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PARALLELIZING THE KOLMOGOROV FOKKER PLANCK EQUATION

Luca Gerardo-Giorda, Minh-Binh Tran[1]

Abstract. We design two parallel schemes, based on Schwarz Waveform Relaxation (SWR) procedures, for the numerical solution of the Kolmogorov equation. The latter is a simplified version of the Fokker-Planck equation describing the time evolution of the probability density of the velocity of a particle. SWR procedures decompose the spatio-temporal computational domain into subdomains and solve (in parallel) subproblems, that are coupled through suitable conditions at the interfaces to recover the solution of the global problem. We consider coupling conditions of both Dirichlet (Classical SWR) and Robin (Optimized SWR) types. We prove well-posedeness of the schemes subproblems and convergence for the proposed algorithms. We corroborate our findings with some numerical tests.

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1. Introduction

The Fokker-Planck equation describes the time evolution of the probability density function of the velocity of a particle. It reads for $(t, x, v) \in \mathbb{R}_+^d \times \mathbb{R}^d \times \mathbb{R}^d$, $(d \geq 1)$

$$\partial_t u + v \cdot \nabla_x u - \nabla_x V(x) \cdot \nabla_v u = \nabla_v \cdot (\nabla_v u + vu), \tag{1.1}$$

where $V(x)$ is the external potential. Together with the theoretical study of the equation ([11], [12]), there are a lot of numerical studies on the Fokker-Plank and related equations ([6], [5], [4], [27], [30], [25], [9]), fractional Fokker-Plank equation ([10]), Wigner-Fokker-Plank equation ([15]), Fokker-Planck-Landau equation ([3], [28], [14]), Vlasov-Fokker-Planck system ([1], [8]), Vlasov-Poisson-Fokker-Planck system ([32]), Maxwell-Fokker-Planck-Landau equation ([13]), Vlasov-Fokker-Planck-Landau equation ([7]). However, up to our knowledge, there has been no domain decomposition scheme to parallelize the numerical resolution of these types of kinetic equations.
Parallel computing is a form of computation in which calculations are carried out in parallel, based on the principle that large problems can be divided into smaller ones. Due to the physical constraints of computers, parallelism has got more and more attention in the recent years. In the

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[1] Basque Center for Applied Mathematics Mazarredo 14, 48009 Bilbao Spain Email: lgerardo@bcamath.org, tbinh@bcamath.org

last two decades, domain decomposition methods have become a very useful tool to parallelize the numerical resolution of partial differential equations numerically. Schwarz waveform relaxation methods, together with its accelerated version optimized Schwarz waveform relaxation algorithms, is a new class of domain decomposition algorithms adapted to the context of studying evolution equations numerically. For a survey on this, we refer to [23] and the pioneering works [22], [18], [21], [19], [20], [17].

The main feature of our present work is to design parallel schemes based on the Schwarz waveform relaxation methods to solve numerically a simplified version of the Fokker-Planck model (1.1): the Kolmogorov equation

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial v^2} = f, \tag{1.2}$$

where f is some external force. As we can see from its form, the Kolmogorov equation diffuses not only in the velocity variable, since it contains the diffusion term $\frac{\partial^2 u}{\partial v^2}$, but also in the space variable, because of the hidden interaction between the transport term $v\frac{\partial u}{\partial x}$ and the diffusion term $\frac{\partial^2 u}{\partial v^2}$. The hypoellipticity and the asymptotic behavior of this operator have been studied in the work of L. Hormander [24] and of C. Villani [38]. Recently, the null controllability property of this operator has been explored deeply by K. Beauchard and E. Zuazua in [2].

Since the principal part of the operator involves the second derivatives in v, we design some Schwarz waveform relaxation algorithms with Dirichlet (classical Schwarz method) or Robin (optimized Schwarz method) transmission condition for this equation, by splitting the domain in the v direction. For the sake of simplicity, we only split the domain into two subdomains, however, the extension to a larger number of subdomains does not present any theoretical difficulties.

We provide some results on the existence and uniqueness of a solution for the Kolmogorov equation with different boundary conditions, in order to prove that our algorithms are well-posed. The convergence proof of Schwarz methods at the continuous level has been a very difficult task. In [35], [36], [33], [34], [37] a new class of techniques has been introduced in order to study this convergence problem of domain decomposition methods. Based on these techniques, we give a new proof of the convergence of our algorithms by some maximum principles and some energy estimates.

The structure of the paper is the following:

Section 2 is devoted to the definition of the equation and the algorithms. In Section 3 we prove existence and uniqueness results for (2.1) with Dirichlet and Robin boundary conditions, ensuring the well-posedenss of the algorithms introduced in Section 2. In Section 4 and 5 we prove convergence for the Classical and Optimized Schwarz Waveform Relaxation algorithms, respectively. Section 6 is devoted to numerical results, while the conclusions of the paper are drawn in Section 7.

2. General Setting

We are interested in the following 2 dimensional Kolmogorov model of [2]

$$\begin{cases} \frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial v^2} = f, \text{for } (t,x,v) \text{ in } (0,\infty)\times\Omega := (0,\infty)\times\mathbb{T}\times(-R,R), \\ u(t,x,-R) = u(t,x,R) = 0, \text{for } (t,x) \text{ in } (0,\infty)\times\mathbb{T}, \\ u(0,x,v) = u_0(x,v), \text{for } (x,v) \text{ in } \mathbb{T}\times(-R,R), \end{cases} \tag{2.1}$$

where $\mathbb{T}$ is the periodic domain $\mathbb{R}/\mathbb{Z}$, $f \in C([0,\infty), L^2(\mathbb{T}, L^2(-R,R))) \cap C^1(0,\infty, L^2(\mathbb{T}, L^2(-R,R))) \cap L^\infty(0,\infty, L^2(\mathbb{T}, L^2(-R,R))) \cap L^\infty((0,\infty)\times\mathbb{T}\times(-R,R)) \cap L^2(0,\infty, L^2(\mathbb{T}, L^2(-R,R)))$, $u_0 \in L^2(\mathbb{T}, H^2(-R,R))$.

It is proved in [2] that the fundamental solution of the 2 dimensional Kolmogorov equation has the form of a Gaussian. Therefore, similar to the heat equation, in order to perform numerical computations, which only work for bounded domains, in this work we do a truncation of the whole space $\mathbb{R}^2$ to a bounded domain $(-R,R)\times\mathbb{T}$ and impose homogeneous Dirichlet and periodic boundary conditions.
Notice that our results are valid for the general, multidimensional case. However, for the sake of simplicity in presentation and to avoid unnecessarily heavy notations, we consider here the 2 dimensional case as studied in [2].
Parallel domain decomposition algorithms consist of dividing the domain Ω into a number of (possibly overlapping) regions, and solve (2.1) in parallel in each subdomain. The solution to the global problem (2.1) in Ω is recovered through suitable coupling conditions at the interfaces between subdomains. For the sake of simplicity in presentation, we consider here the domain Ω divided into two parts $\Omega_1 := \mathbb{T}\times(-R,L_2)$ and $\Omega_2 := \mathbb{T}\times(L_1,R)$, where $-R<L_1<L_2<R$, and solve (2.1) parallely on each subdomain Ω_1 and Ω_2.
The *classical Schwarz waveform relaxation algorithm* for (2.1) is then written

$$\begin{cases} \frac{\partial u_1^n}{\partial t} + v\frac{\partial u_1^n}{\partial x} - \frac{\partial^2 u_1^n}{\partial v^2} = f, \text{for } (t,x,v) \text{ in } (0,\infty)\times\Omega_1, \\ u_1^n(t,x,-R) = 0, \text{for } (t,x) \text{ in } (0,\infty)\times\mathbb{T}, \\ u_1^n(0,x,v) = u_0(x,v), \text{for } (x,v) \text{ in } \Omega_1, \\ u_1^n(t,x,L_1) = u_2^{n-1}(t,x,L_1), \text{ for } (t,x) \text{ in } (0,\infty)\times\mathbb{T}, \end{cases} \tag{2.2}$$

and

$$\begin{cases} \frac{\partial u_2^n}{\partial t} + v\frac{\partial u_2^n}{\partial x} - \frac{\partial^2 u_2^n}{\partial v^2} = f, \text{for } (t,x,v) \text{ in } (0,\infty)\times\Omega_2, \\ u_2^n(t,x,R) = 0, \text{for } (t,x) \text{ in } (0,\infty)\times\mathbb{T}, \\ u_2^n(0,x,v) = u_0(x,v), \text{for } (x,v) \text{ in } \Omega_2, \\ u_2^n(t,x,L_2) = u_1^{n-1}(t,x,L_2), \text{for } (t,x) \text{ in } (0,\infty)\times\mathbb{T}, \end{cases}$$

the initial guess $u_1^0(t,x,L_2)$ and $u_2^0(t,x,L_1)$ are chosen arbitrarily in $L^\infty(0,\infty,L^\infty(\mathbb{T}))\cap L^2(0,\infty,L^\infty(\mathbb{T}))\cap C^1(0,\infty,L^\infty(\mathbb{T}))\cap C([0,\infty),L^\infty(\mathbb{T}))$ and satisfy the compatibility conditions of the equations:

$$\begin{cases} u_1^0(0,x,L_2) = u_0(x,L_2), \text{ on } \mathbb{T} \\ u_2^0(0,x,L_1) = u_0(x,L_1), \text{ on } \mathbb{T}. \end{cases}$$

When n tends to ∞, u_1^n and u_2^n are expected to converge to u on Ω_1 and Ω_2.
For any two positive numbers p,q, the *optimized Schwarz waveform relaxation algorithm* for (2.1) is defined by replacing the Dirichlet transmission condition in (2.2)

$$\begin{cases} \frac{\partial u_1^n}{\partial t} + v\frac{\partial u_1^n}{\partial x} - \frac{\partial^2 u_1^n}{\partial v^2} = f, \text{ for } (t,x,v) \text{ in } (0,\infty)\times\Omega_1, \\ u_1^n(0,x,v) = u_0(x,v), \text{ for } (x,v) \text{ in } \Omega_1, \\ u_1^n(t,x,-R) = 0, \text{ for } (t,x) \text{ in } (0,\infty)\times\mathbb{T}, \\ (p+\frac{\partial}{\partial v})u_1^n(t,x,L_2) = (p+\frac{\partial}{\partial v})u_2^{n-1}(t,x,L_2), \text{ for } (t,x) \text{ in } (0,\infty)\times\mathbb{T}, \end{cases} \tag{2.3}$$

and

$$\begin{cases} \frac{\partial u_2^n}{\partial t} + v\frac{\partial u_2^n}{\partial x} - \frac{\partial^2 u_2^n}{\partial v^2} = f, \text{ in } (0,\infty)\times\Omega_2, \\ u_2^n(0,x,v) = u_0(x,v), \text{ for } (x,v) \text{ in } \Omega_2, \\ u_2^n(t,x,R) = 0, \text{ for } (t,x) \text{ in } (0,\infty)\times\mathbb{T}, \\ (q-\frac{\partial}{\partial v})u_2^n(t,x,L_1) = (q-\frac{\partial}{\partial v})u_1^{n-1}(t,x,L_1), \text{ for } (t,x) \text{ in } (0,\infty)\times\mathbb{T}, \end{cases}$$

where at the first iteration the initial guesses u_1^0, $u_2^0 \in L^\infty(0,\infty,L^\infty(\mathbb{T},H^2(-R,R)))\cap L^2(0,\infty,L^\infty(\mathbb{T},H^2(-R,R)))\cap C^1(0,\infty,L^\infty(\mathbb{T},H^2(-R,R)))\cap C([0,\infty),L^\infty(\mathbb{T},H^2(-R,R)))$ are chosen such that $(p+\frac{\partial}{\partial v})u_1^0(t,x,L_2)$ and $(q-\frac{\partial}{\partial v})u_2^0(t,x,L_1)$ are in $C([0,\infty),L^\infty(\mathbb{T}))\cap C^1(0,\infty,L^\infty(\mathbb{T}))\cap L^\infty(0,\infty,L^\infty(\mathbb{T}))\cap L^2(0,\infty,L^\infty(\mathbb{T}))$ and satisfy the compatibility conditions of the equations:

$$\begin{cases} u_1^0(0,x,L_2) = u_0(x,L_2), \text{ on } \mathbb{T}, \\ u_2^0(0,x,L_1) = u_0(x,L_1), \text{ on } \mathbb{T}. \end{cases}$$

Compared with the classical Schwarz waveform relaxation algorithm, optimized ones require less iterations to converge to the solution of (2.1). Moreover, optimized Schwarz algorithms converge also in the non-overlapping case, a feature not shared by the classical ones.

3. Existence and Uniqueness Results for the Kolmogorov Equations

In this section, we will prove the existence and uniqueness of a solution of the Kolmogorov equation

$$\begin{cases} \frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial v^2} = f \text{ for } (t,x,v)\in(0,\infty)\times\mathbb{T}\times(a,b)\subset\Omega, \\ u(0,x,v) = u_0 \text{ in } \mathbb{T}\times(a,b). \end{cases} \tag{3.1}$$

where (a,b) could be $(-R,R)$, $(-R,L_2)$ or (L_1,R), $f\in L^\infty_{loc}((0,\infty),L^2(\mathbb{T}\times(a,b)))\cap C^1((0,\infty),L^2(\mathbb{T}\times(a,b)))\cap C([0,\infty),L^2(\mathbb{T}\times(a,b)))$, $u_0\in L^2(\mathbb{T},H^2(a,b))$.
Depending on each type of domains $(-R,R)$, $(-R,L_2)$ or (L_1,R), the boundary conditions are of the following types:

- For the problem on $(a,b)=(-R,R)$

$$\begin{cases} u(t,x,-R) = 0, \text{ on } (0,\infty)\times\mathbb{T}, \\ u(t,x,R) = 0, \text{ on } (0,\infty)\times\mathbb{T}. \end{cases} \tag{3.2}$$

- Dirichlet boundary condition
  For $(a,b)=(-R,L_2)$

$$\begin{cases} u(t,x,-R) = 0, \text{ on } (0,\infty)\times\mathbb{T}, \\ u(t,x,L_2) = h_0(t,x), \text{ on } (0,\infty)\times\mathbb{T}, \end{cases} \tag{3.3}$$

  and for $(a,b)=(L_1,R)$

$$\begin{cases} u(t,x,R) = 0, \text{ on } (0,\infty)\times\mathbb{T}, \\ u(t,x,L_1) = h_0(t,x), \text{ on } (0,\infty)\times\mathbb{T}. \end{cases} \tag{3.4}$$

- Robin boundary condition
  For $(a,b)=(-R,L_2)$

$$\begin{cases} u(t,x,-R)=0, \text{ on } (0,\infty)\times\mathbb{T}, \\ p\,u(t,x,L_2)+\frac{\partial u(t,x,L_2)}{\partial v}=h_1(t,x), \text{ on } (0,\infty)\times\mathbb{T}, \end{cases} \tag{3.5}$$

  and for $(a,b)=(L_1,R)$

$$\begin{cases} u(t,x,R)=0, \text{ on } (0,\infty)\times\mathbb{T}, \\ q\,u(t,x,L_1)-\frac{\partial u(t,x,L_1)}{\partial v}=h_1(t,x), \text{ on } (0,\infty)\times\mathbb{T}, \end{cases} \tag{3.6}$$

  for h_0, $h_1\in L^\infty_{loc}((0,\infty),L^2(\mathbb{T}))\cap C^1((0,\infty),L^2(\mathbb{T}))\cap C([0,\infty),L^2(\mathbb{T}))$ and p, q are positive constants.

