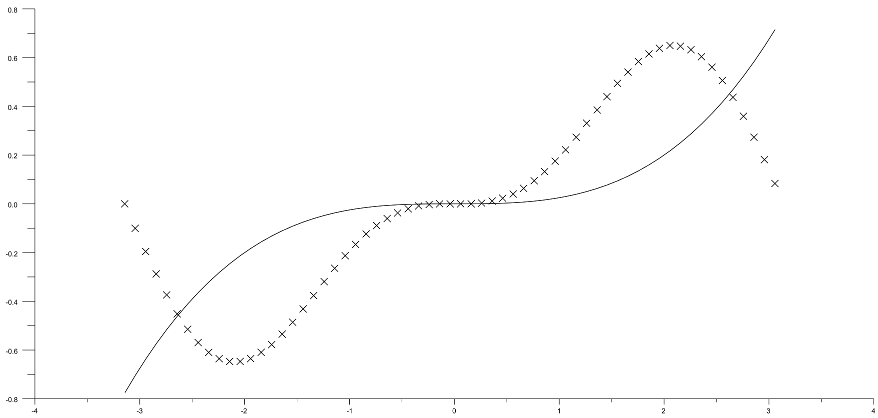

Figure 1: The connected line is the graph of the symbol of $-\partial_x^3$, the other one is the graph of the discrete symbol p for $h=1$. The 'bad' points are the inflexion points ξ_0, ξ_1 and the local extremas $\pm 2\pi/3$.

essential case of interpolation operators. If ψ cancels at ξ_0, ξ_1, then for any $u_0 \in l^2(Nh\mathbb{Z})$,

$$\|V_h(t)\Pi u_0\|_{l^4_j L^\infty_t} \leq C\||D|^{1/4} u_0\|_{l^2}, \tag{3.4}$$

For any $f \in l^{4/3}_{Nh} L^1$, we have the estimates

$$\||D|^{-1/2} \int_0^t V_h(t-s)\Pi f(s) ds\|_{l^4 L^\infty} \leq C\|f\|_{l^{4/3} L^1}, \tag{3.5}$$

In both estimates the constant C does not depend on h.

Proof. The proof follows the one of the continuous case, up to some supplementary technical difficulties.
By duality and elementary calculus the inequality (3.4) can be reduced to

$$\| \int_{\mathbb{R}} |D|^{-1/4} V_h(t)\Pi\Pi^* V(-t') D^{-1/4} g_j(t') dt' \|_{l^4_j L^\infty_t} \lesssim \|g\|_{l^{4/3}_j L^1_t}.$$

(for a detailed argument see [13] Lemma 7.3). By repeating the argument of the proof of (2.6), we see that it is sufficient to prove the above estimate for the function $\int_{\mathbb{R}} |D|^{-1/2} V_h(t-$

$t')\Pi\Pi^* g_j(t')dt'$, but then we have

$$\begin{aligned}\int_{\mathbb{R}}(\cdots)dt' &= \int_{-\pi/h}^{\pi/h}\int_{\mathbb{R}}\frac{e^{i(t-t')p_h(\xi)+ij\xi}}{|\xi|^{1/2}}\widehat{\Pi\Pi^* g}(\xi,t')dt'd\xi \\ &= \int_{-\pi/h}^{\pi/h}\int_{\mathbb{R}}\psi\frac{e^{i(t-t')p_h(\xi)+ij\xi}}{|\xi|^{1/2}}\widehat{\Pi^* g}(\xi,t')dt'd\xi \\ &\leq \int_{\mathbb{R}}|(\Pi^* g)_j(t')|dt' *_j \sup_t\Big|\int_{-\pi/h}^{\pi/h}\frac{e^{itp_h(\xi)+ij\xi}}{|\xi|^{1/2}}\psi d\xi\Big| \\ &\lesssim \|\max_{|j-k|\leq 5}|g_k|\,\|_{L^1_t} *_j \sup_t\Big|\int_{-\pi/h}^{\pi/h}\frac{e^{itp_h(\xi)+ij\xi}}{|\xi|^{1/2}}\psi d\xi\Big|,\end{aligned}$$

the last inequality being a direct consequence of the explicit form of Π^*. Thus we are reduced to prove

$$\Big\|\,\|\max_{|j-k|\leq 5}|g_k|\,\|_{L^1(\mathbb{R})} *_j \sup_t\Big|\int_{-\pi/h}^{\pi/h}\frac{e^{itp_h(\xi)+ij\xi}}{|\xi|^{1/2}}\psi d\xi\Big|\,\Big\|_{l^4_j} \lesssim \|g\|_{l^{4/3}L^1} \tag{3.6}$$

Similar arguments show that (3.5) amounts to

$$\Big\|\sup_t\|\max_{|j-k|\leq 5}|f_k|\,\|_{L^1(0,t)} *_j \sup_s\Big|\int_{-\pi/h}^{\pi/h}\frac{e^{isp_h(\xi)+ij\xi}}{|\xi|^{1/2}}\psi d\xi\Big|\,\Big\|_{l^4_j} \lesssim \|f\|_{l^{4/3}L^1}$$

which is implied by (3.6). According to proposition 23 (discrete Hardy-Littlewood-Sobolev), it is sufficient to prove

$$\sup_t\Big|\int_{-\pi/h}^{\pi/h}\frac{e^{4it/h^3\sin^2(\xi h/2)\sin(\xi h)+ijh\xi}}{|\xi|^{1/2}}\psi(h\xi)d\xi\Big| \lesssim \frac{1}{\big(h(1+|j|)\big)^{1/2}}. \tag{3.7}$$

By homogeneity we may reduce it to $h=1$, indeed the change of variable $\xi=\eta/h$ shows that (3.7) is equivalent to

$$\frac{1}{\sqrt{h}}\sup_t\Big|\int_{-\pi}^{\pi}\frac{e^{4it/h^3\sin^2(\eta/2)\sin(\eta)+ij\eta}}{|\eta|^{1/2}}\psi(\eta)d\eta\Big| \lesssim \frac{1}{(h(1+|j|))^{1/2}},$$

which amounts to

$$\sup_t\Big|\int_{-\pi}^{\pi}\frac{e^{4it\sin^2(\eta/2)\sin(\eta)+ij\eta}}{|\eta|^{1/2}}\psi(\eta)d\eta\Big| \lesssim \frac{1}{1+|j|^{1/2}}.$$

We remind that $p=4\sin^2(\xi/2)\sin\xi$, we perform the analysis only on $[0,\pi]$, $t\geq 0$, the proof being the same on $[-\pi,0]$ or $t\leq 0$. However no assumption is made on j. We split the interval $[0,\pi]$ in several parts:

$$\begin{aligned} A_1 &= [0, t^{-1/3}],\\ A_2 &= A_1^c \cap \{|tp'+j| \geq |j|/2\},\\ A_3 &= (A_1\cup A_2)^c. \end{aligned}$$

Note that if , $t^{-1/3}\geq \pi$ we only need to work on A_1, moreover in this case the integrand only runs over $[0,\pi]$, though we do not write it explicitly.
If $|j|\leq C_0 t^{1/3}$, $C_0>0$ fixed, one has trivially $|\int_{A_1}(\cdots)|\lesssim 1/\sqrt{j}$, else

$$\int_{A_1} e^{i(tp+j\xi)}\psi/\sqrt{\xi}d\xi = \int_0^{1/|j|} e^{i(tp+j\xi)}\psi/\sqrt{\xi}d\xi + \int_{1/|j|}^{1/t^{1/3}} e^{i(tp+j\xi)}\psi/\sqrt{\xi}d\xi,$$

the first integral is obviously bounded by $1/\sqrt{|j|}$. After an integration by part we obtain for the second one

$$\begin{aligned} |\int_{1/|j|}^{1/t^{1/3}} e^{i(tp+j\xi)}\psi/\sqrt{\xi}d\xi| &\leq \frac{t^{1/6}}{|j+tp'(t^{-1/3})|} + \frac{\sqrt{|j|}}{j+tp'(1/j)}\\ &+ \left|\int_{1/|j|}^{1/t^{1/3}} e^{i(j\xi+tp)}\frac{1}{\xi^{1/2}(j+tp')}\left(\frac{-\psi}{2\xi} - \frac{\psi tp''}{(j+tp')} + \psi'\right)\right|\\ &\lesssim \frac{1}{\sqrt{|j|}}, \end{aligned}$$

provided C_0 is chosen large enough, for $|p'|\lesssim \xi^2$ and $p'\sim_0 3\xi^2$. We implicitly used the fact that the integration interval is bounded to get the estimate

$$\left|\int_{1/|j|}^{1/t^{1/3}} e^{i(j\xi+tp)}\frac{1}{\xi^{1/2}|j+tp'|}\psi'\right| \lesssim \int_{1/|j|}^{1/t^{1/3}} \frac{|j|^{1/2}}{|j+tp'|} \lesssim \frac{1}{|j|^{1/2}}.$$

Let us note that the key point in the previous analysis was $j\geq C_0 t^{1/3} \Rightarrow |j+tp'|\gtrsim |p|$. Since on A_2 this last inequality is automatically satisfied, the same integration by part argument gives

$$|\int_{A_2} e^{i(tp+j\xi)}\psi/\sqrt{\xi}d\xi| \lesssim \frac{1}{\sqrt{|j|}}.$$

We still have to estimate the integral on A_3. Let us consider $\mathcal{V},\widetilde{\mathcal{V}}$ two disjoint neighbourhoods of the points $\{\xi_0,\pi\}$ where p'' cancels, and set $A_4 = A_3\cap(\mathcal{V}\cup\widetilde{\mathcal{V}})^c$, $A_5 = A_3\cap\mathcal{V}$, $\widetilde{A_5} = A_3\cap\widetilde{\mathcal{V}}$. On A_4, $p''\asymp\xi,\ p'\leq\xi^2$ and $tp'\asymp j/2$, thus $\xi\gtrsim\sqrt{|j|/t}$. The Van der Corput lemma (see [13] Corollary 1.1) implies

$$\int_{A_4} e^{i(tp+j\xi)}\psi/\sqrt{\xi}d\xi| \lesssim \frac{1}{\sqrt{t}}\sqrt[4]{\frac{t}{|j|}}\sup_{A_4}\frac{1}{\sqrt{\xi}} \lesssim \frac{1}{\sqrt{t}}\sqrt[4]{\frac{t}{|j|}}^2 = \frac{1}{\sqrt{|j|}}.$$

We then have on the neighbourhood of ξ_0

$$p' - p'(\xi_0) = \frac{\beta}{2}(\xi - \xi_0)^2 + o(\xi - \xi_0)^2,\ p'' = \beta(\xi - \xi_0) + o(\xi - \xi_0),\ \beta \in \mathbb{R}^*.$$

For c (resp C) chosen small enough (resp large enough), if $t/|j| \notin [c, C]$, one obtains easily by integrations by parts

$$|\int_{A_5} e^{i(tp+j\xi)}\psi/\sqrt{\xi}d\xi| \lesssim 1/|j|,$$

thus we may assume that $t \asymp |j|$, and all that remains is to prove

$$|\int_{A_5} e^{i(tp+j\xi)}\psi/\sqrt{\xi}d\xi| \lesssim 1/\sqrt{t}.$$

Since ψ cancels at ξ_0, we have

$$|\int_{\xi_0-1/t^{1/3}}^{\xi_0+1/t^{1/3}} e^{i(tp+j\xi)}\psi/\sqrt{\xi}d\xi| \lesssim t^{-2/3} \lesssim 1/\sqrt{t},$$

because t is bounded away from 0.
(note here that it would be enough that ψ simply cancelled at the order 1/2)
On $A_5' := A_5 \cap \{|p'+j/t| \leq |(j+p'(\xi_0)t)/2t|,\ |\xi-\xi_0| \geq t^{-1/3}\}$ one has $|\xi-\xi_0| \asymp \sqrt{(j+p'(\xi_0)t)/2t}$ so that by using again the Van der Corput's lemma and the fact that $\psi \lesssim \max(|\xi - \xi_0|, 1) \Rightarrow \psi \lesssim \sqrt{|\xi - \xi_0|}$

$$\begin{aligned} |\int_{A_5'} e^{i(tp+j\xi)}\psi/\sqrt{\xi}d\xi| \lesssim \frac{1}{\sqrt{t}}\left(\frac{j+p'(\xi_0)t}{2t}\right)^{-1/4}\sqrt{\frac{j+p'(\xi_0)t}{2t}} &= \frac{1}{\sqrt{t}}\left(\frac{j+p'(\xi_0)t}{2t}\right)^{1/4} \\ &\lesssim \frac{1}{\sqrt{t}} \lesssim \frac{1}{\sqrt{|j|}}. \end{aligned}$$

Finally on $A_5'' := A_5 \cap \{|p' + j/t| \geq |(j + p'(\xi_0)t)/2t|,\ |\xi - \xi_0| \geq t^{-1/3}\}$, one has $|p' + j/t| \gtrsim |p'| \gtrsim |\xi - \xi_0|^2 \geq t^{-2/3}$.
We may conclude again by integration by parts, since the calculus are very similar we shall only detail the case of one of the terms appearing then:

$$\frac{1}{t}\int_{A_5''} \frac{e^{i(tp+x\xi)}\psi}{(p'+j/t)\xi^{3/2}} \lesssim \frac{1}{t}\int_{A_5''} \frac{|\xi-\xi_0|}{|\xi-\xi_0|^2}d\xi \lesssim \frac{1}{t}\max_{A_5''}|\ln(|\xi-\xi_0|)| \lesssim \frac{1}{\sqrt{|j|}}.$$

The integral estimate on $\widetilde{A_5}$ can be performed in the same way, and this concludes the proof. □

Remark 5. In the previous proof, the case $t \asymp j$ may be seen as a particular case of lemma 2.7 in [9].

Finally some other useful estimates are obtained by interpolation of the estimates in Prop. 4 and those of section 2 with $M = |D|$. For the precise argument on interpolation we refer to [13] prop. 7.4.

