CONTROLLABILITY OF FAST DIFFUSION COUPLED PARABOLIC SYSTEMS

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Abstract. In this work we are concerned with the null controllability of coupled parabolic systems depending on a parameter and converging to a parabolic-elliptic system. We show the uniform null controllability of the family of coupled parabolic systems with respect to the degenerating parameter.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded connected open set whose boundary $\partial\Omega$ is regular enough ($N \geq 1$). Let $T > 0$ and let ω_1 and ω_2 be two nonempty subsets of Ω, which will be referred as *control domains*. We will use the notation $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$.

The main objective of this paper is to analyze the controllability of linear coupled parabolic systems in which one of the equations is degenerating into an elliptic equation.

In order to state our problem, we introduce the following system

$$\left| \begin{array}{ll} u_t - \Delta u = au + bv + f\chi_{\omega_1} & \text{in } Q, \\ \epsilon v_t - \Delta v = cu + dv + g\chi_{\omega_2} & \text{in } Q, \\ u = v = 0 & \text{on } \Sigma, \\ u(0) = u_0; \ v(0) = v_0 & \text{in } \Omega, \end{array} \right. \tag{1.1}$$

where $a = a(x, t)$, $b = b(x, t)$, $c = c(x, t)$ and $d = d(x, t)$ are functions belonging to $C^3(\overline{Q})$, f and g are internal controls and ϵ is a small positive parameter, intended to tend to zero. In particular we want to study this problem when only one control is active, namely when $g \equiv 0$ or $f \equiv 0$ and analyze the dependence of the cost of the null controllability of system (1.1) with respect to the parameter ϵ.

Our interest in this problem comes from the fact that in many physical situations system (1.1) is formally approximated by the following parabolic-elliptic system

$$\left| \begin{array}{ll} u_t - \Delta u = au + bv + f\chi_{\omega_1} & \text{in } Q, \\ -\Delta v = cu + dv + g\chi_{\omega_2} & \text{in } Q, \\ u = v = 0 & \text{on } \Sigma, \\ u(0) = u_0 & \text{in } \Omega. \end{array} \right. \tag{1.2}$$

This is the case for instance of biological systems modeling aggregation phenomena or chemical systems having two different concentrations, see [8] and references therein. However, even if this approximation is consistent with the existence and uniqueness point of view, it is not clear at all what can be done from a control theory point of view. The main reason for that arises from

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the fact that we are considering systems having different physical properties and therefore, at least a priori, different control properties.

The analysis of existence and uniqueness of solutions to system (1.1) is done in [10]. In this work the author proves existence and uniqueness of solution when the initial data are in $L^2(\Omega)$. The author also studied the asymptotic expansion of solutions of (1.1) in terms of ϵ.

It is important to mention that this question of approximating an equation by another having different physical properties was already studied in the case of a hyperbolic equation degenerating into a parabolic one and vice-versa. In fact, it was proved in [9] that system

$$\left|\begin{array}{ll} \epsilon u_{tt} - \Delta u + u_t = f\chi_\omega & \text{in } \ \Omega\times(0,T), \\ u = 0 & \text{on } \ \partial\Omega\times(0,T), \\ u(0) = u_0;\ u_t(0) = u_1 & \text{in } \ \Omega, \end{array}\right. \tag{1.3}$$

is null controllable, for each ϵ fixed, and the controls remains bounded when $\epsilon \to 0$ if we impose some geometric condition on Ω. Furthermore, the control sequence converges, when $\epsilon \to 0$, to a control for the heat equation

$$\left|\begin{array}{ll} u_t - \Delta u = f\chi_\omega & \text{in } \ \Omega\times(0,T), \\ u = 0 & \text{on } \ \partial\Omega\times(0,T), \\ u(0) = u_0 & \text{in } \ \Omega. \end{array}\right. \tag{1.4}$$

Another relevant work is [6], in which the authors consider the linear transport diffusion equation

$$\left|\begin{array}{ll} y_t - \epsilon\Delta y + M(x,t)\cdot\nabla y = f\chi_\omega & \text{in } \ \Omega\times(0,T), \\ y = 0 & \text{on } \ \partial\Omega\times(0,T), \\ y(0) = y_0 & \text{in } \ \Omega, \end{array}\right. \tag{1.5}$$

and investigate the cost of the control in the vanishing viscosity limit $\epsilon \to 0^+$ and, in particular, they try to determine in which situation it is possible to obtain a control which remains bounded as $\epsilon \to 0^+$. In that paper the authors are able to prove boundedness of controls by assuming some conditions on the vector field M and the time T. See also [2] and [7] for the analysis of (1.5) in the 1-d case, with M constant.

Regarding the case of parabolic systems converging to parabolic-elliptic systems, as far as we know, the first time this problem was addressed was in [1], where the authors considered the case of a nonlinear parabolic-elliptic system appearing in electrocardiology as a simplification of a coupled parabolic system modeling electrical activities in the heart and, combining Carleman estimates and weighted energy inequalities, the authors are able to prove that the control properties of the parabolic-elliptic system can be viewed as a limit process of the control properties of a family of parabolic systems.

Let us denote by $\big(u(t;\epsilon,(u_0,v_0),f,g), v(t;\epsilon,(u_0,v_0),f,g)\big)$ the solution of (1.1) at time $t \in [0,T]$ associated to $(u_0,v_0) \in L^2(\Omega)^2$ and $(f,g) \in L^2(Q)^2$.

Our first main result in this paper is given by the following Theorem.