Theorem 3.1. *Suppose that h_0, $h_1\in L^\infty_{loc}((0,\infty),L^2(\mathbb{T}))\cap C^1((0,\infty),L^2(\mathbb{T}))\cap C([0,\infty),L^2(\mathbb{T}))$, $f\in L^\infty_{loc}((0,\infty),L^2(\mathbb{T}\times(a,b)))\cap C^1((0,\infty),L^2(\mathbb{T}\times(a,b)))\cap C([0,\infty),L^2(\mathbb{T}\times(a,b)))$, $u_0\in L^2(\mathbb{T}\times(a,b))$, Equation* (3.1)*, with one of the boundary conditions* (3.2)*,* (3.3)*,* (3.4)*,* (3.5)*,* (3.6) *has a unique solution in $L^\infty_{loc}(0,\infty,L^2(\mathbb{T},H^2(a,b)))\cap C^1((0,\infty),L^2(\mathbb{T}\times(a,b)))\cap C([0,\infty),L^2(\mathbb{T}\times(a,b)))$.*

Proof. Since there exist functions $\tilde u_1$, $\tilde u_2$, $\tilde u_3$ and $\tilde u_4$ in $L^\infty_{loc}((0,\infty),L^2(\mathbb{T},H^2(a,b)))\cap C^1((0,\infty),L^2(\mathbb{T}))\cap C([0,\infty),L^2(\mathbb{T}))$ such that

$$\begin{cases} \tilde u_1(t,x,-R)=0, \text{ on } (0,\infty)\times\mathbb{T}, \\ \tilde u_1(t,x,L_2)=h_0(t,x), \text{ on } (0,\infty)\times\mathbb{T}, \end{cases} \quad \begin{cases} \tilde u_2(t,x,R)=0, \text{ on } (0,\infty)\times\mathbb{T}, \\ \tilde u_2(t,x,L_1)=h_0(t,x), \text{ on } (0,\infty)\times\mathbb{T}, \end{cases}$$

and

$$\begin{cases} \tilde u_3(t,x,-R)=0, \text{ on } (0,\infty)\times\mathbb{T}, \\ p\,\tilde u_3(t,x,L_2)+\frac{\partial \tilde u_3(t,x,L_2)}{\partial v}=h_1(t,x), \end{cases} \quad \begin{cases} \tilde u_4(t,x,R)=0, \text{ on } (0,\infty)\times\mathbb{T}, \\ q\,\tilde u_4(t,x,L_1)-\frac{\partial \tilde u_4(t,x,L_1)}{\partial v}=h_1(t,x), \text{ on } (0,\infty)\times\mathbb{T}, \end{cases}$$

then by subtracting u with $\tilde u_1$, $\tilde u_2$, $\tilde u_3$ or $\tilde u_4$, we can suppose that $h_0=h_1=0$.
Take the Fourier transform in x of (3.1),

$$\frac{\partial \hat u}{\partial t}+iv\zeta\hat u-\frac{\partial^2\hat u}{\partial v^2}=\hat f, \text{ for } (t,\zeta,v) \text{ in}(0,\infty)\times\mathbb{R}\times(a,b). \tag{3.7}$$

Split $\hat u$ and $\hat f$ into their real and imaginary parts

$$\hat u=\hat u_1+i\hat u_2,\ \hat f=\hat f_1+i\hat f_2.$$

Equation (3.7) becomes

$$\begin{cases} \frac{\partial \hat u_1(\zeta)}{\partial t}-v\zeta\hat u_2(\zeta)-\frac{\partial^2\hat u_1(\zeta)}{\partial v^2}=\hat f_1(\zeta), \text{ on } (0,\infty)\times(a,b), \\ \frac{\partial \hat u_2(\zeta)}{\partial t}+v\zeta\hat u_1(\zeta)-\frac{\partial^2\hat u_2(\zeta)}{\partial v^2}=\hat f_2(\zeta), \text{ on } (0,\infty)\times(a,b), \end{cases} \tag{3.8}$$

the four boundary conditions remain the same after this transformation.

- For the problem on $(a,b)=(-R,R)$

$$\begin{cases} \hat u_1(t,\zeta,-R)=\hat u_2(t,\zeta,-R)=0, \text{ on } (0,\infty)\times\mathbb{R}, \\ \hat u_1(t,\zeta,R)=\hat u_2(t,\zeta,R)=0, \text{ on } (0,\infty)\times\mathbb{R}. \end{cases} \tag{3.9}$$

- Dirichlet boundary condition
  For $(a,b)=(-R,L_2)$

$$\begin{cases} \hat u_1(t,\zeta,-R)=\hat u_2(t,\zeta,-R)=0, \text{ on } (0,\infty)\times\mathbb{R}, \\ \hat u_1(t,\zeta,L_2)=\hat u_2(t,\zeta,L_2)=0, \text{ on } (0,\infty)\times\mathbb{R}, \end{cases} \tag{3.10}$$

  and for $(a,b)=(L_1,R)$

$$\begin{cases} \hat u_1(t,\zeta,R)=\hat u_2(t,\zeta,R)=0, \text{ on } (0,\infty)\times\mathbb{R} \\ \hat u_1(t,\zeta,L_1)=\hat u_2(t,\zeta,L_1)=0, \text{ on } (0,\infty)\times\mathbb{R}. \end{cases} \tag{3.11}$$

- Robin boundary condition
  For $(a,b)=(-R,L_2)$

$$\begin{cases} \hat u_1(t,\zeta,-R)=\hat u_2(t,\zeta,-R)=0, \text{ on } (0,\infty)\times\mathbb{R}, \\ p\,\hat u_1(t,\zeta,L_2)+\frac{\partial\hat u_1(t,\zeta,L_2)}{\partial v}=p\,\hat u_2(t,\zeta,L_2)+\frac{\partial\hat u_2(t,\zeta,L_2)}{\partial v}=0, \text{ on } (0,\infty)\times\mathbb{R}, \end{cases} \tag{3.12}$$

  and for $(a,b)=(L_1,R)$

$$\begin{cases} \hat u_1(t,\zeta,R)=\hat u_2(t,\zeta,R)=0, \text{ on } (0,\infty)\times\mathbb{R}, \\ q\,\hat u_1(t,x,L_1)-\frac{\partial\hat u_1(t,\zeta,L_1)}{\partial v}=q\,\hat u_2(t,x,L_1)-\frac{\partial\hat u_2(t,\zeta,L_1)}{\partial v}=0, \text{ on } (0,\infty)\times\mathbb{R}. \end{cases} \tag{3.13}$$

For any given ζ, since $\hat f_1(\zeta),\,\hat f_2(\zeta)\in L^\infty_{loc}((0,\infty),L^2(a,b))\cap C^1([0,\infty),L^2(a,b))\cap C((0,\infty),L^2(a,b))$, there exists a solution $(\hat u_1(\zeta),\hat u_2(\zeta))$ in $L^\infty_{loc}((0,\infty),H^2(a,b))\cap C^1((0,\infty),L^2(a,b))\cap C([0,\infty),L^2(a,b))$ of (3.8) (see, for example [26, Chapter VII]).
Choose ζ to be an integer n and use $\hat u_1(n)$ and $\hat u_2(n)$ as test functions for the system (3.8),

$$\begin{cases} \frac12\int_a^b\frac{\partial|\hat u_1(n)|^2}{\partial t}dv-\int_a^b vn\hat u_2(n)\hat u_1(n)dv-\int_a^b\frac{\partial^2\hat u_1(n)}{\partial v^2}\hat u_1(n)dv=\int_a^b\hat f_1(n)\hat u_1(n)dv, \\ \frac12\int_a^b\frac{\partial|\hat u_2(n)|^2}{\partial t}dv+\int_a^b vn\hat u_1(n)\hat u_2(n)dv-\int_a^b\frac{\partial^2\hat u_2(n)}{\partial v^2}\hat u_2(n)dv=\int_a^b\hat f_2(n)\hat u_2(n)dv, \end{cases}$$

which implies

$$\begin{cases} \frac12\int_a^b\frac{\partial|\hat u_1(n)|^2}{\partial t}dv-\int_a^b vn\hat u_2(n)\hat u_1(n)dv+\int_a^b\left|\frac{\partial\hat u_1(n)}{\partial v}\right|^2dv=\int_a^b\hat f_1(n)\hat u_1(n)dv \\ +\frac{\partial\hat u_1(n)(b)}{\partial v}\hat u_1(n)(b)-\frac{\partial\hat u_1(n)(a)}{\partial v}\hat u_1(n)(a), \\ \frac12\int_a^b\frac{\partial|\hat u_2(n)|^2}{\partial t}dv+\int_a^b vn\hat u_1(n)\hat u_2(n)dv+\int_a^b\left|\frac{\partial\hat u_2(n)}{\partial v}\right|^2dv=\int_a^b\hat f_2(n)\hat u_2(n)dv \\ +\frac{\partial\hat u_2(n)(b)}{\partial v}\hat u_2(n)(b)-\frac{\partial\hat u_2(n)(a)}{\partial v}\hat u_2(n)(a). \end{cases} \tag{3.14}$$

For the boundary conditions (3.9), (3.10), (3.11), we have

$$\frac{\partial\hat u_j(n)(b)}{\partial v}\hat u_j(n)(b)=\frac{\partial\hat u_j(n)(a)}{\partial v}\hat u_j(n)(a)=0,\ j=\{1,2\}.$$

For the boundary condition (3.12), (3.13), the quantity

$$\frac{\partial\hat u_j(n)(b)}{\partial v}\hat u_j(n)(b)-\frac{\partial\hat u_j(n)(a)}{\partial v}\hat u_j(n)(a) \text{ is either } -p|\hat u_j(n)(b)|^2 \text{ or } -q|\hat u_j(n)(a)|^2,\ j=\{1,2\}.$$

Adding the two equations (3.14), and taking into account the fact that p and q are positive, we get

$$\begin{aligned}
&\frac{1}{2}\int_a^b \frac{\partial|\hat{u}_1(n)|^2}{\partial t}dv + \frac{1}{2}\int_a^b \frac{\partial|\hat{u}_2(n)|^2}{\partial t}dv + \int_a^b \left|\frac{\partial \hat{u}_1(n)}{\partial v}\right|^2 dv + \int_a^b \left|\frac{\partial \hat{u}_2(n)}{\partial v}\right|^2 dv \\
\leq\ & \int_a^b \hat{f}_1(n)\hat{u}_1(n)dv + \int_a^b \hat{f}_2(n)\hat{u}_2(n)dv \qquad (3.15)\\
\leq\ & \frac{1}{2}\int_a^b |\hat{f}_1(n)|^2 dv + \frac{1}{2}\int_a^b |\hat{u}_1(n)|^2 dv + \frac{1}{2}\int_a^b |\hat{f}_2(n)|^2 dv + \frac{1}{2}\int_a^b |\hat{u}_2(n)|^2 dv,
\end{aligned}$$

then

$$\int_a^b \left(\frac{\partial|\hat{u}_1(n)|^2}{\partial t}dv + \frac{\partial|\hat{u}_2(n)|^2}{\partial t}\right)dv - \int_a^b \left(|\hat{u}_1(n)|^2 + |\hat{u}_2(n)|^2\right)dv \quad \leq \quad \int_a^b (|\hat{f}_1(n)|^2 dv + |\hat{f}_2(n)|^2)dv.$$

The previous inequality implies

$$\frac{\partial}{\partial t}\left(\int_a^b (|\hat{u}_1(n)|^2 dv + |\hat{u}_2(n)|^2)dv \exp(-t)\right) \quad \leq \quad \int_a^b \left(|\hat{f}_1(n)|^2 dv + |\hat{f}_2(n)|^2\right) dv \exp(-t).$$

Thus

$$\begin{aligned}
&\int_a^b (|\hat{u}_1(n,t)|^2 + |\hat{u}_2(n,t)|^2)dv \\
\leq\ & \int_0^t \int_a^b \exp(t-s)(|\hat{f}_1(n)|^2 + |\hat{f}_2(n)|^2)dvds + \exp(t)\int_a^b (|\hat{u}_1(n,0)|^2 + |\hat{u}_2(n,0)|^2)dv \\
\leq\ & \exp(t)\int_0^t \int_a^b (|\hat{f}_1(n)|^2 + |\hat{f}_2(n)|)dvds + \exp(t)\int_a^b (|\hat{u}_1(n,0)|^2 + |\hat{u}_2(n,0)|^2)dv,
\end{aligned}$$

Summing up in $\mathbb{Z}$ the previous inequalities yields

$$\begin{aligned}
\int_a^b \sum_{n\in\mathbb{Z}}(|\hat{u}_1(n,t)|^2 + |\hat{u}_2(n,t)|^2)dv \quad \leq \quad & \exp(t)\int_0^t \int_a^b \sum_{n\in\mathbb{Z}}(|\hat{f}_1(n)|^2 + |\hat{f}_2(n)|^2)dvds \\
& + \exp(t)\int_a^b \sum_{n\in\mathbb{Z}}(|\hat{u}_1(n,0)|^2 + |\hat{u}_2(n,0)|^2)dvds,
\end{aligned}$$

which together with the Parseval's theorem implies

$$\int_a^b \int_{\mathbb{R}} |\hat{u}(\zeta,t)|^2 d\zeta dv \leq \exp(t)\int_a^b \int_{\mathbb{R}} \|\hat{f}(\zeta)\|^2_{L^2(0,t)} d\zeta dv + \exp(t)\int_a^b \int_{\mathbb{R}} |\hat{u}_0(\zeta)|^2 d\zeta dv.$$

Therefore, the inverse Fourier transform u of $\hat{u}_1$ and $\hat{u}_2$ exists and

$$\int_{\mathbb{T}} \int_a^b |u(t)|^2 dvdx \leq \exp(t)\int_a^b \int_{\mathbb{T}} \|f(x)\|^2_{L^2(0,t)} dvdx + \exp(t)\int_a^b \int_{\mathbb{T}} |u_0|^2 dvdx. \qquad (3.16)$$

The existence and uniqueness of a solution of (3.1) with one of the above boundary conditions then follow by a classical argument as in [26, Chapter VII]. □

By a classical induction argument as in [16], we have also the well-posedness of the algorithm.