Corollary 2. *For $u_0 \in l^2$, $f \in l^{5/4}L^{10/9}$*

$$\|V_t(t)\Pi u_0\|_{l^5L^{10}} \lesssim \|\Pi u_0\|_{l^2}, \tag{3.8}$$

$$\|\int_0^t V_t(t-s)\Pi f(s)ds\|_{l^5L^{10}} \lesssim \|f\|_{l^{5/4}L^{10/9}}. \tag{3.9}$$

Remark 6. It is apparent from the proofs that all the dispersive estimates only rely on the good spectral localization of the symbol of Π, namely its cancellation at the points $(\xi_0/h, \xi_1/h, \pm 2\pi/(3h), \pm\pi/h)$. Since this cancellation is not changed by the composition with a Fourier multiplier M_1, all our estimates remain true for sequences $M_1\Pi u_0$ or $M_1\Pi f$. For example, if we replace Πf by $\partial_h \Pi f$ we have the inequality

$$\|\int_0^t V_t(t-s)\partial_h\Pi f(s)ds\|_{l^5L^{10}} \lesssim \|\partial_h\Pi f\|_{l^{5/4}L^{10/9}}.$$

4 Existence of a solution

This section is devoted to the existence of a global in time solution to the discrete dispersive scheme, that admits bounds only depending on the $l^2(h\mathbb{Z})$ of the initial data, and in particular independent of h. We E_k to be the canonical projector $\mathcal{S}(h\mathbb{Z}) \to \mathcal{S}(kh\mathbb{Z})$, $(u_n) \to (u_{kn})$. When there is no ambiguity we will simply write it E.
The first result of this section is a simple but useful lemma.

Lemma 1. *Every result stated here is uniform in h.*
In $H^1(h\mathbb{Z})$, the norm $(\int_{-\pi/h}^{\pi/h} |\widehat{u}|^2(1+|\xi|^2)d\xi)^{1/2}$ is equivalent to $\|u\|_{l^2} + \dfrac{\|u_{j+1}-u_j\|_{l^2}}{h}$.
The operator $\Pi \circ E_k$ is continuous $H^s \to H^s$, $0 \leq s \leq 1$, where Π is an interpolation operator $\mathcal{S}(kh\mathbb{Z}) \to \mathcal{S}(h\mathbb{Z})$.

Proof. A short computation shows that

$$\widehat{u_{j+1} - u_j} = 2ie^{ih\xi/2}\sin\frac{\xi h}{2}\widehat{u},$$

and since on $[-\pi/h, \pi/h]$, $2/h \sin\frac{\xi h}{2} \asymp \xi$, one has

$$\|u\|_{H^s}^2 = \int_{-\pi/h}^{\pi/h} |\widehat{u}|^2(1+|\xi|^2)d\xi \asymp \int |\widehat{u}|^2(1+\frac{2^{2s}\sin^{2s}\frac{\xi h}{2}}{h^{2s}})d\xi, \tag{4.1}$$

in particular for $s=1$

$$\|u\|_{H^1} \asymp \int_{-\pi/h}^{\pi/h} |\widehat{u}|^2 (1+4\frac{\sin^2\frac{\xi h}{2}}{h^2})d\xi = \|u\|_{l^2}^2 + \frac{\|u_{j+1}-u_j\|_{l^2}^2}{h^2}. \tag{4.2}$$

It is easily checked that $\|\Pi\circ E_k u\|_{l^2} \lesssim \|u\|_{l^2}$. Moreover for $kn\le j<k(n+1)$, by hypothesis $u_j=\alpha_j u_{kn}+(1-\alpha_j)u_{k(n+1)}$, so that

$$|u_{j+1}-u_j| \le |\alpha_{j+1}-\alpha_j|u_{kn}-u_{k(n+1)}| \le (\max|\alpha_{j+1}-\alpha_j|)\sum_1^5 |u_{kn+j}-u_{kn+j-1}|.$$

This implies $\|(\Pi\circ E_k u)_{j+1}-(\Pi\circ E_k u)_j\|_{l^2} \lesssim \|u_{j+1}-u_j\|_{l^2}$, the operator $\Pi\circ E_k$ is thus continuous $l^2\to l^2$ et $H^1\to H^1$, so that by interpolation it is continuous $H^s\to H^s$, $0\le s\le 1$. □

As an application of the estimates of the previous sections we shall prove the existence of a solution for the following semi-discrete problem

$$\begin{cases} \partial_t u+\partial_h^3 u+\partial_h \Pi E u^5=0, \\ u|_{t=0}=\Pi u_0. \end{cases} \tag{DcKdV}$$

We say that u is a solution of (DcKdV) if it is $C(\mathbb{R}_t,l^2)$ and satisfies

$$u(t)=V_h(t)\Pi u_0-\int_0^t V_h(t-s)\partial_h \Pi E u^5 ds.$$

Theorem 7. *We define the space $X(\mathbb{R}\times h\mathbb{Z})$ as the set of functions $t\to u(t)\in\mathcal{S}(h\mathbb{Z})$ such that*

$$\begin{aligned} u\in\mathcal{C}(\mathbb{R};l^2)\cap L^\infty(\mathbb{R};l^2),\\ \|\partial_h u\|_{l^\infty L^2}<\infty,\\ \|u\|_{l^5L^{10}}<\infty. \end{aligned}$$

with the corresponding norm $\|u\|_X$ associated.
We assume that Π is an interpolation operator whose symbol cancels at the points $\pm 2\pi/3$, $\pm\pi$, ξ_0, ξ_1 (for such operators, see the construction before Remark 3). There exists $\delta>0$ independent of h such that for $\|\Pi u_0\|_{l^2}<\delta$ small enough there exists a unique solution u of (DcKdV) such that

$$\|u\|_X<\infty, \tag{4.3}$$

Moreover the solution map $\{u_0\in l^2(Nh\mathbb{Z}):\ \|u_0\|_{l^2}<\delta\}\to X(h\mathbb{Z})$ is Lipschitz.

Proof. The proof is based on the Picard-Banach fixed point theorem applied in the space X. We recall that $V_h(t)$ is the group corresponding to the third order discretized derivative, it is a Fourier multiplier of symbol $e^{itp_h(\xi)}$. We check that for $\|u_0\| \le \delta$ and a small enough,

$$T:\ u \to V(t)\Pi u_0 - \int_0^t V_h(t-s)\partial_h \Pi E u^5(s)ds$$

sends $\{u \in X:\ \|u\|_X < a\}$ to itself.

Control of the $L^\infty l^2$ norm: we have $\|V_h(t)\Pi u_0\|_{l^2} \le \|u_0\|_{l^2}$ and (2.8) with $M = \partial_h$ implies

$$\|\int_0^t V_h(t-s)\partial_h \Pi E u^5(s)ds\|_{l^2} \lesssim \|\Pi E u^5\|_{l^1 L^2} \lesssim \|u\|^5_{l^5 L^{10}} \le \|u\|^5_X.$$

Control of $\|\partial_h T u\|_{l^\infty L^2}$: using (2.3) we have

$$\|\partial_h V_h(t)\Pi u_0\|_{l^\infty L^2} \lesssim \|u_0\|_{l^2},$$

while (2.4) implies

$$\|\partial_h^2 \int_0^t V_h(t-s)\Pi E u^5(s)ds\|_{l^\infty L^2} \lesssim \|\Pi E u^5\|_{L^1 l^2} \lesssim \|u\|^5_{l^5 L^{10}} \le \|u\|^5_X.$$

Control of the $l^5 L^{10}$ norm: according to (3.8) and (3.9) (see also Remark 6) we have

$$\|V_h(t)\Pi u_0\|_{l^5 L^{10}} \lesssim \|u_0\|_{l^2},\ \text{and}\ \ \|\int_0^t V_h(t-s)\partial_h \Pi E u^5(s)ds\|_{l^5 L^{10}} \lesssim \|\partial_h \Pi E u^5\|_{l^{5/4} L^{10/9}}.$$

Since

$$(\Pi E u)^5_{j+1} - (\Pi E u)^5_j = ((\Pi E u)_{j+1} - (\Pi E u)_j)(\sum_0^4 (\Pi E u)^k_{j+1}(\Pi E u)^{4-k}_j), \tag{4.4}$$

Hölder inequality then implies

$$\|\partial_h \Pi E u^5\|_{l^{5/4} L^{10/9}} \lesssim \|\partial_h \Pi E u\|_{l^\infty L^2} \|\Pi E u^4\|_{l^{5/4} L^{10/4}},$$

then using the continuity of $\Pi \circ E:\ H^1 \to H^1$ (Lemma 1)

$$(\cdots) \lesssim \|\partial_h u\|_{l^\infty L^2} \|u\|^4_{l^5 L^{10}} \le \|u\|^5_X.$$

Finally we have obtained

$$\|Tu\|_X \le c\|\Pi u_0\|_{l^2} + c\|u\|^5_X.$$

Let a be fixed such that $ca^4 < 1/2$, then if we choose $\delta \leq \frac{a}{2c}$ we get

$$\|Tu\|_X \leq c\|\Pi u_0\|_{l^2} + a/2 \leq a,$$

that means $T: \ B_X(0,a) \to B_X(0,a)$. Following the same arguments, we see that (up to diminishing a), the operator is a contraction. By the Picard-Banach's fixed point theorem this ensures existence and uniqueness of a solution (in X) for any u_0 such that $\|u_0\|_{l^2} \leq \delta$. The smoothness of the solution operator follows from the classical fixed point theory. □

5 Convergence to the solution of the Cauchy problem

We denote by P the interpolation operator from l^2 to the set of continuous affine by parts functions:

$$\forall\, u \in l^2(h\mathbb{Z}),\ \forall\, x \in [jh,(j+1)h]:\ Pu(x) = u_j + (x-jh)\frac{u_{j+1}-u_j}{h}.$$

Although our natural level of regularity is only l^2, we need to use this operator for the discrete dispersive smoothing on u_h to have its counterpart on Pu_h. The operator P is particularly handy since it commutes with the operator ∂_h. Let u_h be the discrete solution such that $P(\Pi u)_{0,h} \to u_0$ (L^2), u be the solution of $cKdV$. The aim of this section is to establish the convergence of Pu_h to u in a precise sense.
We start with a lemma that links the "smoothness" of a sequence v to the smoothness of Pv.

Lemma 2. *For any $v \in \mathcal{S}(h\mathbb{Z})$, we have*

$$\|Pv\|_{L^2(\mathbb{R})} = \|v\|_{l^2(h\mathbb{Z})},\ \|Pv\|_{H^1(\mathbb{R})} = \left(\|v\|_{l^2}^2 + \left\|\frac{v_{j+1}-v_j}{h}\right\|_{l^2}^2\right)^{1/2},$$

and the same equalities are true if we replace $\mathbb{R}$ by $[kh,lh]$ and $h\mathbb{Z}$ by $kh\cdots lh$ for $(k,l)\in\mathbb{Z}^2$. The Fourier transform of Pu is

$$\widehat{Pv}(\xi) = \frac{4\sin^2(h\xi/2)}{h^2\xi^2}\widehat{v}(\xi).$$

If $P\Pi v_h \to v$ (L^2), then for any fixed t $PV_h(t)\Pi v_h \to V(t)v\,(L^2)$, where V is the group corresponding to the operator ∂_x^3, moreover the convergence is uniform (in t for fixed u) if t remains in a compact.

Proof. The norm equalities are elementary. A simple calculation gives for the Fourier transform

$$\begin{aligned}
\widehat{Pv}(\xi) &= \sum_j \int_{jh}^{(j+1)h} e^{-ix\xi}\big(v_j + \frac{v_{j+1}-v_j}{h}(x-jh)\big)dx \\
&= \sum_j v_j e^{-ijh\xi}\frac{e^{-ih\xi}-1}{-i\xi} + \frac{v_{j+1}-v_j}{h}\left(\frac{he^{-ij\xi}}{-i\xi} + \frac{e^{-ijh\xi}(e^{-ih\xi}-1)}{\xi^2}\right) \\
&= \sum_j \frac{v_{j+1}-v_j}{h}\frac{e^{-ijh\xi}(e^{-ih\xi}-1)}{\xi^2} \\
&= \frac{(e^{ih\xi}-1)(e^{-ih\xi}-1)}{h^2\xi^2}\widehat{v}(\xi) = \frac{4\sin^2(h\xi/2)}{h^2\xi^2}\widehat{v}(\xi).
\end{aligned}$$

For the norm convergence we may write

$$\begin{aligned}
\widehat{PV_h(t)\Pi v_h} &= \frac{4\sin^2(h\xi/2)}{h^2\xi^2}e^{itp_h(\xi)}\widehat{\Pi v_h}(\xi) \\
\Rightarrow \|PV_h(t)\Pi v_h - V(t)v\|_{L^2} &= \left(\int_{\mathbb{R}} |\frac{4\sin^2(h\xi/2)}{h^2\xi^2}e^{itp_h}\widehat{\Pi v_h}(\xi) - e^{it\xi^3}\widehat{v}|^2 d\xi\right)^{1/2} \\
&\leq \left(\int_{\mathbb{R}} |\frac{4\sin^2(h\xi/2)}{h^2\xi^2}e^{itp_h}\widehat{\Pi v_h}(\xi) - e^{itp_h}\widehat{v}|^2 d\xi\right)^{1/2} \\
&\quad + \left(\int_{\mathbb{R}} |e^{itp_h}\widehat{v}(\xi) - e^{it\xi^3}\widehat{v}|^2 d\xi\right)^{1/2}
\end{aligned}$$

Since $|e^{itp_h}| = 1$, the first term tends to 0 according the the hypothesis $P\Pi v_h \to v$ (L^2), moreover $|e^{tp_h} - e^{it\xi^3}| \leq 2$ and $\to_{h\to 0} 0$, so that by Lebesgue's dominated convergence theorem the second term also tends to 0.
If t, ξ remain in a compact, $|t(p_h(\xi) - \xi^3)| \to 0$ uniformly. Since $\lim_{A\to\infty} \|\widehat{v}\|_{L^2(|\xi|\geq A)} = 0$, the uniform convergence of $PV_h(t)\Pi v_h$ to $V(t)v$ follows easily. □

Remark 8. Note that the convergence $PV_h(t)\Pi u_h \to V(t)u$ (L^2) is uniform *in* t for t bounded, however it does depend on u_0, preventing any rate of convergence in h.