Theorem 1.1. *Let $(u_0,v_0) \in L^2(\Omega)^2$ and a, b, c and d be $C^3(\overline{Q})$ functions. Then*

(1) *If $c \neq 0$ in $\overline{\omega}$, for some $\omega \subset\subset \omega_1$ and $||d||_{L^\infty(Q)} < \mu_1$ (μ_1 being the first eigenvalue of the Laplacian), then system (1.1) is uniformly null controllable, with respect to ϵ, with control*

only in the first equation. More precisely, for each $\epsilon > 0$ *there exists* $f = f(\epsilon) \in L^2(Q)$ *such that*

$$\big(u(T;\epsilon,(u_0,v_0),f(\epsilon),0),v(T;\epsilon,(u_0,v_0),f(\epsilon),0)\big)=(0,0). \tag{1.6}$$

Moreover, we have the following estimate on the control

$$||f(\epsilon)\chi_{\omega_1}||_{L^2(Q)} \leq C(||u_0||_{L^2(\Omega)} + \epsilon||v_0||_{L^2(\Omega)}), \tag{1.7}$$

where C *is a constant that does not depend on* ϵ*,* u_0 *and* v_0*.*

(2) *If* $b \neq 0$ *in* $\overline{\omega}$*, for some* $\omega \subset\subset \omega_2$ *and* $d < \mu_1$*, then system (1.1) is uniformly null controllable, with respect to* ϵ*, with control acting only in the second equation. More precisely, for each* $\epsilon > 0$ *there exists* $g = g(\epsilon) \in L^2(Q)$ *such that*

$$\big(u(T;\epsilon,(u_0,v_0),0,g(\epsilon)),v(T;\epsilon,(u_0,v_0),0,g(\epsilon))\big)=(0,0). \tag{1.8}$$

Moreover, we have the following estimate on the control

$$||g(\epsilon)\chi_{\omega_2}||_{L^2(Q)} \leq C(||u_0||_{L^2(\Omega)} + \epsilon||v_0||_{L^2(\Omega)}), \tag{1.9}$$

where C *does not depend on* ϵ*,* u_0 *and* v_0*.*

In order to prove Theorem 1.1 we are led to consider the adjoint system of (1.1),

$$\left|\begin{array}{ll} -\varphi_t - \Delta\varphi = a\varphi + c\xi & \text{in } Q, \\ -\epsilon\xi_t - \Delta\xi = b\varphi + d\xi & \text{in } Q, \\ \varphi = \xi = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi_T;\ \xi(T) = \xi_T & \text{in } \Omega, \end{array}\right. \tag{1.10}$$

with $(\varphi_T, \xi_T) \in L^2(\Omega)^2$.

It is well known that case 1 of Theorem 1.1 is equivalent to prove the existence of an universal constant C, which does not depend on ϵ, such that the observability inequality

$$||\varphi(0)||^2_{L^2(\Omega)} + \epsilon||\xi(0)||^2_{L^2(\Omega)} \leq C\iint_{Q_{\omega_1}} |\varphi|^2 dxdt, \tag{1.11}$$

holds for all solutions (φ, ξ) of (1.10). Analogously, one can prove that case 2 of Theorem 1.1 is equivalent to show that

$$||\varphi(0)||^2_{L^2(\Omega)} + \epsilon||\xi(0)||^2_{L^2(\Omega)} \leq C\iint_{Q_{\omega_2}} |\xi|^2 dxdt, \tag{1.12}$$

for all solutions (φ, ξ) of (1.10).

The study of the controllability of systems of parabolic equations has obtained a lot of attention in the recent years. For instance, in [4] the authors analyze the controllability of a reaction diffusion system of a system of two parabolic equations coupled by zero-order terms, obtaining the null controllability for the linear system and the local null controllability of the semilinear system. In [5] the controllability of a quite general system of two coupled linear parabolic equations is studied and, combining Carleman inequalities and some energy inequalities, null controllability is proved.

Following [4] or [5] one can prove that for each $\epsilon > 0$ system (1.1) is null controllable in $L^2(\Omega)$. A carefully analysis on booth proofs shows that uniform null controllability with respect to ϵ can be obtained only in the case of a control on the second equation of (1.1), i.e., following [4] or

[5] it is possible to prove that the cost of the null controllability is bounded, with respect to ϵ, when the control is acting on the second equation of (1.1). The same is not true if one is trying to control (1.1) by means of a control in the first equation. Indeed, in that case the cost of the null controllability is of order ϵ^{-1}.

Thus, in this paper we obtain a uniform estimate on the cost of controllability of (1.1) in the case of a control acting only on the first equation, Theorem 1.1, case 1. Our proof can also be applied in order to obtain the boundedness of the cost of the null controllability of (1.1) when the control is acting on the second equation, see Theorem 2.3.

2. Carleman estimates and an extended adjoint system

In this section we deduce Carleman type estimates that will be used to prove observability inequalities (1.11) and (1.12). To this end we first define several weight functions which will be useful in the sequel.

The basic weight will be a function $\psi \in C^2(\overline{\Omega})$ verifying

$$\psi(x) > 0, \text{ in } \Omega, \quad \psi \equiv 0 \text{ on } \partial\Omega, \quad |\nabla\psi(x)| > 0, \ \forall x \in \overline{\Omega\backslash\omega_0},$$

where $\omega_0 \subset\subset \omega_1$ ($\omega_0 \subset\subset \omega_2$ in the case 2) is a nonempty open set. The existence of such a function ψ is proved in [3]. Then, for some positive real numbers s and λ we introduce:

$$\begin{aligned} &\phi(x,t) = \frac{e^{\lambda\psi(x)}}{t(T-t)}; \ \alpha(x,t) = \frac{e^{\lambda\psi(x)} - e^{2\lambda\|\psi\|_\infty}}{t(T-t)} \\ &\hat{\phi}(t) = \min_{x\in\overline{\Omega}} \phi(x,t); \ \phi^*(t) = \max_{x\in\overline{\Omega}} \phi(x,t); \ \alpha^*(t) = \min_{x\in\overline{\Omega}} \alpha(x,t). \end{aligned} \tag{2.1}$$

The following Carleman inequality will be very important to our purposes:

Lemma 2.1. *Let $\beta \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$ and $\omega_0 \subset\subset \omega$. There exists a constant $\lambda_0 = \lambda_0(\Omega,\omega) \geq 1$ such that for every $\lambda \geq \lambda_0$ there exists $s_0 = s_0(\Omega,\omega,\lambda)$ and $C = C(\Omega,\omega,\lambda)$ such that, for every $s \geq s_0(T+T^2)$, the following inequality holds:*

$$\begin{aligned} &s^{\beta-1}\iint_Q e^{2s\alpha}\phi^{\beta-1}(\sigma^2|q_t|^2 + |\sum_{i,j=1}^N \frac{\partial^2 q}{\partial x_i \partial x_j}|^2)dxdt \\ &+ s^{\beta+1}\iint_Q e^{2s\alpha}\phi^{\beta+1}|\nabla q|^2 dxdt + s^{\beta+3}\iint_Q e^{2s\alpha}\phi^{\beta+3}|q|^2 dxdt \\ &\leq C\Big(s^\beta \iint_Q e^{2s\alpha}\phi^\beta|\sigma\partial_t q + \Delta q|^2 dxdt + s^{\beta+3}\iint_{\omega\times(0,T)} e^{2s\alpha}\phi^{\beta+3}|q|^2 dxdt\Big), \end{aligned}$$

for all $q \in C^2(\overline{Q})$, with $q = 0$ on Σ.