Theorem 3.2. *Suppose that* $f \in L^\infty(0,\infty,L^2(\mathbb{T},L^2(-R,R))) \cap L^\infty((0,\infty)\times\mathbb{T}\times(-R,R)) \cap L^2(0,\infty,L^2(\mathbb{T},L^2(-R,R)))\cap C^1((0,\infty),L^2(\mathbb{T}\times(-R,R)))\cap C([0,\infty),L^2(\mathbb{T}\times(-R,R)))$, $u_0 \in L^2(\mathbb{T}\times(-R,R))$ *and the initial guesses for the Dirichlet transmission condition* u_1^0, $u_2^0 \in L^\infty_{loc}((0,\infty),L^2(\mathbb{T}))\cap C^1((0,\infty),L^2(\mathbb{T}))\cap C([0,\infty),L^2(\mathbb{T}))$, *the initial guesses for the Robin transmission condition* u_1^0, $u_2^0 \in L^\infty(0,\infty,L^\infty(\mathbb{T},H^2(-R,R))) \cap L^2(0,\infty,L^\infty(\mathbb{T},\ H^2(-R,R))) \cap C^1(0,\infty,L^2(\mathbb{T}\times(-R,R))) \cap C([0,\infty),L^2(\mathbb{T}\times(-R,R)))$ *are chosen such that* $(p+\frac{\partial}{\partial v})u_1^0(t,x,L_2)$ *and* $(q-\frac{\partial}{\partial v})u_2^0(t,x,L_1)$ *are in* $L^\infty_{loc}(0,\infty,L^2(\mathbb{T}))$, *Equations* (2.2) *and* (2.3) *have unique solutions in* $L^\infty_{loc}(0,\infty,L^2(\mathbb{T},H^2(-R,L_2))) \cap C^1(0,\infty,\ L^2(\mathbb{T}\times(-R,L_2))) \cap C([0,\infty),L^2(\mathbb{T}\times(-R,L_2)))$ *and* $L^\infty_{loc}(0,\infty,L^2(\mathbb{T},H^2(L_1,R)))\cap C^1(0,\infty,\ L^2(\mathbb{T}\times(L_1,R))) \cap C([0,\infty),L^2(\mathbb{T}\times(L_1,R)))$.

4. Convergence of the Classical Schwarz Waveform Relaxation Algorithm

Theorem 4.1. *Suppose that* $L_1 < L_2$. *For all positive number* T, *the algorithm converges in the following sense*

$$\lim_{n\to\infty}\|u_1^n - u\|_{L^\infty((0,T)\times\Omega_1)} = 0,$$

and

$$\lim_{n\to\infty}\|u_2^n - u\|_{L^\infty((0,T)\times\Omega_2)} = 0.$$

Proof. Since the problems are linear in u, we can prove the convergence on the error equation by letting $e_1^n = u_1^n - u$ and $e_2^n = u_2^n - u$, then

$$\begin{cases} \frac{\partial e_1^n}{\partial t} + v\frac{\partial e_1^n}{\partial x} - \frac{\partial^2 e_1^n}{\partial v^2} = 0, \text{ in } (0,\infty)\times\Omega_1,\\ e_1^n(0,x,v) = 0, \text{ on } \Omega_1,\\ e_1^n(t,x,-R) = 0, \text{ on } (0,\infty)\times\mathbb{T},\\ e_1^n(t,x,L_2) = e_2^{n-1}(t,x,L_2), \text{ on } (0,\infty)\times\mathbb{T}, \end{cases} \quad \begin{cases} \frac{\partial e_2^n}{\partial t} + v\frac{\partial e_2^n}{\partial x} - \frac{\partial^2 e_2^n}{\partial v^2} = 0, \text{ in } (0,\infty)\times\Omega_2,\\ e_2^n(0,x,v) = 0, \text{ on } \Omega_2,\\ e_2^n(t,x,R) = 0, \text{ on } (0,\infty)\times\mathbb{T},\\ e_2^n(t,x,L_1) = e_1^{n-1}(t,x,L_1), \text{ on } (0,\infty)\times\mathbb{T}. \end{cases} \tag{4.1}$$

Let α be a constant to be chosen later, following the classical strategy in [33], [34], [37], we define

$$\Phi_1^n = (e_1^n)^2\exp(-\alpha^2 t)\exp(\alpha v),\ \Phi_2^n = (e_2^n)^2\exp(-\alpha^2 t)\exp(\alpha v).$$

Again, following the classical routine of [33], [34], [37], we develop

$$\begin{aligned} \frac{\partial}{\partial t}\Phi_1^n &= -\alpha^2(e_1^k)^2\exp(-\alpha^2 t)\exp(\alpha v) + 2\partial_t e_1^k e_1^k\exp(-\alpha^2 t)\exp(\alpha v),\\ \frac{\partial}{\partial x}\Phi_1^n &= 2\partial_x e_1^k e_1^k\exp(-\alpha^2 t)\exp(\alpha v),\\ \frac{\partial}{\partial v}\Phi_1^n &= \alpha(e_1^k)^2\exp(-\alpha^2 t)\exp(\alpha v) + 2\partial_v e_1^k e_1^k\exp(-\alpha^2 t)\exp(\alpha v),\\ \frac{\partial^2}{\partial v^2}\Phi_1^n &= \alpha^2(e_1^k)^2\exp(-\alpha^2 t)\exp(\alpha v) + 4\alpha\partial_v e_1^k e_1^k\exp(-\alpha^2 t)\exp(\alpha v)\\ &\quad +2(\partial_v e_1^k)^2\exp(-\alpha^2 t)\exp(\alpha v) + 2\partial_{vv}e_1^k e_1^k\exp(-\alpha^2 t)\exp(\alpha v) \end{aligned}$$

to get

$$\frac{\partial}{\partial t}\Phi_1^n - \frac{\partial^2}{\partial v^2}\Phi_1^n + v\frac{\partial}{\partial x}\Phi_1^n + 2\alpha\frac{\partial}{\partial v}\Phi_1^n = -2(\partial_v e_1^n)^2\exp(-\alpha^2 t)\exp(\alpha v) \le 0, \tag{4.2}$$

and similarly

$$\frac{\partial}{\partial t}\Phi_2^n - \frac{\partial^2}{\partial v^2}\Phi_2^n + v\frac{\partial}{\partial x}\Phi_2^n + 2\alpha\frac{\partial}{\partial v}\Phi_2^n \leq 0. \tag{4.3}$$

Step 1: The maximum principle.
We prove that the solution u of (2.1) belongs to $L^\infty([0,T]\times\mathbb{T}\times(-R,R))$. Let K be greater than $\|f\|_{L^\infty([0,T]\times\mathbb{T}\times(-R,R))}$ and $\|u_0\|_{L^\infty(\mathbb{T}\times(-R,R))}$, then

$$\begin{cases} \frac{\partial(u-Kt)}{\partial t} + v\frac{\partial(u-Kt)}{\partial x} - \frac{\partial^2(u-Kt)}{\partial v^2} = f-K, \text{ in } (0,T)\times\Omega, \\ (u-Kt)(0,x,v) = u_0(x,v), \text{ on } \mathbb{T}\times(-R,R). \end{cases}$$

Set $(u-Kt)_+ = u-Kt$ for $u \geq Kt$ and 0 for $u < Kt$. Using $(u-Kt)_+$ as a test function for the above equation, we get

$$\begin{aligned} 0 \;\geq\; & \int_0^T\int_{-R}^R\int_{\mathbb{T}} \frac{\partial}{\partial t}(u-Kt)(u-Kt)_+ dxdvdt + \int_0^T\int_{-R}^R\int_{\mathbb{T}} v\frac{\partial}{\partial x}(u-Kt)(u-Kt)_+ dxdvdt \\ & + \int_0^T\int_{-R}^R\int_{\mathbb{T}} \frac{\partial}{\partial v}(u-Kt)\frac{\partial}{\partial v}(u-Kt)_+ dxdvdt, \end{aligned}$$

which yields

$$\begin{aligned} 0 \;\geq\; & \int_0^T\int_{-R}^R\int_{\mathbb{T}} \frac{\partial}{\partial t}(u-Kt)_+(u-Kt)_+ dxdvdt + \int_0^T\int_{-R}^R\int_{\mathbb{T}} v\frac{\partial}{\partial x}(u-Kt)_+(u-Kt)_+ dxdvdt \\ & + \int_0^T\int_{-R}^R\int_{\mathbb{T}} \frac{\partial}{\partial v}(u-Kt)_+\frac{\partial}{\partial v}(u-Kt)_+ dxdvdt. \end{aligned}$$

Therefore

$$0 \;\geq\; \int_{-R}^R\int_{\mathbb{T}} \frac{(u-Kt)_+^2}{2}\Big|_0^T dxdv + \int_0^T\int_{-R}^R\int_{\mathbb{T}} \left(\frac{\partial}{\partial v}(u-Kt)_+\right)^2 dxdvdt.$$

Hence $(u-Kt)_+ = 0$, then $u \leq KT$ or u is bounded from above. By a similar argument, we can prove that u is bounded from below, and $u \in L^\infty([0,T]\times\mathbb{T}\times(-R,R))$.
Let $M = \sup_{(t,x)\in(0,T)\times\mathbb{T}}\{\Phi_2^{n-1}(t,x,L_2)\}$ and suppose that $M < \infty$. Notice that $u_2^0 \in L^\infty([0,T]\times\mathbb{T})$ and $u \in L^\infty([0,T]\times\mathbb{T}\times(-R,R))$, then $M < \infty$ for $n = 1$. Set $(\Phi_1^n - M)_+ = \Phi_1^n - M$ for $\Phi_1^n \geq M$ and 0 for $\Phi_1^n < M$. Using it as a test function for (4.2), we obtain

$$\begin{aligned} 0 \;\geq\; & \int_0^T\int_{-R}^{L_2}\int_{\mathbb{T}} \frac{\partial}{\partial t}(\Phi_1^n - M)(\Phi_1^n - M)_+ dxdvdt + \int_0^T\int_{-R}^{L_2}\int_{\mathbb{T}} v\frac{\partial}{\partial x}(\Phi_1^n - M)(\Phi_1^n - M)_+ dxdvdt \\ & + \int_0^T\int_{-R}^{L_2}\int_{\mathbb{T}} \frac{\partial}{\partial v}(\Phi_1^n - M)\frac{\partial}{\partial v}(\Phi_1^n - M)_+ dxdvdt + 2\alpha\int_0^T\int_{-R}^{L_2}\int_{\mathbb{T}} \frac{\partial}{\partial v}(\Phi_1^n - M)(\Phi_1^n - M)_+ dxdvdt. \end{aligned}$$

This leads to

$$\begin{aligned} 0 \;\geq\; & \int_{-R}^{L_2}\int_{\mathbb{T}} \frac{(\Phi_1^n - M)_+^2}{2}\Big|_0^T dxdv + \int_0^T\int_{-R}^{L_2}\int_{\mathbb{T}} \left(\frac{\partial}{\partial v}(\Phi_1^n - M)_+\right)^2 dxdvdt \\ & + \alpha\int_0^T\int_{\mathbb{T}} (\Phi_1^n - M)_+^2\Big|_{-R}^{L_2} dxdt, \end{aligned}$$

which gives $(\Phi_1^n - M)_+ = 0$. As a consequence,

$$\Phi_1^n \leq M,$$

or

$$\Phi_1^n(t,x,v) \leq \sup_{(t,x')\in(0,T)\times\mathbb{T}} \{\Phi_2^{n-1}(t,x',L_2)\} \text{ on } \Omega_1. \tag{4.4}$$

A similar argument leads to

$$\Phi_2^n(t,x,v) \leq \sup_{(t,x')\in(0,T)\times\mathbb{T}} \{\Phi_1^{n-1}(t,x',L_1)\} \text{ on } \Omega_2. \tag{4.5}$$

Step 2: The convergence estimates.
Denote

$$E^n = \max_{i\in\{1,2\}} \left(\sup_{(t,x)\in((0,T)\times\Omega_i)} (e_i^n)^2 \exp(-\alpha^2 t) \right).$$

Since e_1^0 and e_2^0 are bounded, E^n is bounded.
Inequality (4.4) implies that for (t,x) in $(0,T)\times\mathbb{T}$

$$(e_1^n(t,x,L_1))^2 \exp(-\alpha^2 t)\exp(\alpha L_1) \leq \sup_{(t,x)\in(0,T)\times\mathbb{T}} (e_2^{n-1}(t,x,L_2))^2 \exp(-\alpha^2 t)\exp(\alpha L_2),$$

which yields

$$(e_1^n(t,x,L_1))^2 \exp(-\alpha^2 t) \leq \exp((L_2-L_1)\alpha) \sup_{(t,x)\in(0,T)\times\mathbb{T}} (e_2^{n-1}(t,x,L_2))^2 \exp(-\alpha^2 t).$$

Choosing $\alpha = -\alpha_0$ where α_0 is a positive constant to get

$$(e_1^n(t,x,L_1))^2 \exp(-\alpha^2 t) \leq \exp((L_1-L_2)\alpha_0) \sup_{(t,x)\in(0,T)\times\mathbb{T}} (e_2^{n-1}(t,x,L_2))^2 \exp(-\alpha^2 t).$$

Similarly, by using the same argument and replacing α by α_0

$$(e_2^n(t,x,L_2))^2 \exp(-\alpha^2 t) \leq \exp((L_1-L_2)\alpha_0) \sup_{(t,x)\in(0,T)\times\mathbb{T}} (e_1^{n-1}(t,x,L_1))^2 \exp(-\alpha^2 t).$$

Choose $\alpha = 0$, (4.4) and (4.5) imply

$$E^{n+1} \leq \max\{\sup_{(t,x)\in(0,T)\times\mathbb{T}} (e_1^n(t,x,L_2))^2 \exp(-\alpha^2 t), \sup_{(t,x)\in(0,T)\times\mathbb{T}} (e_2^n(t,x,L_1))^2 \exp(-\alpha^2 t)\}.$$

The above inequality implies

$$\begin{aligned} E^{n+1} &\leq \exp((L_1-L_2)\alpha_0)\max\{\sup_{(t,x)\in(0,T)\times\mathbb{T}} (e_2^{n-1}(t,x,L_2))^2 \exp(-\alpha^2 t), \\ &\quad \sup_{(t,x)\in(0,T)\times\mathbb{T}} (e_2^{n-1}(t,x,L_2))^2 \exp(-\alpha^2 t)\}. \\ &\leq \exp((L_1-L_2)\alpha_0)E^{n-1}, \end{aligned}$$

which shows that the errors converge geometrically

$$\lim_{k\to\infty} E^k = 0.$$

□

5. Convergence of the Schwarz Waveform Relaxation Methods with Robin Transmission Conditions

Again, we prove the convergence on the error equation by letting $e_1^n = u_1^n - u$ and $e_2^n = u_2^n - u$, we consider

$$\begin{cases} \frac{\partial e_1^n}{\partial t} + v\frac{\partial e_1^n}{\partial x} - \frac{\partial^2 e_1^n}{\partial v^2} = 0, \text{ in } (0,\infty)\times\Omega_1, \\ e_1^n(0,x,v) = 0, \text{ on } \Omega_1, \\ e_1^n(t,x,-R) = 0, \text{ on } (0,\infty)\times\mathbb{T}, \\ (p+\frac{\partial}{\partial v})e_1^n(t,x,L_2) = (p+\frac{\partial}{\partial v})e_2^{n-1}(t,x,L_2), \text{ on } (0,\infty)\times\mathbb{T}, \end{cases} \tag{5.1}$$

$$\begin{cases} \frac{\partial e_2^n}{\partial t} + v\frac{\partial e_2^n}{\partial x} - \frac{\partial^2 e_2^n}{\partial v^2} = 0, \text{ in } (0,\infty)\times\Omega_2, \\ e_2^n(0,x,v) = 0, \text{ on } \Omega_2, \\ e_2^n(t,x,R) = 0, \text{ on } (0,\infty)\times\mathbb{T}, \\ (q-\frac{\partial}{\partial v})e_2^n(t,x,L_1) = (q-\frac{\partial}{\partial v})e_1^{n-1}(t,x,L_1), \text{ on } (0,\infty)\times\mathbb{T}. \end{cases}$$

Let α be a constant larger than 1. For φ in $L^2(0,\infty)$, we recall the following norm first introduced in [35]

$$|||\varphi|||_\alpha = \sup_{\alpha'>\alpha}\left[\int_{\alpha'}^{\alpha'+1}\left(\int_0^\infty \varphi(x)\exp(-yx)dx\right)^2 dy\right]^{\frac{1}{2}},$$

and the space

$$\mathbb{L}^2_\alpha(0,\infty) = \{\varphi \ : \ \varphi\in L^2(0,\infty), |||\varphi|||_\alpha < \infty\}.$$

Remark 5.1. *Notice that in order to check $|||.|||_\alpha$ is a norm, the properties*

$$|||\varphi_1+\varphi_2|||_\alpha \le |||\varphi_1|||_\alpha + |||\varphi_2|||_\alpha, \ \forall \varphi_1,\varphi_2\in\mathbb{L}^2_\alpha(0,\infty),$$

$$|||\lambda\varphi|||_\alpha = |\lambda||||\varphi|||_\alpha, \ \forall\varphi\in\mathbb{L}^2_\alpha(0,\infty), \lambda\in\mathbb{R},$$

are easy to check. And the fact that $|||\varphi|||_\alpha = 0$ if and only if $\varphi = 0$ is classical (see, for example the book [39]). The norm was introduced in [35] to overcome, provided α is large enough, the lack of energy structure in the equation due to the coexistence of different boundary conditions.