In what follows we shall assume that $u_{0h} \in l^2(Nh\mathbb{Z})$ is such that

$$\|P\Pi u_{0h} - u_0\|_{L^2} \to_h 0,$$

which is clearly the minimal assumption for convergence, and we consider the family u_h of solutions to

$$\begin{cases} \partial_t u_h + \partial_h^3 u_h + \partial_h \Pi u_h^5 = 0, \\ u_h|_{t=0} = \Pi u_{0h}. \end{cases} \tag{5.1}$$

According to the theorem 7, the family $(Pu_h)_{0\leq h\leq 1}$ is bounded in $L_t^\infty L_x^2 \cap L_x^5 L_t^{10}$, and $(u_h(\cdot + h) - u_h(\cdot))/h$ is bounded in $L_x^\infty L_t^2$. By weak star (resp weak) compactness, we may extract u_h

weakly converging toward $u \in L_t^\infty L_x^2 \cap L_x^5 L_t^{10}$.
It is slightly more delicate to check that $\partial_x u \in L_x^\infty L_t^2$. We use that

$$\forall\, \varphi \in C_c^\infty,\ \int_{\mathbb{R}^2} \frac{Pu_h(x+h) - Pu_h(x)}{h} \varphi dxdt = \int_{\mathbb{R}^2} Pu_h(x) \frac{\varphi(x-h) - \varphi(x)}{h} dxdt.$$

Since $\frac{\varphi(x-h)-\varphi(x)}{h} \to -\varphi'\ L_x^1 L_t^2$, we have

$$\int_{\mathbb{R}^2} Pu_h(x) \frac{\varphi(x-h) - \varphi(x)}{h} dxdt \to -\int_{\mathbb{R}} u\varphi' dxdt,$$

Moreover up to extracting again we may assume that $(Pu_h(\cdot + h) - Pu_h(\cdot))/h \rightharpoonup^* v \in L_x^\infty L_t^2$, so that

$$\int_{\mathbb{R}^2} v\varphi dxdt = -\int_{\mathbb{R}^2} u\partial_x \varphi dxdt,$$

that is $\partial_x u = v \in L_x^\infty L_t^2$ (in the sense of distributions).

Theorem 9. *Let $\Pi u_{0,h}$ be the initial data of the discrete problem (DcKdV). Assume that $P\Pi u_{0,h} \to_h u_0$ in L^2. Let $u \in L_t^\infty L_x^2 \cap L_x^5 L_t^{10}$ be the weak limit of a sequence extracted from Pu_h as above. Then u is the solution of cKdV with initial data u_0 and the whole sequence Pu_h converges to u in the following sense*

$$Pu_h \rightharpoonup^* u\ (L_t^\infty L_x^2),\ Pu_h \rightharpoonup u\ (L_x^5 L_t^{10}),$$
$$\partial_x Pu_h \rightharpoonup^* \partial_x u\ (L_x^\infty L_t^2),\ Pu_h \to u\ \big(L_{loc}^2(\mathbb{R}^2)\big).$$

Proof. Let $T > 0$ be fixed, $J = [-T, T]$, $\Omega = [-C, C]$. Up to increasing by at most $2h$ the size of Ω we may always assume that C is a multiple of h and apply the norm equalities of Lemma 2 (a fact that we will not mention in the rest of the proof). Since

$$\partial_t Pu_h = -P\partial_h^3 u_h - P\partial_h \Pi u_h^5/5,$$

we have according to the injection $L^1(\Omega) \hookrightarrow H^{-1}(\Omega)$ (dual of $H^1 \hookrightarrow L^\infty$)

$$\begin{aligned} \|\partial_t Pu_h\|_{L^1(J,H^{-3}(\Omega))} &\leq \|Pu_h\|_{L^1(J,L^2)} + \|P\partial_h \Pi u_h^5/5\|_{L^1(J,H^{-3}(\Omega))} \\ &\lesssim \|u_0\|_{L^2} + \|P\partial_h \Pi u_h^5/5\|_{L^1(J\times\Omega)}. \end{aligned}$$

As J and Ω are bounded, we may write

$$\|P\partial_h \Pi u_h^5\|_{L^1(J\times\Omega)} \lesssim \|P\partial_h \Pi u_h^5\|_{L_x^{5/4}(\mathbb{R}_x, L_t^{10/9}(J))},$$

then by application of Hölder's inequality

$$\|P\partial_h u_h^5\|_{L_x^{5/4}(\mathbb{R}_x, L_t^{10/9}(J))} \lesssim \|\partial_h u_h\|_{l^\infty L^2} \|u_h\|_{l^5 L^{10}}^4 \lesssim \|u_0\|_{L^2}^5.$$

Thus $\partial_t Pu_h$ is bounded in $L^2(J, H^{-3}(\Omega))$. On the other hand

$$\|Pu_h\|_{L^2(J;H^1(\Omega))} \lesssim \|Pu_h\|_{L^\infty(J;L^2)} + \|\partial_x Pu_h\|_{L^\infty_x L^2_t},$$

and thus Pu_h is bounded in $L^2(J, H^1(\Omega))$. Since we have the sequence of inclusions $H^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-3}(\Omega)$, with the first injection being compact, Aubin-Lions's lemma implies that Pu_h is precompact in $L^2(J, L^2(\Omega))$. Using a diagonal extraction argument, we find that a subsequence of u_h is strongly converging in $L^2_{loc}(\mathbb{R}_x \times \mathbb{R}_t)$ as $h \to 0$.
This list of convergences is now sufficient to check that the limit is the solution of the continuous problem. In order to do so we introduce the 'variational' formulation

$$\forall \varphi \in C_c^\infty(\mathbb{R}^2), \int_{\mathbb{R}^2} Pu_h(-\partial_t\varphi - \partial_h^3\varphi) - P\Pi u_h^5 \partial_h \varphi dxdt = 0$$

Passing to the limit in h we will obtain that u is a solution of cKdV. The first term does not rise any difficulties, indeed for $\varphi \in C_c^\infty$, $\partial_h^3\varphi \to \partial_x^3\varphi$ (L^2), thus using it with $Pu_h \to u$ (L^2_{loc}) we find

$$\int_{\mathbb{R}^2} Pu_h(-\partial_t\varphi - \partial_h^3\varphi) \to \int_{\mathbb{R}^2} u(-\partial_t\varphi - \partial_x^3\varphi).$$

For the nonlinear term, let us fix R such that supp$\varphi \subset]-R, R[^2$ and $\varepsilon > 0$ to be small. Let us define

$$A_{h,\varepsilon} := \{(x,T) \in [-R,R]^2 : \ |u_h^5 - u^5| > \varepsilon\}. \tag{5.2}$$

Since $Pu_h \to u$ $L^2([-R,R]^2)$, we have for fixed ε (up to extracting again) $\lambda(A_{h,\varepsilon}) \to_h 0$.
For h small enough we may write

$$\int_{[-R,R]^2} \partial_h P\Pi u_h^5 \varphi dxdt = \int_{[-R,R]^2} P\Pi u_h^5 \partial_{-h}\varphi dxdt.$$

But $\partial_{-h}\varphi \to -\partial_x\varphi$ uniformly, so that is is sufficient to prove

$$\int_{[-R,R]^2} (P\Pi u_h^5 - u^5)\partial_{-h}\varphi dxdt \to 0.$$

Let $N \in \mathbb{N}^*$ be such that $\Pi: \ Nh\mathbb{Z} \to h\mathbb{Z}$, $\mathbb{Z} = \sqcup_{p=0}^{N-1}\{Nh\mathbb{Z} + ph\} =: \sqcup \mathbb{Z}_p$:

$$\begin{aligned}
|\int_{[-R,R]^2} (P\Pi u_h^5 - u^5)\partial_{-h}\varphi dxdt| &\lesssim \int_{-R}^{R} \sum_{k\in\mathbb{Z}} \sum_{p=0}^{N-1} \int_{Nkh+p}^{Nkh+p+1} |\Pi u_{Nkh+ph}^5 - u^5|\,|\partial_{-h}\varphi| dxdt \\
&\leq \int_{-R}^{R} \sum_{p,k} \int_{Nkh+p}^{Nkh+p+1} \sum_{j=0}^{1} \alpha_{j,p} |u_{N(k+j)h}^5 - u^5|\,|\partial_{-h}\varphi| dxdt,
\end{aligned}$$

where the $\alpha_{j,p}$ only depend on the operator Π. This implies

$$
\begin{aligned}
|\int_{[-R,R]^2}(P\Pi u_h^5 - u^5)\partial_{-h}\varphi dxdt| &\leq \sum_{p=0}^{N-1}\int_{-R}^{R}\sum_{\mathbb{Z}_p}\sum_{j=0}^{1}\alpha_{j,p}|u^5_{N(k+j)h} - u^5|\,|\partial_{-h}\varphi|dxdt \\
&\leq \sum_{p=0}^{N-1}\sum_{j=0}^{1}\int_{-R}^{R}\int_{\mathbb{R}}\alpha_{j,p}|Pu_h^5(x) - u^5(x+\tau_{j,p}h)|\,|\partial_{-h}\varphi|dxdt
\end{aligned}
$$

where $\tau_{j,p} = p$ if $j = 0$, $p - N$ if $j = 1$. By using the standard result

$$\limsup_{\varepsilon\to 0}\int_{\mathbb{R}}|u^5(x+\varepsilon) - u^5(x)|dxdt = 0,$$

we are reduced to prove

$$\int_{-R}^{R}\int_{\mathbb{R}}|Pu_h^5(x) - u^5(x)|\,|\partial_{-h}\varphi|dxdt = \int_{[-R,R]^2}|Pu_h^5(x) - u^5(x)|\,|\partial_{-h}\varphi|dxdt \to 0. \tag{5.3}$$

But

$$
\begin{aligned}
\int_{[-R,R]^2}|Pu_h^5(x) - u^5(x)|\,|\partial_{-h}\varphi|dxdt &= \int_{A_{h,\varepsilon}}|Pu_h^5(x) - u^5(x)|\,|\partial_{-h}\varphi|dxdt \\
&\quad + \int_{A^c_{h,\varepsilon}}|Pu_h^5(x) - u^5(x)|\,|\partial_{-h}\varphi|dxdt
\end{aligned}
$$

The second term is simply bounded by

$$|\int_{A^c_{h,\varepsilon}}|Pu_h^5(x) - u^5(x)|\,|\partial_{-h}\varphi|dxdt| \leq \varepsilon 4R^2 \sup_{0\leq h\leq 1,\ (x,t)\in[-R,R]^2}|\partial_h\varphi|.$$

For the other term we use a discrete version of the Sobolev embedding $H^1 \hookrightarrow L^\infty$ (the proof is similar)

$$Pu_h, P\partial_h u_h \in L^2([-R,R]^2) \Rightarrow Pu_h \in L^2_tL^\infty_x([-R,R]^2) \Rightarrow Pu_h \in L^\infty_xL^2_t([-R,R]^2),$$

with bound independent of h. By interpolating with $Pu_h \in L^5_xL^{10}_t$ we get $Pu_h \in L^6([-R,R]^2)$. The same (simpler) argument implies $u \in L^6([-R,R]^2)$, so that

$$\int_{A_{h,\varepsilon}}|P\Pi u_h^5 - u^5|\,|\partial_{-h}\varphi|\,dxdt \leq \big(\|Pu_h^5\|^{5/6}_{L^{6/5}(A_{h,\varepsilon})} + \|u^5\|^{5/6}_{L^{6/5}(A_{h,\varepsilon})}\big)\|\partial_{-h}\varphi\|^{1/6}_{L^6(A_{h,\varepsilon})},$$

but since $\lambda(A_{h,\varepsilon}) \to 0$, we get

$$\int_{A_{h,\varepsilon}}|Pu_h^5 - u^5|\,|\partial_{-h}\varphi|\,dxdt \to 0.$$

Finally we have obtained

$$\forall\,\varepsilon>0,\ \limsup_{h\to 0}\Big|\int_{[-R,R]^2}(P\Pi u_h^5-u^5)\partial_h\varphi dxdt\Big|\leq 4R^2\varepsilon\sup_{0\leq h\leq 1,\ (x,t)\in[-R,R]^2}|\partial_h\varphi|,$$

which means as desired $\partial_h P\Pi u_h^5 \rightharpoonup \partial_x u^5\ (\mathcal{D}')$.
The last point to check is that this solution has indeed u_0 for initial data, that is $u\in C_tL^2,\ u|_{t=0}=u_0$:

$$u_h(t)=V_h(t)\Pi u_{0,h}-\int_0^t\partial_h V_h(t-s)\Pi u_h^5(s)ds,\ \lim_{h,t\to 0}PV_h(t)\Pi u_{0,h}=u_0\ (L^2)\ \text{(see lemma 2)}.$$

It is thus sufficient to prove $\lim_{h,t\to 0}\|\int_0^t\partial_h V_h(t-s)\Pi u_h^5(s)ds\|_{L^2}=0$. According to the estimates of section 4, we have

$$\|\int_0^t\partial_h V_h(t-s)\Pi u_h^5(s)ds\|_{L^2}\leq\|u_h^5\|_{l^1L^2([0,t])}=\|u_h\|^5_{l^5L^{10}([0,t])}.$$

Moreover (following the proof of theorem 7)

$$\begin{aligned}\|u_h\|_{l^5L^{10}} &\leq \|V_hu_{0,h}\|_{l^5L^{10}}+\|\int_0^t\partial_hV_h(t-s)\Pi u_h^5(s)ds\|_{l^5L^{10}}\\ &\leq \|V_hu_{0,h}\|_{l^5L^{10}}+\|\partial_h\Pi u_h^5\|_{l^{5/4}L^{10/9}}\\ &\leq \|V_hu_{0,h}\|_{l^5L^{10}}+C\|\partial_hu_h\|_{l^\infty L^2}\|u_h\|^4_{l^5L^{10}}\\ &\leq \|V_hu_{0,h}\|_{l^5L^{10}}+\frac{1}{2}\|u_h\|_{l^5L^{10}},\end{aligned}$$

so that $\|u_h\|_{l^5L^{10}}\leq 2\|V_hu_{0,h}\|_{l^5L^{10}}$ and it is sufficient to prove that this last quantity goes to 0 when $h,t\to 0$. Let $\varepsilon>0$ be fixed, we introduce $v_0\in H^1$ such that $\|u_0-v_0\|_{L^2}\leq\varepsilon$, and we set $v_{0,h,n}=v_0(nh)$. It is a standard approximation result that $\|Pv_{0,h}-v_0\|_{H^1}\to 0$, this implies both $\|v_{0,h}-u_{0,h}\|_{l^2}\to 0$ and $\|Pv_{0,h}\|_{H^1}\leq C\|v_0\|_{H^1}$. We get then, using the continuity of $V_h:\ H^1(h\mathbb{Z})\to H^1(h\mathbb{Z})$,

$$\begin{aligned}\|V_h\Pi u_{0,h}\|_{l^5L^{10}([0,t])}&\leq\|V_h(\Pi u_{0,h}-v_{0,h})\|_{l^5L^{10}([0,t])}+\|V_hv_{0,h}\|_{l^5L^{10}([0,t])}\\ &\leq\|V_h(\Pi u_{0,h}-v_{0,h})\|_{l^5L^{10}([0,t])}+t^{1/10}\|V_hv_{0,h}\|_{L^\infty[0,T]l^\infty\cap l^2}\\ &\lesssim\|\Pi u_{0,h}-v_{0,h})\|_{l^2}+t^{1/10}\|v_0\|_{H^1}.\end{aligned}$$

As $\limsup_h\|\Pi u_{0,h}-v_{0,h})\|_{l^2}\leq\varepsilon$, by chosing h, (then) t small enough we have obtained

$$\forall\,\varepsilon>0,\ \exists t_0,h_0>0:\ \forall\,\varepsilon\leq t\leq t_0,\ h\leq h_0,\ \|V_h\Pi u_{0,h}\|_{l^5L^{10}([0,t])}\leq 2\varepsilon,$$

which is the expected convergence. □

Remark 10. The convergence of any extracted subsequence to the solution u actually proves that any sequence Pu_{h_n} converges to u if $h_n\to 0$.

