\textbf{ν-lossofderivativesforanevolutiontypemodel}

Daoyuan Fang\textsuperscript{a}, Xiaojun Lu\textsuperscript{a,∗}, Michael Reissig\textsuperscript{b}

\textsuperscript{a}Department of Mathematics, Zhejiang University, 310027 Hangzhou, PR China
\textsuperscript{b}Faculty for Mathematics and Computer Science, TU Bergakademie Freiberg, Prüferstr. 9, 09596 Freiberg, Germany

\begin{abstract}
This paper is devoted to the study of loss of derivatives for an evolution type model, which allows for non-integer powers of \((-\Delta)\). On the one hand, we describe the influence from the orders of principle symbols. On the other hand, we explore the significant impact of \(\nu\)-related oscillation from the pseudo-differential operators. In the final analysis, to demonstrate the sharpness of our estimates, counter-examples are constructed through the application of Floquet theory and instability arguments.
\end{abstract}

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1. Introduction

The phenomenon of loss of derivatives is an interesting topic in the discussion of \(H^\infty\) well-posedness for several famous linear differential operator models. For instance, both classical wave model and Klein–Gordon model with Cauchy data have no loss of derivatives, which means that, for large \(s\) and given initial data \((u(0, x), \partial_t u(0, x)) \in H^{s+1}(\mathbb{R}^N) \times H^s(\mathbb{R}^N)\), there exists a unique global solution \(u \in C([0, \infty) ; H^{s+1}(\mathbb{R}^N) \cap C^1([0, \infty) ; H^s(\mathbb{R}^N))\) depending continuously on the initial data \((u(0, x), \partial_t u(0, x))\). In [1], the author introduced several types of loss of derivatives, including no loss, arbitrarily small loss and finite loss, and gave a refined classification of oscillating coefficients up to the second-order derivatives for the wave type model \(\mathcal{L} = \partial_t^2 - a(t)\Delta\). This example shows, in detail, that the oscillating behavior of coefficients of the self-adjoint operator \((-\Delta)\) plays a significant role in the loss of derivatives for solutions in the Sobolev spaces. As to the discussion of difference of derivatives, in [2], the authors considered the regularity behavior of \(p\)-evolution type model \(\mathcal{L} = \partial_t^2 + \sum_{p=0}^{2p} a_0(t)D_t^{2p}\) \((p, h \in \mathbb{N}, p \geq 1)\) in \([0, T] \times \mathbb{R}\). Through studying the Log-Lipschitz continuity of the coefficients, we know that the principal oscillating in time part \(a_{2p}(t)D_t^{2p}\) determines the difference of derivatives \(p\). So far, many authors have concentrated on the study of integer index case for \(p\)-evolution models, while, in this paper, we are interested in exploring the influence of loss of derivatives from the order of principle symbol classes by filling the integer gap. In addition, \(\nu\)-related oscillating in time behavior will be introduced as in [3], in order to show the so-called \(\nu\)-loss arising from the singular behavior of derivatives up to the second order with respect to time. Now we turn to the main result of this paper. First, we introduce the key notions and notations used in the description of the model:

- To measure the oscillating behavior of the model, we introduce a positive, monotonously decreasing and continuous function \(\nu(t), t \in (0, T]\) satisfying \(\lim_{t \to 0} \inf \nu(t) \geq C > 0\).

\textsuperscript{∗}Corresponding author.

\textit{E-mail address: lvxiaojun1119@hotmail.de} (X. Lu).
• The extended phase space is divided into the following low frequency zone $Z_{low}(M)$, pseudo-differential zone $Z_{pd}(P, M)$ and p-evolution type zone $Z_{pe}(P, M)$: ($\sigma$, $M$ and $P$ will be given later)

$$
Z_{low}(M) = \{(t, \xi) \in [0, T] \times \{|\xi| \leq M\} \};
Z_{pd}(P, M) = \{(t, \xi) \in [0, T] \times \{|\xi| \geq M\} : t(\xi)^{\sigma} < 2^P v(t) \};
Z_{pe}(P, M) = \{(t, \xi) \in [0, T] \times \{|\xi| \geq M\} : t(\xi)^{\sigma} > 2^P v(t) \}.
$$

• The following classes of symbols are defined only in the p-evolution type zone for large frequencies; ($\ell \in \mathbb{N}$, ($m_1, m_2 \in \mathbb{R} \times \mathbb{R}$)

$$S_{\ell}[m_1, m_2] := \{a \in C^\ell((0, T); C^\infty(|\xi| \geq M)) : |D_x^\ell D_\xi^k a(t, \xi)| \leq C_{k,\alpha} (|\xi|)^{m_1-|\alpha|} (v(t)/t)^{m_2+k} \quad \text{for all } k \in \mathbb{N}, k \leq \ell \text{ and all multi-indices } \alpha, (t, \xi) \in Z_{pe}(P, M) \}.$$ 

Then we introduce the main evolution type model to be considered:

$$\mathcal{L} = \beta_1^2 + A_0(t, D_x) + A_1(t, D_x) + A_2(t, D_x), \quad (1.1)$$

with $A_0(t, D_x) = b(t)(-\Delta)^\sigma, \sigma \in \mathbb{R}_+$. In particular, $A_i(t, D_x), i = 0, 1, 2$ are linear pseudo-differential operators defined on the space of tempered distributions $\mathcal{S}'$ by

$$A_i(t, D_x)u = (2\pi)^{-N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \exp(i(x-y, \xi)) A_i(t, \xi) u(y) dy d\xi.$$ 

For large frequencies, $A_1(t, \xi) \in S_2(\sigma_1, 0), A_2(t, \xi) \in S_1(\sigma_2, 0), \sigma_2 \in [0, \sigma), \sigma_1 \in (\sigma, 2\sigma)$; while for small frequencies, $A_1(t, \xi) \in \alpha(|\xi|^{\sigma}), A_2(t, \xi) \in L^{\infty}(|\xi| \leq M)$. Furthermore, we make the following assumptions for the model:

- There exists a positive constant $C_1$ such that $C_1|\xi|^{2\sigma} \leq A_0(t, \xi) \leq C_1^{-1}|\xi|^{2\sigma}$;
- $\lim_{t \to 0} \sup |A_i(t, \xi)| \leq |\xi|^{\sigma}, t = 1, 2$;
- $A_1 \in C^2((0, T]; \mathbb{R}), A_2 \in C^1((0, T]; \mathbb{C})$.

**Remark 1.1.** Actually, from the viewpoint of Fourier multipliers, $|D_x|^\sigma$ is well-defined for a real positive power $\sigma$; while for a positive integer power $p$, $|D_x|^p, |\alpha| = p$ corresponds to the general p-evolution type differential operators. Consequently, our model (1.1) generalizes the wave case in [1] and p-evolution case in [2]. It is evident that Petrovsky type and Schr"odinger type equations are also included in this model. Moreover, this model fills the integer gaps with the application of Fourier multipliers, which means that, for example, $\sigma$ can be fractional and irrational as well. In particular, when we treat the model in $[0, T] \times \mathbb{R}^N$, $(\sqrt{-\Delta})^\sigma, \sigma \in \mathbb{R}_+$, is defined by using the spectral theory for self-adjoint operators, which will be discussed in detail in Section 3.2.

**Remark 1.2.** The useful functions indicate useful information of the singular behavior of the model. An example of such functions in the form of $v(t) = (\log(1/t))^{\gamma}, \gamma \in (0, \infty)$ is given in [4]. As a matter of fact, more delicate examples can be introduced, such as the polynomial $v(t) = \sum_{j=1}^m \prod_{i=1}^n (\log (1/t))^{\gamma_i}$, $\gamma_i \in [0, \infty)$, and $\log^m(1/t) = \log \log^{m-1}(1/t)$ and $\log^0(1/t) = 1/t$. By comparing various measure functions we obtain significant influences of loss of derivatives from different kinds of oscillations.