For $\varsigma\in H^1(-R,L_2)$, $\varrho\in H^1(L_1,R)$ there exist the extensions $\bar\varsigma\in H^1(L_2,R)$, $\bar\varrho\in H^1(-R,L_1)$ and a constant $\mathcal{C}$ not depending on ς and ϱ such that $\varsigma(L_2)=\bar\varsigma(L_2)$, $\varrho(L_1)=\bar\varrho(L_1)$, and

$$\begin{aligned} &\|\bar\varsigma\|_{H^1(L_2,R)} \le \mathcal{C}\|\varsigma\|_{H^1(-R,L_2)}, \ \|\bar\varsigma\|_{L^2(L_2,R)} \le \mathcal{C}\|\varsigma\|_{L^2(-R,L_2)}, \\ &\|\bar\varrho\|_{H^1(-R,L_1)} \le \mathcal{C}\|\varrho\|_{H^1(L_1,R)}, \ \|\bar\varrho\|_{L^2(-R,L_1)} \le \mathcal{C}\|\varrho\|_{L^2(L_1,R)}. \end{aligned} \tag{5.2}$$

Define $\mathcal{C}^*$ to be the constant in the trace theorem

$$|\varsigma(L_1)| \le \mathcal{C}^*\|\varsigma\|_{H^1(-R,L_1)}, \ |\varrho(L_2)| \le \mathcal{C}^*\|\varrho\|_{H^1(L_2,R)}.$$

Let f_3, f_4 be strictly positive functions in $C^2([-R,R])$ such that $f_3, f_4 > \beta$, where β is some positive constant. Suppose that f_3, f_4 satisfy the following assumptions on $[-R,R]$

$$\begin{aligned} & f_3'(L_2) = f_3'(L_1) = f_4'(L_1) = f_4'(L_2) = 0, \\ & \frac{f_4(L_1)}{f_3(L_1)} \max\{1,\mathcal{C}\}\,(3+q\mathcal{C}^*) < \frac{1}{8}, \frac{f_3(L_2)}{f_4(L_2)} \max\{1,\mathcal{C}\}\,(3+p\mathcal{C}^*) < \frac{1}{8}. \end{aligned} \tag{5.3}$$

Let k be an integer, set

$$\begin{aligned} \alpha_k \geq A_k : &= (6R\pi|k|+2)^3 + 2\left(\left\|\frac{f_3'}{f_3}\right\|_\infty + \left\|\frac{f_4'}{f_4}\right\|_\infty + 1\right)^2, \\ f_{1,k}(t) &= \exp(-2\alpha_k t), \\ f_{2,k}(x) &= \exp(2i\pi kx). \end{aligned} \tag{5.4}$$

For $\theta \in L^2((0,\infty)\times\mathbb{T})$, decompose θ under the form of a Fourier series in x

$$\theta(t,x) = \sum_{-\infty}^{\infty} \hat{\theta}(t,k)\exp(2i\pi kx),$$

where

$$\hat{\theta}(t,k) = \int_{\mathbb{T}} \theta(t,x)\exp(-2i\pi kx)dx.$$

Define the norm

$$\|\theta\|_{t,x}^2 = \sum_{k=-\infty}^{\infty} |||\hat{\theta}(t,k)|||_{A_k}^2,$$

and the space

$$\mathbb{H} = \{\theta \ : \ \theta \in L^2((0,\infty)\times\mathbb{T}), \|\theta\|_{t,x} < \infty\}.$$

For a function $f(t,x,v)$ with $(t,x,v) \in (0,\infty)\times\mathbb{T}\times(a,b)$, with $(a,b) = (-\infty, L_2)$ or (L_1,∞), we define the following norm and space

$$\|f\|_{\mathcal{L}((0,\infty)\times\mathbb{T}\times(a,b))} = \left(\int_a^b \|f(.,.,v)\|_{t,x}^2 dv\right)^{1/2},$$

$$\mathcal{L}((0,\infty)\times\mathbb{T}\times(a,b)) = \{f(t,x,v) \mid \|f\|_{\mathcal{L}((0,\infty)\times\mathbb{T}\times(a,b))} < 0\}.$$

Theorem 5.1. *The algorithm converges in the following sense*

$$\lim_{n\to\infty} \|u_1^n - u\|_{\mathcal{L}((0,\infty)\times\mathbb{T}\times(-\infty,L_2))} = 0,$$

and

$$\lim_{n\to\infty} \|u_2^n - u\|_{\mathcal{L}((0,\infty)\times\mathbb{T}\times(L_1,\infty))} = 0.$$

Proof. Define

$$\Phi_{1,k}^{n+1}(v) = \int_0^\infty \int_{\mathbb{T}} e_1^{n+1}(t,x,v) f_{1,k}(t) f_{2,k}(x) f_3(v) dxdt,$$

$$\Phi_{2,k}^{n+1}(v) = \int_0^\infty \int_{\mathbb{T}} e_2^{n+1}(t,x,v) f_{1,k}(t) f_{2,k}(x) f_4(v) dxdt.$$

Following the strategy in [35], [36], [33], [34], [37], we compute

$$\begin{aligned}
\int_0^\infty \int_{\mathbb{T}} \frac{\partial}{\partial t} e_1^{n+1} f_{1,k} f_{2,k} f_3 dx dt &= -\int_0^\infty \int_{\mathbb{T}} e_1^{n+1} f'_{1,k} f_{2,k} f_3 dx dt = 2\alpha_k \Phi_{1,k}^{n+1} \\
\int_0^\infty \int_{\mathbb{T}} \frac{\partial}{\partial x} e_1^{n+1} f_{1,k} f_{2,k} f_3 dx dt &= -\int_0^\infty \int_{\mathbb{T}} e_1^{n+1} f_{1,k} f'_{2,k} f_3 dx dt = -i2\pi k \Phi_{1,k}^{n+1}, \\
\frac{\partial}{\partial v} \Phi_{1,k}^{n+1} &= \int_0^\infty \int_{\mathbb{T}} \frac{\partial}{\partial v} e_1^{n+1} f_{1,k} f_{2,k} f_3 dx dt + \frac{f'_3}{f_3} \Phi_{1,k}^{n+1}, \\
\frac{\partial^2}{\partial v^2} \Phi_{1,k}^{n+1} = \frac{f''_3}{f_3} \Phi_{1,k}^{n+1} &+ \int_0^\infty \int_{\mathbb{T}} \frac{\partial^2}{\partial v^2} e_1^{n+1} f_{1,k} f_{2,k} f_3 dx dt + 2\frac{f'_3}{f_3} \frac{\partial}{\partial v} \Phi_{1,k}^{n+1} - 2\left(\frac{f'_3}{f_3}\right)^2 \Phi_{1,k}^{n+1},
\end{aligned}$$

which implies

$$\int_0^\infty \int_{\mathbb{T}} e_1^{n+1} f_{1,k} f_{2,k} f_3 \left(2\alpha_k - iv2\pi k - 2\left(\frac{f'_3}{f_3}\right)^2 + \frac{f''_3}{f_3}\right) dx dt - \frac{\partial^2}{\partial v^2} \Phi_{1,k}^{n+1} + 2\frac{f'_3}{f_3} \frac{\partial}{\partial v} \Phi_{1,k}^{n+1} = 0, \quad (5.5)$$

and

$$\int_0^\infty \int_{\mathbb{T}} e_2^{n+1} f_{1,k} f_{2,k} f_3 \left(2\alpha_k - iv2\pi k - 2\left(\frac{f'_4}{f_4}\right)^2 + \frac{f''_4}{f_4}\right) dx dt - \frac{\partial^2}{\partial v^2} \Phi_{2,k}^{n+1} + 2\frac{f'_4}{f_4} \frac{\partial}{\partial v} \Phi_{2,k}^{n+1} = 0. \quad (5.6)$$

The Robin boundary conditions become

$$\begin{aligned}
\left(p + \frac{\partial}{\partial v}\right) \Phi_{1,k}^{n+1}(L_2) &= \int_0^\infty \int_{\mathbb{T}} \left(p + \frac{\partial}{\partial v}\right) e_1^{n+1} f_{1,k} f_{2,k} f_3 dx dt + \int_0^\infty \int_{\mathbb{T}} e_1^{n+1} f_{1,k} f_{2,k} f'_3 dx dt \\
&= \frac{f_3(L_2)}{f_4(L_2)} \left(p + \frac{\partial}{\partial v}\right) \Phi_{2,k}^{n}(L_2),
\end{aligned}$$

and

$$\left(-q + \frac{\partial}{\partial v}\right) \Phi_{2,k}^{n+1}(L_1) = \frac{f_4(L_1)}{f_3(L_1)} \left(-q + \frac{\partial}{\partial v}\right) \Phi_{1,k}^{n}(L_1). \quad (5.7)$$

Define

$$S_n = \sum_{k=-\infty}^{\infty} \left(\int_{-R}^{L_2} |\Phi_{1,k}^n|^2 dv + \frac{1}{\alpha_k} \int_{-R}^{L_2} \left| \frac{\partial}{\partial v} \Phi_{1,k}^n \right|^2 dv + \int_{L_1}^{R} |\Phi_{2,k}^n|^2 dv + \frac{1}{\alpha_k} \int_{L_1}^{R} \left| \frac{\partial}{\partial v} \Phi_{2,k}^n \right|^2 dv \right),$$

we will prove that

$$S_n \leq \frac{1}{3} S_{n-1}, \quad (5.8)$$

which implies $S_n \leq \frac{1}{3^n} S_0$, Therefore, if S_0 is bounded, then S_n is also bounded for all n. Moreover, S_n converges geometrically to 0 with the rate $\frac{1}{3^n}$. We divide the rest of the proof into three steps:
Step 1: We prove that S_0 is bounded. Denote

$$S_0 = S_0^1 + S_0^2;$$

$$S_0^1 = \sum_{k=-\infty}^{\infty} \left(\int_{-R}^{L_2} |\Phi_{1,k}^0|^2 dv + \frac{1}{\alpha_k} \int_{-R}^{L_2} \left| \frac{\partial}{\partial v} \Phi_{1,k}^0 \right|^2 dv \right);$$

$$S_0^2 = \sum_{k=-\infty}^{\infty} \left(\int_{L_1}^{R} |\Phi^0_{2,k}|^2 dv + \frac{1}{\alpha_k} \int_{L_1}^{R} \left| \frac{\partial}{\partial v} \Phi^0_{2,k} \right|^2 dv \right).$$

We prove that S_0^1 is bounded, and the fact that S_0^2 is bounded will follow by the same argument. We have that

$$\begin{aligned} S_0^1 &= \sum_{k=-\infty}^{\infty} \int_{-R}^{L_2} \left(\left| \int_0^{\infty} \int_{\mathbb{T}} e_1^0 \exp(-2\alpha_k t) \exp(i2\pi kx) f_3(v) dx dt \right|^2 \right. \\ &\quad \left. + \frac{1}{\alpha_k} \left| \int_0^{\infty} \int_{\mathbb{T}} \left(\frac{\partial}{\partial v} e_1^0 f_3(v) + e_1^0 f_3'(v) \right) \exp(-2\alpha_k t) \exp(i2\pi kx) dx dt \right|^2 \right) dv \\ &= \sum_{k=-\infty}^{\infty} \int_{-R}^{L_2} \left(\left| \int_0^{\infty} \hat{e_1^0}(k) \exp(-2\alpha_k t) f_3(v) dt \right|^2 \right. \\ &\quad \left. + \frac{1}{\alpha_k} \left| \int_0^{\infty} \left(\frac{\partial}{\partial v} \hat{e_1^0}(k) f_3(v) + \hat{e_1^0}(k) f_3'(v) \right) \exp(-2\alpha_k t) dt \right|^2 \right) dv, \end{aligned} \tag{5.9}$$

where $\hat{e_1^0}(k)$ denotes the Fourier transform of e_1^0 in the x variable.
Since f_3 and f_3' are continuous, they are bounded on $[-R, L_2]$, then bounding f_3 and f_3' by $\|f_3\|_\infty$ and $\|f_3'\|_\infty$ in (5.9) and using the fact that $\alpha_k > 1$ we have

$$\begin{aligned} S_0^1 &\leq C \sum_{k=-\infty}^{\infty} \int_{-R}^{L_2} \left(\left| \int_0^{\infty} |\hat{e_1^0}(k)| \exp(-2\alpha_k t) dt \right|^2 \right. \\ &\quad \left. + C \frac{1}{\alpha_k} \left| \int_0^{\infty} \left(\left| \frac{\partial}{\partial v} \hat{e_1^0}(k) \right| + |\hat{e_1^0}(k)| \right) \exp(-2\alpha_k t) dt \right|^2 \right) dv \\ &\leq C \sum_{k=-\infty}^{\infty} \int_{-R}^{L_2} \left(\left| \int_0^{\infty} |\hat{e_1^0}(k)| \exp(-2\alpha_k t) dt \right|^2 + \frac{1}{\alpha_k} \left| \int_0^{\infty} \left| \frac{\partial}{\partial v} \hat{e_1^0}(k) \right| \exp(-2\alpha_k t) dt \right|^2 \right) dv, \end{aligned} \tag{5.10}$$

where C is some constant varying from lines to lines. Now, applying Holder's inequality for the integrals in t in (5.10), we obtain

$$\begin{aligned} S_0^1 &\leq C \sum_{k=-\infty}^{\infty} \int_{-R}^{L_2} \left(\int_0^{\infty} |\hat{e_1^0}(k)|^2 \exp((-2\alpha_k + 1)t) dt \int_0^{\infty} \exp(-t) dt \right. \\ &\quad \left. + \frac{1}{\alpha_k} \int_0^{\infty} \left| \frac{\partial}{\partial v} \hat{e_1^0}(k) \right|^2 \exp((-2\alpha_k + 1)t) dt \int_0^{\infty} \exp(-t) dt \right) dv \\ &\leq C \sum_{k=-\infty}^{\infty} \int_{-R}^{L_2} \left(\int_0^{\infty} |\hat{e_1^0}(k)|^2 \exp((-2\alpha_k + 1)t) dt + \frac{1}{\alpha_k} \int_0^{\infty} \left| \frac{\partial}{\partial v} \hat{e_1^0}(k) \right|^2 \exp((-2\alpha_k + 1)t) dt \right) dv \\ &= C \int_0^{\infty} \int_{-R}^{L_2} \sum_{k=-\infty}^{\infty} \left(|\hat{e_1^0}(k)|^2 \exp((-2\alpha_k + 1)t) + \frac{1}{\alpha_k} \left| \frac{\partial}{\partial v} \hat{e_1^0}(k) \right|^2 \exp((-2\alpha_k + 1)t) \right) dv dt \\ &\leq C \int_0^{\infty} \int_{-R}^{L_2} \sum_{k=-\infty}^{\infty} \left(|\hat{e_1^0}(k)|^2 + \left| \frac{\partial}{\partial v} \hat{e_1^0}(k) \right|^2 \right) dv \exp(-8t) dt = C \int_0^{\infty} \|e_1^0\|^2_{L^2(\mathbb{T}, H^1(-R, L_2))} \exp(-8t) dt, \end{aligned} \tag{5.11}$$

where in the last inequality, we use the fact that $\alpha_k \geq 8$. Since according to the hypothesis (2.1) $f \in L^2(0,\infty, L^2(\mathbb{T}, L^2(-R, L_2))$, Inequality (3.16) implies

$$\int_{\mathbb{T}} \int_a^b |u(t)|^2 dv dx \leq \exp(t) \int_a^b \int_{\mathbb{T}} \|f\|^2_{L^2(0,\infty)} dv dx + \exp(t) \int_a^b \int_{\mathbb{T}} |u_0|^2 dv dx.$$

