# A SEMI-DISCRETE LARGE-TIME BEHAVIOR PRESERVING SCHEME FOR THE AUGMENTED BURGERS EQUATION

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Abstract. In this paper we analyze the large-time behavior of the augmented Burgers equation. We first study the well-posedness of the Cauchy problem and obtain $L^1$-$L^p$ decay rates. The asymptotic behavior of the solution is obtained by showing that the convolution term $K * u_{xx}$ behaves as $u_{xx}$ for large times. Then, we propose a semi-discrete numerical scheme that preserves this asymptotic behavior, by introducing two corrector factors in the discretization of the non-local term. Numerical experiments illustrating the accuracy of the results of the paper are also presented.

## 1. Introduction and main results

Together with linear theory, Burgers-type equations have been one of the main tools to model the propagation of finite-amplitude plane waves. The classical viscous Burgers equation was first considered for wave propagation in a lossy medium. Successive generalizations included other effects such as geometrical spreading and inhomogeneous mediums (generalized Burgers equation [2, 7, 14]) or relaxation processes (augmented Burgers equation [15]). Moreover, the extended Burgers equation [4] (sometimes also called augmented Burgers equation, as in [16]) has been recently used to model the propagation of the sonic-boom produced by supersonic aircrafts from their near-field down to the ground level, taking into account all those phenomena mentioned above.

In this paper we consider the augmented Burgers equation with constant parameters and a unique relaxation process. We focus on the following equation:

$$\begin{cases} u_t = uu_x + \nu\, u_{xx} + c\, K_\theta * u_{xx}, & (t,x) \in (0,\infty)\times\mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $*$ denotes the convolution in the $x$ variable, the parameters $\nu, c, \theta$ are positive and

$$K_\theta(z) = \begin{cases} \frac{1}{\theta} e^{-z/\theta}, & z > 0, \\ 0, & \text{elsewhere}. \end{cases} \tag{1.2}$$

Industrial applications of this kind of models, such as the aforementioned sonic-boom phenomena, need to approximate solutions for large time. Therefore, they need a good understanding of the behavior of the solutions in these extended regimes in order to be able to simulate them accurately. This issue needs to be treated carefully, as it was already shown in [9]. In that work, the authors proved that a numerical scheme with an acceptable accuracy in short-time intervals could completely disturb the large-time behavior of solutions due to the numerical viscosity introduced by the numerical approximation. In our case, (1.1) is not a hyperbolic equation and, hence, the asymptotic profile is not an N-wave, but a diffusive wave. Nevertheless, in our simulations we show that small values for $\nu$ and $c$ require a similar treatment from the numerical point of view, as if the equation was a hyperbolic conservation law. In fact, in those situations, the solution may develop very steep regions (in what follows we refer to these as quasi-shocks), which numerically behave almost like shocks.

For the shake of simplicity, the asymptotical analysis done in the first sections is focused only on the case $\nu = c = \theta = 1$, but the extension to any positive value of the parameters is immediate. We will omit the subindex $\theta$ whenever its value is one. In this case, we have that

$$K * u_{xx} = K * u - u + u_x.$$

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Thus, (1.1) can be rewritten in a more suitable manner as follows:

$$\begin{cases} u_t = uu_x + u_{xx} + K * u - u + u_x, & (t,x) \in (0,\infty) \times \mathbb{R}, \\ u(t=0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.3}$$

The main goals of the present paper are to analyze the asymptotic behavior of the solutions to (1.3) as $t \to \infty$ and to build a semi-discrete numerical scheme that preserves this behavior. In what concerns the large-time behavior of solutions of system (1.3), the main result is stated in the following theorem.

**Theorem 1.1.** *Let $u_0 \in L^1(\mathbb{R})$. For any $p \in [1,\infty]$, the solution $u$ to (1.3) satisfies*

$$t^{\frac{1}{2}(1-\frac{1}{p})} \|u(t) - u_M(t)\|_p \longrightarrow 0, \quad \text{as } t \to \infty,$$

*where $u_M(t,x)$ is the solution of the following equation:*

$$\begin{cases} u_t = uu_x + 2u_{xx}, & x \in \mathbb{R}, t > 0, \\ u(0) = M\delta_0. \end{cases}$$

*Here $\delta_0$ denotes the Dirac delta at the origin and $M$ is the mass of the initial data, $M = \int_{\mathbb{R}} u_0(x)dx$.*

**Remark 1.** *We emphasize that function $u_M$ in Theorem 1.1 is given by (see [6])*

$$u_M(t,x) = 2\sqrt{2}\, t^{-\frac{1}{2}} \exp\left(\frac{-x^2}{8t}\right) \left[ C_M + \int_{-\infty}^{x/\sqrt{2t}} \exp\left(\frac{-s^2}{4}\right) ds \right]^{-1}, \tag{1.4}$$

*where $C_M \in \mathbb{R}$ is a constant such that $\int_{\mathbb{R}} u_M(t,x)dx = M$, for all $t > 0$. This shows that $u_M$ is of the form $t^{-\frac{1}{2}} f_M\left(\frac{x}{\sqrt{t}}\right)$ for some function $f_M$ and, hence, self-similar.*

In the cases when $\nu$, $c$ and $\theta$ are no longer equal to one, the asymptotic profile does not depend on $\theta$. Moreover, the coefficient in front of the viscosity term in the equation satisfied by the profile is $\nu + c$:

$$\begin{cases} u_t = uu_x + (\nu + c)u_{xx}, & x \in \mathbb{R}, t > 0, \\ u(0) = M\delta_0. \end{cases}$$

This will be particularly important at the numerical level. On the one hand, when choosing the numerical flux to discretize the nonlinearity, we need to handle thoroughly the numerical viscosity that is introduced. In [9], it is shown that in the hyperbolic case, the N-wave asymptotic profile could be destroyed if the numerical flux is not chosen carefully. In our case, if $\nu$ and $c$ are much smaller than $\Delta x^2/(2\Delta t)$ ($\Delta x$ being the mesh-size and $\Delta t$, the time-step), the Lax-Friedrichs scheme would make the diffusion start dominating much earlier due to the numerical viscosity. On the other hand, we need to treat the truncation of the integral term in such a manner that we do not introduce undesired pathologies in the large-time behavior of the numerical solutions. We do this by means of two corrector factors for the terms $u$ and $u_x$ in (1.3).

Let us denote by $u_\Delta$ an approximation to the solution $u$ of (1.3). We define this piecewise constant in space function as follows:

$$u_\Delta(t,x) = u_j(t), \quad x \in (x_{j-1/2}, x_{j+1/2}), t \geq 0, \tag{1.5}$$

where $x_{j+1/2} = (j + \frac{1}{2})\Delta x$, for all $j \in \mathbb{Z}$, and $\Delta x > 0$ is a given mesh-size. We will also denote by $x_j = j\Delta x$ the intermediate points of the spatial cells. For each $j \in \mathbb{Z}$ we need to compute a function $u_j(t)$ that approximates the value of the solution in the cell. Taking into account the issues enumerated above, we choose the following discretization of (1.3): the Engquist-Osher scheme for the flux, centered finite differences for the laplacian and the composite rectangle rule for the integral:

$$\begin{cases} u_j'(t) = \dfrac{g_{j+1/2}(t) - g_{j-1/2}(t)}{\Delta x} + \dfrac{u_{j-1}(t) - 2u_j(t) + u_{j+1}(t)}{\Delta x^2} \\ \qquad + \displaystyle\sum_{m=1}^{N} \omega_m u_{j-m}(t) - F_0^\Delta u_j(t) + F_1^\Delta \frac{u_{j+1}(t) - u_j(t)}{\Delta x}, & j \in \mathbb{Z}, t \geq 0, \\ u_j(0) = \dfrac{1}{\Delta x} \displaystyle\int_{x_{j-1/2}}^{x_{j+1/2}} u_0(x)dx, & j \in \mathbb{Z}, \end{cases} \tag{1.6}$$

where

$$\omega_m = e^{-m\Delta x}\left(e^{\Delta x}-1\right), \quad m=1,\dots,N, \tag{1.7}$$

and

$$g_{j+1/2}(t) = \frac{u_j(t)\big(u_j(t)-|u_j(t)|\big)}{4} + \frac{u_{j+1}(t)\big(u_{j+1}(t)+|u_{j+1}(t)|\big)}{4}, \quad j\in\mathbb{Z}, t\geq 0.$$

The parameter $N=N(\Delta x)\in\mathbb{N}$ denotes the number of nodes considered in the quadrature formula of the integral. The corrector factors $F_0^\Delta$ and $F_1^\Delta$ in front of the approximations of $u$ and $u_x$, given by

$$F_0^\Delta = \sum_{m=1}^N \omega_m \quad \text{and} \quad F_1^\Delta = \Delta x \sum_{m=1}^N m\omega_m, \tag{1.8}$$

handle, from the asymptotic behavior point of view, the correct truncation of the nonlocal term .

Finally, for $\Delta x$ fixed we study the asymptotic behavior as $t\to\infty$ of these semi-discrete solutions $u_\Delta$.

**Theorem 1.2.** *Let $u_0\in L^1(\mathbb{R})$, $\Delta x>0$ and $u_\Delta$ be the corresponding solution of the semi-discrete scheme* (1.6) *for the augmented Burgers equation* (1.3)*. For any $p\in[1,\infty]$, the following holds*

$$t^{\frac{1}{2}(1-\frac{1}{p})}\|u_\Delta(t)-u_M(t)\|_p \longrightarrow 0, \quad \text{as } t\to\infty, \tag{1.9}$$

*where $u_M(t,x)$ is the unique solution of the following viscous Burgers equation:*

$$\begin{cases} v_t = vv_x + (1+F_2^\Delta)v_{xx}, & x\in\mathbb{R}, t>0,\\ v(x,0)=M\delta_0. \end{cases}$$

*Here, $M=\int_{\mathbb{R}} u_0(x)dx$ is the mass of the initial data and*

$$F_2^\Delta = \frac{\Delta x^2}{2}\left(\sum_{m=1}^N m(m-1)\omega_m\right).$$

Let us observe that if $N$ is taken such that $N\to\infty$ and $N\Delta x\to\infty$ when $\Delta x\to 0$, then $F_2^\Delta\to 1$, which is, precisely, the value that we should expect from the continuous model. Besides, let us remark that in the case where $\nu$, $c$ and $\theta$ are not necessarily equal to one, the asymptotic profile is the unique solution of:

$$\begin{cases} v_t = vv_x + (\nu + c\,F_2^{\Delta,\theta})v_{xx}, & x\in\mathbb{R}, t>0,\\ v(x,0)=M\delta_0. \end{cases}$$

In this case, we take

$$\omega_m^\theta = e^{-m\Delta x/\theta}\left(e^{\Delta x/\theta}-1\right)$$

and

$$F_0^{\Delta,\theta} = \sum_{m=1}^N \omega_m^\theta, \qquad F_1^{\Delta,\theta} = \frac{\Delta x}{\theta}\sum_{m=1}^N m\omega_m^\theta \quad \text{and} \quad F_2^{\Delta,\theta} = \frac{\Delta x^2}{2\theta^2}\left(\sum_{m=1}^N m(m-1)\omega_m^\theta\right).$$

In the same conditions as above, for a fixed $\theta$ we still have that $F_2^{\Delta,\theta}$ converges to one.