**Remark 1.3.** As in [5], we name $A_0(t, D_x)$ the principle part in the sense of Petrovsky, and $A_0(t, D_x) + A_1(t, D_x)$ the principle part of the p-evolution problem. Actually, we can choose $b(t) = 2 + \sin((\log(1/t))^k), k \in (1, \infty)$ which satisfies our assumptions with $v(t) = (\log(1/t))^{k-1}$. Under the above assumptions we have the following statement:

**Theorem 1.1.** Let us consider the Cauchy problem of model (1.1) in $[0, T] \times \mathbb{R}^N$,

$$\mathcal{L} u = 0, \quad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x).$$

Define $\mu(s)$ as $s/v(s)$. If $u_0 \in H^s(\mathbb{R}^N), u_1 \in H^{s-\sigma}(\mathbb{R}^N)$, and $P$ is a fixed appropriate positive integer, then there exists a unique solution $u$ belonging to the following function spaces:

$$C([0, T]; \exp(c_1 v(\mu^{-1}(P/(D_x)^\sigma))) H^{s}(\mathbb{R}^N)) \cap C^1([0, T]; \exp(c_1 v(\mu^{-1}(P/(D_x)^\sigma))) H^{s-\sigma}(\mathbb{R}^N))$$

with a positive constant $c_1$. Moreover, $\mu^{-1}$ denotes the inverse function of $\mu$.

**Remark 1.4.** Actually, Theorem 1.1 also holds when we replace $2^P$ with a general sufficiently large real number. We choose $2^P$ since this is very convenient for us to construct sequences in the counter-examples in Section 3. Moreover, one has to notice that when $v(t) = (\log(1/t))^{\gamma}, \gamma \in (0, 1)$, increasing $P$ increases the Sobolev spaces. For more details, please see the typical examples in Remark 2.3.
Remark 1.5. The function $\mu^{-1}$ is uniquely determined since $\mu$ is a monotonously increasing function. Particularly, when $v(t) \leq C$, there is no loss of derivatives. This theorem shows explicitly the so-called (at most) $v$-loss of derivatives which arises from the singular behavior of the symbols in $t$. This result generalizes the Log-effect in [5].

Remark 1.6. Together with the conclusions in [3], one can easily unify the results of $v$-loss of derivatives for the weakly or strictly hyperbolic model $\mathcal{L} = \partial_x^2 - t^{2\ell} b(t) \Delta$ with $\ell \geq 0$ and $b(t)$ satisfying the same assumptions as in Theorem 1.1.

Recall the Fourier series defined for functions on $\mathbb{T}^N = \mathbb{R}^N / 2\pi \mathbb{Z}^N$, we know

$$u(x) = \sum_{k \in \mathbb{Z}^N} \hat{u}(k) \exp(i k \cdot x),$$

where

$$\hat{u}(k) = (2\pi)^{-N} \int_{\mathbb{R}^N} u(x) \exp(-ix \cdot x) dx.$$

For a continuous function $F : \mathbb{R} \to \mathbb{R}$, by applying the spectral theory of self-adjoint operator $\sqrt{-\Delta}$ on the compact manifold $\mathbb{T}^N$, we define the following pseudo-differential operators as in [6,7]:

$$F(\sqrt{-\Delta})u = \sum_{k \in \mathbb{Z}^N} \hat{u}(k) F(|k|) \exp(ik \cdot x).$$

From Theorem 1.1, actually we have the following corollary:

**Corollary 1.2.** Let us consider the Cauchy problem in $[0, T] \times \mathbb{T}^N$ for $\sigma \in \mathbb{R}_+$,

$$\partial_t^2 u + b(t) (-\Delta)^\sigma u = 0, \quad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x).$$

Define $\mu(s)$ as $s/\nu(s)$. If $u_0 \in H^s(\mathbb{T}^N)$, $u_1 \in H^{s-\sigma}(\mathbb{T}^N)$, and $P$ is a fixed appropriate positive integer, then there exists a unique solution $u$ belonging to the following function spaces:

$$C([0, T]; \exp(c_1 \nu(\mu^{-1}(2^P/(1-\Delta)^{s/2})))H^s(\mathbb{T}^N)) \cap C([0, T]; \exp(c_1 \nu(\mu^{-1}(2^P/(1-\Delta)^{s/2})))H^{s-\sigma}(\mathbb{T}^N))$$

with a positive constant $c_1$. Moreover, $\mu^{-1}$ denotes the inverse function of $\mu$.

The rest of the paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.1. Some powerful tools from micro-local analysis and WKB analysis will be used to obtain the precise $v$-loss of derivatives. Several examples with respect to $\nu(t)$ complete this part. In Section 3.1 we discuss the optimality of our statement for general $\sigma \in \mathbb{R}_+$ on the torus $\mathbb{T}$ by the application of instability arguments, while in the whole space $\mathbb{R}$, the question for the optimality in the case of $\sigma = 1$ is considered through Floquet theory in Section 3.2. Some concluding remarks complete the paper.

2. Proof of Theorem 1.1

In this section, we divide our proof into three parts; namely, the micro-energy estimates are considered in $Z_{low}(M)$, $Z_{pd}(P, M)$ and $Z_{pe}(P, M)$ respectively. Finally, we combine the estimates in different zones to obtain the $v$-loss of derivatives.

2.1. Estimates in $Z_{low}(M)$

In this compact zone we introduce the micro-energy $V(t, \xi) = (V_1, V_2)^T = (\hat{u}, D_t \hat{u})^T$, and $\hat{u}(\xi) := \mathcal{F}(u)(\xi)$.

**Lemma 2.1.** Actually, for all $(t, \xi) \in Z_{low}(M)$, we have the following energy estimate:

$$\left| \begin{pmatrix} \hat{u}(t, \xi) \\ D_t \hat{u}(t, \xi) \end{pmatrix} \right| \leq |\hat{u}_0(\xi)| + |\hat{u}_1(\xi)|.$$

**Proof.** After partial Fourier transformation we have the following new equation by applying the definitions of $A_i(t, D_x)$, $i = 0, 1, 2$:

$$D_t^2 \hat{u}(t, \xi) - (A_0(t, \xi) + A_1(t, \xi) + A_2(t, \xi)) \hat{u}(t, \xi) = 0.$$

As in the case of ordinary differential equations we study the system of first order

$$D_t V = \mathcal{A}(t, \xi) V := \begin{pmatrix} 0 & A_0(t, \xi) + A_1(t, \xi) + A_2(t, \xi) \\ 0 & 1 \end{pmatrix} V.$$

In fact, the method of successive approximation enables us to construct the fundamental solution of the system $D_t \mathcal{E}(t, s, \xi) = \mathcal{A}(t, \xi) \mathcal{E}(s, \xi), \mathcal{E}(s, s, \xi) = I$. More precisely, $\mathcal{E}(t, s, \xi)$ is given in the form of matrizen representation:

$$\mathcal{E}(t, s, \xi) = I + \sum_{k=1}^{\infty} \int_s^t \mathcal{A}(t_1, \xi) \int_s^{t_1} \mathcal{A}(t_2, \xi) \cdots \int_s^{t_{k-1}} \mathcal{A}(t_k, \xi) dt_k \cdots dt_1.$$
Actually, we have

**Lemma 2.2.** For \( k \in \mathbb{N}_+ \) it holds

\[
\left\| \int_s^t \alpha(t_1, \xi) \int_s^{t_1} \alpha(t_2, \xi) \cdots \int_s^{t_{k-1}} \alpha(t_k, \xi) dt_k \cdots dt_1 \right\| \leq \frac{1}{k!} \left( \int_s^t \| \alpha(r, \xi) \| \, dr \right)^k.
\]

**Proof.** In fact,

\[
\int_s^t \| \alpha(t_1, \xi) \| \int_s^{t_1} \| \alpha(t_2, \xi) \| dt_2 \, dt_1 = \int_s^t \frac{\partial}{\partial t_1} \left( \int_s^{t_1} \| \alpha(t_2, \xi) \| \, dt_2 \right) \left( \int_s^{t_1} \| \alpha(t_2, \xi) \| \, dt_2 \right) \, dt_1
\]

\[
= \int_s^t \frac{1}{2} \frac{\partial}{\partial t_1} \left( \int_s^{t_1} \| \alpha(t_2, \xi) \| \, dt_2 \right)^2 \, dt_1 = \frac{1}{2} \left( \int_s^t \| \alpha(r, \xi) \| \, dr \right)^2.
\]

By the induction method the statement follows immediately. \( \square \)

Consequently, by applying the definition of the zone \( Z_{low}(M) \), we have

\[
\| \alpha(t, s, \xi) \| \leq \exp \left( \int_s^t \| \alpha(r, \xi) \| \, dr \right) \leq \exp \left( \int_0^T C(M) \, ds \right) \leq C(M, T).
\]

And this estimate leads to our main conclusion. \( \square \)

**Remark 2.1.** The constant \( M \) is chosen in order to separate large frequencies from small frequencies. Actually, we can choose any large constant as \( M \), since Lemma 2.1 indicates that the small frequencies play an insignificant role in the discussion of loss of derivatives, and only large frequencies are of interest.