6 Scattering

Scattering of the solutions of the continuous $cKdV$ equation for small initial data has been well known for almost 20 years [12]. Roughly, it means that the solution of the nonlinear equation asymptotically behaves like a solution of the linearized equation. This is a phenomenon opposed to the existence of solitons, which are the canonical example of nonlinear behavior. It has the simple following statement.

Theorem 11. *[12, Kenig-Ponce-Vega] Let $u(t)$ be the solution of $cKdV$ with initial data u_0 small enough in $L^2(\mathbb{R})$. Then $w(x) = u_0 - \int_0^\infty V(-s)u^5(s)ds$ is in $L^2(\mathbb{R})$ and is such that*

$$\|u(t) - V(t)w\|_{L^2} \to_{t\to+\infty} 0. \tag{6.1}$$

Remark 12. A similar result holds for $t \to -\infty$, but the functions obtained w, $\widetilde{w}$ have no reason to be equal.

We check here as a first step the existence of a discrete analog, and then discuss the convergence of the discrete functions w_h such that $u_h \sim_t V_h(t)w_h$ to the continuous function w such that $u \sim_t V(t)w$.

Proposition 13. *For $\|u_{0,h}\|_{l^2}$ satisfying the smallness condition of Theorem 7, we denote u_h the solution of $(DcKdV)$. Then $w_h = u_{0,h} - \int_0^\infty V_h(-s)\Pi u_h^5(s)ds$ is in $l^2(\mathbb{Z})$ and is such that*

$$\|u_h(t) - V_h(t)w_h\|_{l^2} \to 0.$$

Proof. According to the formula

$$u_h(t) = V_h(t)u_0 - \int_0^t \partial_h V_h(t-s)\Pi u_h^5(s)ds = V_h(t)\big(u_0 - \int_0^t \partial_h V_h(-s)\Pi u_h^5(s)ds\big),$$

it is sufficient to check that $\int_0^t \partial_h V_h(-s)\Pi u^5(s)ds$ converges in l^2 as $t \to \infty$, thus to check that it is a Cauchy sequence. The inequality (2.8) implies

$$\| \int_T^t \partial_h V_h(-s)\Pi u_h^5(s)ds\|_{l^2} \le \|\Pi u_h^5\|_{l^1L^2([T,\infty[)} \lesssim \|1_{t\ge T}u_h\|^5_{l^5L^{10}([T,\infty[)}.$$

But by dominated convergence we have $\|1_{t\ge T}u_h\|_{l^5L^{10}([T,\infty[)} \to 0$, we can conclude:

$$\|u_h(t) - V_h(t)v_h\|_{l^2} \to 0, \text{ where } v_h = v_0 + \int_0^\infty \partial_h V_h(-s)\Pi u^5(s)ds.$$

□

At this point, it should be emphasized that there is no reason for the discrete solution to behave asymptotically as the continuous one. In fact even in linear settings it is not hard to see that in general

$$\|PV_h(t)u_{0,h} - V(t)u_0\| \nrightarrow_{t\to\infty,h\to 0} 0.$$

Thus at the very best one may prove that $Pw_h \to w$ (or equivalently $PV_h(-t)u_h \to_t V(-t)u$) in a sense that we will clarify.

Proposition 14. *Assume that* $Pu_{0,h} \to_h u_0$ L^2 *and satisfies the* l^2 *smallness condition of Theorem 7, then* $Pw_h \rightharpoonup_h w$ L^2_w.

Proof. According to the previous Proposition,

$$w_h = u_{0,h} - \int_0^\infty V_h(-s)\Pi u_h^5(s)ds,$$

the improper integral being seen as a strong limit in l^2. Since $w = u_0 - \int_0^\infty V(-s)u^5(s)ds$ it is sufficient to check that

$$\int_0^\infty PV_h(-s)\Pi u_h^5(s)ds \rightharpoonup_h \int_0^\infty V(-s)u^5(s)ds.$$

Note that the convergence in h of $\int_0^t V_h(-s)\Pi u_h^5(s)ds$ for fixed t is actually rather easy but leaves open the problem of interversion $\lim_t \lim_h = \lim_h \lim_t$. More directly we shall write

$$\|\int_0^\infty PV_h(-s)\Pi u_h^5(s)ds\|_{L^2} \leq \|P\Pi u_h^5(s)\|_{L^1L^2} \leq \|Pu_h\|L^5L^{10} \leq C,$$

so that we may extract an L^2 weakly converging subsequence. Up to further extraction we may also assume (see the proof of theorem 9) that $Pu_h \to u$ a.e. . Let φ be an L^2 function with compactly supported Fourier transform,

$$\int_{\mathbb{R}}\int_0^\infty P\partial_h V_h \Pi u_h^5 \varphi ds dx = \int_{\mathbb{R}}\int_0^T P\partial_h V_h \Pi u_h^5 \varphi ds dx + \int_{\mathbb{R}}\int_T^\infty P\partial_h V_h \Pi u_h^5 \varphi ds dx.$$

As a first step, we prove that the limit as $T \to \infty$ of the second term in the above right hand side is 0 uniformly in h. In what follows we use the fact that the Fourier multiplier $V_h(t)$ is formally defined by its symbol on $l^2(h\mathbb{Z})$ as well as on $L^2(\mathbb{R})$, and we have $PV_h = V_h P$. Specifically $\int_{\mathbb{R}}\int_T^\infty P\partial_h V_h(-t)\Pi u_h^5 \varphi dt dx = \int_{\mathbb{R}}\int_T^\infty \partial_h P u_h^5 V_h(t)\varphi dx dt$. Hölder's inequality implies

$$|\int_{\mathbb{R}}\int_T^\infty \partial_h P u_h^5 V_h(t)\varphi dx dt| \lesssim \|\partial_h u_h\|_{\infty L^2([T,\infty])} \|u\|^4_{l^5 L^{10}([T,\infty])} \|V_h(t)\varphi\|_{L^5 L^{10}([T,\infty])}$$

Now using the fact that φ has a Fourier transform compactly supported, for h small enough we have [1] $\|V_h(t)\varphi\|_{L^5_x L^{10}_t} \lesssim \|\varphi\|_{L^2}$, thus

$$\lim_T \|V_h(t)\varphi\|_{L^5_x L^{10}([T,\infty]} = 0.$$

This implies our first step: for any $\varepsilon > 0$, there exists $T_0 > 0$, $h_0 > 0$ such that for $T \geq T_0$, $h \leq h_0$,

$$|\int_{\mathbb{R}}\int_T^\infty P\partial_h V_h \Pi u_h^5 \varphi ds dx| \leq \varepsilon. \tag{6.2}$$

[1] This fact may be seen as a simple corollary of the dispersive estimates proved previously: supp$\widehat{\varphi} \subset [-C, C] \Rightarrow \widehat{\varphi} = \widehat{\varphi} 1_{[-\pi/6h, \pi/6h]}$ for h small enough. It is thus supported away from the "bad" points $\pm 2\pi/3$, ξ_0, ξ_1.

We now focus on the convergence of $\int_{\mathbb{R}}\int_0^T P\partial_h V_h \Pi u_h^5 \varphi ds dx$:

$$\begin{aligned}\int_{\mathbb{R}}\int_0^T P\partial_h V_h \Pi u_h^5 \varphi ds dx &= \int_{\mathbb{R}}\int_0^T P\Pi u_h^5 \partial_{-h} V_h(-s)\varphi ds dx \\ &= \int_{\mathbb{R}}\int_0^T P\Pi u_h^5 \Big((V-V_h)\partial_x\varphi + V_h(\partial_x+\partial_{-h})\varphi - V\partial_x\varphi\Big) ds dx.\end{aligned}$$

From the proof of theorem 9, we know that $P\Pi u_h^5 \rightharpoonup u^5$ $(\mathcal{D}'(\mathbb{R}^2))$, and thus weakly in $L_x^1 L_t^2$ since it is bounded in that space: indeed by uniqueness of the distributional limit any extracted subsequence converging weakly in L^1L^2 has limit u^5, so by a standard argument the whole sequence is weakly converging to u^5. In particular

$$\int_{\mathbb{R}}\int_0^T P\Pi u_h^5 V(s)\partial_x\varphi ds dx \to \int_{\mathbb{R}}\int_0^T u^5 V(s)\partial_x\varphi ds dx = -\int_{\mathbb{R}}\int_0^T \partial_x V(-s) u^5\, \varphi ds dx.$$

Moreover for fixed T, one easily sees that $\sup_{s\in[0,T]}\|(V-V_h)(-s)\partial_x\varphi\|_{H_x^1} \to 0$ (it is a simple consequence of the convergence $p_h \to \xi^3$ and the fast decrease of $\widehat{\varphi}$). Thus

$$\|(V-V_h)(-s)\partial_x\varphi\|_{L_x^\infty L_T^2} \to 0, \text{ and } \|(\partial_x+\partial_{-h})\varphi\|_{H^1} \to 0,$$

so that $\|V_h(\partial_x+\partial_{-h})\varphi\|_{L_x^\infty L_T^2} \to 0$, and we finally get

$$\int_{\mathbb{R}}\int_0^T P\partial_h V_h(-s)\Pi u_h^5 \varphi ds dx \to \int_{\mathbb{R}}\int_0^T u^5 \partial_x V(s)\varphi ds dx = -\int_{\mathbb{R}}\int_0^T \partial_x V(-s) u^5 \varphi ds dx. \quad (6.3)$$

Using (6.2), (6.3) we find that for any $\varepsilon > 0$, there exists h_0 small enough such that for $h \le h_0$:

$$\begin{aligned}\Big|\int_{\mathbb{R}}\int_0^\infty P\partial_h V_h \Pi u_h^5 \varphi ds dx - \int_{\mathbb{R}}\int_0^\infty \partial_x V u^5 \varphi ds dx\Big| &\le \Big|\int_{\mathbb{R}}\int_T^\infty P\partial_h V_h \Pi u_h^5 \varphi ds dx \\ +\Big|\int_{\mathbb{R}}\int_T^\infty \partial_x V u^5 \varphi ds dx\Big| &+ \Big|\int_{\mathbb{R}}\int_0^T P\partial_h V_h \Pi u_h^5 \varphi - \partial_x V u^5 \varphi ds dx\Big| \\ &\le 3\varepsilon.\end{aligned}$$

We have proved that for any φ with compactly supported Fourier transform,

$$\int_{\mathbb{R}}\int_0^\infty P\partial_h V_h \Pi u_h^5 \varphi ds dx \to \int_{\mathbb{R}}\int_0^\infty \partial_x V u^5 \varphi ds dx,$$

using the density of these functions in L^2 and the L^2 boundedness of $\int_0^\infty P\partial_h V_h \Pi u_h^5 dt$, we obtain the weak L^2 convergence.

□

7 Rates of convergence

Since the previous sections were only devoted to the critical case, it is very unlikely that any rate of convergence may be obtained for L^2 initial data . However if $u_0 \in H^s$, $s > 0$ the problem becomes subcritical and more quantitative estimates are expected. We first define discrete initial data converging to the continuous one in the following way. Let T_h be defined as

$$T_h : \ L^2 \to l^2(h\mathbb{Z}),$$
$$(T_h u)_n = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} \widehat{u} e^{inh\xi} d\xi,$$

where $\widehat{\cdot}$ denotes the Fourier transform on $\mathbb{R}$ (opposed to the *discrete* Fourier transform). If u_0 is the initial continuous data, we simply set $u_{0,h} = T_h u_0$. By dominated convergence, it can be seen (though it is not absolutely obvious) that for any $f \in L^2$, $\|PT_h f - f\|_{L^2} \to 0$, and more precisely $\|PT_h f - f\|_{L^2} \lesssim h^s \|f\|_{H^s}$. For this reason we will only study the convergence to 0 of $u_h - T_h u$ rather than $Pu_h - u$. Note that the operator $T_h$is particularly convenient since $\widehat{V(t)T_h u_0}|_{[-\pi/h,\pi/h]} = e^{it\xi^3}\widehat{u_0} = T_h\widehat{V(t)u_0}|_{[-\pi/h,\pi/h]}$. Consequently - as in previous section- *in what follows we will abusively write $V(t)$ for both multipliers of symbol $e^{it\xi^3}$ acting on $L^2(\mathbb{R})$ and $l^2(h\mathbb{Z})$.*

Convergence rates were obtained for the (subcritical) nonlinear Schrödinger equation in [5] by using “discrete Littlewood-Paley analysis“. Our case is slightly more complicated, for essentially three reasons; critical regularity, lack of ”choice“ for the Strichartz estimates, and the nonlinearity involving derivatives. The section is divided in two parts. The first one establishes a list of linear estimates with rates of convergence. Unfortunately this list is not sufficient to obtain actual rates of convergence on $T_h u - u_h$ but we prove such results for a simpler semi-linear problem. Though it is not entirely satisfactory we believe that the result and the technics used are interesting by themselves.