Moreover, as we can see in the numerical experiments, the numerical flux needs to be chosen carefully, to avoid adding an extra viscosity term to the equation of the asymptotic profile. This has already been observed in [9] in the context of the numerical approximation of the inviscid Burgers equation. That extra viscosity term, of the order of $\Delta x^2/(2\Delta t)$, would affect critically the numerical solution if both parameters $\nu$ and $c$ were much smaller. Note also that taking $F_0^\Delta = F_1^\Delta = 1$ would add undesired phenomena, such as a transport, to the equation too.

Let us conclude this section by adding a final comment on the time discretization, which we do not address in this paper. At the continuous/semi-discrete level, we obtain estimates on the solution that allow us to prove the compactness of a family of rescaled solutions. Then, the asymptotic behavior is obtained as in (1.9). The analogous step for the fully time-explicit discrete scheme requires further development.

The paper is organized as follows. In Section 2, we deal with the well-posedness of equation (1.3) and the asymptotical behavior of its solutions, showing that $K*u_{xx}$ behaves like $u_{xx}$ as $t\to\infty$. In Section 3, we focus on the semi-discrete numerical scheme (1.6), showing its convergence and analyzing for a fixed $\Delta x$ the large-time behavior of the numerical solutions. To illustrate the main results of this work, we conclude with some numerical simulations in Section 4.

## 2. Analysis of the augmented Burgers equation

In this section we study the well-posedness of the Cauchy problem for (1.3) with initial data in $L^1(\mathbb{R})$. We also obtain estimates in the $L^p$-norms of its solution, which we subsequently denote $\|\cdot\|_p$. We mainly proceed as in [6] and [12]. .

### 2.1. Existence and uniqueness of solutions.

The following theorem concerns the global existence of solutions and specifies their regularity. Let us remark that the result coincides with the one for the classical convection-diffusion equation [6].

**Theorem 2.1.** *For any $u_0 \in L^1(\mathbb{R})$, there exists a unique solution $u \in C([0,\infty), L^1(\mathbb{R}))$ of (1.3). Moreover, it also satisfies*

$$u \in C((0,\infty), \mathcal{W}^{2,p}(\mathbb{R})) \cap C^1((0,\infty), L^p(\mathbb{R})), \quad \forall p \in (1,\infty).$$

*Additionally, equation (1.3) generates a contractive semigroup in $L^1(\mathbb{R})$.*

*Proof. Existence in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.* The local existence of the solution follows by a classical Banach fixed point argument as in [6] or [10]. To extend the solution globally, we deduce a priori estimates on the $L^1(\mathbb{R})$ and $L^\infty(\mathbb{R})$ norms of the solution. Let us first focus on the $L^1$-norm. Multiplying (1.3) by $\operatorname{sign}(u)$ and integrating in $\mathbb{R}$, it follows that

$$\frac{d}{dt}\int_{\mathbb{R}} |u| dx \le \int_{\mathbb{R}} (K*u - u)\operatorname{sign}(u) dx \le \int_{\mathbb{R}} K dx \int_{\mathbb{R}} |u| dx - \int_{\mathbb{R}} |u| dx \le 0 \tag{2.1}$$

and, consequently, $\|u(t)\|_1 \le \|u_0\|_1$.

To estimate the $L^\infty$-norm similar arguments apply. We define $\mu = \|u_0\|_\infty$, multiply equation (1.3) by $\operatorname{sign}[(u-\mu)^+]$, where $z^+ := \max\{0, z\}$, and integrate it in $\mathbb{R}$. We obtain

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}} (u-\mu)^+ dx &\le \int_{\mathbb{R}} (K*u - u + u_x)\operatorname{sign}(u-\mu)^+ dx = \int_{\mathbb{R}} (K*(u-\mu) - (u-\mu))\operatorname{sign}(u-\mu)^+ dx \\ &\le \int_{\mathbb{R}} K*(u-\mu)^+ - \int_{\mathbb{R}} (u-\mu)^+ \le 0. \end{aligned} \tag{2.2}$$

We conclude that $(u-\mu)^+ \le (u_0-\mu)^+ = 0$ and, consequently, $u(t) \le \mu$ almost everywhere. The same argument for $(u+\mu)^-$, where $z^- := -\max\{0, -z\}$, shows that $u \ge -\mu$. Therefore, if $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, then $\|u(t)\|_\infty \le \|u_0\|_\infty$ for all $t > 0$. Lastly, since both $L^1$-norm and $L^\infty$-norm remain bounded in time, the solution $u$ exists globally.

*Regularity.* It follows from classical regularity arguments (e.g., [11]) that

$$u \in C((0,T), W^{2,p}(\mathbb{R})) \cap C^1((0,T), L^p(\mathbb{R}))$$

for every $p \in (1,\infty)$. This also holds for $T = \infty$. Let us remark that this regularity makes the integrals in the previous steps be well defined.

*Uniqueness.* To prove the uniqueness of solution it is enough to check that (1.3) generates a contractive semigroup in $L^1(\mathbb{R})$; that is, for any initial datum $u_0, v_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$

$$\|u(t) - v(t)\|_1 \le \|u_0 - v_0\|_1, \quad \forall t > 0, \tag{2.3}$$

where $u$ and $v$ are the corresponding solutions. An analogous argument as in (2.1), applied to the equation verified by $u - v$, shows

$$\frac{d}{dt}\int_{\mathbb{R}} |u-v| dx \le 0,$$

hence the contraction property in $L^1(\mathbb{R})$.

*Existence and uniqueness in $L^1(\mathbb{R})$.* The extension of the result to a general $u_0 \in L^1(\mathbb{R})$ can be done following the same arguments as in [6]. □

2.2. **Decay estimates and large-time behavior.** Now we obtain $L^p$-decay rates for the solution to (1.3). These are the same as the ones for the viscous Burgers equation [6].

**Theorem 2.2.** *For all $p \in [1,\infty]$, there exists a constant $C = C(p) > 0$ such that*

$$\|u(t)\|_p \leq C\|u_0\|_1 t^{-\frac{1}{2}(1-\frac{1}{p})}, \quad \forall t > 0, \tag{2.4}$$

*for all solutions of equation (1.3) with initial data $u_0 \in L^1(\mathbb{R})$.*

*Proof.* The case $p = 1$ is an immediate consequence of Theorem 2.1. In the case $p \in [2,\infty)$, we multiply equation (1.3) by $|u|^{p-2}u$ and integrate it in $\mathbb{R}$. We obtain:

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\left(\|u\|_p^p\right) &= \int_{\mathbb{R}} |u|^{p-2}uu_t dx = \int_{\mathbb{R}} |u|^p u_x dx + \int_{\mathbb{R}} |u|^{p-2}uu_{xx}dx + \int_{\mathbb{R}} |u|^{p-2}u(K*u - u + u_x)dx \\ &= -\frac{4(p-1)}{p^2}\left\|\left(|u|^{p/2}\right)_x\right\|_2^2 - \|u\|_p^p + \int_{\mathbb{R}} |u|^{p-2}u(K*u)dx. \end{aligned} \tag{2.5}$$

Let us focus on the last term, so that we can compare it with the $L^p$-norm of $u$. Young's inequality gives us that

$$\left||u(t,x)|^{p-2}u(t,x)u(t,y)\right| = |u(t,x)|^{p-1}|u(t,y)| \leq \frac{p-1}{p}|u(t,x)|^p + \frac{1}{p}|u(t,y)|^p.$$

Thus, using that $K$ has mass one, it follows:

$$\left|\int_{\mathbb{R}} |u|^{p-2}u(K*u)dx\right| \leq \int_{\mathbb{R}}\int_{\mathbb{R}} K(x-y)|u(t,x)|^{p-1}|u(t,y)|dydx \leq \|u\|_p^p.$$

Plugging this last estimate in (2.5) we have

$$\frac{d}{dt}\left(\|u(t)\|_p^p\right) + \frac{4(p-1)}{p}\left\|\left(|u(t)|^{p/2}\right)_x\right\|_2^2 \leq 0. \tag{2.6}$$

Finally, with the same arguments as in [6] we obtain the desired estimate (2.4) for any $p \in [2,\infty)$, as well as for $p = \infty$. The case $p \in (1,2)$ follows by applying Hölder's inequality and (2.4) with $p = 1$ and $p = 2$. □

Similar estimates can be found for the derivative of the solution of (1.3). Let us define the re-scaled function $u_\lambda$, which will also be used in the following section to obtain the asymptotic profile. For $\lambda > 0$ we define

$$u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x). \tag{2.7}$$

The scales are the same as for the Burgers or heat equations. Clearly, $u_\lambda$ is the solution of the following equation:

$$\begin{cases} u_{\lambda,t} = u_\lambda\, u_{\lambda,x} + u_{\lambda,xx} + \lambda^2(K_\lambda * u_\lambda - u_\lambda) + \lambda u_{\lambda,x}, & (t,x) \in (0,\infty)\times\mathbb{R}, \\ u_\lambda(0,x) = u_{\lambda,0}(x) = \lambda u_0(\lambda x), & x \in \mathbb{R}, \end{cases} \tag{2.8}$$

where $K_\lambda(z) = \lambda K(\lambda z)$, $z \in \mathbb{R}$.

**Theorem 2.3.** *For each $p \in [1,\infty]$, there exists a constant $C = C(p) > 0$, such that the solution of equation (1.3) satisfies*

$$\|u_x(t)\|_p \leq C\|u_0\|_1 t^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}, \quad \forall t > 0. \tag{2.9}$$

*Proof.* First, let us denote by $D_\lambda^t$ the semigroup associated to the linear problem

$$\begin{cases} v_t = \lambda^2(K_\lambda * v - v) + \lambda v_x, & (t,x) \in (0,\infty)\times\mathbb{R}, \\ v(0,x) = v_0(x), & x \in \mathbb{R}. \end{cases}$$

It is immediate that $D_\lambda^t$ is stable in $L^1(\mathbb{R})$:

$$\frac{d}{dt}\int_{\mathbb{R}} |v|dx = \lambda^2 \int_{\mathbb{R}} (K_\lambda * v - v)\,\mathrm{sign}(v)dx \leq 0.$$

On the other hand, for all $\tau > 0$, function $u_\lambda$ solution of (2.8) verifies the following integral equation:

$$u_\lambda(t+\tau) = G(t) * D_\lambda^t u_\lambda(\tau) + \int_0^t G(t-s) * D_\lambda^s \left(\left(\frac{u_\lambda^2(s+\tau)}{2}\right)_x\right) ds,$$

where $G(t)$ is the heat kernel, given by

$$G(t,x)=(4\pi t)^{-1/2}e^{-\frac{x^2}{4t}}.$$

If we differentiate it with respect to $x$, we obtain:

$$u_{\lambda,x}(t+\tau)=G_x(t)*D^t_\lambda u_\lambda(\tau)+\int_0^t G_x(t-s)*D^s_\lambda\left(\left(\frac{u^2_\lambda(s+\tau)}{2}\right)_x\right)ds. \tag{2.10}$$