Proof. See [1] or [3]. □

Now we state the second main result of this paper, a Carleman type estimate for the adjoint system (1.10) given by the following Theorem:

Theorem 2.2. *There exists* $\lambda_0 = \lambda_0(\Omega, \omega_1) \geq 1$ *such that for every* $\lambda \geq \lambda_0$*, there exists* $s_0 = s_0(\Omega, \omega_1, \lambda_0) > 0$ *such that, for every* $s > s_0(T + T^2)$*, the solution* (φ, ξ) *of system* (1.10) *satisfies*

$$s^4 \iint_Q e^{2s\alpha}\phi^4|\varphi|^2 dxdt + \iint_Q e^{2s\alpha}\varphi^4|\xi|^2 dxdt \leq Cs^{14}\iint_{\omega_1\times(0,T)}(e^{2s\alpha^*} + e^{4s\alpha^* - 2s\alpha})(\phi^*)^{14}|\varphi|^2 dxdt, \tag{2.2}$$

with C *depending on* Ω*,* ω_1 *and* λ*.*

For the purpose of proving Theorem 2.2, we extend our adjoint system to a system of 4 equations. We set the notation:

$$\mathcal{L}_{\gamma,\theta} = \gamma\partial_t - \Delta - \theta, \text{ with } \gamma \in \mathbb{R} \text{ and } \theta \in L^\infty(Q).$$

With this notation, we define the following function

$$w = \mathcal{L}_{-\epsilon,d}\varphi$$

and, if φ_T and ξ_T are smooth and (φ, ξ) is the solution of (1.10) associated to this initial data, a simple calculation gives

$$-w_t - \Delta w - aw = \varphi(cb + \mathcal{L}_{-\epsilon,0}a - \mathcal{L}_{-1,0}d) + \xi\mathcal{L}_{-\epsilon,0}c - 2\nabla\xi\nabla c + 2\nabla\varphi(\nabla d - \nabla a).$$

Therefore, we can add two more equations to our adjoint system, going from a system of 2 equations to a system of 4 equations, namely

$$\left|\begin{array}{ll} \mathcal{L}_{-1,a}w = \varphi(cb + \mathcal{L}_{-\epsilon,0}a - \mathcal{L}_{-1,0}d) + \xi\mathcal{L}_{-\epsilon,0}c - 2\nabla\xi\nabla c + 2\nabla\varphi(\nabla d - \nabla a) & \text{in } Q, \\ \mathcal{L}_{-\epsilon,d}\varphi = w & \text{in } Q, \\ \mathcal{L}_{-1,a}\varphi = c\xi & \text{in } Q, \\ \mathcal{L}_{-\epsilon,d}\xi = b\varphi & \text{in } Q, \\ \varphi = \xi = w = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi_T;\ \xi(T) = \xi_T;\ w(T) = -\epsilon\varphi_T - \Delta\varphi_T - d\varphi_T & \text{in } \Omega. \end{array}\right. \tag{2.3}$$

The plan of the proof of Theorem 2.2 contains five parts:

First part: We see equations of (2.3) as heat equations and apply the Carleman estimate for the heat equation given in Lemma 2.1. This yields a global estimate for φ, w and ξ in terms of local terms of φ, w and ξ.

Second part: Using the second equation in (2.3) we eliminate the local integral of w appearing in the Carleman estimate obtained in *Step 1*.

Third part: We estimate a local integral of ξ in terms of a local integral of φ, a local integral of φ_t and some small order terms.

Fourth part: Using the extend adjoint system, we show that we can estimate φ_t locally in terms of a local integral of φ and some small order terms.

Fifth part: We gather the estimates of previous steps and absorb the small order terms, obtaining our desired Carleman estimate.

Along the proof we will use the notation:

$$I_\beta(s,\sigma;\rho) = s^{\beta+3}\iint_Q e^{2s\alpha}\phi^{\beta+3}|\rho|^2dxdt + s^{\beta+1}\iint_Q e^{2s\alpha}\phi^{\beta+1}|\nabla\rho|^2dxdt$$
$$+ s^{\beta-1}\iint_Q e^{2s\alpha}\phi^{\beta-1}(\sigma^2|\rho_t|^2 + |\sum_{i,j=1}^N \frac{\partial^2\rho}{\partial x_i\partial x_j}|^2)dxdt, \quad (2.4)$$

where s,β and σ are real numbers and $\rho=\rho(x,t)$.

Proof of Theorem 2.2. For an easier comprehension, we divide the proof in several steps:

Step 1. *First Carleman inequalities.*

Let ω' be a nonempty set such that $\omega_0 \subset\subset \omega' \subset\subset \omega$. We apply Lemma 2.1 to $(2.3)_1$, with $\beta=2$, and to $(2.3)_3$ and $(2.3)_4$, with $\beta=1$. Then

$$I_2(s,1;w) \le C\big(s^5\iint_{\omega'\times(0,T)} e^{2s\alpha}\phi^5|w|^2dxdt + s^2\iint_Q \phi^2e^{2s\alpha}(|\varphi|^2+|\nabla\varphi|^2+|\xi|^2+|\nabla\xi|^2)dxdt\big), \quad (2.5)$$

$$I_1(s,1;\varphi) \le C\big(s^4\iint_{\omega'\times(0,T)} e^{2s\alpha}\phi^4|\varphi|^2dxdt + s\iint_Q e^{2s\alpha}\phi|\xi|^2dxdt\big) \quad (2.6)$$

and

$$I_1(s,\epsilon;\xi) \le C\big(s^4\iint_{\omega'\times(0,T)} e^{2s\alpha}\phi^4|\xi|^2dxdt + s\iint_Q e^{2s\alpha}\phi|\varphi|^2dxdt\big). \quad (2.7)$$

Adding (2.5), (2.6), (2.7) and absorbing the lower order terms, we get

$$I_2(s,1;w) + I_1(s,\epsilon;\xi) + I_1(s,1;\varphi)$$
$$\le C\big(s^4\iint_{\omega'\times(0,T)} e^{2s\alpha}\phi^4|\varphi|^2dxdt + s^4\iint_{\omega'\times(0,T)} e^{2s\alpha}\phi^4|\xi|^2dxdt + s^5\iint_{\omega'\times(0,T)} e^{2s\alpha}\phi^5|w|^2dxdt\big). \quad (2.8)$$

Step 2. *Estimate of the local integral of w.*

In this step we estimate the local integral of w in the right-hand side of (2.8) in terms of a local integral of φ and a small order term involving w. In order to do that, we introduce a cut-off function θ with

$$\theta \in C_0^\infty(\omega''), \text{with } 0\le\theta\le 1 \text{ and } \theta\equiv 1 \text{ on } \omega',$$

where $\omega' \subset\subset \omega'' \subset\subset \omega$.