Therefore

$$\int_0^\infty \|u\|^2_{L^2(\mathbb{T}, L^2(-R,L_2))} \exp(-8t) dt < \infty, \tag{5.12}$$

Moreover, (3.15) implies

$$\int_a^b \left(\frac{\partial |\hat{u}_1(n)|^2}{\partial t} dv + \frac{\partial |\hat{u}_2(n)|^2}{\partial t} \right) dv + \int_a^b \left(\left| \frac{\partial \hat{u}_1(n)}{\partial v} \right|^2 + \left| \frac{\partial \hat{u}_2(n)}{\partial v} \right|^2 \right) dv$$
$$- \int_a^b \left(|\hat{u}_1(n)|^2 dv + |\hat{u}_2(n)|^2 \right) dv \leq \int_a^b (|\hat{f}_1(n)|^2 dv + |\hat{f}_2(n)|^2) dv.$$

We argue similarly as to obtain (3.16)

$$\frac{\partial}{\partial t} \left(\int_a^b (|\hat{u}_1(n)|^2 dv + |\hat{u}_2(n)|^2) dv \exp(-t) \right) + \int_a^b \left(\left| \frac{\partial \hat{u}_1(n)}{\partial v} \right|^2 + \left| \frac{\partial \hat{u}_2(n)}{\partial v} \right|^2 \right) dv \exp(-t)$$
$$\leq \int_a^b \left(|\hat{f}_1(n)|^2 dv + |\hat{f}_2(n)|^2 \right) dv \exp(-t),$$

which leads to

$$\exp(-t) \int_a^b (|\hat{u}_1(n,t)|^2 + |\hat{u}_2(n,t)|^2) dv + \int_0^t \int_a^b \left(\left| \frac{\partial \hat{u}_1(n)}{\partial v} \right|^2 + \left| \frac{\partial \hat{u}_2(n)}{\partial v} \right|^2 \right) \exp(-s) dv ds$$
$$\leq \int_0^t \int_a^b \exp(-s)(|\hat{f}_1(n)|^2 + |\hat{f}_2(n)|) dv ds + \int_a^b (|\hat{u}_1(n,0)|^2 + |\hat{u}_2(n,0)|^2) dv.$$

Therefore

$$\int_0^t \int_a^b \left(\left| \frac{\partial \hat{u}_1(n)}{\partial v} \right|^2 + \left| \frac{\partial \hat{u}_2(n)}{\partial v} \right|^2 \right) \exp(-s) dv ds$$
$$\leq \int_0^\infty \int_a^b \exp(-s)(|\hat{f}_1(n)|^2 + |\hat{f}_2(n)|^2) dv ds + \int_a^b (|\hat{u}_1(n,0)|^2 + |\hat{u}_2(n,0)|^2) dv$$
$$\leq \int_0^\infty \int_a^b (|\hat{f}_1(n)|^2 + |\hat{f}_2(n)|^2) dv ds + \int_a^b (|\hat{u}_1(n,0)|^2 + |\hat{u}_2(n,0)|^2) dv. \ \forall t \in \mathbb{R}_+.$$

Let t tend to infinity, we deduce from the above inequality that

$$\int_0^\infty \|\nabla u\|^2_{L^2(\mathbb{T}, L^2(-R,L_2))} \exp(-t) dt < \infty. \tag{5.13}$$

Since u_1^0, $u_2^0 \in L^\infty(0,\infty, L^\infty(\mathbb{T}, H^2(-R,R))) \cap L^2(0,\infty, L^\infty(\mathbb{T}, H^2(-R,R)))$, we deduce from the two inequalities (5.12) and (5.13) that

$$\int_0^\infty \|e_1^0\|^2_{L^2(\mathbb{T}, H^1(-R,L_2))} \exp(-8t) dt < \infty,$$

which means (5.11) implies

$$S_0^1 < C \int_0^\infty \|e_1^0\|^2_{L^2(\mathbb{T},H^1(-R,L_2))} \exp(-8t)dt < \infty.$$

Similarly

$$S_0^2 < C \int_0^\infty \|e_2^0\|^2_{L^2(\mathbb{T},H^1(L_1,R))} \exp(-8t)dt < \infty.$$

Step 2: We prove (5.8).
Consider (5.5) with the index n instead of $n+1$ on $\mathbb{T}\times(-R,L_1)$ and take φ_1^n in $H^1(-R,L_1)$ as a test function, then

$$\begin{aligned} 0 \quad = \quad & \int_{-R}^{L_1}\int_0^\infty\int_{\mathbb{T}} e_1^n f_1 f_2 f_3 \left(2\alpha - iv2\pi k - 2\left(\frac{f_3'}{f_3}\right)^2 + \frac{f_3''}{f_3}\right)\varphi_1^n dxdtdv \\ & -\int_{-R}^{L_1}\frac{\partial^2}{\partial v^2}\Phi_{1,k}^n\varphi_1^n dv + \int_{-R}^{L_1} 2\frac{f_3'}{f_3}\frac{\partial}{\partial v}\Phi_{1,k}^n\varphi_1^n dv. \end{aligned}$$

Intergrating by parts the term $\int_{-R}^{L_1}\frac{\partial^2}{\partial v^2}\Phi_{1,k}^n\varphi_1^n dv$ in the above intergral, we get

$$\begin{aligned} & \frac{\partial}{\partial v}\Phi_{1,k}^n(L_1)\varphi_1^n(L_1) - q\Phi_{1,k}^n(L_1)\varphi_1^n(L_1) \\ = \quad & \int_{-R}^{L_1}\int_0^\infty\int_{\mathbb{T}} e_1^n f_{1,k} f_{2,k} f_3 \left(2\alpha_k - iv2\pi k - 2\left(\frac{f_3'}{f_3}\right)^2 + \frac{f_3''}{f_3}\right)\varphi_1^n dxdtdv \\ & + \int_{-R}^{L_1}\frac{\partial}{\partial v}\Phi_{1,k}^n\frac{\partial}{\partial v}\varphi_1^n dv + \int_{-R}^{L_1} 2\frac{f_3'}{f_3}\frac{\partial}{\partial v}\Phi_{1,k}^n\varphi_1^n dv - q\Phi_{1,k}^n(L_1)\varphi_1^n(L_1). \end{aligned} \tag{5.14}$$

Considering (5.6) on $\mathbb{T}\times(L_1,R)$ and taking φ_2^{n+1} in $H^1(L_1,R)$ as a test function satisfying $\varphi_2^{n+1}(L_1)=\varphi_1^n(L_1)$, we get

$$\begin{aligned} & -\frac{\partial}{\partial v}\Phi_{2,k}^{n+1}(L_1)\varphi_2^{n+1}(L_1) + q\Phi_{2,k}^{n+1}(L_1)\varphi_2^{n+1}(L_1) \\ = \quad & \int_{L_1}^{R}\int_0^\infty\int_{\mathbb{T}} e_2^{n+1} f_{1,k} f_{2,k} f_3 \left(2\alpha_k - iv2\pi k - 2\left(\frac{f_3'}{f_3}\right)^2 + \frac{f_3''}{f_3}\right)\varphi_2^{n+1} dxdtdv \\ & + \int_{L_1}^{R}\frac{\partial}{\partial v}\Phi_{2,k}^{n+1}\frac{\partial}{\partial v}\varphi_2^{n+1} dv + \int_{L_1}^{R} 2\frac{f_3'}{f_3}\frac{\partial}{\partial v}\Phi_{2,k}^{n+1}\varphi_2^{n+1} dv + q\Phi_{2,k}^{n+1}(L_1)\varphi_2^{n+1}(L_1). \end{aligned} \tag{5.15}$$

Since $\varphi_2^{n+1}(L_1)=\varphi_1^n(L_1)$, equation (5.7) leads to

$$-\frac{\partial}{\partial v}\Phi_{2,k}^{n+1}(L_1)\varphi_2^{n+1}(L_1)+q\Phi_{2,k}^{n+1}(L_1)\varphi_2^{n+1}(L_1) = -\frac{f_4(L_1)}{f_3(L_1)}\left[\frac{\partial}{\partial v}\Phi_{1,k}^n(L_1)\varphi_1^n(L_1) - q\Phi_{1,k}^n(L_1)\varphi_1^n(L_1)\right],$$

which, together with (5.14) and (5.15) imply

$$-\frac{f_4(L_1)}{f_3(L_1)}\left[\int_{-R}^{L_1}\int_0^\infty\int_{\mathbb{T}} e_1^n f_{1,k} f_{2,k} f_3 \left(2\alpha_k - iv2\pi k - 2\left(\frac{f_3'}{f_3}\right)^2 + \frac{f_3''}{f_3}\right)\varphi_1^n dxdtdv\right.$$

$$
\begin{aligned}
&+\int_{-R}^{L_1}\frac{\partial}{\partial v}\Phi^n_{1,k}\frac{\partial}{\partial v}\varphi^n_1 dv+\int_{-R}^{L_1}2\frac{f_3'}{f_3}\frac{\partial}{\partial v}\Phi^n_{1,k}\varphi^n_1 dv-q\Phi^n_1(L_1)\varphi^n_1(L_1)\Bigg]\\
=\;&\int_{L_1}^{R}\int_0^\infty\int_{\mathbb{T}}e_2^{n+1}f_{1,k}f_{2,k}f_3\left(2\alpha_k-iv2\pi k-2\left(\frac{f_3'}{f_3}\right)^2+\frac{f_3''}{f_3}\right)\varphi_2^{n+1}dxdtdv\\
&+\int_{L_1}^{R}\frac{\partial}{\partial v}\Phi^{n+1}_{2,k}\frac{\partial}{\partial v}\varphi^{n+1}_2 dv+\int_{L_1}^{R}2\frac{f_3'}{f_3}\frac{\partial}{\partial v}\Phi^{n+1}_{2,k}\varphi^{n+1}_2 dv+q\Phi^{n+1}_{2,k}(L_1)\varphi^{n+1}_2(L_1).
\end{aligned}
$$

In the above equality, choose φ_2^{n+1} to be $\overline{\Phi^{n+1}_{2,k}}$, and φ_1^n to be the extension of $\overline{\Phi^{n+1}_{2,k}}$ over $(-R,L_1)$ like in (5.2) such that there exists a constant $\mathcal{C}$ satisfying

$$
\|\varphi_1^n\|_{H^1(-R,L_1)}\le\mathcal{C}\|\Phi^{n+1}_{2,k}\|_{H^1(L_1,R)},\ \ \|\varphi_1^n\|_{L^2(-R,L_1)}\le\mathcal{C}\|\Phi^{n+1}_{2,k}\|_{L^2(L_1,R)},\tag{5.16}
$$

to get

$$
\begin{aligned}
&-\frac{f_4(L_1)}{f_3(L_1)}\Bigg(\int_{-R}^{L_1}\int_0^\infty\int_{\mathbb{T}}e_1^{n}f_{1,k}f_{2,k}f_3\left(2\alpha_k-iv2\pi k-2\left(\frac{f_3'}{f_3}\right)^2+\frac{f_3''}{f_3}\right)\varphi_1^{n}dxdtdv\\
&+\int_{-R}^{L_1}\frac{\partial}{\partial v}\Phi^n_{1,k}\frac{\partial}{\partial v}\varphi^n_1 dv+\int_{-R}^{L_1}2\frac{f_3'}{f_3}\frac{\partial}{\partial v}\Phi^n_{1,k}\varphi^n_1 dv-q\Phi^n_{1,k}(L_1)\varphi^n_1(L_1)\Bigg)\\
=\;&\int_{L_1}^{R}\int_0^\infty\int_{\mathbb{T}}e_2^{n+1}f_{1,k}f_{2,k}f_3\left(2\alpha_k-iv2\pi k-2\left(\frac{f_3'}{f_3}\right)^2+\frac{f_3''}{f_3}\right)\overline{\Phi^{n+1}_{2,k}}dxdtdv\\
&+\int_{L_1}^{R}\left|\frac{\partial}{\partial v}\Phi^{n+1}_{2,k}\right|^2 dv+\int_{L_1}^{R}2\frac{f_3'}{f_3}\frac{\partial}{\partial v}\Phi^{n+1}_{2,k}\overline{\Phi^{n+1}_{2,k}}dv+q(\Phi^{n+1}_{2,k}(L_1))^2.
\end{aligned}\tag{5.17}
$$