Now, let us first estimate the $L^1$ norm of $u_{\lambda,x}(t+\tau)$. Note that there exists a constant $C=C(p)$ such that

$$\|G_x(t)\|_p\le Ct^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}},\quad \forall t>0.$$

Moreover, by Theorem 2.2 we have

$$\|u_\lambda(\tau)\|_1=\|u(\lambda^2\tau)\|_1\le\|u_0\|_1$$

and

$$\|u_\lambda(s+\tau)\|_\infty=\lambda\|u(\lambda^2(s+\tau))\|_\infty\le C\tau^{-\frac{1}{2}}\|u_0\|_1.$$

Thus, it follows:

$$\begin{aligned}\|u_{\lambda,x}(t+\tau)\|_1&\le\|G_x(t)\|_1\|D^t_\lambda u_\lambda(\tau)\|_1+\int_0^t\|G_x(t-s)\|_1\left\|D^s_\lambda\left(\left(\frac{u^2_\lambda(s+\tau)}{2}\right)_x\right)\right\|_1ds\\&\le Ct^{-\frac{1}{2}}\|u_\lambda(\tau)\|_1+C\int_0^t(t-s)^{-\frac{1}{2}}\|(u^2_\lambda(s+\tau))_x\|_1ds\\&\le Ct^{-\frac{1}{2}}\|u_0\|_1+C\tau^{-1/2}\|u_0\|_1\int_0^t(t-s)^{-\frac{1}{2}}\|u_{\lambda,x}(s+\tau)\|_1ds.\end{aligned}$$

Applying Gronwall's Lemma, we deduce for $t=\tau$ that

$$\|u_{\lambda,x}(2\tau)\|_1\le C_\tau,\quad\forall\lambda>0, \tag{2.11}$$

for some constant $C_\tau>0$ that only depends on $\tau$ and $\|u_0\|_1$. This is equivalent to (2.9) for $p=1$.

The case $p\in(1,\infty)$ is an immediate consequence of (2.11):

$$\begin{aligned}\|u_{\lambda,x}(t+\tau)\|_p&\le\|G_x(t)\|_p\|D^t_\lambda u_\lambda(\tau)\|_1+\int_0^t\|G_x(t-s)\|_p\left\|D^s_\lambda\left(\left(\frac{u^2_\lambda(s+\tau)}{2}\right)_x\right)\right\|_1ds\\&\le Ct^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}\|u_\lambda(\tau)\|_1+C\int_0^t(t-s)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}\|(u^2_\lambda(s+\tau))_x\|_1ds\\&\le Ct^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}\|u_0\|_1+C_\tau\int_0^t(t-s)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}\|u_{\lambda,x}(s+\tau)\|_1ds\\&\le Ct^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}\|u_0\|_1+C_\tau\int_0^t(t-s)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}ds\end{aligned}$$

Taking $t=\tau$, we conclude that

$$\|u_{\lambda,x}(2\tau)\|_p\le C_\tau,\quad\forall\lambda>0,$$

which is equivalent to (2.9) for $p\in(1,\infty)$.

Finally, we repeat a similar argument for the case $p=\infty$:

$$\begin{aligned}\|u_{\lambda,x}(t+\tau)\|_\infty&\le\|G_x(t)\|_\infty\|D^t_\lambda u_\lambda(\tau)\|_1+\int_0^t\|G_x(t-s)\|_q\left\|D^s_\lambda\left(\left(\frac{u^2_\lambda(s+\tau)}{2}\right)_x\right)\right\|_{q'}ds\\&\le Ct^{-1}\|u_\lambda(\tau)\|_1+C\int_0^t(t-s)^{-\frac{1}{2}(1-\frac{1}{q})-\frac{1}{2}}\|(u^2_\lambda(s+\tau))_x\|_{q'}ds\\&\le Ct^{-1}\|u_0\|_1+C_\tau\int_0^t(t-s)^{-\frac{1}{2}(1-\frac{1}{q})-\frac{1}{2}}\|u_{\lambda,x}(s+\tau)\|_{q'}ds\\&\le Ct^{-1}\|u_0\|_1+C_\tau\int_0^t(t-s)^{-\frac{1}{2}(1-\frac{1}{q})-\frac{1}{2}}ds\end{aligned}$$

where $q\in(1,\infty)$ and $1/q+1/q'=1$. It is now enough to take $t=\tau$ to conclude the proof. □

2.3. **Large-time behavior.** The decay rates of the previous section will allow us to obtain the asymptotic profile of solutions for (1.3). The aim is to compute the limit $\lambda \to \infty$ in (2.8), which is equivalent to taking the limit $t \to \infty$ in (1.3).

Let us first observe that the estimates in Theorem 2.2 and Theorem 2.3 are also valid for $u_\lambda$ defined in (2.7). The mass is conserved too. We state this in the following lemma.

**Lemma 2.1.** *For each $p \in [1, \infty]$, there exists a constant $C = C(p) > 0$ such that, for all $\lambda > 0$, the solution of* (2.8) *satisfies*

$$\|u_\lambda(t)\|_p \le C\|u_0\|_1 t^{-\frac{1}{2}(1-\frac{1}{p})} \qquad \text{and} \qquad \|u_{\lambda,x}(t)\|_p \le C\|u_0\|_1 t^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}, \quad \forall t > 0.$$

*Moreover, the mass of $u_\lambda$ is conserved in time.*

*Proof.* We just have to use the definition of $u_\lambda$ in (2.7) and apply Theorem 2.2. For all $t > 0$ and $\lambda > 0$ we have

$$\|u_\lambda(t)\|_p = \lambda^{1-\frac{1}{p}}\|u(\lambda^2 t)\|_p \le Ct^{-\frac{1}{2}(1-\frac{1}{p})}.$$

Same procedure applies to $u_{\lambda,x}$, concerning Theorem 2.3. Regarding the last result, it is easy to see that:

$$\int_{\mathbb{R}} u_\lambda(t, x)dx = \int_{\mathbb{R}} u(\lambda^2 t, x)dx = \int_{\mathbb{R}} u_0(x)dx,$$

which proves the mass conservation. □

In particular, this lemma implies that, for any finite time interval $[\tau, T]$ with $0 < \tau < T < \infty$, the set $\{u_\lambda\}_{\lambda>0}$ is uniformly bounded in $L^\infty([\tau, T], L^p(\mathbb{R}))$.

2.3.1. *Compactness of the family $\{u_\lambda\}_{\lambda>0}$.* As we said at the beginning, we would like to pass to the limit $\lambda \to \infty$. We need the following theorem due to J. Simon ([17]), as an extension of the Aubin-Lions Lemma, to assure the compactness of the set $\{u_\lambda\}_{\lambda>0}$.

**Theorem 2.4** ([17, Theorem 5])**.** *Let $X$, $Z$ and $Y$ be Banach spaces satisfying $X \subset Z \subset Y$ with compact embedding $X \subset Z$. Assume, for $p \in [1, \infty]$ and $T > 0$, that $F$ is bounded in $L^p(0, T; X)$ and $\{\partial_t f : f \in F\}$ is bounded in $L^p(0, T; Y)$. Then, $F$ is relatively compact in $L^p(0, T; Z)$ and, in the case of $p = \infty$, also in $\mathcal{C}(0, T; Z)$.*

Applying this result we can prove the following theorem regarding the relative compactness of the set $\{u_\lambda\}_{\lambda>0}$. In the sequel, for any functions $f$ and $g$, we denote $f \lesssim g$ if there exists a constant $C > 0$, not depending on the scaling parameter nor the time, such that $f \le Cg$.

**Theorem 2.5.** *For every $0 < \tau < T < \infty$, the set $\{u_\lambda\}_{\lambda>0} \subset C([\tau, T], L^1(\mathbb{R}))$ is relatively compact.*

*Proof. Step 1.* First, for any $r > 0$ we will show the relative compactness in $C([\tau, T], L^2(I))$, with $I = [-r, r]$. Let us consider the spaces $X = H^1(I)$, $Z = L^2(I)$ and $Y = H^{-1}(I)$. We would like to apply Theorem 2.4 to the set $F = \{u_\lambda\}_{\lambda>0}$.

From Lemma 2.1 we know that $\{u_\lambda\}_{\lambda>0}$ and $\{u_{\lambda,x}\}_{\lambda>0}$ are bounded in $L^\infty([\tau, T], L^2(I))$. In particular, the first condition on $F$ is fulfilled. Therefore, it suffices to check that $u_{\lambda,t}$ is bounded in $L^\infty([\tau, T], H^{-1}(I))$. Using (2.8), for every $\varphi \in \mathcal{C}_c^\infty(I)$, we have:

$$\begin{aligned}
(2.12) \qquad \left|\int_{\mathbb{R}} u_{\lambda,t}\varphi dx\right| &\le \left|\int_{\mathbb{R}} u_\lambda\, u_{\lambda,x}\varphi dx\right| + \left|\int_{\mathbb{R}} u_{\lambda,xx}\varphi dx\right| + \left|\int_{\mathbb{R}} \big(\lambda^2(K_\lambda * u_\lambda - u_\lambda) + \lambda u_{\lambda,x}\big)\varphi dx\right| \\
&\lesssim \|\varphi_x\|_2\|u_\lambda\|_4^2 + \|\varphi_x\|_2\|u_{\lambda,x}\|_2 + \left|\int_{\mathbb{R}} \big(\lambda^2(K_\lambda * u_\lambda - u_\lambda) + \lambda u_{\lambda,x}\big)\varphi dx\right|.
\end{aligned}$$

Obviously, the first and second terms on the right hand side of (2.12) are uniformly bounded in $[\tau, T]$, so let us focus on the third one:

$$\mathcal{I}_\lambda = \left|\int_{\mathbb{R}} \big(\lambda^2(K_\lambda * u_\lambda - u_\lambda) + \lambda u_{\lambda,x}\big)\varphi dx\right| = \left|\int_{\mathbb{R}} \Big(\lambda^2\big(\widehat{K}(\xi/\lambda) - 1\big) + i\lambda\xi\Big)\widehat{u_\lambda}(\xi)\widehat{\varphi}(\xi)d\xi\right|.$$

Let us denote

$$m_\lambda(\xi) = \lambda^2\Big(\widehat{K}(\frac{\xi}{\lambda}) - 1\Big) + i\lambda\xi.$$

We claim that

$$|m_\lambda(\xi)| \le \xi^2, \quad \forall \xi \in \mathbb{R}, \forall \lambda > 0. \tag{2.13}$$

Using the Cauchy-Schwartz inequality, we have:

$$\mathcal{I}_\lambda = \left| \int_{\mathbb{R}} m_\lambda(\xi) \widehat{u_\lambda}(\xi) \widehat{\varphi}(\xi) d\xi \right| \lesssim \|\varphi\|_{H^1(\mathbb{R})} \|u_\lambda\|_{H^1(\mathbb{R})}. \tag{2.14}$$

Hence, going back to (2.12) and replacing (2.14), we obtain

$$\left| \int_{\mathbb{R}} u_{\lambda,t} \varphi dx \right| \lesssim \|\varphi\|_{H^1(\mathbb{R})} \left( \|u_\lambda\|_4^2 + \|u_\lambda\|_{H^1(\mathbb{R})} \right).$$

By Lemma 2.1, all the quantities in the right-hand side are uniformly bounded in $[\tau, T]$. Consequently, the set $\{u_\lambda\}_{\lambda>0}$ is relatively compact in $C([\tau,T], L^2(I))$.