7.1 Linear estimates

As a warm up, we first treat the control of $\|T_h V(t)u_0 - V_h(t)\Pi T_h u_0\|_{l^2}$, which is quite simple but gives a good idea of the technics used in this subsection.

Proposition 15. *Let Π be an interpolator as in theorem 7, of symbol m such that $m(0) = 1$. For $u_0 \in L^2$, we have the homogeneous estimate*

$$\|V(t)T_h u_0 - V_h(t)T_h \Pi T_{Nh} u_0\|_{L^\infty([0,T];l^2)} \leq Ch^{2s/5}\|u_0\|_{H^s(\mathbb{R})}. \tag{7.1}$$

The constant C only depends on p and Π.

Proof. First we note that the scheme is of order 2 in the sense that

$$\xi^3 - p_h = \xi^3 - \frac{4\sin^2(\xi h/2)\sin(\xi h)}{h^3} = O(h^2\xi^5) \text{ (by Taylor expansion).} \tag{7.2}$$

We split the left hand side of (7.1) as follows

$$\begin{aligned}\|V(t)T_hu_0 - V_h(t)\Pi T_{Nh}u_0\|_{L^\infty([0,T];l^2)} &\leq \|V(t)\Pi T_{Nh}u_0 - V_h(t)\Pi T_{Nh}u_0\|_{L^\infty([0,T];l^2)} \\ &\quad +\|V(t)(T_h - \Pi T_{Nh})u_0\|_{L^\infty([0,T];l^2)} \\ &= N_1 + N_2.\end{aligned}$$

Using the inequality

$$\forall\, 0 \leq \alpha \leq 1,\ a, b \geq 0,\ \min(a, b) \leq a^\alpha b^{1-\alpha} \tag{7.3}$$

we find

$$N_1^2 \lesssim h^{4s/5} T \int_{\mathbb{R}} \xi^{2s} |\widehat{\Pi T_{Nh} u_0}|^2 d\xi \lesssim h^{4s/5} \|u_0\|_{H^s}^2. \tag{7.4}$$

On the other hand if we denote m the symbol of Π

$$N_2^2 \leq \|(T_h - \Pi T_{Nh})u_0\|_{l^2)} = \int_{-\pi/h}^{\pi/h} |\widehat{u_0} - m(h\xi)\widetilde{u_0}(\xi)|^2 d\xi,$$

where $\widetilde{u_0}$ is the $2\pi/Nh$ periodic function such that $\widetilde{u_0}|_{[-\pi/Nh,\pi/Nh]} = \widehat{u_0}$. But since m is bounded and $m(0) = 1$, $|m(h\xi) - 1| \leq Ch^s|\xi|^s$. We also remind (see Remark 3) that m satisfies $m(2k\pi\xi/N) = 0,\ 1 \leq k \leq N-1$, thus $m(2k\pi/N + h\xi) = O(h^s|\xi|^s)$. This gives

$$\begin{aligned}\int_{-\pi/h}^{\pi/h} |\widehat{u_0} - m(h\xi)\widetilde{u_0}(\xi)|^2 d\xi &= \int_{-\pi/Nh}^{\pi/Nh} |\widehat{u_0} - m(h\xi)\widehat{u_0}(\xi)|^2 d\xi \\ &\quad + \int_{\pi/Nh \leq |\xi| \leq \pi/h} |\widehat{u_0} - m(h\xi)\widetilde{u_0}(\xi)|^2 d\xi \\ &\lesssim \int_{-\pi/h}^{\pi/h} h|\xi|^2 |\widehat{u_0}|^2 d\xi + \sum_{k=1}^{N-1} \int_{(2k-1)\pi/Nh}^{(2k+1)\pi/Nh} |m(h\xi)|\widehat{u_0}(\xi)|^2 d\xi. \\ &\lesssim h^{2s} \int_{-\pi/Nh}^{\pi/Nh} |\xi|^{2s} |\widehat{u_0}|^2 d\xi + h^{2s} \int_{-\pi/h}^{\pi/h} |\xi|^{2s} |\widehat{u_0}|^2 d\xi \\ &\leq 2h^{2s} \|u_0\|_{H^s}^2.\end{aligned} \tag{7.5}$$

Summing (7.4), (7.5) we get

$$\|V(t)T_hu_0 - V_h(t)\Pi T_{Nh}u_0\|_{L^\infty([0,T];l^2)} \leq CT(h^{2s/5} + h^{s/2})\|u_0\|_{H^s} \leq 2CTh^{2s/5}\|u_0\|_{H^s}.$$

□

Proposition 16. *Under the same assumptions as Prop. 15, and for any $g \in l^{4/3}L_T^1$,*

$$\||D|^{-1/4} V(t)T_hu_0 - V_h(t)\Pi T_{Nh}u_0\|_{l^4 L^\infty([0,T])} \leq C(T)h^{2s/5}\|u_0\|_{H^s(\mathbb{R})}, \tag{7.6}$$

$$\||D|^{-1/2} \int_0^t (V(t-s)g - V_h(t-s))\Pi T_{Nh}g\|_{l^4 L_T^\infty} \leq C(T)h^{2s/5}\||D|^s \Pi T_{Nh}g\|_{l^{4/3}L^1([0,T])}. \tag{7.7}$$

Proof. As in the proof of Prop 15, we write

$$\| |D|^{-1/4}(V(t)T_hu_0 - V_h(t)\Pi T_{Nh}u_0)\|_{l^4L^\infty([0,T])} \leq \| |D|^{-1/4}(V - V_h)\Pi T_{Nh}u_0\|_{l^4L^\infty} \quad (7.8)$$
$$+ \quad \| |D|^{-1/4}V(T_h - \Pi T_{Nh})u_0\|_{l^4L^\infty}. \quad (7.9)$$

We have directly using (7.5)

$$\||D|^{-1/4}V(t)(T_h - \Pi T_{Nh})u_0\|_{l^4L^\infty([0,T])} \lesssim \|(T_h - \Pi T_{Nh})u_0\|_{l^2} \lesssim h^s\|u_0\|_{H^s},$$

so that it suffices to prove the (more precise) estimate

$$\|V(t)\Pi T_{Nh}u_0 - V_h(t)\Pi T_{Nh}u_0\|_{l^4L^\infty([0,T])} \lesssim h^{2s/5}T^{s/5}\| |D_x|^s\Pi T_{Nh}u_0\|_{L^2}. \quad (7.10)$$

A careful look at the proof of Prop 4 shows that it amounts to the estimate

$$\left| \int_{-\pi/h}^{\pi/h} \psi_h \frac{e^{itp_h(\eta)} - e^{it\eta^3}}{|\eta|^{1/2+s}} e^{ijh\eta} d\eta \right| \leq C \frac{h^{2s/5}T^{s/5}}{(h(1+|j|))^{1/2}}, \quad (7.11)$$

where ψ_h is the symbol of Π and cancels at $\pm 2\pi/(3h)$, ξ_0/h, ξ_1/h. Since (7.11) also implies by the duality argument of Prop 4 the estimate (7.7), the rest of the proof is devoted to its derivation. It is equivalent after setting $\eta = \xi/h$ to

$$\left| \int_{-\pi}^{\pi} \psi(\eta) \frac{e^{it/h^3 p(\eta)} - e^{it/h^3\eta^3}}{|\xi|^{1/2+s}} e^{ijh\eta} d\xi \right| \leq C \frac{h^{-3s/5}T^{s/5}}{(1+|j|)^{1/2}}.$$

By parity, we may also reduce it to

$$\left| \int_{0}^{\pi} \psi(\xi) \frac{e^{it/h^3 p(\xi)} - e^{it/h^3\xi^3}}{|\xi|^{1/2+s}} e^{ijh\xi} d\xi \right| \leq C \frac{h^{-3s/5}T^{s/5}}{(1+|j|)^{1/2}}.$$

The proof of this estimate is rather delicate, in fact it follows the proof of Prop 4 with some non trivial modifications. Since a lot of quantities which will appear are estimated by similar technics, we will often skip details.

We will use repeatedly the fact that ξ lies in a bounded set, thus the inequality $|p - \xi^3| \lesssim |\xi|^5$ implies $|p - \xi^3| \lesssim |\xi|^r$ for $0 \leq r \leq 5$. Similarly $|e^{it/h^3p} - e^{it/h^3\xi^3}| \leq |t/h^3\xi^5|^r$ for $0 \leq r \leq 1$. First note that for $j = 0$ the result is trivial since

$$\left| \int_{0}^{\pi} \psi(\xi) \frac{e^{it/h^3 p(\xi)} - e^{it/h^3\xi^3}}{|\xi|^{1/2+s}} e^{ijh\xi} d\xi \right| \leq C \left| \int_{-\pi}^{\pi} \frac{(t/h^3)^{s/5}|\xi|^s}{|\xi|^{1/2+s}} d\xi \right|. \leq Ch^{-3s/5}T^{s/5}$$

We split $[0,\pi]$ as

$$A_1 = [0, ht^{-1/3}] \cap [0,\pi],$$
$$A_2 = A_1^c \cap \{|3t/h^3\xi^2 + j| \geq |j|/2\},$$
$$A_3 = (A_1 \cup A_2)^c.$$

If $ht^{-1/3} \leq C/J$ with C large enough such that on A_1 $\min(|j+3t/h^3\xi^2|, |j+t/h^3p'|) \geq |j|/2$,

$$\Big|\int_{A_1} \psi(\xi)\frac{e^{it/h^3 p(\xi)} - e^{it/h^3\xi^3}}{|\xi|^{1/2+s}} e^{ijh\xi} d\xi\Big| \leq \int_{-c/j}^{C/j} \frac{(t/h^3)^{s/5}}{|\xi^{1/2}} d\xi \leq \frac{h^{-3s/5}T^{s/5}}{|j|^{1/2}}$$

and the estimate on A_1 is complete. Else

$$\begin{aligned}\Big|\int_{A_1} \psi(\xi)\frac{e^{it/h^3 p(\xi)} - e^{it/h^3\xi^3}}{|\xi|^{1/2+s}} e^{ijh\xi} d\xi\Big| &\leq \Big|\int_{-1/j}^{1/j} \psi(\xi)\frac{e^{it/h^3 p(\xi)} - e^{it/h^3\xi^3}}{|\xi|^{1/2+s}} e^{ijh\xi} d\xi\Big| \\ &\quad + \Big|\int_{1/j\leq|\xi|\leq ht^{-1/3}} \psi(\xi)\frac{e^{it/h^3 p(\xi)} - e^{it/h^3\xi^3}}{|\xi|^{1/2+s}} e^{ijh\xi} d\xi\Big| \\ &\lesssim \Big|\int_{1/j\leq|\xi|\leq ht^{-1/3}} \psi(\xi)\frac{e^{it/h^3 p(\xi)} - e^{it/h^3\xi^3}}{|\xi|^{1/2+s}} e^{ijh\xi} d\xi\Big| \\ &\quad + \frac{h^{-3s/5}T^{s/5}}{|j|^{1/2}}\end{aligned}$$

On $\xi \geq 0$, an integration by part gives

$$\begin{aligned}\int_{1/|j|}^{ht^{-1/3}} \psi(\xi)\frac{e^{it/h^3 p(\xi)} - e^{it/h^3\xi^3}}{|\xi|^{1/2+s}} e^{ijh\xi} d\xi &= \frac{1}{|ht^{-1/3}|^{1/2+s}}\left(\frac{e^{it/h^3 p(ht^{-1/3})}}{j+\frac{t}{h^3}p'(ht^{-1/3})} - \frac{e^i}{j+3\frac{t^{1/3}}{h}}\right) \\ &- j^{1/2+s}\left(\frac{e^{it/h^3 p(1/j)}}{j+\frac{t}{h^3}p'(1/j)} - \frac{e^{it/(hj)^3}}{j+3\frac{t}{h^3 j^2}}\right) \\ &+ \int_{1/j}^{ht^{-1/3}} e^{it\xi^3/h^3}\left(\frac{\psi}{(j+3t/h^3\xi^2)|\xi|^{1/2+s}}\right)' \\ &- \int_{1/j}^{ht^{-1/3}} e^{itp/h^3}\left(\frac{\psi}{(j+t/h^3p')|\xi|^{1/2+s}}\right)'.\end{aligned}$$

Set $y = ht^{-1/3}$, note that $1/|y| \leq j$. The first term is estimated by using the fact that $|p'/\xi^2 - 3| \leq \min(|\xi|^2, 1)$:

$$\begin{aligned}\Big|\frac{1}{(ht^{-1/3})^{1/2+s}}\left(\frac{e^{it/h^3 p(ht^{-1/3})}}{j+\frac{t}{h^3}p'(ht^{-1/3})} - \frac{e^i}{j+3\frac{t^{1/3}}{h}}\right)\Big| &= \Big|y^{-(1/2+s)}\left(\frac{e^{ip(y)/y^3}}{j+p'(y)/y^3} - \frac{e^i}{j+3/y}\right)\Big| \\ &= |y|^{-(1/2+s)}\frac{e^{ip(y)/y^3}(j+3/y) - e^i(j+p'/y^3)}{(j+p'/y^3)(j+3/y)} \\ &= |y|^{-(1/2+s)}\Big|\frac{j(e^{ip/y^3}-e^i) + 1/y\big(3e^{ip/y^3} - 3e^i + 3e^i - p'/y^2 e^i\big)}{(j+p'/y^3)(j+3/y)}\Big| \\ &\lesssim |y|^{-(1/2+s)}\frac{|jy^{2s/5}|}{j^2} \lesssim \frac{1}{\sqrt{j}}y^{-3s/5} = \frac{h^{-3s/5}T^{s/5}}{\sqrt{|j|}}.\end{aligned}$$