We now bound the right hand side of (5.17) from below and the left hand side of (5.17) from above. Consider the first term on the right hand side of (5.17)

$$
\begin{aligned}
&\int_{L_1}^{R}\int_0^\infty\int_{\mathbb{T}}e_2^{n+1}f_{1,k}f_{2,k}f_3\left(2\alpha_k-iv2\pi k-2\left(\frac{f_3'}{f_3}\right)^2+\frac{f_3''}{f_3}\right)\overline{\Phi^{n+1}_{2,k}}dxdtdv\\
=\;&2\alpha_k\int_{L_1}^{R}\int_0^\infty\int_{\mathbb{T}}e_2^{n+1}f_{1,k}f_{2,k}f_3\overline{\Phi^{n+1}_{2,k}}dxdtdv\\
&+\int_{L_1}^{R}\int_0^\infty\int_{\mathbb{T}}e_2^{n+1}f_{1,k}f_{2,k}f_3\left(-iv2\pi k-2\left(\frac{f_3'}{f_3}\right)^2+\frac{f_3''}{f_3}\right)\overline{\Phi^{n+1}_{2,k}}dxdtdv\\
=\;&2\alpha_k\int_{L_1}^{R}\int_0^\infty\int_{\mathbb{T}}e_2^{n+1}f_{1,k}f_{2,k}f_3\overline{\Phi^{n+1}_{2,k}}dxdtdv\\
&+\int_{L_1}^{R}\left[\left(-iv2\pi k-2\left(\frac{f_3'}{f_3}\right)^2+\frac{f_3''}{f_3}\right)\left(\int_0^\infty\int_{\mathbb{T}}e_2^{n+1}f_{1,k}f_{2,k}f_3dxdt\right)\overline{\Phi^{n+1}_{2,k}}\right]dv\\
=\;&2\alpha_k\int_{L_1}^{R}|\Phi^{n+1}_{2,k}|^2dv+\int_{L_1}^{R}\left[\left(-iv2\pi k-2\left(\frac{f_3'}{f_3}\right)^2+\frac{f_3''}{f_3}\right)|\Phi^{n+1}_{2,k}|^2\right]dv\\
\ge\;&2\alpha_k\int_{L_1}^{R}|\Phi^{n+1}_{2}|^2dv-\left(|v2\pi k|+2\left\|\frac{f_3'}{f_3}\right\|^2_{L^\infty}+\left\|\frac{f_3''}{f_3}\right\|_{L^\infty}\right)\int_{L_1}^{R}\int_0^\infty\int_{\mathbb{T}}|e_2^{n+1}f_{1,k}f_{2,k}f_3\overline{\Phi^{n+1}_{2,k}}|dxdtdv
\end{aligned}
$$

$$
\begin{aligned}
&= \left(2\alpha_k - |v2\pi k| - 2\left\|\frac{f_3'}{f_3}\right\|_{L^\infty}^2 - \left\|\frac{f_3''}{f_3}\right\|_{L^\infty}\right)\int_{L_1}^R |\Phi_{2,k}^{n+1}|^2 dv \\
&\geq \alpha_k \int_{L_1}^R |\Phi_{2,k}^{n+1}|^2 dv,
\end{aligned}
$$

where in the last inequality, we use (5.3), (5.4). Therefore the right hand side of (5.17) could be bounded from below by the use of Cauchy inequality

$$
\begin{aligned}
& \int_{L_1}^R \alpha_k |\Phi_{2,k}^{n+1}|^2 dv + \int_{L_1}^R \left|\frac{\partial}{\partial v}\Phi_{2,k}^{n+1}\right|^2 dv + \int_{L_1}^R 2\frac{f_3'}{f_3}\frac{\partial}{\partial v}\Phi_{2,k}^{n+1}\overline{\Phi_2^{n+1}} dv \\
\geq\ & \int_{L_1}^R \alpha_k |\Phi_{2,k}^{n+1}|^2 dv + \int_{L_1}^R \left|\frac{\partial}{\partial v}\Phi_{2,k}^{n+1}\right|^2 dv - \left\|\frac{f_3'}{f_3}\right\|_{L^\infty}\int_{L_1}^R 2\left|\frac{\partial}{\partial v}\Phi_{2,k}^{n+1}\right|\left|\Phi_2^{n+1}\right| dv \\
\geq\ & \int_{L_1}^R \alpha_k |\Phi_{2,k}^{n+1}|^2 dv + \int_{L_1}^R \left|\frac{\partial}{\partial v}\Phi_{2,k}^{n+1}\right|^2 dv - \left\|\frac{f_3'}{f_3}\right\|_{L^\infty}\int_{L_1}^R \left(\epsilon\left|\frac{\partial}{\partial v}\Phi_{2,k}^{n+1}\right|^2 + \frac{1}{\epsilon}|\Phi_{2,k}^{n+1}|^2\right) dv \\
\geq\ & \left(\alpha_k - \frac{1}{\epsilon}\left\|\frac{f_3'}{f_3}\right\|_{L^\infty}\right)\int_{L_1}^R |\Phi_{2,k}^{n+1}|^2 dv + \left(1 - \epsilon\left\|\frac{f_3'}{f_3}\right\|_{L^\infty}\right)\int_{L_1}^R \left|\frac{\partial}{\partial v}\Phi_{2,k}^{n+1}\right|^2 dv \\
\geq\ & \int_{L_1}^R \frac{\alpha_k}{2}|\Phi_{2,k}^{n+1}|^2 dv + \frac{1}{2}\int_{L_1}^R \left|\frac{\partial}{\partial v}\Phi_{2,k}^{n+1}\right|^2 dv, \qquad (5.18)
\end{aligned}
$$

the last inequality follows from (5.3), (5.4), ϵ is a positive constant could be chosen to be $(2\left\|\frac{f_3'}{f_3}\right\|_{L^\infty})^{-1}$. By Cauchy inequality, the Sobolev imbedding theorem, Holder inequality and (5.2) with the notice that $(-R, L_1) \subset (-R, L_2)$

$$
\begin{aligned}
\int_{-R}^{L_1} \alpha_k |\Phi_{1,k}^n||\varphi_1^n| dv &\leq \frac{\alpha_k}{2}\int_{-R}^{L_1} |\Phi_{1,k}^n|^2 dv + \frac{\alpha_k}{2}\int_{-R}^{L_1} |\varphi_1^n|^2 dv \\
&\leq \frac{\alpha_k}{2}\|\Phi_{1,k}^n\|_{L^2(-R,L_2)}^2 + \mathcal{C}\frac{\alpha_k}{2}\|\Phi_{2,k}^{n+1}\|_{L^2(L_1,R)}^2, \\
\int_{-R}^{L_1} \left|\frac{\partial}{\partial v}\Phi_{1,k}^n\right|\left|\frac{\partial}{\partial v}\varphi_1^n\right| dv, &\leq \frac{1}{2}\int_{-R}^{L_1} \left|\frac{\partial}{\partial v}\Phi_{1,k}^n\right|^2 dv + \frac{1}{2}\int_{-R}^{L_1} \left|\frac{\partial}{\partial v}\varphi_1^n\right|^2 dv, \\
&\leq \frac{1}{2}\|\Phi_{1,k}^n\|_{H^1(-R,L_2)}^2 + \frac{\mathcal{C}}{2}\|\Phi_{2,k}^{n+1}\|_{H^1(L_1,R)}^2 \\
\int_{-R}^{L_1} 2\left|\frac{f_3'}{f_3}\right|\left|\frac{\partial}{\partial v}\Phi_{1,k}^n\right| |\varphi_1^n| dv &\leq \left\|\frac{f_3'}{f_3}\right\|_{L^\infty}\int_{-R}^{L_1} 2\left|\frac{\partial}{\partial v}\Phi_{1,k}^n\right||\varphi_1^n| dv, \\
&\leq \int_{-R}^{L_1} \left|\frac{\partial}{\partial v}\Phi_{1,k}^n\right|^2 + \left\|\frac{f_3'}{f_3}\right\|_{L^\infty}^2\int_{-R}^{L_1} |\varphi_1^n|^2 dv \\
&\leq \|\Phi_{1,k}^n\|_{H^1(-R,L_2)}^2 + \mathcal{C}\alpha_k\|\Phi_{2,k}^{n+1}\|_{L^2(L_1,R)}^2,
\end{aligned}
$$

and by the trace theorem (5.3)

$$
\begin{aligned}
q|\Phi_{1,k}^n(L_1)||\varphi_1^n(L_1)| &\leq q\mathcal{C}^*\|\Phi_{1,k}^n\|_{H^1(-R,L_1)}^2 + q\mathcal{C}^*\|\varphi_1^n\|_{H^1(-R,L_1)}^2 \\
&\leq q\mathcal{C}^*\|\Phi_{1,k}^n\|_{H^1(-R,L_2)}^2 + q\mathcal{C}^*\mathcal{C}\|\Phi_{2,k}^{n+1}\|_{H^1(L_1,R)}^2.
\end{aligned}
$$

Summing all of the above inequalites, we infer that the left hand side of (5.17) is bounded from above by

$$\begin{aligned}
&\frac{f_4(L_1)}{f_3(L_1)}\left(\int_{-R}^{L_1}\alpha_k|\Phi_1^n||\varphi_1^n|dv+\int_{-R}^{L_1}\left|\frac{\partial}{\partial v}\Phi_{1,k}^n\right|\left|\frac{\partial}{\partial v}\varphi_1^n\right|dv\right. \qquad (5.19)\\
&\left.+\int_{-R}^{L_1}2\left|\frac{f_3'}{f_3}\right|\left|\frac{\partial}{\partial v}\Phi_{1,k}^n\right||\varphi_1^n|dv+q|\Phi_{1,k}^n(L_1)||\varphi_1^n(L_1)|\right)\\
\leq\;&\frac{f_4(L_1)}{f_3(L_1)}\left[\left(\frac{\alpha_k+3}{2}+q\mathcal{C}^*\right)\int_{-R}^{L_2}\alpha_k|\Phi_{1,k}^n|^2dv+\left(\frac{3}{2}+q\mathcal{C}^*\right)\int_{-R}^{L_2}\left|\frac{\partial}{\partial v}\Phi_{1,k}^n\right|^2dv\right]\\
&+\frac{f_4(L_1)}{f_3(L_1)}\mathcal{C}\left[\left(\frac{3\alpha_k+1}{2}+q\mathcal{C}^*\right)\int_{L_1}^{R}|\Phi_{2,k}^{n+1}|^2dv+\left(\frac{3}{2}+q\mathcal{C}^*\right)\int_{L_1}^{R}\left|\frac{\partial}{\partial v}\Phi_{2,k}^{n+1}\right|^2dv\right]\\
&\leq\frac{f_4(L_1)}{f_3(L_1)}\left(\frac{3}{2}+q\mathcal{C}^*\right)\left(\int_{-R}^{L_2}|\Phi_{1,k}^n|^2dv+\int_{-R}^{L_2}\left|\frac{\partial}{\partial v}\Phi_{1,k}^n\right|^2dv\right)\\
&+\frac{f_4(L_1)}{f_3(L_1)}\mathcal{C}\left(3+q\mathcal{C}^*\right)\left(\int_{L_1}^{R}\alpha_k|\Phi_{2,k}^{n+1}|^2dv+\int_{L_1}^{R}\left|\frac{\partial}{\partial v}\Phi_{2,k}^{n+1}\right|^2dv\right)\\
&\leq\frac{1}{8}\left(\int_{-R}^{L_2}\alpha_k|\Phi_{1,k}^n|^2dv+\int_{-R}^{L_2}\left|\frac{\partial}{\partial v}\Phi_{1,k}^n\right|^2dv\right)\\
&+\frac{1}{8}\left(\int_{L_1}^{R}\alpha_k|\Phi_{2,k}^{n+1}|^2dv+\int_{L_1}^{R}\left|\frac{\partial}{\partial v}\Phi_{2,k}^{n+1}\right|^2dv\right),
\end{aligned}$$

where we have used (5.4).
Combine (5.17), (5.18) and (5.19)

$$\begin{aligned}
\frac{1}{2}\left(\int_{L_1}^{R}\alpha_k|\Phi_{2,k}^{n+1}|^2dv+\int_{L_1}^{R}\left|\frac{\partial}{\partial v}\Phi_{2,k}^{n+1}\right|^2dv\right)\quad&\leq\frac{1}{8}\left(\int_{-R}^{L_2}\alpha_k|\Phi_{1,k}^n|^2dv+\int_{-R}^{L_2}\left|\frac{\partial}{\partial v}\Phi_{1,k}^n\right|^2dv\right)\\
&+\frac{1}{8}\left(\int_{L_1}^{R}\alpha_k|\Phi_{2,k}^{n+1}|^2dv+\int_{L_1}^{R}\left|\frac{\partial}{\partial v}\Phi_{2,k}^{n+1}\right|^2dv\right).
\end{aligned}$$

Therefore

$$\int_{L_1}^{R}|\Phi_{2,k}^{n+1}|^2dv+\frac{1}{\alpha_k}\int_{L_1}^{R}\left|\frac{\partial}{\partial v}\Phi_{2,k}^{n+1}\right|^2dv\qquad\leq\frac{1}{3}\left(\int_{-R}^{L_2}|\Phi_{1,k}^n|^2dv+\frac{1}{\alpha_k}\int_{-R}^{L_2}\left|\frac{\partial}{\partial v}\Phi_{1,k}^n\right|^2dv\right).$$

Similarly

$$\int_{-R}^{L_2}|\Phi_{1,k}^{n+1}|^2dv+\frac{1}{\alpha_k}\int_{-R}^{L_2}\left|\frac{\partial}{\partial v}\Phi_{1,k}^{n+1}\right|^2dv\leq\frac{1}{3}\left(\int_{L_1}^{R}|\Phi_{2,k}^n|^2dv+\frac{1}{\alpha_k}\int_{L_1}^{R}\left|\frac{\partial}{\partial v}\Phi_{2,k}^n\right|^2dv\right).$$

Take the sum of the previous two inequalities to get

$$\int_{-R}^{L_2}|\Phi_{1,k}^{n+1}|^2dv+\frac{1}{\alpha_k}\int_{-R}^{L_2}\left|\frac{\partial}{\partial v}\Phi_{1,k}^{n+1}\right|^2dv+\int_{L_1}^{R}|\Phi_{2,k}^{n+1}|^2dv+\frac{1}{\alpha_k}\int_{L_1}^{R}\left|\frac{\partial}{\partial v}\Phi_{2,k}^{n+1}\right|^2dv$$

$$\leq \frac{1}{3}\left(\int_{-R}^{L_2}|\Phi^n_{1,k}|^2dv+\frac{1}{\alpha_k}\int_{-R}^{L_2}\left|\frac{\partial}{\partial v}\Phi^n_{1,k}\right|^2dv+\int_{L_1}^{R}|\Phi^n_{2,k}|^2dv+\frac{1}{\alpha_k}\int_{L_1}^{R}\left|\frac{\partial}{\partial v}\Phi^n_{2,k}\right|^2dv\right),$$

which leads to

$$\sum_{k=-\infty}^{\infty}\left(\int_{-R}^{L_2}|\Phi^{n+1}_{1,k}|^2dv+\frac{1}{\alpha_k}\int_{-R}^{L_2}\left|\frac{\partial}{\partial v}\Phi^{n+1}_{1,k}\right|^2dv+\int_{L_1}^{R}|\Phi^{n+1}_{2,k}|^2dv+\frac{1}{\alpha_k}\int_{L_1}^{R}\left|\frac{\partial}{\partial v}\Phi^{n+1}_{2,k}\right|^2dv\right)$$
$$\leq \frac{1}{3}\sum_{k=-\infty}^{\infty}\left(\int_{-R}^{L_2}|\Phi^n_{1,k}|^2dv+\frac{1}{\alpha_k}\int_{-R}^{L_2}\left|\frac{\partial}{\partial v}\Phi^n_{1,k}\right|^2dv+\int_{L_1}^{R}|\Phi^n_{2,k}|^2dv+\frac{1}{\alpha_k}\int_{L_1}^{R}\left|\frac{\partial}{\partial v}\Phi^n_{2,k}\right|^2dv\right).$$