It remains to prove claim (2.13). Observe that

$$\widehat{K}(\xi) = \frac{1 - i\xi}{1+\xi^2}, \quad \xi \in \mathbb{R}, \tag{2.15}$$

and, therefore,

$$|m_\lambda(\xi)| = \left| \lambda^2 \left( \frac{1 - i\xi/\lambda}{1 + (\xi/\lambda)^2} - 1 \right) + i\lambda\xi \right| = \frac{\lambda \xi^2}{\sqrt{\lambda^2 + \xi^2}} \le \xi^2, \quad \forall \lambda > 0.$$

*Step 2.* The next step consists in proving the compactness in $C([\tau,T], L^1(I))$. Since $L^2(I)$ is continuously embedded in $L^1(I)$, the compactness in $C([\tau,T], L^2(I))$ is clearly transferred to $C([\tau,T], L^1(I))$.

*Step 3.* Now we need to extend the result to $C([\tau,T], L^1(\mathbb{R}))$. We do that by proving uniform, with respect to $\lambda$, estimates on the tails of $u_\lambda$.

For every $r > 0$, let us define function $\psi_r(z) = \psi(z/r)$, where $\psi$ is a nonnegative $C^\infty(\mathbb{R})$ function such that

$$\psi(z) = \begin{cases} 0, & |z| < 1, \\ 1, & |z| > 2. \end{cases} \tag{2.16}$$

Since $\{u_\lambda\}_{\lambda>0}$ is relatively compact in $C([\tau,T], L^1(I))$, it suffices to show that

$$\sup_{t \in [\tau,T]} \|u_\lambda(t)\psi_r\|_1 \longrightarrow 0 \quad \text{as } r \to \infty, \text{ uniformly for } \lambda > 0. \tag{2.17}$$

We first observe that it is enough to consider nonnegative initial data. Indeed, the same argument as in Theorem 2.1 shows that for any $u_0, v_0 \in L^1(\mathbb{R})$ the corresponding solutions $u^\lambda, v^\lambda$ to (2.8) satisfy $\|u_\lambda - v_\lambda\|_1 \le \|u_0 - v_0\|_1$. As a consequence, due to Lemma 2.1 and Crandall-Tartar Lemma (see, for instance, [8, Chapter II]), we know that $u \le v$ if $u_0 \le v_0$. Thus, choosing $v_0 = |u_0|$ and $w_0 = -|u_0|$ as initial data implies that $|u_\lambda(t,x)| \le |v_\lambda(t,x)| + |w_\lambda(t,x)|$, where $u_\lambda$, $v_\lambda$ and $w_\lambda$ are the solutions corresponding to $u_0$, $v_0$ and $w_0$ respectively. In conclusion, it is sufficient to show (2.17) for nonnegative initial data and solutions.

Let us assume that $u_\lambda$ is a nonnegative solution. We multiply (2.8) by $\psi_r$ and integrate it over $(0,t) \times \mathbb{R}$. We obtain:

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} u_{\lambda,s} \psi_r dx ds = &-\frac{1}{2} \int_0^t \int_{\mathbb{R}} u_\lambda^2 \psi_r' dx ds + \int_0^t \int_{\mathbb{R}} u_\lambda \psi_r'' dx ds \\ &+ \int_0^t \int_{\mathbb{R}} \Big( \lambda^2 (K_\lambda * u_\lambda - u_\lambda) + \lambda u_{\lambda,x} \Big) \psi_r dx ds. \end{aligned}$$

and, therefore,

$$\begin{aligned} \int_{\mathbb{R}} u_\lambda(t) \psi_r dx \le &\int_{\mathbb{R}} u_{\lambda,0} \psi_r dx + \frac{\|\psi'\|_\infty}{2r} \int_0^t \|u_\lambda(s)\|_2^2 ds + \frac{\|\psi''\|_\infty}{r^2} \int_0^t \|u_\lambda(s)\|_1 ds \\ &+ \int_0^t \int_{\mathbb{R}} \Big( \lambda^2 \big(K_\lambda * u_\lambda(s) - u_\lambda(s)\big) + \lambda u_{\lambda,x}(s) \Big) \psi_r dx ds. \end{aligned} \tag{2.18}$$

We have to obtain an estimate on the last term in the integral, uniformly on $\lambda$. Let us denote

$$I = \int_{\mathbb{R}} \Big( \lambda^2 \big(K_\lambda * u_\lambda(s) - u_\lambda(s)\big) + \lambda u_{\lambda,x}(s) \Big) \psi_r dx.$$

A change of variables and integration by parts give us that

(2.19)

$$
\begin{aligned}
I &= \lambda^2 \int_{\mathbb{R}} \int_{\mathbb{R}} K(x-y) u(\lambda^2 s, y) \psi_{\lambda r}(x) dy dx - \lambda^2 \int_{\mathbb{R}} u(\lambda^2 s, x) \psi_{\lambda r}(x) dx - \lambda^2 \int_{\mathbb{R}} u_\lambda(\lambda^2 s, x) \psi'_{\lambda r}(x) dx \\
&= \lambda^2 \int_{\mathbb{R}} u(\lambda^2 s, y) \Big( \int_{\mathbb{R}} K(x-y) \big( \psi_{\lambda r}(x) - \psi_{\lambda r}(y) - (x-y)\psi'_{\lambda r}(y) \big) dx \Big) dy \\
&\leq \lambda^2 \|u_0\|_1 \|\psi''_{\lambda r}\|_\infty \leq \frac{\|u_0\|_1 \|\psi''\|_\infty}{r^2}
\end{aligned}
$$

**Remark 2.** *Note that the first moment of $K$ plays an important role here. The fact that*

$$
\int_{\mathbb{R}} K(z) dz = \int_{\mathbb{R}} z K(z) dz = 1
$$

*is critical to be able to find a bound for $I$ and, hence, to show that $K_\lambda * u_{\lambda,xx} \to u_{xx}$ as $t \to \infty$. This is also much related with the decomposition of $K$ in Dirac delta functions as in* [5]. *Moreover, taking into account this identity at the numerical level will be essential to preserve the large-time behavior correctly. As a matter of fact, the corrector factors $F_0^\Delta$ and $F_1^\Delta$ are related to this observation.*

So, plugging (2.19) into (2.18) and using Theorem 2.2, we get:

$$
\int_{\mathbb{R}} |u_\lambda(t)| \, \psi_r dx \leq \int_{\mathbb{R}} |u_0| \, \psi_{\lambda r} dx + C \left( \frac{\sqrt{t}}{r} + \frac{t}{r^2} \right)
$$

where $C > 0$ depends only on $\|u_0\|_1$ and $\|\psi\|_{W^{2,\infty}(\mathbb{R})}$, which are both bounded. For $\lambda > 1$, since $\psi_r(x) > \psi_{\lambda r}(x)$, we get

$$
\int_{\mathbb{R}} |u_\lambda(t,x)| \, \psi_r(x) dx \leq \int_{\mathbb{R}} |u_0(x)| \, \psi_r(x) dx + C \left( \frac{\sqrt{t}}{r} + \frac{t}{r^2} \right),
$$

which tends to zero uniformly on $\lambda$ when $r \to \infty$. Therefore, we proved (2.17) and, consequently, we can assure that $\{u_\lambda\}_{\lambda > 0}$ is relatively compact in $C([\tau, T], L^1(\mathbb{R}))$. □

Modifying slightly the previous proof, we can also conclude the following lemma, regarding the initial condition $u_{\lambda,0}$.

**Lemma 2.2.** *For every test function $\varphi \in C_c^\infty(\mathbb{R})$, there exists a constant $C = C(\varphi, u_0) > 0$, such that*

$$
\left| \int_{\mathbb{R}} u_\lambda(t,x) \varphi(x) dx - \int_{\mathbb{R}} u_{\lambda,0}(x) \varphi(x) dx \right| \leq C(t + \sqrt{t}), \quad \forall t > 0,
$$

*holds uniformly on $\lambda > 0$.*

*Proof.* We multiply (2.8) by $\varphi \in C_c^\infty(\mathbb{R})$ and integrate it over $(0,t) \times \mathbb{R}$. We get:

$$
\int_0^t \int_{\mathbb{R}} u_{\lambda,t} \varphi = \int_0^t \int_{\mathbb{R}} u_\lambda u_{\lambda,x} \varphi + \int_0^t \int_{\mathbb{R}} u_{\lambda,xx} \varphi + \int_0^t \int_{\mathbb{R}} \big( \lambda^2 (K_\lambda * u_\lambda - u_\lambda) + \lambda u_{\lambda,x} \big) \varphi.
$$

Integrating by parts and making use of Lemma 2.1, we have

$$
\begin{aligned}
\left| \int_{\mathbb{R}} u_\lambda(t) \varphi dx - \int_{\mathbb{R}} u_{\lambda,0} \varphi dx \right| \leq \frac{\|\varphi'\|_\infty}{2} \int_0^t \|u_\lambda(s)\|_2^2 ds + \|\varphi''\|_\infty \int_0^t \|u_\lambda(s)\|_1 ds \\
+ \left| \int_0^t \int_{\mathbb{R}} \Big( \lambda^2 \big( K_\lambda * u_\lambda(s) - u_\lambda(s) \big) + \lambda u_{\lambda,x}(s) \Big) \varphi dx ds \right|.
\end{aligned}
$$

To conclude the proof, it is enough to apply a similar argument as for (2.18) to get:

$$
\left| \int_{\mathbb{R}} u_\lambda(t) \varphi dx - \int_{\mathbb{R}} u_{\lambda,0} \varphi dx \right| = \left| \int_{\mathbb{R}} u(\lambda^2 t, x) \varphi(\frac{x}{\lambda}) dx - \int_{\mathbb{R}} u_0(x) \varphi(\frac{x}{\lambda}) dx \right| \leq C(\|\varphi\|_{W^{2,\infty}(\mathbb{R})}, \|u_0\|_1)(\sqrt{t} + t).
$$

□

2.3.2. *Passing to the limit.* Now we have all the ingredients that we need to prove our main result on the large-time behavior of solutions to problem (1.3), stated in Theorem 1.1.

*Proof of Theorem 1.1.* By Theorem 2.5, we know that for every $0<\tau<T<\infty$, the family $\{u_\lambda\}_{\lambda>0}$ is relatively compact in $C([\tau,T],L^1(\mathbb{R}))$. Consequently, there exists a subsequence of it (which we will not relabel) and a function $\bar{u}\in C((0,\infty),L^1(\mathbb{R}))$ such that

$$u_\lambda \longrightarrow \bar{u}\in C([\tau,T],L^1(\mathbb{R})), \quad \text{as } \lambda\to\infty. \tag{2.20}$$

We can also assume that $u_\lambda(t,x)\to\bar{u}(t,x)$ almost everywhere in $(0,\infty)\times\mathbb{R}$ as $\lambda\to\infty$.