2.2. Estimates in \( Z_{pd}(P, M) \)

In this zone, we introduce the micro-energy \( V(t, \xi) = (V_1, V_2)^T = (|\xi|^\sigma \hat{u}, D_t \hat{u})^T \).

**Lemma 2.3.** For all \( (t, \xi) \in Z_{pd}(P, M) \), we have the following energy estimate with a positive constant \( c_1 \) depending upon \( P, M \):

\[
\left( \begin{array}{c}
|\xi|^\sigma \hat{u}(t, \xi) \\
D_t \hat{u}(t, \xi)
\end{array} \right) \leq \exp \left( c_1 \nu(t_\xi) \right) \left( |\xi|^\sigma |\hat{u}_0(\xi)| + |\hat{u}_1(\xi)| \right)
\]

\( t_\xi \) is defined by the separating line of the two zones, \( t_\xi(|\xi|^\sigma) = 2^\beta \nu(t_\xi) \) for \( |\xi| \geq M \).

**Proof.** As in the proof of Lemma 2.1 we study the system of first order

\[
D_t V = \mathcal{A}(t, \xi)V := \begin{pmatrix}
0 \\
A_0(t, \xi) + A_1(t, \xi) + A_2(t, \xi)/|\xi|^\sigma \end{pmatrix} |\xi|^\sigma V.
\]

Similarly, we construct the fundamental solution of the system

\[
D_t \mathcal{E}(t, s, \xi) = \mathcal{A}(t, \xi) \mathcal{E}(t, s, \xi), \mathcal{E}(s, s, \xi) = I.
\]

More precisely, \( \mathcal{E}(t, s, \xi) \) is given in the form of matrizon representation

\[
\mathcal{E}(t, s, \xi) = I + \sum_{k=1}^\infty i^k \int_s^t \mathcal{A}(t_1, \xi) \int_s^{t_1} \mathcal{A}(t_2, \xi) \cdots \int_s^{t_{k-1}} \mathcal{A}(t_k, \xi) dt_k \cdots dt_1.
\]

By the induction method, as in the proof of Lemma 2.1, we have

\[
\| \mathcal{E}(t, s, \xi) \| \leq \exp \left( \int_s^t \| \mathcal{A}(r, \xi) \| \, dr \right) \leq \exp \left( \int_0^{t_\xi} c_1 |\xi|^\sigma \, ds \right) \leq \exp(c_1 \nu(t_\xi)).
\]

The final inequality holds when we take account of the definition of \( t_\xi \). And this estimate leads to our conclusion. \( \square \)

2.3. Estimates in \( Z_{pe}(P, M) \)

In this zone, we introduce the following \( p \)-evolution type micro-energy \( V(t, \xi) = (V_1, V_2)^T = (\sqrt{A_0(t, \xi)} \hat{u}, D_t \hat{u})^T \). To explain our approach, we need some basic rules in hierarchies of non-standard symbol classes.

- For large frequencies, \( A_0(t, \xi)^{-1} \in S_2(-2\sigma, 0) \);
- \( S_l(m_1, m_2) \subset S_{c}(m_1 + \sigma k, m_2 - k) \) for all \( k \in \mathbb{N} \).
• if $a \in S_t(m_1, m_2)$ and $b \in S_t(k_1, k_2)$, then $ab \in S_t(m_1 + k_1, m_2 + k_2)$;
• if $a \in S_t(m_1, m_2)$, then $D_t a \in S_{t-k}(m_1, m_2 + k)$, $D_k^2 a \in S_t(m_1 - |a|, m_2)(k \leq \ell)$;
• if $a(t, \xi) \in S_t(-\sigma, 2)$, then for all $(t, \xi) \in Z_{pe}(P, M)$, $|\int_{t k}^t a(\tau, \xi) d\tau| \lesssim v(t_k)$.

**Proof.** The first four rules are obvious when we notice the definition of the classes of symbols. The last statement follows from

$$\left|\int_{t k}^t a(\tau, \xi) d\tau\right| \lesssim \int_{t k}^t \frac{v^2(\tau)}{\tau^2} d\tau \lesssim v^2(t_k) |\xi|^{-1} \lesssim v(t_k) \,. \quad \square$$

**Lemma 2.4.** For all $(t, \xi) \in Z_{pe}(P, M)$ we have the following energy estimate with a positive constant $c_1$ depending upon $P, M$:

$$\left|\begin{bmatrix} \sqrt{A_0(t, \xi)} \hat{u}(t, \xi) \\ D_t \hat{u}(t, \xi) \end{bmatrix}\right| \lesssim \exp\left(c_1 v(t_k)\right) \left(\sqrt{A_0(t_k, \xi)} |\hat{u}(t_k, \xi)| + |D_t \hat{u}(t_k, \xi)|\right).$$

As before, $t_k$ is defined by the separating line of the two zones.

**Proof.** The proof is based on the application of two steps of a diagonalization procedure and the construction of the fundamental solution. Taking account of the definition of micro-energy in this zone, we study the following first order system:

$$D_t V = \begin{pmatrix} 0 & \frac{1}{\sqrt{A(0, t, \xi)}} \sqrt{A_0(t, \xi)} \\ (A_0(t, \xi) + A_1(t, \xi)) / \sqrt{A_0(t, \xi)} & \frac{1}{\sqrt{A_0(t, \xi)}} \end{pmatrix} \begin{pmatrix} \sqrt{A_0(t, \xi)} \\ 0 \end{pmatrix} V + \begin{pmatrix} D_t \sqrt{A_0(t, \xi)} \\ A_0(t, \xi) / \sqrt{A_0(t, \xi)} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} V.$$

We choose $\mathcal{M} = \begin{pmatrix} 1 & -1 \\ \sqrt{1 + A_1(t, \xi) / A_0(t, \xi)} & 1 + A_1(t, \xi) / A_0(t, \xi) \end{pmatrix}$. Obviously, $\mathcal{M}^{-1}$ exists and belongs to $S_t[0, 0]$, since

$$2 - \varepsilon \leq \det \mathcal{M} \leq 2 \sup_{t \in (0, T], |\xi| \geq M} \sqrt{1 + |A_1(t, \xi)| / A_0(t, \xi)},$$

and

$$1 - \varepsilon \leq \|\mathcal{M}\| \leq \sup_{t \in (0, T], |\xi| \geq M} \sqrt{1 + |A_1(t, \xi)| / A_0(t, \xi)},$$

with a sufficiently small $\varepsilon$.

**Step 1: First step of diagonalization**

Applying the transformation $V = \mathcal{M} V_0$, we obtain the following representation after the first step of diagonalization:

$$0 = D_t V_0 - \mathcal{M} V_0 + B_1 V_0 + B_2 V_0$$

$$= D_t V_0 - \begin{pmatrix} 0 & \frac{1}{\sqrt{A(0, t, \xi)}} \sqrt{A_0(t, \xi)} \\ (A_0(t, \xi) + A_1(t, \xi)) / \sqrt{A_0(t, \xi)} & \frac{1}{\sqrt{A_0(t, \xi)}} \end{pmatrix} \begin{pmatrix} \sqrt{A_0(t, \xi)} \\ 0 \end{pmatrix} V_0$$

$$- \frac{1}{2} \begin{pmatrix} D_t \sqrt{A_0(t, \xi)} + A_2(t, \xi) & D_t \sqrt{A_0(t, \xi)} + A_2(t, \xi) \\ -D_t \sqrt{A_0(t, \xi)} + A_2(t, \xi) & -D_t \sqrt{A_0(t, \xi)} + A_2(t, \xi) \end{pmatrix} \begin{pmatrix} \sqrt{A_0(t, \xi)} \\ 0 \end{pmatrix} V_0$$

$$+ \frac{1}{2} \begin{pmatrix} D_t \sqrt{1 + A_1(t, \xi) / A_0(t, \xi)} & D_t \sqrt{1 + A_1(t, \xi) / A_0(t, \xi)} \\ D_t \sqrt{1 + A_1(t, \xi) / A_0(t, \xi)} & D_t \sqrt{1 + A_1(t, \xi) / A_0(t, \xi)} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} V_0,$$

where $\mathcal{M} \in S_t[\sigma, 0]$ and $B \coloneqq B_1 + B_2 \in S_t[0, 1]$.