Step 3: We have proved that

$$S_n\leq\frac{1}{3^n}S_0<\frac{C}{3^n}\int_0^\infty\left(\|e_1^0\|^2_{L^2(\mathbb{T},H^1(-R,L_2))}+\|e_2^0\|^2_{L^2(\mathbb{T},H^1(-R,L_2))}\right)\exp(-8t)dt=:\frac{C^*}{3^n},$$

which leads to

$$\sum_{k=-\infty}^{\infty}\left(\int_{-R}^{L_2}|\Phi^n_{1,k}|^2dv+\int_{L_1}^{R}|\Phi^n_{2,k}|^2dv\right)<\frac{C^*}{3^n}.$$

Since according to our hypothesis $f_3,f_4>\beta>0$ on $[-R,R]$, we have that

$$|\Phi^n_{1,k}(v)|=\left|\int_0^\infty\hat{e_1^n}(k)(v)\exp(-\alpha_kt)dt\right|f_3(v)>\beta\left|\int_0^\infty\hat{e_1^n}(k)(v)\exp(-\alpha_kt)dt\right|,$$

$$|\Phi^n_{2,k}(v)|=\left|\int_0^\infty\hat{e_2^n}(k)(v)\exp(-\alpha_kt)dt\right|f_4(v)>\beta\left|\int_0^\infty\hat{e_2^n}(k)(v)\exp(-\alpha_kt)dt\right|.$$

Therefore

$$\sum_{k=-\infty}^{\infty}\left(\int_{-R}^{L_2}\left|\int_0^\infty\hat{e_1^n}(k)\exp(-\alpha_kt)dt\right|^2dv+\int_{L_1}^{R}\left|\int_0^\infty\hat{e_2^n}(k)\exp(-\alpha_kt)dt\right|^2dv\right)<\frac{C^*}{3^n\beta}. \quad (5.20)$$

Recall that in (5.4), we choose $\alpha_k\geq A_k$. Therefore (5.20) holds true for all $\alpha_k\geq A_k$. Moreover, since the functions

$$F_k(\alpha_k)=\int_{-R}^{L_2}\left|\int_0^\infty\hat{e_1^n}(k)\exp(-\alpha_kt)dt\right|^2dv, G_k(\alpha_k)=\int_{L_1}^{R}\left|\int_0^\infty\hat{e_2^n}(k)\exp(-\alpha_kt)dt\right|^2dv,$$

are bounded and continous on $[0,\infty)$, there exists α_k^* such that $F_k(\alpha_k^*)=\sup_{[A_k,\infty)}F_k(\alpha_k)$, $G_k(\alpha_k^{**})=\sup_{[A_k,\infty)}G_k(\alpha_k)$. Then choose α_k to be α_k^* and α_k^{**} respectively in (5.20), we get

$$\sum_{k=-\infty}^{\infty}\sup_{\alpha_k\in[A_k,\infty)}\left(\int_{-R}^{L_2}\left|\int_0^\infty\hat{e_1^n}(k)\exp(-\alpha_kt)dt\right|^2dv\right)<\frac{C^*}{3^n\beta},$$

$$\sum_{k=-\infty}^{\infty}\sup_{\alpha_k\in[A_k,\infty)}\left(\int_{L_1}^{R}\left|\int_0^\infty\hat{e_2^n}(k)\exp(-\alpha_kt)dt\right|^2dv\right)<\frac{C^*}{3^n\beta},$$

As a consequence, we get the convergence in the norm of $\mathcal{L}$. □

6. Numerical experiments

In this section we provide some numerical tests to support the theoretical analysis of the previous sections.

6.1. Model problem

We consider the initial boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial v^2} &= f && \text{in } (0,T)\times[0,1]\times[-1,1] \\ u(t,0,v) &= u(t,1,v) && \text{on } (0,T)\times[-1,1] \\ u(t,x,-1) &= 0 && \text{on } (0,T)\times[0,1] \\ u(t,x,1) &= 0 && \text{on } (0,T)\times[0,1] \end{aligned} \tag{6.1}$$

equipped with homogeneous Dirichlet boundary conditions in v and periodic boundary conditions in x. We claim that different choices of boundary condition in $v=-1$ and $v=1$ do not affect the results we show in what follows. Since the problem is linear, we can directly test the convergence on the error equation (*i.e.* letting $f\equiv 0$) whose unknown, with a little abuse of notation, we still denote by u.

6.2. Finite dimensional approximation on a single domain

We briefly describe here the numerical approximation of equation (6.1), and we focus for presentation purposes on a single domain. We discretize equation (6.1) by an operator splitting technique (see e.g. [29]), where we first solve a parabolic problem in (t,v) for half the time step, and we correct it by explicitly advancing the transport part of the equation in (t,x). Let then Δt be the time discretization step, and set $\tau = \Delta t/2$.

Step 1. Solve, in $[t,t+\tau]$, for all $x\in[0,1]$, $\dfrac{\partial}{\partial t}w(t,x,v) - \dfrac{\partial^2}{\partial v^2}w(t,x,v) = 0.$

Step 2. For all $x\in[0,1]$, $u(t+\Delta t,x,v) = w(t,x-\tau v,v)$.

We discretize the parabolic part of equations (6.1) with an implicit Euler scheme in t, and by finite elements in the v direction (see e.g. [31]). The transport part is solved explicitly by interpolation on the solution computed at Step 1. We denote by h_x and h_v the discretization steps in the x and v variable, respectively, and by N_x and $2N_v$ the corresponding grid point numbers. We let $x_m = m\,h_x$ $(m=0,..,N_x-1)$, $v_i = -1 + i\,h_v$ $(i=0,..,2N_v-1)$, we denote by $\{\varphi_j\}_{j=0,..,2N_v-1}$ a nodal basis for the finite element space associated to v, and we can approximate $u(t^n,x_m,v)$ by

$$u(t^n,x_m,v) \ \sim\ u_m(t^n,v) \ =\ \sum_{j=0}^{N_v} u_{j,m}(t^n)\varphi_j(v).$$

For the sake of compactness in notations, for all $m=0,..,N_x$, we let $\boldsymbol{u}_m(t) = [u_{1,m}(t),...,u_{2N_v,m}(t)]^T$ and $\boldsymbol{u}_m^n = \boldsymbol{u}_m(t^n)$.

The numerical approximation of (6.1) is then computed by the following operator splitting scheme.

Given $\{u^n_{i,m}\}_{i=1,..,2N_v,m=1,..,N_x}$

Step 1. For $m = 0, ..., N_x - 1$, solve

$$\frac{1}{\tau}\, M\boldsymbol{u}^{n+1/2}_m + S\boldsymbol{u}^{n+1/2}_m = \frac{1}{\tau}\, M\boldsymbol{u}^n_m, \tag{6.2}$$

where M and S are the mass and stiffness matrices, whose entries (i, j) are given by

$$[M]_{ij} = \int_0^1 \varphi_j \varphi_i \, dv \qquad\qquad [S]_{ij} = \int_0^1 \frac{d\varphi_j}{dv}\frac{d\varphi_i}{dv}\, dv. \tag{6.3}$$

Let then $\boldsymbol{u}^{n+1/2}_m = \left[u^{n+1/2}_{1,m}, ..., u^{n+1/2}_{2N_v,m}\right]^T$.

Step 2. For $i = 0, .., N_v - 1$, set

$$\begin{aligned} & u^{n+1}_{i,m} = (1 - |v_i|\,\tau)\, u^{n+1/2}_{i,m} + (|v_i|\tau\,)\, u^{n+1/2}_{i,m+1} && \text{for } m = 1, ..., N_x - 1 \\ & u^{n+1}_{i,N_x} = u^{n+1}_{i,1}. \end{aligned} \tag{6.4}$$

For $i = N_v, .., 2N_v - 1$, set

$$\begin{aligned} & u^{n+1}_{i,1} = (1 - |v_i|\,\tau)\, u^{n+1/2}_{i,m} + (|v_i|\tau\,)\, u^{n+1/2}_{i,m-1} && \text{for } m = 1, ..., N_x - 1 \\ & u^{n+1}_{i,1} = u^{n+1}_{i,N_x}. \end{aligned} \tag{6.5}$$

Remark 6.1. *In the numerical tests of the following section, we use linear finite elements and a Cavalieri-Simpson quadrature rule to evaluate these entries. Since the Cavalieri-Simpson rule is third order accurate, the matrices M and S are computed exactly. The numerical procedure described above is a classical operator splitting technique (see [29]).*

6.3. **Schwarz Waveform Relaxation**

We decompose the computational domain $\Omega = [0, T] \times [0, 1] \times [-1, 1]$ into two subdomains

$$\Omega_1 = [0, T] \times [0, 1] \times [-1, \beta] \qquad\qquad \Omega_2 = [0, T] \times [0, 1] \times [\alpha, 1], \tag{6.6}$$

which may or may not overlap ($\beta - \alpha \geq 0$). As a matter of fact, even if the analysis was carried on in the case of overlapping subdomains only, the use of Robin interface conditions in an Optimized Schwarz Waveform Relaxation (OSWR) algorithm guarantees convergence also in the absence of overlap, a feature not shared by the Classical Schwarz Waveform Relaxation (CSWR) one. In what follows we denote by $L = \beta - \alpha$ the size of the overlap between the two subdomains.
We introduce the interface variables

$$\lambda_1(t, x, \beta) = \mathcal{Q}_1\, u_2(t, x, \beta) \qquad\qquad \lambda_2(t, x, 0) = \mathcal{Q}_2\, u_1(t, x, 0), \tag{6.7}$$

where the operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are given by

$$\mathcal{Q}_1\, w \,=\, w \qquad\qquad \mathcal{Q}_2\, w \,=\, w$$

for the CSWR, and by

$$\mathcal{Q}_1\, w = \left(p + \frac{\partial}{\partial v}\right) w \qquad\qquad \mathcal{Q}_2\, w = \left(q - \frac{\partial}{\partial v}\right) w$$

for the OSWR. With these positions, the Schwarz Waveform Relaxation algorithms read as follows.

Given $\lambda_1^0(t,x,\beta)$ on $[0,T]\times[0,1]$, solve for $k \geq 1$ until convergence

$$\begin{aligned} \frac{\partial u_1^k}{\partial t} + v\frac{\partial u_1^k}{\partial x} - \frac{\partial^2 u_1^k}{\partial v^2} &= 0 && \text{in } \Omega_1 \\ u_1^k(t,0,v) &= u_1^k(t,1,v) && \text{on } [0,1] \\ u_1^k(t,x,-1) &= 0 && \text{on } [0,T]\times[0,1] \\ \mathcal{Q}_1\, u_1^k(t,x,\beta) &= \lambda_1^{k-1}(t,x,\beta) && \text{on } [0,T]\times[0,1], \end{aligned} \tag{6.8}$$

$$\lambda_2^k(t,x,\alpha) = \mathcal{Q}_2\, u_1^k(t,x,\alpha) \qquad \text{on } [0,T]\times[0,1], \tag{6.9}$$

$$\begin{aligned} \frac{\partial u_2^k}{\partial t} + v\frac{\partial u_2^k}{\partial x} - \frac{\partial^2 u_2^k}{\partial v^2} &= 0 && \text{in } \Omega_2 \\ u_2^k(t,0,v) &= u_2^k(t,1,v) && \text{on } [0,T]\times[0,1] \\ u_2^k(t,x,1) &= 0 && \text{on } [0,T]\times 0,1], \\ \mathcal{Q}_2\, u_2^k(t,x,\alpha) &= \lambda_2^k(t,x,\alpha) && \text{on } [0,T]\times[0,1] \end{aligned} \tag{6.10}$$

$$\lambda_1^k(t,x,\beta) = \mathcal{Q}_1\, u_2^k(t,x,\beta) \qquad \text{on } [0,T]\times[0,1]. \tag{6.11}$$

For a given tolerance $\varepsilon > 0$, the Schwarz Waveform Relaxation algorithm (6.8)-(6.11) is considered to have reached convergence when

$$\left\| u_1^k(t,x,v) - u_2^k(t,x,v) \right\|_{L^\infty([0,T]\times[0,1])\times(\alpha,\beta)} \;<\; \varepsilon. \tag{6.12}$$

Remark 6.2. *The Schwarz waveform relaxation algorithm is serial in the form presented in* (6.8)-(6.11)*, but it can be easily parallelized by just replacing* $\lambda_2^k(t,x,\alpha)$ *with* $\lambda_2^{k-1}(t,x,\alpha)$ *in* (6.10)*.*

6.4. **Optimization of the Robin parameters**

Since an analytical optimization of the Robin parameters (p,q) is not available, we perform an empirical optimization both in the case of one-sided $(p = q)$ and two-sided $(p \neq q)$ interface conditions. We let $T = 2$, and for the linearity of the problem we test directly the convergence on the error equation. We discretize the domains Ω_1 and Ω_2 by a uniform grid. Since the mesh size in v is not affecting the size of the interface problem, we use the same step h_v in both Ω_1 and Ω_2, with $h_v = h_x = \Delta t = 0.01$. As a consequence, the interface problems features $20,200$ unknowns. We choose an overlap of three elements $(L = 3h_v)$. We initialize the interface variable with a random value for $\lambda_1^0(t,x,\beta)$, in order to have all the frequencies represented in the initial error. Finally, we consider the algorithm to have converged when the error (6.12) drops below $\varepsilon = 10^{-6}$.

6.4.1. *One-sided Optimized Schwarz Waveform Relaxation: OSWR(p)*

In Figure 1 (left) we plot the iteration counts needed to achieve convergence, as the parameter p varies. In Figure 1 (right) we plot the error after 15 iteration for different values of p. The optimal parameter is numerically identified as $p^* = 4.23$, by sampling the interval $(4,5)$ with step 0.001. Although the iteration counts is the same as for $p = 4$, the Robin parameter p^* features a steeper convergence history. This is the case also for $p = 5$, which requires 2 more iterations to converge, but has a smaller error than $p = 4$ after 15 iterations. Finally, we consider the algorithm to have converged when the error (6.12) drops below $\varepsilon = 10^{-6}$.

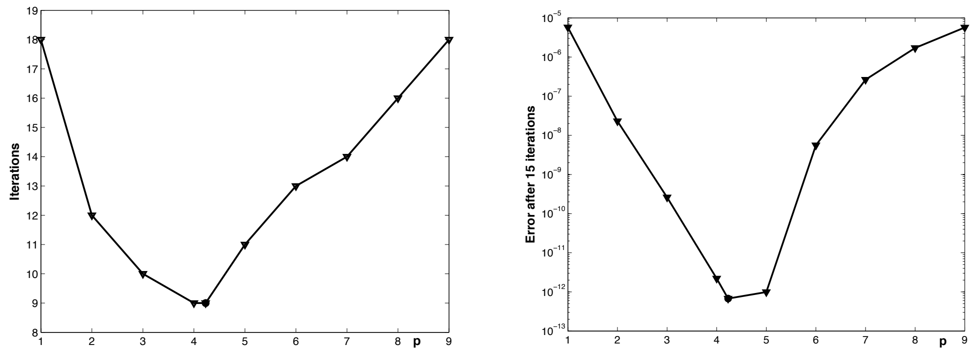

FIGURE 1. OSWR(p). Left: iteration counts to reach convergence as a function of the Robin parameter p. Right: error after 15 iterations as a function of p.