Our claim is that, passing to the limit $\lambda\to\infty$, we obtain that $\bar{u}$ is a weak solution of the equation:

$$\begin{cases}\bar{u}_t=\bar{u}\bar{u}_x+2\bar{u}_{xx}, & (t,x)\in(0,\infty)\times\mathbb{R},\\ \bar{u}(0)=M\delta_0.\end{cases} \tag{2.21}$$

Let us multiply equation (2.8) by a test function $\phi\in C_c^\infty((0,\infty)\times\mathbb{R})$ and integrate it over $(0,\infty)\times\mathbb{R}$. We have:

$$\int_0^\infty\int_{\mathbb{R}} u_{\lambda,t}\phi=\int_0^\infty\int_{\mathbb{R}} u_\lambda\ u_{\lambda,x}\phi+\int_0^\infty\int_{\mathbb{R}} u_{\lambda,xx}\phi+\int_0^\infty\int_{\mathbb{R}}(\lambda^2(K_\lambda*u_\lambda-u_\lambda)+\lambda u_{\lambda,x})\phi.$$

Using the properties of $\{u_\lambda\}_{\lambda>0}$ shown in the previous section, it is sufficient to check that

$$\lim_{\lambda\to\infty}\int_0^\infty\int_{\mathbb{R}}\Big(\lambda^2\big(K_\lambda*u_\lambda(t)-u_\lambda(t)\big)+\lambda u_{\lambda,x}(t)\Big)\phi(t)dxdt=\int_0^\infty\int_{\mathbb{R}}\bar{u}(t)\phi_{xx}(t)dxdt.$$

Let us focus on the integral over the spatial domain. Taking into account the definition of $K_\lambda$ and that $\int_{\mathbb{R}} z^mK(z)dz=m!$ for $m\in\mathbb{N}\cup\{0\}$:

$$\begin{aligned}\mathcal{L}_\lambda(t)&=\int_{\mathbb{R}}\Big(\lambda^2\big(K_\lambda*u_\lambda(t)-u_\lambda(t)\big)+\lambda u_{\lambda,x}(t)\Big)\phi(t)dx\\ &=\lambda^2\int_{\mathbb{R}}\int_{\mathbb{R}}\Big(\phi(t,x+\frac{y}{\lambda})-\phi(t,x)\Big)K(y)u_\lambda(t,x)dydx-\lambda\int_{\mathbb{R}}\int_{\mathbb{R}}\phi_x(t,x)yK(y)u_\lambda(t,x)dydx,\end{aligned} \tag{2.22}$$

Now, because of Taylor's Theorem, we know that there exists a point $\zeta\in(x,x+y/\lambda)$ such that

$$\phi(t,x+\frac{y}{\lambda})-\phi(t,x)=\frac{y}{\lambda}\phi_x(t,x)+\frac{1}{2}\frac{y^2}{\lambda^2}\phi_{xx}(t,x)+\frac{1}{6}\frac{y^3}{\lambda^3}\phi_{xxx}(t,\zeta).$$

We introduce this in (2.22):

$$\begin{aligned}\mathcal{L}_\lambda(t)&=\frac12\int_{\mathbb{R}}\phi_{xx}(t,x)u_\lambda(t,x)dx\int_{\mathbb{R}}y^2K(y)dy+O(\|\phi_{xxx}(t)\|_\infty)\frac{1}{6\lambda}\int_{\mathbb{R}}u_\lambda(t,x)dx\int_{\mathbb{R}}y^3K(y)dy\\ &=\int_{\mathbb{R}}\phi_{xx}(t,x)u_\lambda(t,x)dx+O(\|\phi_{xxx}(t)\|_\infty)\frac{1}{\lambda}\int_{\mathbb{R}}u_\lambda(t,x)dx\\ &=\int_{\mathbb{R}}\phi_{xx}(t,x)u_\lambda(t,x)dx+\lambda^{-1}O(\|\phi_{xxx}(t)\|_\infty).\end{aligned}$$

Since $u_\lambda(t)\to\bar{u}(t)$ in $C([\tau,T],L^1(\mathbb{R}))$, we obtain that

$$\lim_{\lambda\to\infty}\int_0^\infty\mathcal{L}_\lambda(t)dt=\int_0^\infty\int_{\mathbb{R}}\phi_{xx}(t,x)\bar{u}(t,x)dxdt.$$

It follows that $\bar{u}$ satisfies

$$-\int_0^\infty\int_{\mathbb{R}}\bar{u}\phi_t=-\frac12\int_0^\infty\int_{\mathbb{R}}\bar{u}^2\phi_x+2\int_0^\infty\int_{\mathbb{R}}\bar{u}\phi_{xx}.$$

It remains to identify the behavior of $\bar{u}$ as $t\to0$. From Lemma 2.2, for any $\varphi\in C_c^\infty(\mathbb{R})$ we have

$$\left|\int_{\mathbb{R}}u_\lambda(t,x)\varphi(x)dx-\int_{\mathbb{R}}u_{\lambda,0}(x)\varphi(x)dx\right|\le C(t+\sqrt{t})$$

and, due to (2.20), we deduce

$$\left|\int_{\mathbb{R}}\bar{u}(t,x)\varphi(x)dx-M\varphi(0)\right|\le C(t+\sqrt{t})$$

by letting $\lambda\to\infty$. Passing to the limit $t\to0$ and using classical approximation arguments, we conclude that $\bar{u}(0)=M\delta_0$ in the sense of bounded measures.

Therefore, we can finally conclude that $\bar{u}$ is the unique solution $u_M$ of (2.21), and that, indeed, the whole family $\{u_\lambda\}_{\lambda>0}$ converges to $u_M$ in $C((0,\infty), L^1(\mathbb{R}))$. In particular, we have:

$$\lim_{\lambda\to\infty} \|u_\lambda(1) - u_M(1)\|_1 = 0.$$

Setting $\lambda = \sqrt{t}$ and using the self-similar form of $u_M$ (see e.g. [6]), we obtain that

$$\lim_{t\to\infty} \|u(t) - u_M(t)\|_1 = 0. \tag{2.23}$$

Finally, the convergence in the $L^p$-norms for $p \in (1,\infty)$ follows from (2.23), the decay estimate given in Lemma 2.1 for $p = \infty$ and the Hölder inequality. In fact, we have:

$$\|u(t) - u_M(t)\|_p \leq (\|u(t)\|_\infty + \|u_M(t)\|_\infty)^{1-\frac{1}{p}} \|u(t) - u_M(t)\|_1^{\frac{1}{p}} \leq o(t^{-\frac{1}{2}(1-\frac{1}{p})}). \tag{2.24}$$

In the case of the $L^\infty$-norm, we use the decay of $u_x(t)$ given by Theorem 2.3 and the estimate $\|u_{M,x}(t)\|_2 \lesssim t^{-\frac{3}{4}}$, resulting from the explicit formula (1.4). Using the Gagliardo-Nirenberg-Sobolev inequality and (2.24), we obtain:

$$\|u(t) - u_M(t)\|_\infty \lesssim (\|u_x(t)\|_2 + \|u_{M,x}(t)\|_2)^{\frac{1}{2}} \|u(t) - u_M(t)\|_2^{\frac{1}{2}} \leq o(t^{-\frac{1}{2}}). \tag{2.25}$$

The proof is now finished. □

## 3. Semidiscrete scheme

In this section, we focus on the semi-discrete numerical scheme for equation (1.3), defined in (1.6). In order to prove Theorem 1.2, we need some preliminary results on the decay of $u_\Delta$ similar to those obtained in Section 2 for the solution of equation (1.3). For simplicity, for every $h > 0$, we define the operators $d_h^+$ and $d_h^-$ as follows:

$$d_h^+ f(x) := \frac{f(x+h) - f(x)}{h} \quad \text{and} \quad d_h^- f(x) := \frac{f(x) - f(x-h)}{h}.$$

As in the continuous case, for $\mu > 0$ we also introduce the family of rescaled solutions

$$u^\mu(t,x) = \mu u_\Delta(\mu^2 t, \mu x), \quad t \geq 0, x \in \mathbb{R}, \tag{3.1}$$

and analyze the behavior of $u^\mu$ when $\mu \to \infty$. Note that function $u^\mu$ is piecewise constant on space intervals of length $\Delta x/\mu$. Moreover, it satisfies the following system:

$$\begin{cases} u_t^\mu(t,x) = \dfrac{1}{4}\Big(d_{\Delta x/\mu}^+\big(u^\mu(t,x)^2\big) + d_{\Delta x/\mu}^-\big(u^\mu(t,x)^2\big)\Big) \\ \qquad + \Delta x\, d_{\Delta x/\mu}^+ R\big(u^\mu(t, x - \frac{\Delta x}{\mu}), u^\mu(t,x)\big) + d_{\Delta x/\mu}^- \Big(d_{\Delta x/\mu}^+ u^\mu(t,x)\Big) \\ \qquad + \mu^2 \displaystyle\sum_{m=1}^N \omega_m u^\mu(t, x - m\tfrac{\Delta x}{\mu}) - \mu^2 F_0^\Delta u^\mu(t,x) + \mu F_1^\Delta d_{\Delta x/\mu}^+ u^\mu(t,x), & t > 0, \text{ a.e. } x \in \mathbb{R}, \\ u^\mu(0,x) = \mu u_\Delta^0(\mu x), & \text{a.e. } x \in \mathbb{R}, \end{cases} \tag{3.2}$$

where

$$R(u,v) = \frac{1}{4\Delta x}(v|v| - u|u|). \tag{3.3}$$

Of course, the approximated solution $u_\Delta$ defined in (1.5) satisfies (3.2) for $\mu = 1$.

### 3.1. $L^1$-$L^p$ estimates.

We are interested in the large-time behavior of $u_\Delta$. The following two propositions are the discrete versions of Theorems 2.2 and 2.3. The way of proceeding will be, indeed, very similar.