The estimate for the second term is similar, and we only give details for two of the remaining integral terms involved:

$$\Big|\int_{1/j}^{y} e^{i\xi^3/y^3}\frac{6\xi/h^3\psi(\xi)}{(j+3\xi^2/y^3)^2|\xi|^{1/2+s})} - e^{ip/y^3}\frac{p''(\xi)/y^3\psi(\xi)}{(j+p'(\xi)/y^3)|\xi|^{1/2+s}}d\xi\Big|$$

$$\lesssim \int_{1/j}^{y}\frac{|p''-6\xi|}{|y|^3j^2|\xi|^{1/2+s}}d\xi + \int_{1/j}^{y}\frac{|p''|}{y^3|\xi|^{1/2+s}}\Big|\frac{1}{(j+p'/y^3)^2}-\frac{1}{(j+3\xi^2/y^2)^2}\Big|$$

$$\lesssim \int_{1/j}^{y}\frac{|\xi|^{1+2s/5}}{j^2|\xi|^{7/2+2s/5}y^{3s/5}}d\xi + \int_{1/j}^{y}\frac{|p''|}{y^3|\xi|^{1/2+s}j^4}\Big|2j/y^3(3\xi^2-p'(\xi))+9\xi^4/y^6-(p')^2/y^6\Big|$$

$$\lesssim \frac{h^{-3s/5}T^{s/5}}{\sqrt{|j|}} + \int_{1/j}^{y}\frac{|\xi|}{y^3|\xi|^{1/2+s}j^4}\Big(|2j|\xi|^{2+2s/5}/y^3| + |\xi^{4+2s/5}|/y^6\Big)d\xi$$

$$\lesssim \frac{h^{-3s/5}T^{s/5}}{\sqrt{|j|}} + \int_{1/j}^{y}\frac{y^{-3s/5}}{|\xi|^{3+1/2}j^3} + \frac{y^{-3s/5}}{|\xi|^{4+1/2}j^4})d\xi$$

$$\lesssim \frac{h^{-3s/5}T^{s/5}}{\sqrt{|j|}}.$$

The analysis on A_2 is similar (and in fact simpler) so we skip it and prove the estimate for the integral on $A_3 = \{|ht^{-1/3}| \le |\xi| \le \pi\} \cap \{j + 3t|\xi|^2/h^3\} \le |j|/2$. Although in the proof of Prop 4 the neighbourhood of the points where p'' cancels was the delicate part, it is here easy. Indeed (the proof of) Prop 4 implies that on a small neighbourhood $\mathcal{V}$ of $\{\xi_0, \pi\}$,

$$\Big|\int_{\mathcal{V}}\psi(\xi)\frac{e^{it/h^3p(\xi)}-e^{it/h^3\xi^3}}{|\xi|^{1/2+s}}e^{ij\xi}d\xi\Big| \lesssim \frac{1}{\sqrt{|j|}} \le \frac{h^{-3s/5}T^{s/5}}{\sqrt{|j|}},$$

(this is due to the fact that $|\xi|$ is bounded away from 0 on $\mathcal{V}$, thus having $|\xi|^{1/2+s}$ on the denominator instead of $|\xi|^{1/2}$ does not change the analysis).
On $A_4 = A_3 \cap \mathcal{V}^c$, we have $|3t\xi^2/h^3 + j| \le j/2$, thus $|\xi| \asymp \sqrt{jh^3/t}$, and the Lebesgue measure of A_4 is dominated by $\sqrt{jh^3/t}$. Moreover ξ is bounded away from $\{\xi_0, \pi\}$, thus $|p''| \gtrsim |\xi|$. The van der Corput lemma implies

$$\Big|\int_{A_4}\frac{e^{it/h^3p}-e^{it/h^3\xi^3}}{|\xi|^{1/2+s}}\psi d\xi\Big| \lesssim \frac{1}{\sqrt{t/h^3\min_{A_4}|\xi|}}\max\frac{1}{|\xi|^{1/2+s}} \lesssim \frac{t^{s/2}h^{-3s/2}}{|j|^{(1+s)/2}}.$$

It is sufficient to prove

$$\Big|\int_{A_4}\frac{e^{it/h^3p}-e^{it/h^3\xi^3}}{|\xi|^{1/2+s}}\psi d\xi\Big| \lesssim \frac{1}{j^{1/2-s/3}},$$

indeed it implies then

$$\Big|\int_{A_4}\frac{e^{it/h^3p}-e^{it/h^3\xi^3}}{|\xi|^{1/2+s}}\psi d\xi\Big| \lesssim \Big(\frac{t^{s/2}h^{-3s/2}}{|j|^{(1+s)/2}}\Big)^{2/5}\Big(\frac{1}{j^{1/2-s/3}}\Big)^{3/5} \le \frac{T^{s/5}h^{-3s/5}}{\sqrt{|j|}}.$$

Since A_4 is the union of at most two intervals, we may assume that A_4 is an interval. Set $f = e^{it/h^3 p} - e^{it/h^3\xi^3}$, F a primitive of f that vanishes at some point of A_4. An integration by parts gives

$$\begin{aligned}\left|\int_{A_4}\frac{e^{it/h^3p}-e^{it/h^3\xi^3}}{|\xi|^{1/2+s}}\right| &\lesssim \frac{\max_{A_4}|F|}{\min_{A_4}|\xi|^{1/2+s}} + \left|\int_{A_4}\frac{F}{|\xi|^{3/2+s}}\psi d\xi\right| + \left|\int_{A_4}\frac{F}{|\xi|^{1/2+s}}\psi' d\xi\right| \\ &\lesssim \frac{\max_{A_4}|F|}{\min_{A_4}|\xi|^{1/2+s}} \\ &\lesssim \frac{\max_{A_4}|F|}{|jh^3/t|^{1/4+s/2}}\end{aligned}$$

It is now easily seen that we "only" need to prove that

$$\forall\, \xi \in A_4,\ |F| \lesssim \frac{(h^3)^{1/4+s/2}t^{-(1/4+s/2)}}{|j|^{1/4-5s/6}}. \tag{7.12}$$

Remind that $|\xi| \asymp \sqrt{jh^3/t}$. The van der Corput lemma implies

$$|F(\xi)| \lesssim \frac{1}{\sqrt{t/h^3\sqrt{jh^3/t}}} = \left(\frac{h^3}{tj}\right)^{1/4}. \tag{7.13}$$

On the other hand the Lipschitz continuity of e^{ix} and $|A_4| \lesssim \sqrt{jh^3/t}$ gives

$$|F| \lesssim t/h^3 \max|\xi|^4 \lesssim j^2h^3/t. \tag{7.14}$$

Finally, "interpolation" of (7.13) and (7.14) implies

$$|F| \lesssim \left(\frac{h^3}{tj}\right)^{1/4(1-2s/3)} (j^2h^3/t)^{2s/3} = \left(\frac{h^3}{t}\right)^{1/4+s/2}\frac{1}{j^{1/4-5s/6}}.$$

which is (7.12). The proof is now complete. □

Remark 17. So far in every estimates one looses s derivatives to gain a rate in $h^{2s/5}$. This is probably optimal considering the inequality $|p - p_h| \le Ch^2|\xi|^5$. On the contrary, the l^5L^{10} estimate will not be optimal, this is due to the fact that it is obtained via the interpolation of the estimates above with the dispersive smoothing results of section 2.
Nevertheless, it should be noticed that without dispersive estimates, one may only obtain for example

$$\|\big(V(t)-V_h(t)\big)\partial_h\Delta_j\Pi T_{Nh}u_0\|_{l^\infty L^2_T} \le h^{2s/5}2^{j(s+3/2)}\|\Delta_j\Pi T_{Nh}u_0\|_{l^2},$$

this would lead to estimates involving $h^{2s/5}\|u_0\|_{H^{3/2+s}}$ that clearly forbid low regularity results.

Using the interpolation argument of [13] prop. 7.4 (as for the Corollary 2) we have the following.

Proposition 18. *Let* $u_0 \in L^2(\mathbb{R})$, $g \in l^{5/4}(h\mathbb{Z};\ L_T^{10/9})$, *we have the following estimates*

$$\|V(t)T_h u_0 - V_h(t)\Pi T_{Nh} u_0\|_{l^5 L_T^{10}} \leq C(T) h^{8s/25} \|u_0\|_{H^s(\mathbb{R})}, \tag{7.15}$$

$$\|\int_0^t V(t-s)g - V_h(t-s)\Pi T_{Nh} g\|_{l^5 L_T^{10}} \leq C(T) h^{8s/25} \||D|^s T_h g\|_{l^{5/4} L^{10/9}(([0,T])}. \tag{7.16}$$

Remark 19. The exponent 8/25 comes from the weights in the interpolation, which are respectively 4/5 for inequality (7.6) and 1/5 for inequality (2.3).

More dispersive estimates (not useful for the next subsection) are given in Appendix B.

7.2 A simpler problem

Though we did not manage to collect enough dispersive estimates to obtain rates of convergence for the approximation of the cKdV problem, we will describe for a simpler problem how these estimates may be succesfully used. Let us consider the semi-linear equation

$$\begin{cases} \partial_t u + \partial_x^3 u + f(u) = 0, \\ u|_{t=0} = u_0 \in L^2, \end{cases} \tag{7.17}$$

where $f(u) = u|u|^{3/2}$. It is quite clear that the existence of an L^2 solution may not be obtained by basic semigroup methods, however using $\|V(t)u_0\|_{L_x^5 L_t^{10}} \lesssim \|u_0\|_{L^2}$ and its inhomogeneous counterpart, we can solve the equation

$$Tu = u \text{ where } Tu(t) = V(t)u_0 - \int_0^t V(t-s)f(u)(s)ds$$

by a fixed point argument for small times or small initial data. Indeed

$$\begin{aligned} \|Tu\|_{L_x^2} \leq \|u_0\|_{L^2} + \|V(t-\cdot)f(u)\|_{L_x^2 L_t^1} &\leq \|u_0\|_{L^2} + t^{1/2}\|V(t-\cdot)f(u)\|_{L_x^2 L_t^2} \\ &\leq \|u_0\|_{L^2} + t^{1/2}\|f(u)\|_{L_x^2 L_t^2} \\ &\leq \|u_0\|_{L^2} + t^{1/2}\|u\|_{L_x^5 L_t^5}^{5/2} \\ &\leq \|u_0\|_{L^2} + t^{3/4}\|u\|_{L_x^5 L_t^{10}}^{5/2}, \end{aligned}$$

similarly

$$\begin{aligned} \|Tu\|_{L_x^5 L_t^{10}} \leq \|u_0\|_{L^2} + \|u\|_{L_x^{25/8} L_t^{25/9}}^{5/2} &\leq \|u_0\|_{L^2} + t^{1/10}\|u\|_{L_{x,t}^{25/8}}^{5/2} \\ &\lesssim \|u_0\|_{L^2} + t^{1/10}(\|u\|_{L_t^{25/8} L_x^2}^{5/2} + \|u\|_{L_t^{25/8} L_x^5}^{5/2}) \\ &\leq \|u_0\|_{L^2} + t^{13/20}\|u\|_{L_x^5 L_t^{10}}^{5/2} + t^{9/10}\|u\|_{L_t^\infty L_x^2}^{5/2}. \end{aligned}$$

For t small enough (or small initial data), these estimates are sufficient to apply the Picard-Banach fixed point theorem in the space $X_T = L^\infty_T L^2_x \cap L^5_x L^{10}_T$, which implies existence and uniqueness of a solution in this space.
We focus now on the derivation of rates of convergence. We define as for $(cKdV)$ the semi-discrete approximation scheme

$$\begin{cases} \frac{d}{dt}u_n + \frac{u_{n+2} - 2u_{n+1} + 2u_{n-1} - u_{n-2}}{h^3} + (\Pi E_{Nh} f(u_h))_n = 0, \\ u_h|_{t=0} = \Pi T_{Nh} u_0, \end{cases} \tag{7.18}$$

Using the discrete version of X_T, $X_{h,T} = L^\infty_T l^2_x \cap l^5_x L^{10}_T$, it can be proved as for the continuous problem that for T small enough there exists an unique solution of this problem admitting bounds in $X_{h,T}$ independent of h. The following theorem establishes a precise convergence of u_h to u as $h \to 0$.

Theorem 20. *Let u be the solution of (7.17) and u_h the solution of (7.18). For T small enough and* $0 < s \leq 1$

$$\|u_h - T_h u\|_{X_{h,T}} \lesssim h^{8s/25}(\,\|u\|_{X_T} + \||D|^s u\|_{X,T} + \|u\|^{5/2}_{X_T} + \||D|^s u\|^{5/2}_{X_T}\,). \tag{7.19}$$

The $W^{s,p}$ spaces are defined here as the usual Bessel potential spaces, namely

$$\{f:\ \mathcal{F}^{-1}\big((1+|\xi|)^s \widehat{f}\big) \in L^p\}.$$

For the proof of the theorem we will need several technical properties on fractional derivation and Fourier multipliers:

- For any $\alpha \in (0,1)$, p, p_1, p_2, such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ then for F differentiable $\||D|^\alpha F(f)\|_{L^p} \leq C\|F'(f)\|_{L^{p_1}}\||D|^\alpha f\|_{L^{p_2}}$ (see [1] section 3 for a proof).
- The l^p norm of $T_h f$ is equivalent to the L^p norm of $\mathcal{F}^{-1}(\chi_{[-\pi/h,\pi/h]}\widehat{f})$, independently of h (see Lemma 2.1 in [5], referring itself to the classical article [15], we include a sketch of proof for the estimate $\|\mathcal{F}^{-1}T_h f\|_{L^p} \leq \|T_h f\|_{l^p}$ in the appendix).
- We have for any $1 < p < \infty$,

  $$\|T_h f - \Pi T_{Nh} f\|_{l^p} \leq C h^s \||D|^s f\|_{L^p(\mathbb{R})} \tag{7.20}$$

  This is proved for a very slightly less general Π in [5], in the end of the proof of their Theorem 4.2. It relies on their Lemma 2.1 combined with the Marcinkiewicz multiplier theorem (that they state in appendix). The main ingredient is that the symbol of $T_h - \Pi T_{Nh}$ is bounded by $h^s|\xi|^s$.