6.4.2. *Optimized two-sided Schwarz Waveform Relaxation: OSWR(p,q)*

In Figure 2 (left) we plot the iteration counts needed to achieve convergence, as the parameters p and q vary. In Figure 2 (right) we plot the error after 15 iteration for different values of p and q. The optimal parameters are numerically identified as $p^* = 11$ and $q^* = 2.5$, by sampling the square $(10.5, 11.5) \times (2, 3)$ with step 0.002 in both directions.

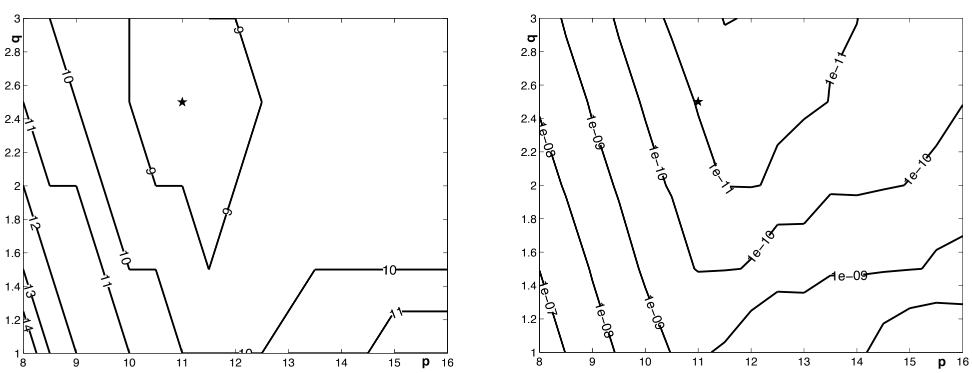

FIGURE 2. OSWR(p,q). Left: iteration counts to reach convergence as a function of the Robin parameters p and q. Right: error after 15 iterations as a function of (p, q).

6.5. **Comparison between Optimized and Classical Schwarz Waveform Relaxation**

We compare in this section the performance of the Classical and Optimized algorithms. We consider both non-overlapping and overlapping decompositions as in (6.6), always with the overlap of three element ($L = 3h_v$). Following the results from the previous Section, we implemented OSWR(p) with $p = 4.23$, and with OSWR(p,q) with $p = 11$ and $q = 2.5$. We consider a reference mesh size $\Delta t = h_v = h_x = 0.01$, and test the behavior of the algorithm in four successive dyadic mesh refinements, $\tau_j = 2^{-j} \times 0.01$ ($\tau = \Delta t, h_x, h_v$), with $j = 0, .., 4$. We report the results in Table 1.

In the overlapping case, both OSWR(p) and OSWR(p,q) algorithm appear to be almost insensitive to the mesh refinement, while the CSWR appears to be very sensitive to it. The two-sided OSWR(p,q) appears globally more robust in terms of iteration counts with respect to the one-sided OSWR(p), whose iteration counts still remain more than reasonable. Both algorithms outperform the CSWR.

In the non-overlapping case, a similar pattern is observed for OSWR(p) and OSWR(p,q). Both algorithms appear to be a little sensitive to the size of the interface problem. However, iteration counts are higher than in the overlapping case, but not significantly higher. The OSWR(p,q) is more robust than the OSWR(p), featuring an increase of around 50% in iterations for the most refined case, while the latter experiences a doubling. For both algorithms, however, the iteration counts remain reasonable in all cases. As expected, CSWR does not converge in the absence of overlap. Finally, we plot in Figure 3 the convergence history of the three overlapping algorithms at level $j = 2$ of refinement.

$\Delta_t = 2^{-j}\times 0.01$ $h_x = 2^{-j}\times 0.01$ $h_v = 2^{-j}\times 0.01$	Overlapping ($L = 3 \times h_v$)				
	$j = 0$	$j = 1$	$j = 2$	$j = 3$	$j = 4$
CSWR	70	105	132	>150	>150
OSWR(p)	9	12	15	17	18
OSWR(p,q)	9	10	10	10	13

$\Delta_t = 2^{-j}\times 0.01$ $h_x = 2^{-j}\times 0.01$ $h_v = 2^{-j}\times 0.01$	Non-overlapping ($L = 0$)				
	$j = 0$	$j = 1$	$j = 2$	$j = 3$	$j = 4$
CSWR	-	-	-	-	-
OSWR(p)	12	17	20	23	26
OSWR(p,q)	11	12	13	14	16

Table 1. Classical vs Optimized Schwarz Waveform Relaxation: iteration counts to achieve convergence for successive dyadic refinements. Overlapping ($L = 3h_v$), and non-overlapping decomposition ($L = 0$).

7. Conclusion

We have designed some new Schwarz waveform relaxation algorithms adapted to the context of the Kolmogorov equations. The domain is split in the v-direction, which is the 'parabolic' direction of the equation. The algorithms are proven to be well-posed, stable and useful in both numerical and theoretical senses. The Kolmogorov operator is hypoelliptic and it has properties of both hyperbolic and parabolic operators. Domain decomposition methods for hyperbolic problems are sometimes unstable, even for optimized algorithms, which means that the hyperbolicity of the operator really affects the convergence rates of the algorithm. In our situation, the algorithms are stable in both cases: classical and optimized algorithms. The theoretical and numerical results in this paper show that the equation is more parabolic than hyperbolic, in the regime of domain decomposition. Moreover, according to our results, the Schwarz waveform relaxation algorithms for the Kolmogorov equation have almost the same properties with an advection diffusion equation or a heat equation.

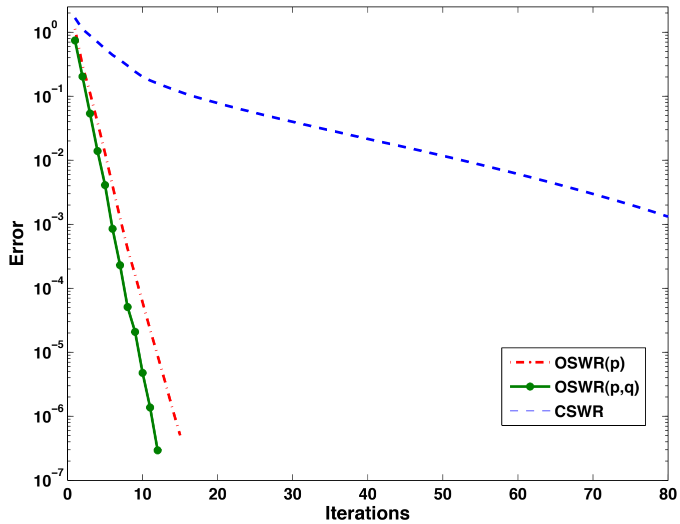

Figure 3. Overlapping Schwarz Waveform Relaxation. Convergence history for the three different algorithm, CSWR (blue dashed line), OSWR(p) (red dot-dashed line), and OSWR(p,q) (green solid line).

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References

[1] M. Asadzadeh and A. Sopasakis. Convergence of a *hp*-streamline diffusion scheme for Vlasov-Fokker-Planck system. *Math. Models Methods Appl. Sci.*, 17(8):1159–1182, 2007.
[2] K. Beauchard and E. Zuazua. Some controllability results for the 2D Kolmogorov equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(5):1793–1815, 2009.
[3] C. Buet, S. Cordier, P. Degond, and M. Lemou. Fast algorithms for numerical, conservative, and entropy approximations of the Fokker-Planck-Landau equation. *J. Comput. Phys.*, 133(2):310–322, 1997.
[4] C. Buet, S. Dellacherie, and R. Sentis. Numerical solution of an ionic Fokker-Planck equation with electronic temperature. *SIAM J. Numer. Anal.*, 39(4):1219–1253 (electronic), 2001.
[5] María J. Cáceres, José A. Carrillo, and Louis Tao. A numerical solver for a nonlinear Fokker-Planck equation representation of neuronal network dynamics. *J. Comput. Phys.*, 230(4):1084–1099, 2011.
[6] J. A. Carrillo, M. P. Gualdani, and A. Jüngel. Convergence of an entropic semi-discretization for nonlinear Fokker-Planck equations in $\mathbb{R}^d$. *Publ. Mat.*, 52(2):413–433, 2008.
[7] N. Crouseilles and F. Filbet. A conservative and entropic method for the Vlasov-Fokker-Planck-Landau equation. In *Numerical methods for hyperbolic and kinetic problems*, volume 7 of *IRMA Lect. Math. Theor. Phys.*, pages 59–70. Eur. Math. Soc., Zürich, 2005.
[8] Nicolas Crouseilles and Francis Filbet. Numerical approximation of collisional plasmas by high order methods. *J. Comput. Phys.*, 201(2):546–572, 2004.
[9] Pierre Degond and Brigitte Lucquin-Desreux. An entropy scheme for the Fokker-Planck collision operator of plasma kinetic theory. *Numer. Math.*, 68(2):239–262, 1994.
[10] Weihua Deng. Finite element method for the space and time fractional Fokker-Planck equation. *SIAM J. Numer. Anal.*, 47(1):204–226, 2008/09.

[11] L. Desvillettes and C. Villani. On the trend to global equilibrium in spatially inhomogeneous entropy-dissipating systems: the linear Fokker-Planck equation. *Comm. Pure Appl. Math.*, 54(1):1–42, 2001.

[12] Jean Dolbeault, Clément Mouhot, and Christian Schmeiser. Hypocoercivity for kinetic equations with linear relaxation terms. *C. R. Math. Acad. Sci. Paris*, 347(9-10):511–516, 2009.

[13] Roland Duclous, Bruno Dubroca, Francis Filbet, and Vladimir Tikhonchuk. High order resolution of the Maxwell-Fokker-Planck-Landau model intended for ICF applications. *J. Comput. Phys.*, 228(14):5072–5100, 2009.

[14] Francis Filbet and Lorenzo Pareschi. Numerical solution of the Fokker-Planck-Landau equation by spectral methods. *Commun. Math. Sci.*, 1(1):206–207, 2003.

[15] Irene M. Gamba, Maria Pia Gualdani, and Richard W. Sharp. An adaptable discontinuous Galerkin scheme for the Wigner-Fokker-Planck equation. *Commun. Math. Sci.*, 7(3):635–664, 2009.

[16] M. J. Gander and L. Halpern. Optimized Schwarz waveform relaxation methods for advection reaction diffusion problems. *SIAM J. Numer. Anal.*, 45(2):666–697 (electronic), 2007.

[17] M. J. Gander, L. Halpern, and F. Magoulès. An optimized Schwarz method with two-sided Robin transmission conditions for the Helmholtz equation. *Internat. J. Numer. Methods Fluids*, 55(2):163–175, 2007.

[18] M. J. Gander, L. Halpern, and F. Nataf. Optimal convergence for overlapping and non-overlapping Schwarz waveform relaxation. In *Eleventh International Conference on Domain Decomposition Methods (London, 1998)*, pages 27–36 (electronic). DDM.org, Augsburg, 1999.

[19] Martin J. Gander and Laurence Halpern. Méthodes de décomposition de domaines pour l'équation des ondes en dimension 1. *C. R. Acad. Sci. Paris Sér. I Math.*, 333(6):589–592, 2001.

[20] Martin J. Gander and Laurence Halpern. Un algorithme discret de décomposition de domaines pour l'équation des ondes en dimension 1. *C. R. Acad. Sci. Paris Sér. I Math.*, 333(7):699–702, 2001.

[21] Martin J. Gander, Laurence Halpern, and Frederic Nataf. Optimized Schwarz methods. In *Domain decomposition methods in sciences and engineering (Chiba, 1999)*, pages 15–27 (electronic). DDM.org, Augsburg, 2001.

[22] Martin J. Gander and Andrew M. Stuart. Space-time continuous analysis of waveform relaxation for the heat equation. *SIAM J. Sci. Comput.*, 19(6):2014–2031, 1998.

[23] Laurence Halpern. Optimized Schwarz waveform relaxation: roots, blossoms and fruits. In *Domain decomposition methods in science and engineering XVIII*, volume 70 of *Lect. Notes Comput. Sci. Eng.*, pages 225–232. Springer, Berlin, 2009.

[24] Lars Hörmander. Hypoelliptic second order differential equations. *Acta Math.*, 119:147–171, 1967.

[25] David J. Knezevic and Endre Süli. Spectral Galerkin approximation of Fokker-Planck equations with unbounded drift. *M2AN Math. Model. Numer. Anal.*, 43(3):445–485, 2009.

[26] O. A. Ladyzenskaja, V. A. Solonnikov, and N. N. Uraceva. *Linear and quasilinear equations of parabolic type*. Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1967.

[27] Mehrdad Lakestani and Mehdi Dehghan. Numerical solution of Fokker-Planck equation using the cubic B-spline scaling functions. *Numer. Methods Partial Differential Equations*, 25(2):418–429, 2009.

[28] Mohammed Lemou and Luc Mieussens. Implicit schemes for the Fokker-Planck-Landau equation. *SIAM J. Sci. Comput.*, 27(3):809–830 (electronic), 2005.

[29] G. I. Marchuk. Splitting and alternating direction methods. In *Handbook of numerical analysis, Vol. I*, Handb. Numer. Anal., I, pages 197–462. North-Holland, Amsterdam, 1990.

[30] Dejan Milić. Explicit method for the numerical solution of the Fokker-Planck equation of filtered phase noise. In *Approximation and computation*, volume 42 of *Springer Optim. Appl.*, pages 401–407. Springer, New York, 2011.

[31] A. Quarteroni and A. Valli. *Numerical Approximation of Partial Differential Equations*. Springer-Verlag, Berlin, 1994.

[32] Jack Schaeffer. Convergence of a difference scheme for the Vlasov-Poisson-Fokker-Planck system in one dimension. *SIAM J. Numer. Anal.*, 35(3):1149–1175 (electronic), 1998.

[33] Minh-Binh Tran. Parallel Schwarz waveform relaxation method for a semilinear heat equation in a cylindrical domain. *C. R. Math. Acad. Sci. Paris*, 348(13-14):795–799, 2010.

[34] Minh-Binh Tran. A parallel four step domain decomposition scheme for coupled forward-backward stochastic differential equations. *J. Math. Pures Appl. (9)*, 96(4):377–394, 2011.

[35] Minh-Binh Tran. Optimized overlapping domain decomposition: Convergence proofs. *Domain Decomposition Methods in Science and Engineering XXI, Lecture Notes in Computational Science and Engineering, Springer-Verlag*, 91:493–500, 2013.

[36] Minh-Binh Tran. Overlapping optimized Schwarz methods for parabolic equations in n dimensions. *Proc. Amer. Math. Soc.*, 141(5):1627–1640, 2013.

[37] Minh-Binh Tran. Parallel schwarz waveform relaxation algorithm for an n-dimensional semilinear heat equation. *ESAIM Math. Model. Numer. Anal.*, 48(03):795–813, 2014.

[38] Cédric Villani. Hypocoercive diffusion operators. In *International Congress of Mathematicians. Vol. III*, pages 473–498. Eur. Math. Soc., Zürich, 2006.

[39] David Vernon Widder. *The Laplace Transform*. Princeton Mathematical Series, v. 6. Princeton University Press, Princeton, N. J., 1941.