**Proposition 3.1.** *For all $p \in [1,\infty]$ there exists a constant $C = C(p) > 0$ such that:*

$$\|u^\mu(t)\|_p \leq C\|u_\Delta^0\|_1 t^{-\frac{1}{2}(1-\frac{1}{p})}, \quad \forall t > 0. \tag{3.4}$$

*for all solutions of (3.2) with initial data $u_\Delta^0 \in L^1(\mathbb{R})$.*

*Proof.* Let us consider first the case $\mu = 1$ and $p \in [2, \infty)$. We multiply (3.2) by $|u_\Delta|^{p-2} u_\Delta$ and integrate it over the whole space domain. We have:

$$\frac{1}{p}\frac{d}{dt}\|u_\Delta(t)\|_p^p \le I_1 + \int_{\mathbb{R}} d^-_{\Delta x}\left(d^+_{\Delta x} u_\Delta(x)\right)|u_\Delta(x)|^{p-2}u_\Delta(x)dx + I_2, \tag{3.5}$$

where

$$I_1 = \frac{1}{4}\int_{\mathbb{R}} \Big(d^+_{\Delta x}\big(u_\Delta(t,x)^2\big) + d^-_{\Delta x}\big(u_\Delta(t,x)^2\big)\Big)|u_\Delta(x)|^{p-2}u_\Delta(x)dx$$
$$+ \Delta x \int_{\mathbb{R}} d^+_{\Delta x} R\big(u_\Delta(t, x-\Delta x), u_\Delta(t,x)\big)|u_\Delta(x)|^{p-2}u_\Delta(x)dx$$

and

$$\begin{aligned} I_2 &= \int_{\mathbb{R}} \left(\sum_{m=1}^N \omega_m u_\Delta(t, x - m\Delta x) - F_0^\Delta u_\Delta(t,x) + F_1^\Delta d^+_{\Delta x} u_\Delta(t,x)\right)|u_\Delta(x)|^{p-2}u_\Delta(x)dx \\ &= \sum_{m=1}^N \omega_m \left(\int_{\mathbb{R}} u_\Delta(x - m\Delta x)|u_\Delta(x)|^{p-2}u_\Delta(x)dx - \int_{\mathbb{R}} |u_\Delta(x)|^p dx\right) \\ &\quad + \frac{F_1^\Delta}{\Delta x}\left(\int_{\mathbb{R}} u_\Delta(x+\Delta x)|u_\Delta(x)|^{p-2}u_\Delta(x)dx - \int_{\mathbb{R}} |u_\Delta(x)|^p dx\right). \end{aligned}$$

On the one hand, for any $k \in \mathbb{Z}$, we know that

$$\int_{\mathbb{R}} u_\Delta(x + k\Delta x)|u_\Delta(x)|^{p-2}u_\Delta(x)dx \le \frac{p-1}{p}\int_{\mathbb{R}} |u_\Delta(x+k\Delta x)|^p dx + \frac{1}{p}\int_{\mathbb{R}}|u_\Delta(x)|^p dx = \int_{\mathbb{R}} |u_\Delta(x)|^p dx.$$

Therefore, $I_2 \le 0$.

On the other hand, for $i \in \{-1, 0, 1\}$ let us denote $U_i^\pm = \{x \in \mathbb{R} : \pm u_\Delta(x + i\Delta x) > 0\}$ and $U_i^0 = \{x \in \mathbb{R} : u_\Delta(x + i\Delta x) = 0\}$. From the definition of $R$ in (3.3), reordering $I_1$ we get:

$$\begin{aligned} I_1 &= \frac{1}{4\Delta x}\int_{\mathbb{R}} \left(u^2_\Delta(x+\Delta x) + u_\Delta(x+\Delta x)|u_\Delta(x+\Delta x)|\right)|u_\Delta(x)|^{p-2}u_\Delta(x)dx - \frac{1}{2\Delta x}\int_{\mathbb{R}}|u_\Delta(x)|^{p+1}dx \\ &\quad + \frac{1}{4\Delta x}\int_{\mathbb{R}} \left(u_\Delta(x-\Delta x)|u_\Delta(x-\Delta x)| - u^2_\Delta(x-\Delta x)\right)|u_\Delta(x)|^{p-2}u_\Delta(x)dx \\ &\le \frac{1}{2\Delta x}\int_{U_0^+\cap U_1^+} u^2_\Delta(x+\Delta x)|u_\Delta(x)|^{p-1}dx - \frac{1}{2\Delta x}\int_{\mathbb{R}}|u_\Delta(x)|^{p+1}dx \\ &\quad + \frac{1}{2\Delta x}\int_{U_{-1}^-\cap U_0^-} u^2_\Delta(x-\Delta x)|u_\Delta(x)|^{p-1}dx. \end{aligned}$$

Using

$$a^2|b|^{p-1} \le \frac{2}{p+1}|a|^{p+1} + \frac{p-1}{p+1}|b|^{p+1}, \quad \forall a, b \in \mathbb{R},$$

we obtain that

$$\begin{aligned} I_1 &\le \frac{1}{2\Delta x}\left(\frac{2}{p+1}\int_{U_0^+\cap U_1^+}|u_\Delta(x+\Delta x)|^{p+1}dx + \frac{p-1}{p+1}\int_{U_0^+\cap U_1^+}|u_\Delta(x)|^{p+1}dx\right) - \frac{1}{2\Delta x}\int_{\mathbb{R}}|u_\Delta(x)|^{p+1}dx \\ &\quad + \frac{1}{2\Delta x}\left(\frac{2}{p+1}\int_{U_{-1}^-\cap U_0^-}|u_\Delta(x-\Delta x)|^{p+1}dx + \frac{p-1}{p+1}\int_{U_{-1}^-\cap U_0^-}|u_\Delta(x)|^{p+1}dx\right) \\ &\le \frac{1}{2\Delta x}\int_{U_0^+}|u_\Delta(x)|^{p+1}dx - \frac{1}{2\Delta x}\int_{\mathbb{R}}|u_\Delta(x)|^{p+1}dx + \frac{1}{2\Delta x}\int_{U_0^-}|u_\Delta(x)|^{p+1}dx \end{aligned}$$

and, hence, $I_1 \le 0$.

Thus, from (3.5) we deduce:

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\|u_\Delta(t)\|_p^p &\le \int_{\mathbb{R}} d^-_{\Delta x}\left(d^+_{\Delta x}u_\Delta(x)\right)|u_\Delta(x)|^{p-2}u_\Delta(x)dx \\ &= -\frac{1}{\Delta x^2}\int_{\mathbb{R}}\left(u_\Delta(x+\Delta x) - u_\Delta(x)\right)\left(|u_\Delta(x+\Delta x)|^{p-2}u_\Delta(x+\Delta x) - |u_\Delta(x)|^{p-2}u_\Delta(x)\right)dx. \end{aligned} \tag{3.6}$$

Moreover, the inequality

$$\left||x|^{p/2}-|y|^{p/2}\right|^2 \le \frac{p^2}{4(p-1)}(x-y)\left(|x|^{p-2}x-|y|^{p-2}y\right), \quad \forall x,y\in\mathbb{R},$$

guarantees that

$$\frac{d}{dt}\|u_\Delta(t)\|_p^p \le -\frac{4(p-1)}{p}\int_{\mathbb{R}}\left|\frac{|u_\Delta(x+\Delta x)|^{p/2}-|u_\Delta(x)|^{p/2}}{\Delta x}\right|^2 = -\frac{4(p-1)}{p}\|d^+_{\Delta x}\left(|u_\Delta|^{p/2}\right)\|_2^2 \le 0. \tag{3.7}$$

This estimate and Lemma A.1 allow us to write

$$\frac{d}{dt}\|u_\Delta(t)\|_p^p + \frac{(p-1)}{p}\frac{\|u_\Delta(t)\|_p^{p(p+1)/(p-1)}}{\|u^0_\Delta(t)\|_1^{2p/(p-1)}} \le 0. \tag{3.8}$$

Following the same arguments as in [6], we conclude that for any $p\in[2,\infty)$

$$\|u_\Delta(t)\|_p \le C(p)\|u^0_\Delta\|_1\, t^{-\frac12(1-\frac1p)}, \quad \forall t>0. \tag{3.9}$$

In the same way, the cases $p=\infty$ and $p\in(1,2)$ are proved too.

Finally, the general case $\mu>0$ is immediate from (3.9) and the definition of $u^\mu$ (3.1), since for any $p\in[1,\infty]$ we have

$$\|u^\mu(t)\|_p = \mu^{1-\frac1p}\|u_\Delta(\mu^2 t)\|_p \le C(p)\|u^0_\Delta\|_1\, t^{-\frac12(1-\frac1p)}.$$

The proof is now complete. ☐

Now that we have estimates on the $L^p$-norms of the solution, we need to obtain a similar result for the discrete gradient. We proceed as in Theorem 2.3.

**Proposition 3.2.** *For all $p\in[1,\infty]$ there exists a constant $C=C(p,\|u^0_\Delta\|_1)>0$ such that:*

$$\|d^+_{\Delta x}u^\mu(t)\|_p \le Ct^{-\frac12(1-\frac1p)-\frac12}, \quad \forall t>0, \tag{3.10}$$

*for all solutions of* (3.2) *with initial data $u^0_\Delta\in L^1(\mathbb{R})$.*

*Proof.* Let us denote by $D^t_\mu$ the semigroup associated to

$$\begin{cases} v_t(t,x) = \mu^2\sum_{m=1}^N \omega_m v(t,x-m\frac{\Delta x}{\mu}) - \mu^2 F_0^\Delta v(t,x) + \mu F_1^\Delta d^+_{\Delta x/\mu}v(t,x), & t>0, \text{ a.e. } x\in\mathbb{R},\\ v(0,x)=v_0(x), & \text{a.e. } x\in\mathbb{R}. \end{cases} \tag{3.11}$$

Multiplying (3.11) by $\operatorname{sign}(v(t,x))$, integrating on $\mathbb{R}$ and using that

$$\int_{\mathbb{R}} v(x-h)\operatorname{sign}(v(x))dx \le \int_{\mathbb{R}}|v(x)|dx, \quad \forall h\in\mathbb{R},$$

one shows that $D^t_\mu$ is stable in $L^1(\mathbb{R})$.

Now, for every $\tau>0$ and $\mu>0$, we know that the solution of (3.2) satisfies:

$$u^\mu(t+\tau) = G^\mu_\Delta(t)*D^t_\mu u^\mu(\tau) + \int_0^t G^\mu_\Delta(t-s)*D^s_\mu\Big(H\big(u^\mu(s+\tau)\big)\Big)ds, \tag{3.12}$$

where

$$H(u^\mu(s,x)) = \frac14\left(d^+_{\Delta x/\mu}\left(u^\mu(s,x)^2\right)+d^-_{\Delta x/\mu}\left(u^\mu(s,x)^2\right)\right) + \Delta x\, d^+_{\Delta x/\mu}R\big(u^\mu(s,x-\frac{\Delta x}{\mu}),u^\mu(s,x)\big)$$

and $G^\mu_\Delta$ is the fundamental solution of the one-dimensional semi-discrete heat equation, defined by

$$\left(G^\mu_\Delta(t)\right)_j = \frac{1}{2\pi}\int_{-\pi\mu/\Delta x}^{\pi\mu/\Delta x} e^{-\frac{4t\mu^2}{\Delta x^2}\sin^2\frac{\xi\Delta x}{2\mu}}e^{ij\xi\frac{\Delta x}{\mu}}d\xi, \quad j\in\mathbb{Z},$$

It is well known (e.g. [1]) that

$$\|G^\mu_\Delta(t)\|_p \le C(p)t^{-\frac12(1-\frac1p)}, \quad t>0,$$

and

$$\|d^+_{\Delta x/\mu}G^\mu_\Delta(t)\|_p \le C(p)t^{-\frac12(1-\frac1p)-\frac12}, \quad t>0.$$

Now let us apply the discrete operator $d^+_{\Delta x/\mu}$ to (3.12). Then

$$d^+_{\Delta x/\mu}u^\mu(t+\tau)=d^+_{\Delta x/\mu}G^\mu_\Delta(t)*D^t_\mu u^\mu(\tau)+\int_0^t d^+_{\Delta x/\mu}G^\mu_\Delta(t-s)*D^s_\mu\Big(H\big(u^\mu(s+\tau)\big)\Big)ds. \tag{3.13}$$