- For $s \in (0,1)$, there exists $C > 0$ independent of h such that
$$\|f(T_h u) - T_h f(u)\|_{l^2} \le Ch^s \|u\|^{5/2}_{W^{s,5}}, \ \|f(T_h u) - T_h f(u)\|_{l^{5/4}} \le Ch^s \|u\|^{5/2}_{W^{s,25/8}}, \quad (7.21)$$
again, the proof is given in [5], Lemma 5.2, for integration exponents different of 2, 5, but the proof can be adapted without significant modifications.

Proof. We begin by writing the difference as
$$u_h - T_h u = V(t)T_h u_0 - V_h(t)\Pi T_{Nh} u_0 - \int_0^t V(t-s)T_h f(u)(s) - V_h(t-s)\Pi E f(u_h)(s) ds.$$
The first linear term is directly controlled by applying inequalities (7.15) and (7.1) :
$$\|V(t)T_h u_0 - V_h(t)\Pi T_{Nh} u_0\|_{X_{h,T}} \le C(T) h^{8s/25} \|u_0\|_{H^s(\mathbb{R})}.$$
We split the second, non-linear, term as follows :
$$\begin{aligned}\int_0^t V(t-s)T_h f(u)(s) - V_h(t-s)\Pi f(u_h)(s) ds &= \int_0^t V(t-s)(T_h f(u) - \Pi T_{Nh} f(u)) \\ +(V - V_h)\Pi T_{Nh} f(u) + V_h \Pi (T_{Nh} - ET_h) f(u) &+ V_h \Pi E(T_h f(u) - f(u_h)) ds \\ &= I_1 + I_2 + I_3 + I_4\end{aligned}$$
Since most of the estimates are obtained in very similar ways, we will only detail how to deal with I_1 and I_4. The L^2 bound for I_1 is obtained by using (7.5) and the fractional chain rule with $p_1 = 10/3$, $p_2 = 5$:
$$\begin{aligned}\|\int_0^t V(t-s)(T_h f(u) - \Pi T_{Nh} f(u)) ds\|_{L^2} &\lesssim t^{1/2} \|T_h f(u) - \Pi T_{Nh} f(u)) ds\|_{L^2_T l^2} \\ &\lesssim h^s \||D|^s f(u)\|_{L^2_T L^2_x} \\ &\lesssim h^s \|u\|^{3/2}_{L^5_x L^{10}_T} \||D|^s u\|_{L^5_x L^{10}_T}.\end{aligned}$$
For the $l^5 L^{10}_T$ bound, we use the estimate (7.20) to get
$$\begin{aligned}\|\int_0^t V(t-s)(T_h f(u) - \Pi T_{Nh} f(u)) ds\|_{l^5 L^{10}} &\lesssim \|T_h f(u) - \Pi T_{Nh} f(u)\|_{L^{5/4} l^{5/4}} \\ &\lesssim h^s \||D|^s f(u)\|_{L^{5/4}_t L^{5/4}_x},\end{aligned}$$
and using again the chain rule with $p_1 = 10/3, \ p_2 = 2$ we find
$$\begin{aligned}\|\int_0^t V(t-s)(T_h f(u) - \Pi T_{Nh} f(u)) ds\|_{l^5 L^{10}} &\lesssim h^s \| \, \|u\|^{3/2}_{L^5_x} \||D|^s u\|_{L^2_x} \|_{L^{5/4}_t} \\ &\lesssim \|u\|^3_{L^5_x L^{10}_T} + \|u\|^2_{L^\infty_T H^s_x}.\end{aligned}$$

For I_4, we have using (7.21)

$$\begin{aligned}\|\int_0^t V_h \Pi E(T_h f(u) - f(u_h))ds\|_{l^2} &\lesssim \|T_h f(u) - f(T_h u)\|_{L^1([0,T];l^2)} + \|f(T_h u) - f(u_h)\|_{L^1 l^2} \\ &\lesssim h^s \|u\|^{5/2}_{L^1_T W^{s,5}} + \|f(T_h u) - f(u_h)\|_{L^1_T l^2}\end{aligned}$$

Since $\|f(T_h u) - f(u_h)\|_{L^1_T l^2} \lesssim \|T_h u - u_h\|_{L^2 l^5}(\|T_h u\|^{3/2}_{L^3 l^5} + \|u_h\|^{3/2}_{L^3 l^5})$, using again Hölder's inequality in time, this term can be absorbed (for T small enough independent of h) in the left hand side. The $l^5 L^{10}$ norm of I_4 is dealt with in the same way:

$$\begin{aligned}\|\int_0^t V_h \Pi E(T_h f(u) - f(u_h))ds\|_{l^5 L^{10}} &\lesssim \|T_h f(u) - f(u_h\|_{L^{5/4} l^{5/4}} \\ &\lesssim \|T_h f(u) - f(T_h u)\|_{L^{5/4} l^{5/4}} + t^{1/10}\|f(T_h u) - f(u_h)\|_{L^{5/4} l^{5/4}} \\ &\lesssim h^s(\||D|^s u\|^{5/2}_{l^5 L^{10}} + \|u\|^{5/2}_{l^5 L^{10}} + \|u\|_{L^\infty H^s}) \\ &\quad + t^{1/10}\|T_h u - u_h\|_{L^{25/8} l^{25/8}}(\|u\|_{L^{25/8} l^{25/8}} + \|u_h\|_{L^{25/8} l^{25/8}}) \\ &\lesssim h^s(\||D|^s u\|^{5/2}_{l^5 L^{10}_T} + \|u\|^{5/2}_{l^5 L^{10}_T} + \|u\|_{L^\infty_T H^s}) \\ &\quad + T^{13/20}\|T_h u - u_h\|_{X_T}(\|u_h\|_{X_{h,T}} + \|u\|_{X_T})\end{aligned}$$

(in this chain of inequality we implicitly used the continuity of $T_h : L^p \to l^p$ before interversion of time and space integration). As previously for t small enough the second term of the right hand side can be absorbed in the left hand side. Gluing all the estimates we have obtained

$$\begin{aligned}\|T_h u - u_h\|_{l^5 L^1 0_t \cap L^\infty_T l^2} \leq Ch^{8s/25}(&\|u\|^{5/2}_{L^\infty_T H^s} + \|u\|_{l^5 L^{10}_T} + \||D|^s u\|_{l^5(L^{10}_T} \\ &+ \|u\|^{5/2}_{L^\infty_T H^s} + \|u\|^{5/2}_{l^5 L^{10}_T} + \||D|^s u\|^{5/2}_{l^5(L^{10}_T}).\end{aligned}$$

□

Remark 21. It seems likely that (similarly to classical results for the nonlinear Schrödinger equation) the estimate (7.19) can be turned into

$$\|u_h - T_h u\|_{X_{h,T}} \lesssim h^{8s/25}\|u_0\|_{H^s}.$$

Remaining questions and perspectives There are several questions left open that we list here in what we believe is their order of difficulty :

- The existence of rates of convergence for the approximation of the quasi-linear cKdV equation is still open. Basically one would need to obtain rates for every linear dispersive estimates, but it seems like the time-space integration may open some other problem (typically it is not clear whether $\|T_h f\|_{l^p L^q_t} \lesssim \|f\|_{L^p_x L^q_t}$ is true, since even for space-time integration it is not a trivial result),

- The schemes studied here are only semi-discrete, it is essential to introduce time discretization that do not break the dispersive estimates (this was done for NLS in [4]), and it would be of particular interest to compare them with more standard schemes when the initial data are sufficiently rough (say not in H^s, $s > 3/2$),
- The construction of dispersive schemes that do not rely in some way to the Fourier transform seems so far way beyond reach.

A Standard results of harmonic analysis

This section is devoted to some results of standard Fourier analysis that are maybe less known in discrete settings. The proofs are elementary adaptations from the continuous settings and we include them only for completeness.

Definition 1. *We define the (discrete) maximal function of a sequence u as*

$$(\mathcal{M}u)_n = \sup_k \frac{1}{2k+1} \sum_{j=n-k}^{n+k} |u_j|. \tag{A.1}$$

The space $l^1(h\mathbb{Z})$ weak is the set of sequences such that $\exists\ C > 0 :\ |\{k :\ |u_k| > \alpha\}| \leq C/\alpha$. Here $|A|$ is the h times the cardinal of A.

Proposition 22. *The maximal function satisfies the following properties:*

- *for $u \in l^1$, $\alpha > 0$,*
$$|\{k :\ (\mathcal{M}u)_k \leq \alpha\}| \lesssim \|u\|_1/\alpha.$$
(ie $\mathcal{M}$ is continuous from $l^1 \to l^1_w$.
- *For $p > 1$, $\mathcal{M}$ is continuous $l^p \to l^p$.*

Proof. The second point is obvious for $p = \infty$. If the first point is proved, the general case for the second point is implied by the Marcinkiewicz interpolation theorem.
Thus we focus on the first point: let N arbitrarily fixed, $A_{\alpha,N} = \{|k| \leq N :\ (\mathcal{M}u)_k > \alpha\}$. For $k \in A_{\alpha,N}$, there exists n_k such that

$$\frac{1}{2n_k+1} \sum_{k-n_k}^{k+n_k} |u_k| > \alpha$$

The set $\cup_{k\in A_{\alpha,N}}[k-n_k, k+n_k]$ is an open cover $A_{\alpha,N}$, from Vitalli's lemma we find that there exists a part $\widetilde{A}$ of $A_{\alpha,N}$ such that $\cup_{k\in\widetilde{A}}[k-3n_k, k+3n_k] \supset A_{\alpha,N}$ and the $[k-n_k, k+n_k]$ are disjoint. Therefore:

$$|A_{\alpha,N}| \leq 3\sum_{k\in\widetilde{A}} 2n_k + 1 \leq 3\sum_{k\in\widetilde{A}}\sum_{k-n_k}^{k+n_k} \frac{|u_k|}{\alpha} \leq 3\frac{\|u\|_{l^1}}{\alpha}.$$

The estimate being uniform in N, this directly implies

$$|\{k:\ (\mathcal{M}u)_k \le \alpha\}| = |A_{\alpha,\infty}| \lesssim \frac{\|u\|_{l^1}}{\alpha}.$$

which end the proof. $\square$

Proposition 23. *(discrete Hardy-Littlewood-Sobolev with parameter)*
Let $u \in l^p(h\mathbb{Z})$, $1 < p < q < \infty$, $0 < \gamma < 1$ *where*

$$\frac{1}{q} = \frac{1}{p} - 1 + \gamma. \tag{A.2}$$

We set

$$u *_h v := h \sum_{\mathbb{Z}} u_{n-k} v_k. \tag{A.3}$$

If $|v| \le \dfrac{C}{\big(h(1+|k|)\big)^{\gamma}}$, *then* $(\widetilde{u_n}) \in l^q(\mathbb{Z})$ *with* $\|\widetilde{u}\|_{l^q} \lesssim C\|u\|_{l^p}$.

Proof. We first reduce the proof to the case $h = 1$: we write $\|u\|_{l^p_h}^p = h\sum |u_n|^p$, while $\|u\|_{l^p} = \sum |u_n|^p$. With those notations, and if the result is proved for $h = 1$

$$\begin{aligned} \|u *_h v\|_{l^q}^q = h\sum_n |h\sum_k u_{n-k}v_k|^q = h^{q+1}\|u * v\|_{l^q}^q &\le h^{q+1}\|u\|_{l^p}^q C^q/h^{\gamma q} \\ &= C^q h^{q+1}\|u\|_{l^p_h}^q h^{-(\gamma q + q/p)} \\ &= C^q \|u\|_{l^p_h}^q. \end{aligned}$$

Secondly, since $|\sum u_{n-k}v_k| \le C\sum |u_{n-k}|/(1+|k|)^\gamma$, we may assume $u_n \ge 0,\ v_n = 1/(1+|n|)^\gamma$. Let us write for N arbitrary

$$\sum_{\mathbb{Z}} u_{n-k}\frac{1}{(1+|k|)^\gamma} = \sum_{|k|\le N} u_{n-k}\frac{1}{(1+|k|)^\gamma} + \sum_{|k|>N} u_{n-k}\frac{1}{(1+|k|)^\gamma}$$

The second term is bounded thanks to the Hölder inequality

$$|\sum_{|k|>N} u_{n-k}\frac{1}{(1+|k|)^\gamma}| \le \|u\|_{l^p}\Big(\sum_{|k|>N}\frac{1}{(1+|k|)^{\gamma p'}}\Big)^{1/p'} \lesssim \|u\|_{l^p} N^{1/p'-\gamma}.$$

For the first term, set

$$\begin{cases} \alpha_j = \dfrac{1}{(1+j)^\gamma} - \dfrac{1}{(2+j)^\gamma},\ 0 \le j < N \\ \alpha_N = \dfrac{1}{(1+N)^\gamma}. \end{cases}$$

One has

$$\begin{aligned}
|\sum_{|k|\le N} u_{n-k}\frac{1}{(1+|k|)^\gamma}| &= |\sum_{|k|\le N}\sum_{j=k}^{N} u_{n-k}\alpha_j| \\
&= |\sum_{j=0}^{N}\sum_{|k|\le j}\alpha_j u_{n-k}| \\
&\le \sum_{j=0}^{N}\alpha_j(2j+1)(\mathcal{M}u)_n \\
&= (\mathcal{M}u)_n\sum_{j=0}^{N}\sum_{|k|\le n}\alpha_j \\
&= (\mathcal{M}u)_n\sum_{|k|\le N}\sum_{j=|k|}^{N}\alpha_j = (\mathcal{M}u)_n\frac{1}{(1+|k|)^\gamma} \lesssim (\mathcal{M}u)_n N^{1-\gamma}.
\end{aligned}$$