**Step 2: Second step of diagonalization**

To carry out this step of diagonalization, we follow the procedure of asymptotic theory of ordinary differential equations. Namely, we look for a matrix $N(t, \xi) \coloneqq I + N(t, \xi)$, where $B(t) \coloneqq B, F(t) \coloneqq \diag B(t)$.

$$N^{-1}(t) \coloneqq \begin{pmatrix} B(t) & q \neq r \end{pmatrix} \frac{1}{\tau_q - \tau_r} \, \quad k = 1, 2,$n(t) \coloneqq \begin{pmatrix} 0 \\ \tau_k \coloneqq (-1)^{k+1} \sqrt{A_0(t, \xi)} + A_1(t, \xi), \quad k = 1, 2,$$n(t) \coloneqq (D_t - \mathcal{M} + B)(I + N(t)) - (I + N(t))(D_t - \mathcal{M} + F(t)).$$
According to the properties of symbols, we have \( N^{(1)} \in S_1(-\sigma, 1) \) and \( F^{(0)} \in S_1(0, 1) \). As for \( B^{(1)} \), we obtain the following relation:

\[
B^{(1)} = B + [N^{(1)}, \varphi] - F^{(0)} + D_t N^{(1)} + B N^{(1)} - N^{(1)} F^{(0)}.
\]

The construction principle implies that the sum of the first three terms vanishes, hence \( B^{(1)} \in S_0(-\sigma, 2) \). Finally, let

\[
R_1 := N_1^{-1} B^{(1)} = N_1^{-1} \left( (D_t - \varphi + B)(I + N^{(1)}) - (I + N^{(1)})(D_t - \varphi + F^{(0)}) \right).
\]

This definition means that \( R_1 = N_1^{-1} B^{(1)} \in S_0(-\sigma, 2) \). Actually, due to the definition of symbols, \( N^{(1)} \in S_1(-\sigma, 1) \) indicates \( |N^{(1)}|^p \leq C/2^p \). Consequently, an appropriate integer \( P \) assures that \( \|N_1 - I\| < 1/2 \) in \( Z_{pe}(P, M) \), which implies the invertibility of \( N_1 \). As a result, we have the following system after the second step of diagonalization:

\[
(D_t - \varphi + B)N_1 = N_1 (D_t - \varphi + F^{(0)} + R_1), \quad \text{where } R_1 \in S_0(-\sigma, 2).
\]

**Step 3: Estimate of the fundamental solution**

Now we consider the system

\[
(D_t - \varphi + F^{(0)} + R_1) V = 0.
\]

The fundamental solution for this equation is \( \mathcal{E} = \mathcal{E}_1 \mathcal{H} \). And \( \mathcal{E}_1(t, s, \xi) \) has the following form,

\[
\begin{align*}
\mathcal{E}_1(t, s, \xi) &= \exp \left( \frac{1}{2} \int_{t}^{s} \frac{\partial_{\tau} \sqrt{A_0(\tau, \xi) + A_1(\tau, \xi)}}{\sqrt{A_0(\tau, \xi) + A_1(\tau, \xi)}} \, d\tau \right) \mathcal{H}(t, s, \xi), \\
\mathcal{E}_2(t, s, \xi) &= \exp \left( \frac{1}{2} \int_{t}^{s} \frac{-\partial_{\tau} \sqrt{A_0(\tau, \xi) + A_1(\tau, \xi)}}{\sqrt{A_0(\tau, \xi) + A_1(\tau, \xi)}} \, d\tau \right) \mathcal{H}(t, s, \xi), \\
\mathcal{E}_3(t, s, \xi) &= \exp \left( \frac{1}{2} \int_{t}^{s} \frac{\partial_{\tau} \sqrt{A_0(\tau, \xi) + A_1(\tau, \xi)}}{\sqrt{A_0(\tau, \xi) + A_1(\tau, \xi)}} \, d\tau \right) \mathcal{H}(t, s, \xi).
\end{align*}
\]

Furthermore, \( \mathcal{H}(t, s, \xi) \) satisfies

\[
D_t \mathcal{H} + \mathcal{E}_1(s, t, \xi) R_1(t, \xi) \mathcal{E}_1(t, s, \xi) \mathcal{H} = 0, \quad \mathcal{H}(s, s, \xi) = I.
\]

Since \( \|\mathcal{E}_1(t, s, \xi)\| \leq C \) for all \( s, t \in \mathbb{T} \), then by applying the same estimation procedure for the matrization representation as in \( Z_{pe}(P, M) \), we have

\[
\|\mathcal{H}(t, s, \xi)\| \leq \exp \left( \int_{t}^{s} \|\mathcal{E}_1(t, \xi)\| \, d\tau \right) \leq \exp(c_1 v(t_s)).
\]

For the fundamental solution, we conclude the estimate

\[
\|\mathcal{E}(t, s, \xi)\| = \|\mathcal{E}_1 \mathcal{H}\| \leq \exp(c_1 v(t_s)).
\]

Using the invertibility of \( \mathcal{H}, N_1 \), and

\[
V_1(t, \xi) = \mathcal{E}(t, t_s, \xi) V(t_s, \xi), \quad V_0 = N_1 V_1, \quad V = \mathcal{H} V_0,
\]

we transform \( V_1(t, \xi) \) back to the original micro-energy \( V(t, \xi) \) and obtain

\[
\|V(t, \xi)\| \leq \exp(c_1 v(t_s)) \|V(t_s, \xi)\|.
\]

This estimate concludes our proof. \( \square \)

### 2.4. Conclusion

**Lemma 2.5.** Combining the estimates in Lemmas 2.3 and 2.4 one gets the following energy estimate with a positive constant \( c_1 \) in \( [0, T] \times \{ |\xi| \geq M \} : 

\[
\left( |\xi|^\alpha \hat{u}(t, \xi) \right)_{t_t} \leq \exp(c_1 v(t_s)) \left( |\xi|^\alpha |\hat{u}_0(\xi)| + |\hat{u}_1(\xi)| \right).
\]
We take account of the definition of $t_\xi$ and Parseval’s formula, and arrive at the statement of Theorem 1.1 immediately by combining the statements from Lemmas 2.1 and 2.5.

Remark 2.2. As is shown in the proofs of Lemmas 2.3 and 2.4, we can choose the same constant $c_1$ depending upon $P$ and $M$.

Remark 2.3. We give some examples to explain the influence from several functions $v$. Assume that $u_0 \in H^k(\mathbb{R}^n)$, $u_1 \in H^{k-\sigma}(\mathbb{R}^n)$, then due to Theorem 1.1, there exists a unique solution $u = u(t, x)$ belonging to the following function spaces:

1. $v(t) \leq C$
   
   $u \in C([0, T]; H^s(\mathbb{R}^n)) \cap C^1([0, T]; H^{s-\sigma}(\mathbb{R}^n))$.

2. $v(t) = \log(1/t)$
   
   $u \in C([0, T]; \langle D_\xi \rangle H^s(\mathbb{R}^n)) \cap C^1([0, T]; \langle D_\xi \rangle H^{s-\sigma}(\mathbb{R}^n))$.

3. $v(t) = \log(1/t)^\gamma, \gamma \in (0, 1)$
   
   $u \in C([0, T]; \langle D_\xi \rangle \log(\langle D_\xi \rangle/2^P)^{\gamma} H^s(\mathbb{R}^n)) \cap C^1([0, T]; \langle D_\xi \rangle \log(\langle D_\xi \rangle/2^P)^{\gamma} H^{s-\sigma}(\mathbb{R}^n))$.