We use $(2.3)_2$ to write

$$s^5\iint_{\omega''\times(0,T)} e^{2s\alpha}\phi^5|w|^2\theta dxdt = s^5\iint_{\omega''\times(0,T)} e^{2s\alpha}\phi^5\theta w(-\epsilon\varphi_t - \Delta\varphi - d\varphi)dxdt$$
$$:= M_1 + M_2 + M_3 \quad (2.9)$$

and we estimate each term in the expression above.

For the first term, we integrate by parts to see that

$$M_1 = 5\epsilon s^5 \iint_{\omega''\times(0,T)} \phi^4\phi_t e^{2s\alpha}\theta w\varphi dxdt + 2\epsilon s^6 \iint_{\omega''\times(0,T)} \alpha_t\phi^5 e^{2s\alpha}\theta w\varphi dxdt + \epsilon s^5 \iint_{\omega''\times(0,T)} \phi^5 e^{2s\alpha}\theta w_t\varphi dxdt \tag{2.10}$$

and use Young's inequality to obtain

$$M_1 \le \epsilon^2\big(s^5 \iint_{\omega''\times(0,T)} \phi^5 e^{2s\alpha}|w|^2 dxdt + s \iint_{\omega''\times(0,T)} \phi e^{2s\alpha}|w_t|^2 dxdt\big) + Cs^9 \iint_{\omega''\times(0,T)} \phi^9 e^{2s\alpha}|\varphi|^2 dxdt. \tag{2.11}$$

Here we have used that $|\alpha_t| \le C\phi^2$.

Next, since

$$M_2 = 5s^5 \iint_{\omega''\times(0,T)} \phi^4 e^{2s\alpha} w\theta(\nabla\phi\cdot\nabla\varphi)dxdt + 2s^6 \iint_{\omega''\times(0,T)} \phi^5 e^{2s\alpha} w\theta(\nabla\alpha\cdot\nabla\varphi)dxdt + s^5 \iint_{\omega''\times(0,T)} \phi^5 e^{2s\alpha} w(\nabla\theta\cdot\nabla\varphi)dxdt + s^5 \iint_{\omega''\times(0,T)} \phi^5 e^{2s\alpha}\theta(\nabla w\cdot\nabla\varphi)dxdt,$$

it is not difficult to see that

$$M_2 + M_3 \le \delta(s^5 \iint_{\omega''\times(0,T)} e^{2s\alpha}\phi^5|w|^2 dxdt + s^3 \iint_{\omega''\times(0,T)} e^{2s\alpha}\phi^3|\nabla w|^2 dxdt) + C(s^7 \iint_{\omega''\times(0,T)} e^{2s\alpha}\phi^7|\nabla\varphi|^2 dxdt + s^5 \iint_{\omega''\times(0,T)} \phi^5 e^{2s\alpha}|\varphi|^2 dxdt), \tag{2.12}$$

for all $\delta > 0$.

Hence

$$s^5 \iint_{\omega'\times(0,T)} e^{2s\alpha}\phi^5|w|^2 dxdt \le C(s^9 \iint_{\omega''\times(0,T)} \phi^9 e^{2s\alpha}|\varphi|^2 dxdt + s^7 \iint_{\omega''\times(0,T)} e^{2s\alpha}\phi^7|\nabla\varphi|^2 dxdt) + (\delta+\epsilon^2)I_2(s,1,w). \tag{2.13}$$

Now we eliminate the local integral of $\nabla\varphi$. For this, we consider a set ω''' with $\omega'' \subset\subset \omega''' \subset\subset \omega$ and a cut-off function $\theta_1 \in C_0^\infty(\omega''')$ satisfying

$$0 \le \theta_1 \le 1,\ \ \theta_1 \equiv 1 \text{ on } \omega''.$$

Integration by parts gives

$$s^7 \iint_{\omega'''\times(0,T)} e^{2s\alpha}\theta_1\phi^7|\nabla\varphi|^2 dxdt = -s^7 \iint_{\omega'''\times(0,T)} e^{2s\alpha}\theta_1\phi^7\Delta\varphi\varphi dxdt - \frac{1}{2}s^7 \iint_{\omega'''\times(0,T)} \Delta(\theta_1 e^{2s\alpha}\phi^7)|\varphi|^2 dxdt. \tag{2.14}$$

Using the fact that

$$|\Delta(\theta_1 e^{2s\alpha}\phi^7)| \le Cs^2\phi^9 e^{2s\alpha} \ \text{ in } \omega'''\times(0,T),$$

together with Young's inequality, we see that

$$s^7\iint_{\omega'''\times(0,T)} e^{2s\alpha}\phi^7|\nabla\varphi|^2dxdt \le \delta\iint_{\omega'''\times(0,T)} e^{2s\alpha}|\Delta\varphi|^2dxdt + Cs^{14}\iint_{\omega'''\times(0,T)} e^{2s\alpha}\phi^{14}|\varphi|^2dxdt, \quad (2.15)$$

for all $\delta>0$.

Therefore

$$s^5\iint_{\omega'\times(0,T)} e^{2s\alpha}\phi^5|w|^2 \le (\delta+\epsilon)I_2(s,1;w)+\delta I_1(s,1;\varphi)+Cs^{14}\iint_{\omega'''\times(0,T)} e^{2s\alpha}\phi^{14}|\varphi|^2dxdt, \quad (2.16)$$

for all $\delta>0$.