Using the decay properties of $G^\mu_\Delta$, Proposition 3.1 and the $L^1$-stability of $D^t_\mu$, we obtain

$$\begin{aligned}\|d^+_{\Delta x/\mu}u^\mu(t+\tau)\|_1 &\leq \Big\|d^+_{\Delta x/\mu}G^\mu_\Delta(t)\Big\|_1\,\big\|D^t_\mu u^\mu(\tau)\big\|_1\\
&\quad+\int_0^t\Big\|d^+_{\Delta x/\mu}G^\mu_\Delta(t-s)\Big\|_1\,\Big\|D^s_\mu\Big(H\big(u^\mu(s+\tau)\big)\Big)\Big\|_1 ds\\
&\leq \Big\|d^+_{\Delta x/\mu}G^\mu_\Delta(t)\Big\|_1\,\|u^\mu(\tau)\|_1+\int_0^t\Big\|d^+_{\Delta x/\mu}G^\mu_\Delta(t-s)\Big\|_1\,\big\|H\big(u^\mu(s+\tau)\big)\big\|_1 ds\\
&\leq C\,\|u_0\|_1\,t^{-\frac12}+C\int_0^t(t-s)^{-\frac12}\,\big\|H\big(u^\mu(s+\tau)\big)\big\|_1\,ds.\end{aligned} \tag{3.14}$$

We now prove that for any $p\in[1,\infty)$, we have

$$\|H(u^\mu(s+\tau))\|_p\leq C\|u^0_\Delta\|_1\tau^{-\frac12}\|d^+_{\Delta x/\mu}u^\mu(s+\tau)\|_p \tag{3.15}$$

Observe that, in view of Proposition 3.1, we have

$$\Big\|d^+_{\Delta x/\mu}\left(u^\mu(s+\tau)^2\right)\Big\|_p\leq 2\,\|u^\mu(s+\tau)\|_\infty\,\Big\|d^+_{\Delta x/\mu}u^\mu(s+\tau)\Big\|_p\leq C\|u^0_\Delta\|_1\tau^{-\frac12}\,\Big\|d^+_{\Delta x/\mu}u^\mu(s+\tau)\Big\|_p.$$

A similar result holds for $d^-_{\Delta x/\mu}$ since $\big\|d^-_h f\big\|_p=\big\|d^+_h f\big\|_p$ for all $f\in L^p(\mathbb{R})$ and $h>0$. Moreover, from the definition of $R$ in (3.3) we have:

$$\begin{aligned}\Delta x\Big\|d^+_{\Delta x/\mu}R\big(u^\mu(s+\tau,x-\frac{\Delta x}{\mu}),u^\mu(s+\tau,x)\big)\Big\|_p&\leq\frac12\,\Big\|d^+_{\Delta x/\mu}\left(u^\mu(s+\tau)|u^\mu(s+\tau)|\right)\Big\|_p\\
&\leq\|u^\mu(s+\tau)\|_\infty\,\Big\|d^+_{\Delta x/\mu}u^\mu(s+\tau)\Big\|_p\leq C\,\big\|u^0_\Delta\big\|_1\,\tau^{-\frac12}\,\Big\|d^+_{\Delta x/\mu}u^\mu(s+\tau)\Big\|_p,\end{aligned}$$

where we have used Proposition 3.1 and that

$$\big|x|x|-y|y|\big|\leq 2|x-y|\max\{|x|,|y|\},\quad \forall x,y\in\mathbb{R}.$$

Therefore, introducing in (3.14) the case $p=1$ of (3.15), we get

$$\|d^+_{\Delta x/\mu}u^\mu(t+\tau)\|_1\leq C\,\|u_0\|_1\,t^{-\frac12}+C\,\big\|u^0_\Delta\big\|_1\,\tau^{-\frac12}\int_0^t(t-s)^{-\frac12}\,\Big\|d^+_{\Delta x/\mu}u^\mu(s+\tau)\Big\|_1\,ds.$$

Applying Gronwall's Lemma and taking $t=\tau$, we conclude that

$$\|d^+_{\Delta x/\mu}(u^\mu(2\tau))\|_1\leq C_\tau,\quad \forall\mu>0, \tag{3.16}$$

for some $C_\tau>0$ depending only on $\tau$ and $\|u_0\|_1$. It is enough now to use the definition of $u^\mu$ in (3.1), taking $\tau=1/2$ and $\mu=\sqrt{t}$ to obtain

$$\|d^+_{\Delta x}(u_\Delta(t))\|_1\leq Ct^{-\frac12},\quad \forall t>0,$$

that is, (3.10) for $\mu=1$ and $p=1$.

The case $\mu=1$ and $p\in(1,\infty)$ is immediate from (3.13), (3.15) and (3.16). Indeed, we have

$$\begin{aligned}\|d^+_{\Delta x/\mu}u^\mu(t+\tau)\|_p&\leq\Big\|d^+_{\Delta x/\mu}G^\mu_\Delta(t)\Big\|_p\,\|u^\mu(\tau)\|_1+\int_0^t\Big\|d^+_{\Delta x/\mu}G^\mu_\Delta(t-s)\Big\|_p\,\big\|H\big(u^\mu(s+\tau)\big)\big\|_1 ds\\
&\leq C_\tau t^{-\frac12(1-\frac1p)-\frac12}+C_\tau\int_0^t(t-s)^{-\frac12(1-\frac1p)-\frac12}ds.\end{aligned}$$

with $C_\tau=C(p,\tau,\|u_0\|_1)$. Taking $t=\tau$ implies that

$$\|d^+_{\Delta x/\mu}u^\mu(2\tau)\|_p\leq C_\tau,\quad\forall\mu>0, \tag{3.17}$$

This is equivalent to (3.10) for $\mu=1$ and $p\in(1,\infty)$.

Furthermore, repeating similar arguments, the case $\mu = 1$ and $p = \infty$ follows from (3.13) and estimates (3.15) and (3.17):

$$\begin{aligned}\|d^+_{\Delta x/\mu} u^\mu(t+\tau)\|_\infty &\leq \left\|d^+_{\Delta x/\mu} G^\mu_\Delta(t)\right\|_\infty \|u^\mu(\tau)\|_1 + \int_0^t \left\|d^+_{\Delta x/\mu} G^\mu_\Delta(t-s)\right\|_q \left\|H\big(u^\mu(s+\tau)\big)\right\|_{q'} ds \\ &\leq C_\tau t^{-1} + C_\tau \int_0^t (t-s)^{-\frac{1}{2}(1-\frac{1}{q})-\frac{1}{2}} ds.\end{aligned}$$

where $q \in (1,\infty)$, $1/q + 1/q' = 1$ and $C_\tau = C(q, q', \tau, \|u_0\|_1)$. It is now enough to take $t = \tau$ to conclude that

$$\|d^+_{\Delta x/\mu} u^\mu(2\tau)\|_\infty \leq C_\tau, \quad \forall \mu > 0,$$

which is equivalent to (3.10) for $\mu = 1$ and $p = \infty$.

Finally, the general case $\mu > 0$ is immediate from the case $\mu = 1$ and the definition of $u^\mu$ (3.1), since for any $p \in [1, \infty]$ we have

$$\|d^+_{\Delta x/\mu} u^\mu(t)\|_p = \mu^{2-\frac{1}{p}} \|d^+_{\Delta x} u_\Delta(\mu^2 t)\|_p \leq C t^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}$$

This concludes the proof. □

To end this part, let us remark that the solution $u^\mu$ of system (3.2) conserves the mass of the initial data $u^0_\Delta$. In fact, note that it is the same as the mass of $u_0$, when $u^0_\Delta$ is defined as in (1.6). Moreover, we show that (3.2) defines a contractive semigroup. This will be useful to obtain the estimates for the compactness of $\{u^\mu\}_{\mu>0}$. For the shake of clarity, we prove this lemma in the Appendix.

**Lemma 3.1.** *For any initial data $u^0_\Delta \in L^1(\mathbb{R})$, the solution $u^\mu$ to* (3.2) *satisfies*

$$\int_{\mathbb{R}} u^\mu(t,x) = \int_{\mathbb{R}} u^0_\Delta(x), \quad \forall t > 0.$$

*Moreover,* (3.2) *defines a contractive semigroup in $L^1(\mathbb{R})$.*

3.2. **Compactness of the set $\{u^\mu\}_{\mu>0}$.** In this section, we prove the compactness of the trajectories of the family $\{u^\mu(t)\}_{\mu>0}$ introduced in the previous section, in order to pass to the limit $\mu \to \infty$. Unlike the continuous case, we do not have estimates of $u^\mu$ in $H^1(\mathbb{R})$, since it is piecewise constant. Nevertheless, the following lemma makes possible the use of the compact embedding of $H^s_{loc}(\mathbb{R})$ into $L^2_{loc}(\mathbb{R})$, with $0 < s < 1/2$. The proof will be given in the Appendix.

**Lemma 3.2.** *For any $0 < s < \frac{1}{2}$, there exists a positive constant $C = C(s)$ such that, for any mesh-size $0 < \Delta x < 1$, the following holds for all piecewise constant functions $w$ as in* (1.5)*:*

$$\|w\|_{H^s(\mathbb{R})} \leq C\left(\|w\|_2 + \|d^+_{\Delta x} w\|_2\right).$$

Let us remark that, as a consequence of this lemma and Proposition 3.1 and Proposition 3.2, we obtain a time-decay estimate for $u^\mu$ in $H^s(\mathbb{R})$:

$$\|u^\mu(t)\|_{H^s(\mathbb{R})} \leq C\left(\|u^\mu(t)\|_2 + \|d^+_{\Delta x/\mu} u^\mu(t)\|_2\right) \leq C\left(t^{-\frac{1}{4}} + t^{-\frac{3}{4}}\right), \quad \forall t > 0, \forall \mu > 0, \tag{3.18}$$

with $0 < s < 1/2$. Thus, we can use Theorem 2.4 to prove the compactness of the family $\{u^\mu\}_{\mu>0}$.

**Theorem 3.1.** *For every $0 < \tau < T < \infty$, the family $\{u^\mu\}_{\mu>0} \subset C([\tau, T], L^1(\mathbb{R}))$ is relatively compact.*

*Proof.* We will proceed in two steps, analogously to Theorem 2.5.

*Step 1.* First we will show the result locally in $C([\tau, T], L^1(I))$, with $I = [-r, r]$ for an arbitrary $r > 0$. Let us consider the spaces $X = H^s(I)$ with $s \in (0, \frac{1}{2})$, $Z = L^2(I)$ and $Y = H^{-1}(I)$.