4. $v(t) = \log(1/t)^\gamma(\log^{2(1/t)}/2^P)^{\gamma} \cdots \log^{n(1/t)^n}, \gamma_1 \in (0, 1), \gamma_k > 0, k = 2, \ldots, n$
   
   $u \in C([0, T]; \langle D_\xi \rangle \log(\langle D_\xi \rangle/2^P)^{\gamma_1} \cdots \log(\langle D_\xi \rangle/2^P)^{\gamma_n} H^s(\mathbb{R}^n)) \cap C^1([0, T]; \langle D_\xi \rangle \log(\langle D_\xi \rangle/2^P)^{\gamma_1} \cdots \log(\langle D_\xi \rangle/2^P)^{\gamma_n} H^{s-\sigma}(\mathbb{R}^n))$.

3. Optimality of the results

In this section, we discuss the optimality of our estimates in Theorem 1.1. The method of instability argument to be used was developed in [8] to show that a Log-type loss really appears for hyperbolic Cauchy problems. Now we further develop this idea to demonstrate that the precise $v$-loss of derivatives really appears for our model in Theorem 1.1. Actually, the operator $\gamma = \partial_t^2 + (-\Delta)^{\sigma}$ has the finite speed of propagation only for $\sigma = 1$. For $\sigma \neq 1$, there is an infinite speed of propagation. As a result, we divide our discussion into two parts. On the one hand, the results of well-posedness on the torus $\mathbb{T}^n = \mathbb{R}^n/2\pi \mathbb{Z}^n$ and in $\mathbb{R}^n$ are essentially the same, therefore we construct a counter-example for general $\sigma \in \mathbb{R}_+$ in $[0, T] \times \mathbb{T}$. On the other hand, due to the property of finite propagation speed for $\sigma = 1$, we can construct a counter-example in $[0, T] \times \mathbb{R}$.

3.1. Counter-example for $\sigma \in \mathbb{R}_+$ in $[0, T] \times \mathbb{T}$

Let us consider the Cauchy problem

$$
\partial_t^2 u + b(t)(-\partial_x^2)^\sigma u = 0, \quad u(0, x) = u_0(x), \quad \partial_t u(0, x) = u_1(x) \tag{3.1}
$$

with 2π-periodic data $u_0, u_1$. For a 2π-periodic solution $u = u(t, x)$ in the $x$ variable, we introduce the homogeneous energy

$$
\tilde{E}_s(u)(t) := \|u(t, \cdot)\|^2_{H^s(\mathbb{T})} + \|\partial_t u(t, \cdot)\|^2_{H^{s-\sigma}(\mathbb{T})}, \quad s \in \mathbb{R},
$$

where $H^s(\mathbb{T})$ denotes the homogeneous Sobolev space of exponent $s$ on the torus $\mathbb{T} = \mathbb{R}/2\pi \mathbb{Z}$.

Actually, Corollary 1.2.2 indicates the following conclusion which shows at most a $v$-loss.

Corollary 3.1. Let us consider the sequence of Cauchy problems in $[0, T] \times \mathbb{T}$,

$$
\partial_t^2 u_k + b_k(t)(-\partial_x^2)^\sigma u_k = 0, \quad u_k(0, x) = u_{0,k}(x), \quad \partial_t u_k(0, x) = u_{1,k}(x).
$$

Define $\mu(s)$ as $s/\nu(s)$ and $\{b_k\}_k$ satisfy all the assumptions in Theorem 1.1 independent of $k$. If $u_{0,k} \in H^s(\mathbb{T}), u_{1,k} \in H^{s-\sigma}(\mathbb{T})$, and $P$ is a fixed appropriate positive integer, then there exists a sequence of solutions $\{u_k\}_k$ belonging to the following function spaces:

$$
C([0, T]; \exp(c_1 \nu(-2^P/\langle D_\xi \rangle^\sigma))H^s(\mathbb{T})) \cap C^1([0, T]; \exp(c_1 \nu(-2^P/\langle D_\xi \rangle^\sigma))H^{s-\sigma}(\mathbb{T}))
$$

with a positive constant $c_1$. Moreover, $\mu^{-1}$ denotes the inverse function of $\mu$.

Step 1: Introduction of auxiliary functions

For $\varepsilon > 0$, let us define

$$
w_\varepsilon(t) := \sin t \exp \left(2\varepsilon \int_0^t \psi(\tau) \sin^2 \tau d\tau\right),
$$

$$
a_\varepsilon(t) := 1 - 4\varepsilon \psi(t) \sin(2t) - 2\varepsilon \psi'(t) \sin^2 t - 4\varepsilon^2 \psi^2(t) \sin^4 t,
$$

and $t_\varepsilon(t)$.
where $\psi$ is a real non-negative $2\pi$-periodic $C^\infty$ function on $\mathbb{R}$ which is identically 0 in a neighborhood of $t = 0$ and satisfies
\[ \int_0^{2\pi} \psi(\tau) \sin^2(\tau) \, d\tau = \pi. \]

From the above definitions, we find out that $a_e$ and $w_e$ belong to $C^\infty(\mathbb{R})$ and satisfy
\[ \partial_t^2 w_e(t) + a_e(t) w_e(t) = 0, \quad w_e(0) = 0, \quad \partial_t w_e(0) = 1. \]

**Step 2:** Introduction of auxiliary sequences.
We define a sequence of intervals $\{I_k\}_k$ by
\[ I_k = [t_k - \rho_k/2, t_k + \rho_k/2]. \]

We choose the following sequences:
- a null sequence $\{t_k\}_k$ such that $(2^p \nu (t_k) t_k^{-1})^\frac{1}{2} \in \mathbb{N}_+$ for each $k \in \mathbb{N}$;
- $\{\rho_k\}_k = \{2^{-p+2} \pi t_k \nu(t_k) / \nu(t_k)\}_k$, $\{\tilde{h}_k\}_k = \{2^p \nu (t_k) t_k^{-1}\}_k$.

**Remark 3.1.** It is easy to see that the sequences $\{t_k\}_k$, $\{\rho_k\}_k$, tend to 0; the sequence $\{\tilde{h}_k\}_k$ tends to $+\infty$. Such choice of $\rho_k$ guarantees that $I_k$ is contained in $(0, T]$. Furthermore, $h_k \rho_k/(4\pi)$, $h_k^\frac{1}{2} \in \mathbb{N}_+$.

**Step 3:** At least a $\nu$-loss
With these auxiliary sequences, our optimality argument can be expressed as the following main theorem.

**Theorem 3.2.** For the Cauchy problem (3.1) there exists a sequence of coefficients $\{b_k(t)\}_k$ satisfying all assumptions of **Theorem 1.1** with constants independent of $k$. Moreover, there exists a sequence of data $\{(u_{0,k}(x), u_{1,k}(x))\}_k \in H^1(\mathbb{T}) \times H^{\nu-\gamma}(\mathbb{T})$ and a sequence of solutions $\{u_k(t, x)\}_k$ from $C^\infty([0, T] \times \mathbb{T})$ such that
\[ \sup_k \hat{E}_\nu(u_k)(0) \leq C(\epsilon), \quad (3.2) \]
\[ \sup_k \hat{E}_\nu(\exp(-c_1(\epsilon)(\mu^{-1}(2^p/(D_k)^\nu))u_k(t)) = +\infty \quad (3.3) \]

for all $t \in (0, T]$ with positive constants $C(\epsilon)$ and $c_1(\epsilon)$ independent of $k$, and $\epsilon$ is a sufficiently small positive constant.

**Proof.** For each $k \in \mathbb{N}$ we define the coefficient $b_k(t)$ as
\[ b_k(t) = \begin{cases} 1, & t \in [0, T] \setminus I_k; \\ a_e(\tilde{h}_k(t - t_k)), & t \in I_k. \end{cases} \]

The above definition indicates, on the one hand, $b_k \in C^\infty(\mathbb{R})$ since $a_e$ is identically equal to 1 in a neighborhood of the boundary of $I_k$; on the other hand,
\[ 0 < b_0 = \inf_{t \in [0, T]} b_k(t) \leq \sup_{t \in [0, T]} b_k(t) \leq b_1 < \infty, \]

where the constants $b_0$ and $b_1$ are independent of $k$ when we choose an appropriate $\epsilon > 0$. Simple calculations show that the coefficient $b_k$ satisfies all assumptions of **Theorem 1.1** in the interval $I_k$. While in $[0, T] \setminus I_k$, it is trivial.