Combining (2.8) and (2.16) we get

$$I_2(s,1;w)+I_1(s,\epsilon;\xi)+I_1(s,1;\varphi) \le C\big(s^{14}\iint_{\omega'''\times(0,T)} e^{2s\alpha}\phi^{14}|\varphi|^2dxdt + s^4\iint_{\omega'\times(0,T)} e^{2s\alpha}\phi^4|\xi|^2dxdt\big). \quad (2.17)$$

Step 3. *Estimate of the local integral of ξ.*

In this step we estimate the local integral of ξ in the right-hand side of (2.17) in terms of a local integral of φ, a local integral of φ_t and some small order terms.

Using equation $(2.3)_3$ we see that

$$s^4\iint_{\omega''\times(0,T)} e^{2s\alpha^*}\theta(\phi^*)^4|\xi|^2dxdt = s^4\iint_{\omega''\times(0,T)} e^{2s\alpha^*}\frac{\theta}{c}(\phi^*)^4\xi(-\varphi_t-\Delta\varphi-a\varphi)dxdt := M_4+M_5+M_6, \quad (2.18)$$

where θ is the cut-off function introduced in Step 2.

As in previous step, we estimate each term in the expression above. We have

$$M_4 \le \frac{s^4}{2}\iint_{\omega''\times(0,T)} e^{2s\alpha^*}\theta(\phi^*)^4|\xi|^2dxdt + \frac{s^4}{2}\iint_{\omega''\times(0,T)} e^{2s\alpha^*}\frac{\theta}{c^2}(\phi^*)^4|\varphi_t|^2dxdt. \quad (2.19)$$

Integration by parts gives

$$M_5 = -\,s^4\iint_{\omega''\times(0,T)} e^{2s\alpha^*}(\phi^*)^4(\Delta(\frac{\theta}{c})\xi+2\nabla(\frac{\theta}{c})\nabla\xi+(\frac{\theta}{c})\Delta\xi)\varphi dxdt.$$

Using this last equality, we can show that

$$M_5+M_6 \le \delta\big(s^4\iint_{\omega''\times(0,T)} e^{2s\alpha}\phi^4|\xi|^2dxdt + s^2\iint_{\omega''\times(0,T)} e^{2s\alpha}\phi^2|\nabla\xi|^2dxdt + \iint_{\omega''\times(0,T)} e^{2s\alpha}|\Delta\xi|^2dxdt\big)+Cs^8\iint_{\omega''\times(0,T)} e^{4s\alpha^*-2s\alpha}(\phi^*)^8|\varphi|^2dxdt. \quad (2.20)$$

Hence

$$s^4 \iint_{\omega' \times (0,T)} e^{2s\alpha^*}\theta(\phi^*)^4|\xi|^2$$
$$\leq\ C\big(s^4 \iint_{\omega'' \times (0,T)} e^{2s\alpha^*}(\phi^*)^4|\varphi_t|^2 dxdt + s^8 \iint_{\omega'' \times (0,T)} e^{4s\alpha^*-2s\alpha}(\phi^*)^8|\varphi|^2 dxdt\big)+\delta I_1(s,\epsilon;\xi). \tag{2.21}$$

From (2.21), our objective is now reduced to estimate a local integral of φ_t in terms of a local integral of φ and small order terms. This will be done in the next steps.

Step 4. *Estimate of the local integral of φ_t.*

In this step we deal with the first term appearing in the right-hand side of (2.21).

First, we integrate by parts to get

$$\begin{aligned} s^4 \iint_{\omega'' \times (0,T)} e^{2s\alpha^*}(\phi^*)^4|\varphi_t|^2 dxdt =& -s^4 \iint_{\omega'' \times (0,T)} e^{2s\alpha^*}(\phi^*)^4 \varphi_{tt}\varphi dxdt \\ &+ \frac{s^4}{2} \iint_{\omega'' \times (0,T)} (e^{2s\alpha^*}(\phi^*)^4)_{tt}|\varphi|^2 dxdt. \end{aligned} \tag{2.22}$$

Since

$$\begin{aligned} s^4 \iint_{\omega'' \times (0,T)} e^{2s\alpha^*}(\phi^*)^4 \varphi_{tt}\varphi dxdt \leq& \frac{s^{-6}}{2} \iint_{\omega'' \times (0,T)} e^{2s\hat{\alpha}}\hat{\phi}^{-5}|\varphi_{tt}|^2 dxdt \\ &+ \frac{s^{14}}{2} \iint_{\omega'' \times (0,T)} e^{4s\alpha^*-2s\hat{\alpha}}(\phi^*)^8\hat{\phi}^5|\varphi|^2 dxdt, \end{aligned} \tag{2.23}$$

we just have to estimate the local integral of φ_{tt} in the right-hand side of (2.23). In order to do that, we use $(2.3)_2$ to see that

$$\begin{aligned} &-\epsilon(e^{s\hat{\alpha}}\hat{\phi}^{-5/2}\varphi_{tt})_t - \Delta(e^{s\hat{\alpha}}\hat{\phi}^{-5/2}\varphi_{tt}) \\ &= e^{s\hat{\alpha}}\hat{\phi}^{-5/2}w_{tt} - \epsilon(e^{s\hat{\alpha}}\hat{\phi}^{-5/2})_t\varphi_{tt} + e^{s\hat{\alpha}}\hat{\phi}^{-5/2}(d_{tt}\varphi + 2d_t\varphi_t + d\varphi_{tt}) \end{aligned} \tag{2.24}$$

with $e^{s\hat{\alpha}}\hat{\phi}^{-5/2}\varphi_{tt} = 0$ in $\partial\Omega$ and $e^{s\hat{\alpha}}\hat{\phi}^{-5/2}\varphi_{tt}(T) = e^{s\hat{\alpha}}\hat{\phi}^{-5/2}\varphi_{tt}(0) = 0$.