Thus

$$|\widetilde{u}_n| \lesssim N^{1/p'-\gamma}\|u\|_{l^p} + N^{1-\gamma}(\mathcal{M}u)_n, \tag{A.4}$$

all that is left is to optimize in N by chosing $N^{1/p'-\gamma}\|u\|_{l^p} \sim N^{1-\gamma}(\mathcal{M}u)_n$, which gives

$$N = \left\lfloor \left(\frac{\|u\|_{l^p}}{(\mathcal{M}u)_n}\right)^p \right\rfloor, \text{ where } \lfloor\cdot\rfloor \text{ denotes the integer part.}$$

Note that $(\mathcal{M}u)_n \le \|u\|_{l^\infty} \le \|u\|_{l^p}$ is always true, thus $N \ge 1$ and

$$\frac{1}{2}\left\lfloor \left(\frac{\|u\|_{l^p}}{(\mathcal{M}u)_n}\right)^p \right\rfloor \le N \le \left\lfloor \left(\frac{\|u\|_{l^p}}{(\mathcal{M}u)_n}\right)^p \right\rfloor.$$

Injecting this in (A.4) we find

$$\begin{aligned}
|\widetilde{u}_n| &\lesssim (\mathcal{M}u)_n\big(\|u\|_{l^p}^{p(1-\gamma)}(\mathcal{M}u)_n^{p(\gamma-1)}\big) + \|u\|_{l^p}(\mathcal{M}u)_n^{-p(1/p'-\gamma)}\|u\|_{l^p}^{p(1/p'-\gamma)} \\
&\lesssim \|u\|_{l^p}^{p(1-\gamma)}(\mathcal{M}u)_n^{p/q}.
\end{aligned}$$

All that remains is to apply proposition 22, which gives:

$$\|\widetilde{u}_n\|_{l^q} \lesssim \|u\|_{l^p}^{p(1-\gamma)}\|\mathcal{M}u\|_{l^p}^{p/q} \lesssim \|u\|_{l^p}.$$

□

Proposition 24. *Let $\mathcal{F}$ be the usual Fourier transform on $L^2(\mathbb{R})$. For $p > 1$, the operator $I:\ (u_n) \to \mathcal{F}^{-1}(\widehat{u}1_{[-\pi/h,\pi/h]})$ is continuous $l^p(h\mathbb{Z}) \to L^p(\mathbb{R})$.*

Proof. (Sketch of) By homogeneity it is sufficient to prove the result for $h = 1$, and we shall only prove that I is continuous from l^1 to L^1 weak, this implies the strong L^p continuity thanks to the Marcinkiewicz interpolation theorem for $1 < p < 2$ and then by duality for $p > 2$ (the proof is derived from the continuity of the Hilbert transform, however the result goes back at least to Plancherel and Polya [15], 1937). A basic calculus shows that for (u_n) rapidly decaying

$$Iu(x) = \frac{1}{\pi}\sum_n u_n \frac{\sin \pi(x-n)}{\pi(x-n)}.$$

For λ fixed we use without justification the decomposition $\mathbb{Z} = (\sqcup I_k) \sqcup (\mathbb{Z} \setminus (\sqcup I_k))$ such that

- for any k, $\lambda \le 1/|I_k| \sum_{I_k} |u_n| \le 2\lambda$, in particular $|\sqcup I_k| \le \|u\|_1$,
- for any $n \notin \sqcup I_k$, $|u_n| \le \lambda$.

(this is a discrete Calderon-Zygmund decomposition, see for example Stein [17] I.3).
We define $g_n = u_n$ if $n \notin \sqcup I_k$, $e^{in\pi}/|I_k| \sum\limits_{I_K} e^{ij\pi} u_j$ if $n \in I_k$. Clearly, $|g_n| \le 2\lambda$, $\|g_n\|_{l^1} \le \|u_n\|_{l^1}$, and we may write

$$u_n = b_n + g_n,$$

where $b = \sum b^k$, each sequence b^k is supported in I_k and satisfies

$$b_n^k = u_n - e^{in\pi}/|I_k| \sum_{I_K} e^{ij\pi} u_j \text{ pour } n \in I_k.$$

In particular, $\sum_{\mathbb{Z}} e^{\pm in\pi} b_n^k = 0$. We have then

$$|\{x \in \mathbb{R} : \ |Iu| \ge \lambda\}| \le |\{x \in \mathbb{R} : \ |Ig| \ge \lambda/2\}| + |\{x \in \mathbb{R} : \ |Ib| \ge \lambda/2\}|.$$

Using Chebychev's inequality for $p = 2$ and the L^2 continuity, we obtain for the first term

$$|\{x \in \mathbb{R} : \ |Ig| \ge \lambda/2\}| \le \|Ig\|_{L^2}^2/\lambda^2 \le 2\lambda\|u\|_1/\lambda^2 = 2\|u\|_1/\lambda.$$

For the second term we shall denote abusively $2I_k$ the interval with same center c_k as I_k and twice larger. Chebychev's inequality implies again

$$\begin{aligned}
|\{x \in \mathbb{R} : \ |Ib| \ge \lambda/2\}| &\le 2\sum |I_k| + |\{x \in \mathbb{R} \setminus (\sqcup 2I_k) : \ |Ib| \ge \lambda/2\}| \\
&\le 2\sum |I_k| + \frac{1}{\lambda}\int_{\mathbb{R}\setminus(\sqcup I_k)} |Ib| dx \\
&\le \frac{2}{\lambda}\|u\|_1 + \frac{1}{\lambda}\sum_k \int_{\mathbb{R}\setminus I_k} |\sum_{I_k} b_n^k \frac{\sin \pi(x-n)}{\pi(x-n)}| dx.
\end{aligned}$$

But using now $\sum_{I_k} e^{\pm in\pi} b_n^k = 0$ we obtain

$$\begin{aligned}
\int_{\mathbb{R}\setminus I_k} |\sum_{I_k} b_n^k \frac{\sin \pi(x-n)}{\pi(x-n)}|dx &= \frac{1}{2}\int_{\mathbb{R}\setminus I_k} |\sum_{I_k} b_n^k \Big(\frac{e^{i\pi(x-n)}}{\pi(x-n)} - \frac{e^{-i\pi(x-n)}}{\pi(x-n)}\Big)|dx \\
&= \frac{1}{2}\int_{\mathbb{R}\setminus 2I_k} \Big|\sum_{I_k} b_n^k \Big(\frac{e^{-i\pi n}}{\pi(x-n)} - \frac{e^{-i\pi n}}{\pi(x-c_k)} - \frac{e^{i\pi n}}{\pi(x-n)} \\
&\quad + \frac{e^{i\pi n}}{\pi(x-c_k)}\Big)\Big|dx \\
&\leq \int_{\mathbb{R}\setminus 2I_k} \sum_{I_k} |b_n^k| \Big(\frac{c_k - n}{\pi(x-n)(x-c_k)}\Big)|dx \\
&\leq 2\sum_{I_k} |b_n^k| \int_{\mathbb{R}\setminus 2I_k} \frac{|I_k|}{\pi(x-c_k)^2} dx \\
&= \frac{4}{\pi}\sum_{I_k} |b_n^k|.
\end{aligned}$$

So that by summing it all

$$|\{x \in \mathbb{R} : \ |Iu| \geq \lambda\}| \leq \frac{4 + 8/\pi}{\lambda} \|u\|_1.$$

□

B Dispersive smoothing and rate of convergence

We did not manage to obtain a version with rates of convergence for the estimate (7.7)

$$\||D|^2 \int_0^t V(t-t')\Pi g(t')dt'\|_{l^\infty L^2} \lesssim \|\Pi g\|_{l^1 L^2},$$

which is one of the main points that prevented us from obtaining rates of convergence for the approximation of the cKdV equation. It seems like the deep technical problem is the fact that in the proof of (7.7), t lies in an unbounded set and thus $V_h(t) - V(t)$ is certainly not small. More modestly this paragraph establishes how one may obtain a non optimal rate of convergence on

$$\|T_h \partial_x V(t) u_0 - \partial_h V_h(t) \Pi T_{Nh} u_0)\|_{l^\infty L^2}.$$

To do so, we outline rapidly the main features of the Littlewood-Paley decomposition for sequences of $h\mathbb{Z}$, which are similar to the usual properties for functions. Let ψ be a smooth compactly supported function such that $\text{supp}(\psi) \subset [-2, 2]$ and $\forall\, |x| \leq 1,\ \psi(x) = 1$. We set $\varphi_0 = \psi$ and for $j \geq 1$

$$\varphi_j := \psi(2^{-j}\cdot) - \psi(2^{-j+1}\cdot).$$

For $u \in \mathcal{S}(h\mathbb{Z})$, the operators Δ_j are defined by

$$\widehat{\Delta_j u}|_{[-\pi/h,\pi/h]} = \varphi_j \widehat{u}, \tag{B.1}$$

as for interpolation operators we chose not to emphasize the dependance on h (which appears notably in the fact that for $2^j \geq 4\pi/h$, $\Delta_j u = 0$).
Using the quasi orthogonality $|j-k| \geq 2 \Rightarrow \varphi_j\varphi_k = 0$, it is easily seen that $\|u\|_{l^2}^2 \asymp \sum \|\Delta_j u\|_{l^2}^2$, and more deeply a key feature of this decomposition are the so-called Bernstein inequalities.

Proposition 25. *(Bernstein inequalities) The following estimates hold:*

$$\forall\, p \geq 2,\ \|\Delta_j u\|_{l^p} \leq C 2^{j(1/2-1/p)} \|\Delta_j u\|_{l^2}, \tag{B.2}$$

$$\forall\, s \geq 0,\ \||D|^s \Delta_j u\|_{l^2} \leq C_s 2^{js} \|\Delta_j u\|_{l^2}, \tag{B.3}$$

with C, C_s independant of h, and the constant C is also independent of $p \in [2,\infty]$.

The main result of this section is the following.

Proposition 26. *Let Π be an interpolator as in theorem 7, of symbol m such that $m(0) = 1$. For $u_0 \in L^2$, we have the homogeneous estimate*

$$\forall\, \varepsilon > 0,\ \|\partial_x V(t) T_h u_0 - \partial_h V_h(t) \Pi T_{Nh} u_0\|_{l^\infty L^2([0,T])} \leq \frac{C h^{4s/13}(1+T)}{\sqrt{\varepsilon}} \|u_0\|_{H^{s+\varepsilon}(\mathbb{R})}. \tag{B.4}$$

(B.5)

The constant C only depends on Π.

Proof. We first split

$$\begin{aligned} \|\partial_x V(t) T_h u_0 - \partial_h V_h(t)\Pi T_{Nh} u_0\|_{l^\infty L^2([0,T])} &\leq \|\big(V(t) - V_h(t)\big)\partial_h \Pi T_{Nh} u_0\|_{l^\infty L^2([0,T])} \\ &\quad + \|V(t)(\partial_x T_h - \partial_h \Pi T_{Nh}) u_0\|_{l^\infty L^2([0,T])} \\ &= M_1 + M_2. \end{aligned}$$

We will only focus on the derivation of an estimate for M_1, the other one being similar and simpler. In a general manner for f defined on $h\mathbb{Z} \times [0,T]$,

$$\begin{aligned} \|f\|_{l^\infty L_T^2}^2 = \sup_n \int_0^T \Big(\sum_j \Delta_j f_n\Big)^2 dt &\lesssim \frac{1}{\varepsilon} \sup_n \int_0^T \sum_j 2^{2\varepsilon j} |\Delta_j f_n|^2 dt \\ &\leq \frac{1}{\varepsilon} \sum_j 2^{2\varepsilon j} \sup_n \int_0^T |\Delta_j f_n|^2 dt = \frac{1}{\varepsilon} \sum_j 2^{2\varepsilon j} \|\Delta_j f_n\|_{l^\infty L_T^2}^2. \end{aligned}$$

Applying it to M_1 we have

$$\|\big(V(t) - V_h(t)\big)\partial_h \Pi T_{Nh} u_0\|_{l^\infty L^2([0,T])}^2 \lesssim \frac{1}{\varepsilon} \sum_j 2^{2\varepsilon j} \|\big(V(t) - V_h(t)\big)\partial_h \Delta_j \Pi T_{Nh} u_0\|_{l^\infty L_T^2}^2.$$

On one side the dispersive estimate (2.3) gives

$$\|\big(V(t)-V_h(t)\big)\partial_h\Delta_j\Pi T_{Nh}u_0\|_{l^\infty L^2_T}\lesssim\|\Delta_j\Pi T_{Nh}u_0\|_{l^2},\tag{B.6}$$

while Bernstein's inequality (B.2) combined with the inequality $|e^{it\xi^3}-e^{itp_h(\xi)}|\lesssim t|\xi|^5h^2$ implies

$$\begin{aligned}\|\big(V(t)-V_h(t)\big)\partial_h\Delta_j\Pi T_{Nh}u_0\|_{l^\infty L^2_T} &\lesssim \|\big(V(t)-V_h(t)\big)\partial_h\Delta_j\Pi T_{Nh}u_0\|_{L^\infty_T l^\infty}\\ &\leq 2^{j/2}\sup_{[0,T]}\|\big(V(t)-V_h(t)\big)\partial_h\Delta_j\Pi T_{Nh}u_0\|_{l^2}\\ &\leq 2^{j/2+6j}h^2T\|\Delta_j\Pi T_{Nh}u_0\|_{l^2},\end{aligned}\tag{B.7}$$

so that by (7.3) with $\alpha=2s/13$ applied to (B.7), (B.6),

$$\|\big(V(t)-V_h(t)\big)\partial_h\Delta_j\Pi T_{Nh}u_0\|_{l^\infty L^2_T}\leq h^{4s/13}2^{js}\|\Delta_j\Pi T_{Nh}u_0\|_{l^2}.\tag{B.8}$$

By summing and using lemma 1 we find as expected

$$M_1^2\leq\sum_j h^{4s/13}\frac{2^{2j(s+\varepsilon)}}{\varepsilon}\|\Delta_j\Pi T_{Nh}u_0\|^2_{l^2}\lesssim\frac{h^{8s/13}}{\varepsilon}\|u_0\|^2_{H^{s+\varepsilon}}.\tag{B.9}$$

□

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