Next we study the family of Cauchy problems in $[t_k - \rho_k/2, t_k + \rho_k/2] \times \mathbb{T}$,
\[ \partial_t^2 u_k + b_k(t)(-\partial_x^2)^\sigma u_k = 0, \quad u_k(t_k, x) = 0, \quad \partial_t u_k(t_k, x) = u_{1,k}(x). \quad (3.4) \]

Let $u_{1,k}(x) = \exp(ih_k^\frac{1}{2} x)$ and apply the following coordinate transformation $s = h_k(t - t_k)$. Define $v_k(s, x)$ as $u_k(t(s), x)$, then for $s \in [-h_k \rho_k/2, h_k \rho_k/2]$, we get
\[ \partial_s^2 v_k + h_k^{-2} a_e(s)(-\partial_x^2)^\sigma v_k = 0, \quad v_k(0, x) = 0, \quad \partial_s v_k(0, x) = u_{1,k}(x)/h_k. \quad (3.5) \]

Applying spectral theory to the self-adjoint operator $\sqrt{-\partial_x^2}$ on the compact manifold $\mathbb{T}$, we obtain the following lemma.

**Lemma 3.3.** For $a \in \mathbb{Z}$, $x \in \mathbb{T}$, and a continuous function $F : \mathbb{R} \to \mathbb{R}$, we have
\[ F\left(\sqrt{-\partial_x^2}\right) \exp(i|a|x) = F(|a|) \exp(i|a|x). \]
Proof. For $u(x) = \exp(i|a|x), a \in \mathbb{Z}$, we know 
$$u(x) = \sum_{k \in \mathbb{Z}} \hat{u}(k) \exp(ikx),$$
where
$$\hat{u}(k) = (2\pi)^{-1} \int_{\mathbb{T}} u(x) \exp(-ikx)dx = \begin{cases} 1, & k = |a|; \\
0, & k \neq |a|. \end{cases}$$

Then apply the definition of pseudo-differential operators from (1.2) on the compact manifold $\mathbb{T}$, we have
$$F\left(-\partial_t^2\right) \exp(i|a|x) = \sum_{k \in \mathbb{Z}} \hat{u}(k) F(|k|) \exp(ikx) = F(|a|) \exp(i|a|x).$$

The statement is proved. \(\square\)

As a matter of fact, we have a unique solution for (3.5) in the form of $u_k(s, x) = h_k^{-1}u_{1,k}(x)w_{c}(s)$. Transforming back to $u_k(t, x)$ we arrive at $u_k(t, x) = h_k^{-1} \exp(\frac{\rho}{2}x)w_{c}(h_k(t - t_k))$ in $h_k$, further calculations lead to
$$u_k(t_k + \rho_k/2, x) = 0, \quad \partial_t u_k(t_k + \rho_k/2, x) = \exp(\frac{\rho}{2}x) \exp(-\rho h_k/2),$$
$$u_k(t_k - \rho_k/2, x) = 0, \quad \partial_t u_k(t_k - \rho_k/2, x) = \exp(\frac{\rho}{2}x) \exp(\rho h_k/2).$$

Lemma 3.4. For the Cauchy problem in $\{(t, x) \in \mathbb{R} \times \mathbb{T}\}$,
$$\partial^2_t u + (-\partial^2_x)^\sigma u = 0, \quad u(t_0, x) = 0, \quad \partial_t u(t_0, x) = c \exp(i|a|x) \quad (3.6)$$
with $\sigma > 0, c \in \mathbb{R}$ and $a \in \mathbb{Z}$, then the energy conservation law holds, that is,
$$\dot{E}_r(u)(t) = \dot{E}_r(u)(t_0).$$

Proof. In effect, we have the explicit representation of the unique solution of (3.6) for $(t, x) \in \mathbb{R} \times \mathbb{T}$,
$$u(t, x) = \frac{c}{|a|^\sigma} \sin(|a|^\sigma(t - t_0)) \exp(i|a|x).$$

By applying the definition of the homogeneous Sobolev spaces $H^s(\mathbb{T}), s \in \mathbb{R}$, we calculate the homogeneous energy for the solution $u = u(t, x)$. It holds by applying Lemma 3.3
$$\dot{E}_r(u)(t) = \|u(t, \cdot)\|^2_{H^s(\mathbb{T})} + \|\partial_t u(t, \cdot)\|^2_{H^{s-\sigma}(\mathbb{T})}$$
$$= \sum_{k \in \mathbb{Z}} |\hat{u}(t, k)|^2 |k|^{2s} + \sum_{k \in \mathbb{Z}} |\partial_t \hat{u}(t, k)|^2 |k|^{2(s-\sigma)}$$
$$= c^2 |a|^{2(s-\sigma)}(\sin^2(|a|^{\sigma}(t - t_0)) + \cos^2(|a|^{\sigma}(t - t_0)))$$
$$= c^2 |a|^{2(s-\sigma)} = \dot{E}_r(u)(t_0). \quad \square$$

From Lemma 3.4 it follows that
$$u_k(t, x) = \frac{\exp(-\rho h_k/2)}{h_k} \sin(h_k(t - t_0)) \exp(ih_k^\frac{1}{2} x) \quad \text{for} \ t \in [0, t_k - \rho_k/2];$$
$$u_k(t, x) = \frac{\exp(\rho h_k/2)}{h_k} \sin(h_k(t - t_0)) \exp(ih_k^\frac{1}{2} x) \quad \text{for} \ t \in [t_k + \rho_k/2, T].$$

Therefore,
$$\dot{E}_r(u_k)(t) = \exp(-\rho h_k), \quad \text{for} \ t \in [0, t_k - \rho_k/2];$$
$$\dot{E}_r(u_k)(t) = \exp(\rho h_k), \quad \text{for} \ t \in [t_k + \rho_k/2, T].$$

(3.9) It is evident that (3.2) follows directly from (3.9). For $t \in [t_k + \rho_k/2, T]$, we apply Lemma 3.3 and obtain
$$\dot{E}_r(\exp(-c_1v(\mu^{-1}(2^p/(D_k)^\sigma)))u_k)(t) = \dot{E}_r(\exp(-c_1v(\mu^{-1}(2^p/(h_k))))u_k)(t)$$
$$= \exp(-2c_1v(\mu^{-1}(2^p/(h_k))))\dot{E}_r(u_k)(t)$$
$$= \exp(-2c_1v(\mu^{-1}(2^p/(h_k)) + \varepsilon \rho h_k)$$
$$= \exp(-2c_1v(t_k + \varepsilon \rho h_k)).$$
Taking into account the choice of $\rho_k$, $h_k$, we can choose a sufficiently small $c_1(\epsilon)$ independent of $k$ such that (3.3) holds. This concludes our proof. \hfill \Box

**Remark 3.2.** Similarly, we define the non-homogeneous energy

$$E_\epsilon(u)(t) := \|u(t, \cdot)\|_{H^s(T)}^2 + \|\partial_t u(t, \cdot)\|_{H^{s-\alpha}(T)}, \quad s \in \mathbb{R},$$

where $H^s(T)$ denotes the standard Sobolev space of exponent $s$ on the torus $T = \mathbb{R}/2\pi \mathbb{Z}$. And notice the facts (3.7) and (3.8) we have

- when $t \in (0, t_k - \rho_k/2]$, $t_0 = t_k - \rho_k/2$,
  $$\|u_k(t, \cdot)\|^2_{L^2(T)} = \frac{\exp(-\epsilon \rho_k h_k)}{h_k^2} \sin^2(h_k(t - t_0)),$$
  $$\|\partial_t u_k(t, \cdot)\|^2_{L^2(T)} = \exp(-\epsilon \rho_k h_k) \cos^2(h_k(t - t_0));$$
- when $t \in (t_k + \rho_k/2, T)$, $t_0 = t_k + \rho_k/2$,
  $$\|u_k(t, \cdot)\|^2_{L^2(T)} = \frac{\exp(\epsilon \rho_k h_k)}{h_k^2} \sin^2(h_k(t - t_0)),$$
  $$\|\partial_t u_k(t, \cdot)\|^2_{L^2(T)} = \exp(\epsilon \rho_k h_k) \cos^2(h_k(t - t_0)).$$

These estimates show that (3.2) and (3.3) hold also for the non-homogeneous energy $E_\epsilon(u)(t)$.