Next, multiplying both sides of (2.24) by $e^{s\hat{\alpha}}\hat{\phi}^{-5/2}\varphi_{tt}$, integrating over Q and using Young's inequality, we get

$$\begin{aligned} \iint_Q |\nabla(e^{s\hat{\alpha}}\hat{\phi}^{-5/2}\varphi_{tt})|^2 dxdt \leq& C\big(\iint_Q |e^{s\hat{\alpha}}\hat{\phi}^{-5/2}w_{tt}|^2 dxdt + \epsilon^2 \int_Q |(e^{s\hat{\alpha}}\hat{\phi}^{-5/2})_t\varphi_{tt}|^2 dxdt \\ &+ \iint_Q |e^{s\hat{\alpha}}\hat{\phi}^{-5/2}\varphi_t|^2 dxdt + \iint_Q |e^{s\hat{\alpha}}\hat{\phi}^{-5/2}\varphi|^2 dxdt\big) \\ &+ \iint_Q (||d||_\infty + \delta)|e^{s\hat{\alpha}}\hat{\phi}^{-5/2}\varphi_{tt}|^2 dxdt. \end{aligned} \tag{2.25}$$

Choosing now δ small enough such that $||d||_\infty + \delta < \mu_1$, we have

$$s^{-6}\iint_Q |\nabla(e^{s\hat\alpha}\hat\phi^{-5/2}\varphi_{tt})|^2dxdt \le Cs^{-6}\Big(\iint_Q |e^{s\hat\alpha}\hat\phi^{-5/2}w_{tt}|^2dxdt + \iint_Q |e^{s\hat\alpha}\hat\phi^{-5/2}\varphi_t|^2dxdt \\ + \iint_Q |e^{s\hat\alpha}\hat\phi^{-5/2}\varphi|^2dxdt + \epsilon^2\iint_Q |(e^{s\hat\alpha}\hat\phi^{-5/2})_t\varphi_{tt}|^2dxdt\Big). \tag{2.26}$$

Step 4.1. *Estimation of the term in φ_{tt}.*
Here, we estimate the last term in the right-hand side of (2.26). Using $(2.3)_2$ and $(2.3)_3$ we can show that

$$-\epsilon\varphi_{tt} = -\epsilon^2\varphi_{tt} - \epsilon w_t - \epsilon d_t\varphi - \epsilon d\varphi_t + \epsilon a_t\varphi + \epsilon a\varphi_t + \epsilon c_t\xi + \epsilon c\xi_t, \tag{2.27}$$

from where we see that

$$\epsilon^2|(e^{s\hat\alpha}\hat\phi^{-5/2})_t\varphi_{tt}|^2 \le \epsilon^4|(e^{s\hat\alpha}\hat\phi^{-5/2})_t|^2|\varphi_{tt}|^2 + \epsilon^2|(e^{s\hat\alpha}\hat\phi^{-5/2})_t|^2|w_t|^2 + C\epsilon^2\big(|(e^{s\hat\alpha}\hat\phi^{-5/2})_t|^2|\varphi|^2 \\ + |(e^{s\hat\alpha}\hat\phi^{-5/2})_t|^2|\varphi_t|^2 + |(e^{s\hat\alpha}\hat\phi^{-5/2})_t|^2|\xi|^2 + |(e^{s\hat\alpha}\hat\phi^{-5/2})_t|^2|\xi_t|^2\big). \tag{2.28}$$

Since

$$|(e^{s\hat\alpha}\hat\phi^{-5/2})_t| \le Cs^2\hat\phi^{-1/2}e^{s\hat\alpha},$$

inequality (2.28) implies

$$\epsilon^2 s^{-6}\iint_Q |(e^{s\hat\alpha}\hat\phi^{-5/2})_t\varphi_{tt}|^2dxdt \le C\epsilon^2 s^{-2}\iint_Q e^{2s\hat\alpha}\hat\phi^{-1}(|\xi|^2+|\xi_t|^2+|\varphi_t|^2+|\varphi|^2+|w_t|^2)dxdt. \tag{2.29}$$

Step 4.2. *Estimation of the term in w_{tt}.*
Here, we estiamte the first term in the right-hand side of (2.26). From $(2.3)_1$ we have

$$-w_{tt} - \Delta w_t - a_t w - a w_t = (cb)_t\varphi + cb\varphi_t - \epsilon c_{tt}\xi - \epsilon c_t\xi_t - \epsilon a_{tt}\varphi - \epsilon a_t\varphi_t - \xi_t\Delta c - \xi\Delta c_t \\ - 2\nabla\xi_t\nabla c - 2\nabla\xi\nabla c_t + \varphi_t\Delta(d-a) + \varphi\Delta(d-a)_t + 2\nabla\varphi_t\nabla(d-a) \\ + 2\nabla\varphi\nabla(d-a)_t + d_{tt}\varphi + d_t\varphi_t. \tag{2.30}$$

We multiply both sides of (2.30) by $e^{2s\hat\alpha}\hat\phi^{-5}w_{tt}$ and integrate over Q, we obtain this way

$$\begin{aligned}\iint_Q e^{2s\hat\alpha}\hat\phi^{-5}|w_{tt}|^2dxdt &= \iint_Q e^{2s\hat\alpha}\hat\phi^{-5}\Delta w_t w_{tt}dxdt + \iint_Q a_t e^{2s\hat\alpha}\hat\phi^{-5}w w_{tt}dxdt + \iint_Q a e^{2s\hat\alpha}\hat\phi^{-5}w_t w_{tt}dxdt\\
&+ \iint_Q e^{2s\hat\alpha}\hat\phi^{-5}w_{tt}\varphi\big((cb)_t - \epsilon a_{tt} + \Delta(d-a)_t + d_{tt}\big)dxdt\\
&+ \iint_Q e^{2s\hat\alpha}\hat\phi^{-5}w_{tt}\varphi_t\big(cb - \epsilon a_t + \Delta(d-a) + d_t\big)dxdt\\
&+ 2\iint_Q e^{2s\hat\alpha}\hat\phi^{-5}w_{tt}\nabla\varphi\nabla(d-a)_t dxdt + 2\int_Q e^{2s\hat\alpha}\hat\phi^{-5}w_{tt}\nabla\varphi_t\nabla(d-a)dxdt\\
&+ \iint_Q e^{2s\hat\alpha}\hat\phi^{-5}w_{tt}\xi\big(-\epsilon c_{tt} - \Delta c_t\big)dxdt + \int_Q e^{2s\hat\alpha}\hat\phi^{-5}w_{tt}\xi_t\big(-\epsilon c_t - \Delta c\big)dxdt\\
&+ 2\iint_Q e^{2s\hat\alpha}\hat\phi^{-5}w_{tt}\nabla\xi\nabla c_t dxdt + 2\int_Q e^{2s\hat\alpha}\hat\phi^{-5}w_{tt}\nabla\xi_t\nabla c dxdt. \qquad (2.31)\end{aligned}$$