### 3.2. Counter-example for $\sigma = 1$ in $[0, T] \times \mathbb{R}$

The technique of Floquet theory was used in [9] in treating the influence of oscillations to weakly hyperbolic Cauchy problems with infinitely degenerating coefficients. In [10,11], the authors extended this technique to strictly hyperbolic cases. Recently, this method is further applied to the discussion of $p$-evolution type models in [5]. All these counter-examples are constructed with respect to $v(t) = (\log(1/t))^{-q}$, $q > 2$. As is shown in Section 2, $v(t) = \log(1/t)$ is the critical case for $C^\infty$ well-posedness. In [3], the authors constructed a more delicate counter-example, which is arbitrarily close to the critical case with the measure function $v(t) = \log(1/t) \log^{\infty}(1/t)$, $n \geq 2$. With the application of Floquet theory, they showed that the infinite loss of derivatives really appears by constructing explicit coefficients with a periodic function $b(s)$, $s \in \mathbb{R}$. However, an explicit structure of coefficients with respect to finite loss or arbitrarily small loss, or even general $v$-loss, still remains open. For this reason, in the next section, we apply another instability argument to study the optimality of conditions for general $v$-loss of derivatives. As a matter of fact, Theorem 1.1 indicates the following conclusion which shows at most a $v$-loss.

**Corollary 3.5.** Let us consider the sequence of Cauchy problems in $[0, T] \times \mathbb{R}$,

$$\partial_t^2 u_k - b_k(t) \partial_x^2 u_k = 0, \quad u_k(t_0, x) = u_{0,k}(x), \quad \partial_t u_k(t_0, x) = u_{1,k}(x).$$

Define $\mu(s)$ as $\nu(s)$ and $[b_k]_k$ satisfy all the assumptions in Theorem 1.1 independent of $k$. If $u_{0,k} \in H^s(\mathbb{R})$, $u_{1,k} \in H^{s-\alpha}(\mathbb{R})$, and $P$ is a fixed appropriate positive integer, then there exists a sequence of solutions $[u_k]_k$ belonging to the following function spaces:

$$C([0, T]; \exp(c_1 \nu(\mu^{-1}(2^p / (\kappa_0)^\sigma))) H^s(\mathbb{R}^N)) \cap C^1([0, T]; \exp(c_1 \nu(\mu^{-1}(2^p / (\kappa_0)^\sigma)))) H^{s-\alpha}(\mathbb{R}^N))$$

with a positive constant $c_1$ which is independent of $k$. Moreover, $\mu^{-1}$ denotes the inverse function of $\mu$.

First, we introduce the following instability conclusion for periodic problems (see [12]):

**Lemma 3.6** *(Floquet Theory).* Let $b(s) \in C^2(\mathbb{R})$ be a non-constant, positive, 1-periodic function. Then there exists a positive real number $\lambda_0 > 0$ such that $\lambda_0$ belongs to an interval of instability for $\partial^2 w + \lambda_0 b^2(s) w = 0$, that is, $X(1, 0)$ has eigenvalues $\mu_0$ and $\mu_0^{-1}$ satisfying $|\mu_0| > 1$.

We are interested in the fundamental solution $X = X(s, s_0)$ of the Cauchy problem

$$\frac{d}{ds} X = A(s) X = \begin{pmatrix} 0 & -\lambda_0 b^2(s) \\ 1 & 0 \end{pmatrix} X, \quad X(s_0, s_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is clear that $X(s_0 + 1, s_0)$ is independent of $s_0 \in \mathbb{N}$ since $b(s)$ is 1-periodic. Set $X(1, 0) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, then Lemma 3.6 implies that the eigenvalues of this matrix are $\mu_0$ and $\mu_0^{-1}$. Hence, $a_{11} + a_{22} = \mu_0 + \mu_0^{-1}$, which gives $|a_{11} - \mu_0| + |a_{22} - \mu_0| \geq |\mu_0 - \mu_0^{-1}|$. From this estimate it follows $\max(|a_{11} - \mu_0|, |a_{22} - \mu_0|) \geq \frac{1}{2} |\mu_0 - \mu_0^{-1}|$. Without loss of generality, we assume
For the case when \(|\alpha - \alpha_0| \geq \frac{1}{2}|\mu_0 - \mu_0^{-1}|\), then we have \(|\alpha_2 - \mu_0^{-1}| \geq \frac{1}{2}|\mu_0 - \mu_0^{-1}|\). In this section we consider the following strictly hyperbolic Cauchy problem in \([0, T] \times \mathbb{R}\):

\[
\partial^2_t u - b(t) \partial_t u = 0, \quad u(t_0, x) = u_0(x), \quad \partial_t u(t_0, x) = u_1(x).
\]  

(3.11)

Actually, we have the following result.

**Theorem 3.7.** For the Cauchy problem (3.11) there exists a sequence of coefficients \(\{b_k(t)\}_k\) satisfying all assumptions of **Theorem 1.1** with constants independent of \(k\), and a sequence of data \(\{u_{0,k}(x), u_{1,k}(x)\}_k \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}), s \in \mathbb{R}\). Correspondingly, we can find two zero sequences \(\{t_k^{(1)}\}_k, \{t_k^{(2)}\}_k\), such that the following estimates hold for the sequence of solutions \(\{u_k(t, x)\}_k\):

\[
\|\exp(-c_1 \nu(\mu_0^{-1}(2^p/(D_0)^{p^*})))u_k(t_k^{(2)}, \cdot)\|_{H^s(\mathbb{R})} \geq C_k \|u_k(t_k^{(1)}, \cdot)\|_{H^s(\mathbb{R})},
\]

where the positive constant \(c_1\) is independent of \(k\) and \(\sup_k C_k = +\infty\).

**Proof.** First we introduce a consequence of **Lemma 3.6**.

**Lemma 3.8.** If \(w = w(s)\) is the solution of \(d_0^2 w + \lambda_0 b^2(s)w = 0\) with initial data \(w(0) = 1, \partial_t w(0) = 0\). Then for every sufficiently large positive \(M \in \mathbb{N}\), the solution satisfies \(|w(M)| \sim \mu_0^M (b(s), \lambda_0\) and \(\mu_0\) are given in **Lemma 3.6**).

**Proof.** Using the fundamental solution \(X(1, 0)\) we have

\[
\begin{pmatrix}
  d_t w(M) \\
  w(M)
\end{pmatrix}
= X(1, 0)^M
\begin{pmatrix}
  d_t w(0) \\
  w(0)
\end{pmatrix}.
\]

The matrix

\[
B := \begin{pmatrix}
  a_{12} & 1 \\
  \mu_0 - a_{11} & a_{21} \\
  \mu_0^{-1} - a_{22}
\end{pmatrix}
\]

is a diagonalizer for \(X(1, 0)\), which means

\[
X(1, 0)B = B \begin{pmatrix}
  \mu_0 & 0 \\
  0 & \mu_0^{-1}
\end{pmatrix}.
\]

Since \(\det X(1, 0) = 1\) and the trace of \(X(1, 0)\) is \(\mu_0 + \mu_0^{-1}\) we obtain

\[
\det B = \frac{\mu_0 - \mu_0^{-1}}{\mu_0^{-1} - a_{22}}.
\]

Straightforward calculations lead to

\[
\begin{pmatrix}
  d_t w(M) \\
  w(M)
\end{pmatrix}
= \frac{1}{\det B}
\begin{pmatrix}
  a_{12}(\mu_0^{-M} - \mu_0^M) \\
  \mu_0 - a_{11} \\
  \mu_0^{-1} - a_{22}
\end{pmatrix}
\begin{pmatrix}
  \mu_0 - a_{11} \\
  a_{12}a_{21}\mu_0^M \\
  (\mu_0 - a_{11})(\mu_0^{-1} - a_{22})
\end{pmatrix}.
\]

Since \(|\mu_0 - a_{11}| \geq \frac{1}{2}|\mu_0 - \mu_0^{-1}|, |\mu_0^{-1} - a_{22}| \geq \frac{1}{2}|\mu_0 - \mu_0^{-1}|, \) and take into consideration \(\|X(1, 0)\| \leq C\), we see that \(|\det B| \sim 1\). When \(M\) is sufficiently large, \(\mu_0^M\) becomes the dominating part, which gives \(|w(M)| \sim \mu_0^M\). \(\Box\)

**Remark 3.3.** For the case when \(|\mu_0 - a_{11}| \leq \frac{1}{2}|\mu_0 - \mu_0^{-1}|, |\mu_0^{-1} - a_{22}| \leq \frac{1}{2}|\mu_0 - \mu_0^{-1}|\), we can choose

\[
B := \begin{pmatrix}
  a_{12} & 1 \\
  \mu_0^{-1} - a_{11} & a_{21} \\
  \mu_0 - a_{22}
\end{pmatrix}
\]

as the diagonalizer and let \(w(0) = 0, d_t w(0) = 1\).

**Step 1:** Introduction of auxiliary sequences

We define a sequence of intervals \(\{l_k\}_k\) by

\[
l_k = [t_k - \rho_k/2, t_k + \rho_k/2],
\]

where \(\rho_k\) is a sequence of positive numbers.
and choose the following sequences
\[
\{\rho_k\}_k = \left\{2^{-\nu t^2 + \frac{1}{\sqrt{\lambda_0 t_k}} v(t_k) / v(t_k)} \right\}, \quad \left\{h_k = 2^\nu \frac{1}{\sqrt{\lambda_0}} v(t_k) t_k^{-1} \right\}_k.
\]

**Remark 3.4.** We require that \(\{t_k\}_k\) is an appropriate zero sequence, such that the sequence \(\{\rho_k\}_k\) tends to 0, while \(\{\sqrt{\lambda_0 h_k}\}_k \in \mathbb{N}\) tends to \(+\infty\). Furthermore, we have \(h_k \rho_k / 2 \in \mathbb{N}_+\), and a large \(P\) assures that \(I_k \subset (0, T]\).

Next we construct a family of coefficients \(\{b_k = b_k^2(t)\}_k\) which are defined by
\[
b_k(t) = \begin{cases} 1, & t \in [0, T] \setminus I_k; \\ b(t_k - t_k), & t \in I_k. \end{cases}
\]

Clearly, the definition of \(b_k(t)\) indicates:
\[
0 < b_0 \leq \inf_{t \in [0, T]} b_k(t) \leq \sup_{t \in [0, T]} b_k(t) \leq b_1 < \infty,
\]
where the constants \(b_0\) and \(b_1\) are independent of \(k\). Straight-forward calculations show that the coefficients \(b_k\) satisfy all assumptions in Theorem 1.1 with constants independent of \(k\).

**Step 2:** At least a \(\nu\)-loss
Let \(\chi = \chi(r) \in [0, 1]\) be a cut-off function from \(C_0^\infty(\mathbb{R})\), where \(\chi \equiv 1\) for \(|r| \leq 1\) and \(\chi \equiv 0\) for \(|r| \geq 2\). Then we choose for large \(k\) the following data:
\[
u_{0,k}(x) = \exp(ih_k \sqrt{\lambda_0 x}) \chi \left(\frac{x}{(v(t_k))^2 P_k}\right), \quad u_{1,k}(x) = 0 \quad \text{for all } x \in \mathbb{R},
\]
where
\[
P_k = \frac{2\pi}{h_k \sqrt{\lambda_0}} \sim t_k v(t_k)^{-1}.
\]
Then for \(s \in \mathbb{N}_+\) the norm \(\|u_{0,k}\|_{L^s(\mathbb{R})}\) can be estimated in the following way:
\[
\|u_{0,k}\|_{L^s(\mathbb{R})} \leq C(h_k^s + 1) v(t_k) \sqrt{P_k}.
\]

**Remark 3.5.** Actually, interpolation theory ensures that our discussion can also be applied to \(H^s(\mathbb{R}), s \in \mathbb{R}\). For convenience’s sake, we treat only the integer case in this paper.

Now we study the family of Cauchy problems
\[
\partial_t^2 u_k - b^2(t_0(t - t_k)) \partial_x^2 u_k = 0, \quad u_k(t_k, x) = u_{0,k}(x), \quad \partial_t u_k(t_k, x) = 0, \quad t \in [t_k - \rho_k/2, t_k + \rho_k/2].
\]
If \(x\) is taken from \(|x| \leq P_k\) on \(t = t_k + \rho_k/2\), then the solution \(u_k(t_k + \rho_k/2, x)\) will be influenced by the data on the set \(|x| \leq P_k + \rho_k/2\). On this set we have \(u_{0,k}(x) = \exp(ih_k \sqrt{\lambda_0 x})\). We apply the transformation \(s = h_k(t - t_k)\), and define \(v_k(s, x) = u_k(t(s), x)\), then get:
\[
\partial_s^2 v_k - h_k^{-2} b^2(s) \partial_x^2 v_k = 0, \quad v_k(0, x) = u_{0,k}(x), \quad \partial_s v_k(0, x) = 0, \quad s \in [-h_k \rho_k/2, h_k \rho_k/2].
\]
In fact, we have a unique solution in the form \(v_k(s, x) = u_{0,k}(x) w(s)\), where \(w = w(s)\) satisfies
\[
\partial_s^2 w(s) + \lambda_0 b^2(s) w(s) = 0, \quad w(0) = 1, \quad \partial_s w(0) = 0, \quad s \in [-h_k \rho_k/2, h_k \rho_k/2].
\]
Transforming back to \(u_k(t, x)\) we arrive at
\[
u_k(t_k + \rho_k/2, x) = \exp(ih_k \sqrt{\lambda_0 x}) w(\rho_k h_k/2), \quad u_k(t_k, x) = \exp(ih_k \sqrt{\lambda_0 x}) w(0),
\]
where \(|w(\rho_k h_k/2)| \sim |\mu_0| |h_k|^{1/2} / 2\).

Now we consider the terms on the torus \(\mathbb{T}_k = \mathbb{R} / (2\pi / (h_k \sqrt{\lambda_0}) \mathbb{Z})\), and Lemma 3.3 indicates
\[
\|\exp(-c_1 v(\mu^{-1}(2^\nu / (\langle D_x \rangle))) \langle D_x \rangle^s u_k(t_k + \rho_k/2, \cdot)\|_{L^s(\mathbb{T}_k)} \sim \|\exp(-c_1 v(t_k)) (h_k^s + 1) \sqrt{P_k |\mu_0|} \rho_k h_k/2.\]

Denote \(t_k^{(1)} = t_k, t_k^{(2)} = t_k + \rho_k/2\), then we get the following estimate from (3.12) and (3.13):
\[
\|\exp(-c_1 v(\mu^{-1}(2^\nu / (\langle D_x \rangle))) \langle D_x \rangle^s u_k(t_k^{(1)}), \cdot)\|_{L^s(\mathbb{T}_k)} \geq \|\exp(-c_1 v(\mu^{-1}(2^\nu / (\langle D_x \rangle))) \langle D_x \rangle^s u_k(t_k^{(2)}, \cdot)\|_{L^s(\mathbb{T}_k)}
\]
\[
\geq C_k \|u_k(t_k^{(1)}, \cdot)\|_{H^s(\mathbb{R})},
\]
where \(\sup_k C_k = +\infty\) and the sufficiently small constant \(c_1\) is independent of \(k\). This shows that the \(\nu\)-loss of derivatives really appears. □
Remark 3.6. Periodic functions are very useful in the construction of coefficients for instability arguments. When we consider the problems in $[0, T] \times \mathbb{R}$, a compact support with respect to the initial data is needed to ensure the $H^1$-boundedness of the periodic functions. And with the property of finite propagation speed for $\sigma = 1$, we are able to determine the compact support of the solution at time $t$ and find the precise representation of the solution in an interval from this support. However, when $\sigma \neq 1$, how to construct a counter-example remains open due to the lack of this property. For more open problems, please see [4].

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