After a long, but straightforward calculation, we can show that

$$\begin{aligned}s^{-6}\iint_Q e^{2s\hat\alpha}\hat\phi^{-5}|w_{tt}|^2dxdt \le Cs^{-6}\Bigg(&\iint_Q e^{2s\hat\alpha}\hat\phi^{-5}\Delta w_t w_{tt}dxdt + \iint_Q e^{2s\hat\alpha}\hat\phi^{-5}(|w|^2+|w_t|^2)dxdt\\
&+ \iint_Q e^{2s\hat\alpha}\hat\phi^{-5}(|\nabla\varphi|^2 + |\nabla\varphi_t|^2 + |\nabla\xi|^2 + |\nabla\xi_t|^2)dxdt\Bigg). \qquad (2.32)\end{aligned}$$

The rest of the proof of this step is devoted to estimate the integrals $\int_Q e^{2s\hat\alpha}\hat\phi^{-5}\Delta w_t w_{tt}dxdt$, $\int_Q e^{2s\hat\alpha}\hat\phi^{-5}|\nabla\varphi_t|^2dxdt$ and $\int_Q e^{2s\hat\alpha}\hat\phi^{-5}|\nabla\xi_t|^2dxdt$ appearing in the right-hand side of (2.32) . This will be done in the next two substeps.

Step 4.2.1. *Estimation of the term in φ_t.*

We use $(2.3)_2$ to see that $-\Delta\varphi_t = w_t + \epsilon\varphi_{tt} + d_t\varphi + d\varphi_t$ and (2.27) to show that

$$s^{-6}\iint_Q e^{2s\hat\alpha}\hat\phi^{-5}|\Delta\varphi_t|^2dxdt \le Cs^{-6}\iint_Q e^{2s\hat\alpha}\hat\phi^{-5}(|\xi|^2 + |\xi_t|^2 + |\varphi_t|^2 + |\varphi|^2 + |w_t|^2)dxdt. \qquad (2.33)$$

The estimate then follows from the fact that $-\Delta$ gives a norm in $H_0^2(\Omega)$

Step 4.2.2. *Estimation of the term in $\Delta w_t w_{tt}$.*

We have

$$s^{-6}\iint_Q e^{2s\hat\alpha}\hat\phi^{-5}\Delta w_t w_{tt}dxdt = \frac{s^{-6}}{2}\iint_Q (e^{2s\hat\alpha}\hat\phi^{-5})_t|\nabla w_t|^2dxdt \le Cs^{-4}\iint_Q e^{2s\hat\alpha}\hat\phi^{-3}|\nabla w_t|^2dxdt, \qquad (2.34)$$

since

$$|(e^{2s\hat\alpha}\hat\phi^{-5})_t| \le Cs^2e^{2s\hat\alpha}\hat\phi^{-3}.$$

Next, we use $(2.3)_1$ to see that

$$
\begin{aligned}
-w_{tt} - \Delta w_t - a_t w - a w_t = {} & (cb - \epsilon a_t + \Delta(d-a) + d_t)\varphi_t - (\epsilon c_{tt} + \Delta c_t)\xi - (\epsilon c_t + \Delta c)\xi_t \\
& + ((cb)_t + \Delta(d-a)_t - \epsilon a_{tt} + d_{tt})\varphi - 2\nabla\xi_t \nabla c - 2\nabla\xi\nabla c_t \\
& + 2\nabla\varphi_t\nabla(d-a) + 2\nabla\varphi\nabla(d-a)_t \qquad (2.35)
\end{aligned}
$$

Multipyling both sides of (2.35) by $s^{-4}e^{2s\hat{\alpha}}\hat{\phi}^{-3}w_t$, integrating by parts and using Young's inequality, we get

$$
\begin{aligned}
s^{-4}\iint_Q e^{2s\hat{\alpha}}\hat{\phi}^{-3}|\nabla w_t|^2 dxdt \le {} & Cs^{-1}\Big(\iint_Q e^{2s\hat{\alpha}}(|\nabla\varphi|^2 + |\nabla\xi|^2 + |\nabla\xi_t|^2)dxdt \\
& + \iint_Q e^{2s\hat{\alpha}}(|w|^2 + |w_t|^2)dxdt\Big), \qquad (2.36)
\end{aligned}
$$

since

$$
|(e^{2s\hat{\alpha}}\hat{\phi}^{-3})_t| \le Cs^2 e^{2s\hat{\alpha}}\hat{\phi}^{-1}
$$

and

$$
s^{-1}\varphi^{-1} \le C. \qquad (2.37)
$$

Step 4.2.3. *Estimation of the term in $\nabla\xi_t$.*
We use $(2.3)_3$ to see that $-\epsilon\xi_{tt} - \Delta\xi_t = b_t\varphi + b\varphi_t + d_t\xi + d\xi_t$. Multiplying both sides by $e^{2s\hat{\alpha}}\xi_t$ and integrating over Q, we obtain

$$
\begin{aligned}
\iint_Q e^{2s\hat{\alpha}}|\nabla\xi_t|^2 \le {} & (||d||_\infty + \delta)\iint_Q e^{2s\hat{\alpha}}|\xi_t|^2 dxdt \\
& + C\Big(\iint_Q e^{2s\hat{\alpha}}|\xi|^2 dxdt + \iint_Q e^{2s\hat{\alpha}}(|\varphi|^2 + |\varphi_t|^2)dxdt\Big), \qquad (2.38)
\end{aligned}
$$

which gives, for $\delta > 0$ small enough,

$$
\iint_Q e^{2s\hat{\alpha}}|\nabla\xi_t|^2 \le C\iint_Q e^{2s\hat{\alpha}}(|\xi|^2 + |\varphi|^2 + |\varphi_t|^2)dxdt. \qquad (2.39)
$$

Step 5. *Last arrangements and conclusion.*
From (2.22), (2.23), (2.26), (2.29), (2.32), (2.33), (2.36) and (2.39), we get

$$
\begin{aligned}
s^4\iint_{\omega''\times(0,T)} e^{2s\alpha^*}(\phi^*)^4|\varphi_t|^2 dxdt \le {} & Cs^4\iint_{\omega''\times(0,T)} (e^{2s\alpha^*}(\phi^*)^4)_{tt}|\varphi|^2 dxdt \\
& + Cs^{-1}\int_Q e^{2s\hat{\alpha}}(|\nabla\varphi|^2 + |w_t|^2 + |w|^2)dxdt + \delta I_1(s,\epsilon;\xi). \qquad (2.40)
\end{aligned}
$$

Putting (2.40) in (2.21), we obtain

$$s^4 \iint_{\omega'\times(0,T)} e^{2s\alpha^*}\theta(\phi^*)^4|\xi|^2 \leq \; Cs^8 \iint_{\omega''\times(0,T)} (e^{2s\alpha^*}+e^{4s\alpha^*-2s\alpha})(\phi^*)^8|\varphi|^2dxdt$$
$$+Cs^{-1}\iint_Q e^{2s\hat{\alpha}}(|\nabla\varphi|^2+|w_t|^2+|w|^2)dxdt+\delta I_1(s,\epsilon;\xi), \quad (2.41)$$

since

$$|(e^{2s\alpha^*}(\phi^*)^4)_{tt}| \leq Cs^4e^{2s\alpha^*}(\phi^*)^8 \quad (2.42)$$

Finally, choosing s large enough and δ small enough, we put (2.41) in (2.17) and absorb the small order terms, we obtain this way

$$I_2(s,1;w)+I_1(s,\epsilon;\xi)+I_1(s,1;\varphi) \leq Cs^{14}\iint_{\omega_1\times(0,T)}(e^{2s\alpha^*}+e^{4s\alpha^*-2s\alpha})(\phi^*)^{14}|\varphi|^2dxdt. \quad (2.43)$$

This finishes the proof of Theorem 1.1. □

Observing that the system formed by the first two equations (2.3) has the same structure as the system formed by the third and fourth equation in (2.3) we can argue as in steps 1 and 2 above in order to prove the following result, which is the third main result of this paper,

Theorem 2.3. *Let ψ, ϕ, α the functions defined above. Then, there exists $\lambda_0=\lambda_0(\Omega,\omega_2)\geq 1$ and $s_0=s_0(\Omega,\omega_2,\lambda_0)>0$ such that, for each $\lambda\geq\lambda_0$ and $s>s_0(T+T^2)$ the solution (φ,ξ) of system* (1.10) *satisfies*

$$s^3\iint_Q e^{2s\alpha}\phi^3|\xi|^2dxdt+s\iint_Q e^{2s\alpha}\phi|\nabla\xi|^2dxdt+s^{-1}\iint_Q e^{2s\alpha}\phi^{-1}(\epsilon^2|\xi_t|^2+|\Delta\xi|^2)dxdt$$
$$+s^3\iint_Q e^{2s\alpha}\xi^3|\varphi|^2dxdt+s\iint_Q e^{2s\alpha}\phi|\nabla\varphi|^2dxdt+s^{-1}\iint_Q e^{2s\alpha}\phi^{-1}(|\varphi_t|^2+|\Delta\varphi|^2)dxdt$$
$$\leq Cs^7\iint_{\omega_2\times(0,T)} e^{2s\alpha}\phi^7|\xi|^2dxdt, \quad (2.44)$$

with C depending on Ω, ω_2, ψ and λ_0.

Remark 2.4. *In the right hand side of (2.44) we have less power in $s\phi$ than in the right hand side of (2.2). This occurs because in the third and fourth equation of* $(2.3)_3$ *we do not have second order terms in the right hand side.*

3. Proof of Theorem 1.1

Now we prove Theorem 1.1. As we said before, it is equivalent to prove an observability inequality for the adjoint system, inequality (1.11) in case 1 or inequality (1.12) in case 2.

As the proof of (1.11) and (1.12) are similar, we just prove the first one. To do this, we first change the orientation in the adjoint system (1.10), i.e., instead of going from T to 0 the system will evolve from 0 to T. Changing t by $T-t$, we obtain the system

$$\left|\begin{array}{ll}\varphi_t-\Delta\varphi=a\varphi+c\xi & \text{in } \ Q,\\ \epsilon\xi_t-\Delta\xi=b\varphi+d\xi & \text{in } \ Q,\\ \varphi=\xi=0 & \text{on } \ \Sigma,\\ \varphi(0)=\varphi_T;\ \xi(0)=\xi_T & \text{in } \ \Omega.\end{array}\right. \tag{3.1}$$

Our desired observability inequality becomes

$$||\varphi(T)||^2+\epsilon||\xi(T)||^2\leq C\iint_{\omega_1\times(0,T)}|\varphi|^2dxdt, \tag{3.2}$$

where C is a constant which does not depend on ϵ.

In fact, multiplying $(3.1)_1$ by φ and $(3.1)_2$ by ξ and integrating on Ω we obtain,

$$\frac{1}{2}\frac{d}{dt}||\varphi(t)||^2+\frac{\epsilon}{2}\frac{d}{dt}||\xi(t)||^2+||\nabla\varphi(t)||^2+||\nabla\xi(t)||^2$$
$$=||a^{1/2}\varphi(t)||^2+||d^{1/2}\xi(t)||^2+((b+c)\varphi(t),\xi(t). \tag{3.3}$$

Using the assumption on d and Poincaré's inequality we get

$$\frac{1}{2}\frac{d}{dt}||\varphi(t)||^2+\frac{\epsilon}{2}\frac{d}{dt}||\xi(t)||^2\leq C||\varphi(t)||^2. \tag{3.4}$$

Then, by Gronwall's inequality we obtain

$$\frac{1}{2}||\varphi(T)||^2+\frac{\epsilon}{2}||\xi(T)||^2\leq C(||\varphi(t)||^2+||\xi(t)||^2), \tag{3.5}$$

where C does not depend on ϵ.

Integrating from $T/4$ to $3T/4$ we get

$$\frac{1}{2}||\varphi(T)||^2+\frac{\epsilon}{2}||\xi(T)||^2\leq C\int_{T/4}^{3T/4}\int_\Omega(|\varphi(t)|^2+|\xi(t)|^2)dxdt. \tag{3.6}$$

Using the Carleman inequality given by Theorem 2.2, we obtain the desired observability inequality

$$\frac{1}{2}||\varphi(T)||^2+\frac{\epsilon}{2}||\xi(T)||^2\leq C\iint_{\omega_1\times(0,T)}|\varphi|^2dxdt. \tag{3.7}$$

where C does not depend on ϵ.

Inequality (3.7) proves case 1 in Theorem 1.1. Using Theorem 2.3 we prove case 2 in Theorem 1.1 . In this way, prove of Theorem 1.1 is stablished. □

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