From (3.18) we know that the set $\{u^\mu\}_{\mu>0}$ is bounded in $L^\infty([\tau, T], H^s_{loc}(\mathbb{R}))$. In particular, first condition of Theorem 2.4 is fulfilled. Thus, it suffices to check that $u^\mu_t$ is bounded in $L^\infty([\tau, T], H^{-1}(I))$. Let us multiply (3.2) by a function $\varphi \in \mathcal{C}^\infty_c(\mathbb{R})$ and integrate it over $\mathbb{R}$. Using the definition of $R$ in (3.3), we have:

$$\begin{aligned}\left|\int_{\mathbb{R}} u_t^\mu \varphi dx\right| &\leq \frac{1}{4}\left|\int_{\mathbb{R}} \Big(d^+_{\Delta x/\mu}\big(u^\mu(x)^2\big) + d^-_{\Delta x/\mu}\big(u^\mu(x)^2\big)\Big)\varphi(x)dx\right| \\ &\quad + \Delta x\left|\int_{\mathbb{R}} d^+_{\Delta x/\mu} R\big(u^\mu(x-\frac{\Delta x}{\mu}), u^\mu(x)\big)\varphi(x)dx\right| + \left|\int_{\mathbb{R}} d^-_{\Delta x/\mu}\big(d^+_{\Delta x/\mu}u^\mu(x)\big)\varphi(x)dx\right| \\ &\quad + \left|\int_{\mathbb{R}}\left(\mu^2\sum_{m=1}^N \omega_m u^\mu(x-m\frac{\Delta x}{\mu}) - \mu^2 F_0^\Delta u^\mu(x) + \mu F_1^\Delta d^+_{\Delta x/\mu}u^\mu(x)\right)\varphi(x)dx\right| \\ &\leq \frac{1}{2}\|d^+_{\Delta x}\varphi\|_2\|u^\mu\|_4^2 + \frac{1}{2}\|d^+_{\Delta x}\varphi\|_2\|u^\mu\|_4^2 + \|d^+_{\Delta x}\varphi\|_2\|d^+_{\Delta x}u^\mu\|_2 \\ &\quad + \left|\int_{\mathbb{R}}\left(\mu^2\sum_{m=1}^N \omega_m\big(u^\mu(x-m\frac{\Delta x}{\mu}) - u^\mu(x)\big) + \mu F_1^\Delta d^+_{\Delta x/\mu}u^\mu(x)\right)\varphi(x)dx\right|.\end{aligned}$$

Obviously, the first three terms on the right hand side of the inequality are uniformly bounded for $\mu > 0$, so let us focus on the last one. Using the Fourier transform and the definition of $F_0^\Delta$ in (1.8), we have

$$\begin{aligned}I_\mu &= \left|\int_{\mathbb{R}}\left(\mu^2\sum_{m=1}^N \omega_m\big(u^\mu(x-m\frac{\Delta x}{\mu}) - u^\mu(x)\big) + \mu F_1^\Delta d^+_{\Delta x/\mu}u^\mu(x)\right)\varphi(x)dx\right| \\ &\leq \mu^2\int_{\mathbb{R}}\left|\sum_{m=1}^N \omega_m\left(e^{-im\frac{\Delta x}{\mu}\xi} - 1\right) + F_1^\Delta\frac{e^{i\frac{\Delta x}{\mu}\xi} - 1}{\Delta x}\right| \left|\widehat{u^\mu}(\xi)\right| |\widehat{\varphi}(\xi)|\, d\xi.\end{aligned}$$

If we take $p = a^{-\Delta x}$ and $b = e^{-i\frac{\Delta x}{\mu}\xi}$ on Lemma A.2 and use the definitions of $\omega_m$ in (1.7) and $F_1^\Delta$ in (1.8), we have:

$$\begin{aligned}&\left|\sum_{m=1}^N \omega_m\left(e^{-im\frac{\Delta x}{\mu}\xi} - 1\right) + F_1^\Delta\frac{e^{i\frac{\Delta x}{\mu}\xi} - 1}{\Delta x}\right| \\ &\qquad = \left|e^{\Delta x} - 1\right|\left|\sum_{m=1}^N e^{-m\Delta x}\left(e^{-im\frac{\Delta x}{\mu}\xi} - 1\right) + \sum_{m=1}^N m e^{-m\Delta x}(e^{i\frac{\Delta x}{\mu}\xi} - 1)\right| \\ &\qquad \leq \left|e^{\Delta x} - 1\right|\left|e^{-i\frac{\Delta x}{\mu}\xi} - 1\right|^2\frac{e^{-\Delta x}}{(1-e^{-\Delta x})^3} \\ &\qquad = \left|e^{-i\frac{\Delta x}{\mu}\xi} - 1\right|^2\frac{1}{(1-e^{-\Delta x})^2}.\end{aligned} \tag{3.19}$$

Therefore, combining this with the Cauchy-Schwartz inequality, we obtain

$$I_\mu \leq \frac{\Delta x^2}{(1-e^{-\Delta x})^2}\|d^+_{\Delta x/\mu}u^\mu\|_2\|d^+_{\Delta x/\mu}\varphi\|_2.$$

Thus, using that $\|d^+_{\Delta x/\mu}\varphi\|_2 \leq \|\varphi'\|_2$, we get

$$\begin{aligned}\left|\int_{\mathbb{R}} u_t^\mu(t)\varphi dx\right| &\leq \|d^+_{\Delta x/\mu}\varphi\|_2\|u^\mu(t)\|_4^2 + \|d^+_{\Delta x/\mu}\varphi\|_2\|d^+_{\Delta x/\mu}u^\mu(t)\|_2 + \frac{\Delta x^2}{(1-e^{-\Delta x})^2}\|d^+_{\Delta x/\mu}u^\mu(t)\|_2\|d^+_{\Delta x/\mu}\varphi\|_2 \\ &\leq C\|\varphi\|_{H^1(\mathbb{R})}\Big(\|u^\mu(t)\|_4^2 + \|d^+_{\Delta x/\mu}u^\mu(t)\|_2\Big).\end{aligned}$$

for any $\varphi \in \mathcal{C}_c^\infty(I)$ and with $C > 0$ independent of $\mu$. In view of Propositions 3.1 and 3.2, both norms of $u^\mu$ in the right-hand side are uniformly bounded in $[\tau, T]$, so $u_t^\mu$ is uniformly bounded in $L^\infty([\tau,T], H^{-1}(I))$. We conclude that the family $\{u^\mu\}_{\mu>0}$ is relatively compact in $C([\tau,T], L^2_{loc}(\mathbb{R}))$. Finally, compactness in $L^2_{loc}(\mathbb{R})$ implies compactness in $L^1_{loc}(\mathbb{R})$, so $\{u^\mu\}_{\mu>0}$ is also relatively compact in $C([\tau,T], L^1_{loc}(\mathbb{R}))$.

*Step 2.* Now we need to extend the result globally. Let us consider again the same function $\psi_r$ defined in the third step of the proof of Theorem 2.5, such that $\psi_r(z) = \psi(z/r)$ with $\psi$ given by (2.16) and $r > 0$. Since we know that $\{u^\mu\}_{\mu>0}$ is relatively compact in $C([\tau,T], L^1_{loc}(\mathbb{R}))$, it suffices to show that

$$\sup_{[\tau,T]}\|u^\mu(t)\psi_r\|_1 \longrightarrow 0 \quad \text{as } r\to\infty,\ \text{uniformly on } \mu \geq 1. \tag{3.20}$$

Note that, because of Lemma 3.1 and Crandall-Tartar Lemma [8, Chapter II], a similar argument as in Theorem 2.5 shows that it is enough to prove (3.20) for nonnegative initial data and solutions.

Thus, we focus on nonnegative solutions. Let us multiply (3.2) by $\psi_r$ and integrate it over $(0,t)\times\mathbb{R}$. We obtain:

$$\begin{aligned}
(3.21)\qquad &\int_{\mathbb{R}} u^\mu(t,x)\psi_r(x)dx = \int_{\mathbb{R}} u_0^\mu(x)\psi_r(x)dx\\
&+\frac{1}{4}\int_0^t\int_{\mathbb{R}}\Big(d^+_{\Delta x/\mu}\big(u^\mu(s,x)^2\big)+d^-_{\Delta x/\mu}\big(u^\mu(s,x)^2\big)\Big)\psi_r(x)dxds\\
&+\Delta x\int_0^t\int_{\mathbb{R}} d^+_{\Delta x/\mu}\Big(R\big(u^\mu(s,x-\frac{\Delta x}{\mu}),u^\mu(s,x)\big)\Big)\psi_r(x)dxds\\
&+\int_0^t\int_{\mathbb{R}} d^-_{\Delta x/\mu}\Big(d^+_{\Delta x/\mu}\big(u^\mu(s,x)\big)\Big)\psi_r(x)dxds\\
&+\int_0^t\int_{\mathbb{R}}\left(\mu^2\sum_{m=1}^N\omega_m\big(u^\mu(s,x-m\frac{\Delta x}{\mu})-u^\mu(s,x)\big)+\mu F_1^\Delta d^+_{\Delta x/\mu}u^\mu(s,x)\right)\psi_r(x)dxds.
\end{aligned}$$

We pass now the discrete derivatives to $\psi_r$ and estimate the right-hand side using time-decay estimates from Proposition 3.1:

$$\begin{aligned}
(3.22)\qquad &\int_{\mathbb{R}} u^\mu(t,x)\psi_r(x)dx \lesssim \int_{\mathbb{R}} u_0^\mu(x)\psi_r(x)dx+\|\psi'\|_\infty\frac{\sqrt{t}}{r}+\|\psi''\|_\infty\frac{t}{r^2}\\
&+\int_0^t\int_{\mathbb{R}}\left(\mu^2\sum_{m=1}^N\omega_m\big(u^\mu(s,x-m\frac{\Delta x}{\mu})-u^\mu(s,x)\big)+\mu F_1^\Delta d^+_{\Delta x/\mu}u^\mu(s,x)\right)\psi_r(x)dxds,
\end{aligned}$$

Let us focus on the last term, for which we have

$$\begin{aligned}
&\int_{\mathbb{R}}\Big(\mu^2\sum_{m=1}^N\omega_m\big(u^\mu(s,x-m\frac{\Delta x}{\mu})-u^\mu(s,x)\big)+\mu F_1^\Delta d^+_{\Delta x/\mu}u^\mu(s,x)\Big)\psi_r(x)dx\\
&\quad=\mu^2\sum_{m=1}^N\omega_m\int_{\mathbb{R}}u^\mu(s,x)\Big(\psi_r(x+m\frac{\Delta x}{\mu})-\psi_r(x)-m\frac{\Delta x}{\mu}\,d^-_{\Delta x/\mu}(\psi_r(x))\Big)dx\\
&\quad\lesssim\left\|\mu^2\sum_{m=1}^N\omega_m\Big(\psi_r(x+m\frac{\Delta x}{\mu})-\psi_r(x)-m\frac{\Delta x}{\mu}\,d^-_{\Delta x/\mu}(\psi_r(x))\Big)\right\|_\infty\|u^\mu(s)\|_1\\
&\quad\lesssim\frac{\|\psi''\|_\infty}{r^2}\|u^0_\Delta\|_1.
\end{aligned}$$

Thus, plugging this into (3.22) and using the non-negativity of the solution, we get

$$(3.23)\qquad \int_{\mathbb{R}}|u^\mu(t,x)|\psi_r(x)dx\lesssim\int_{\mathbb{R}}|u_0^\mu(x)|\psi_r(x)dx+\|\psi'\|_\infty\frac{\sqrt{t}}{r}+\|\psi''\|_\infty\frac{t}{r^2},$$

which tends to 0 uniformly on $\mu>0$ when $r\to\infty$. Therefore, we proved (3.20) and, consequently, we can assure that $\{u^\mu\}_{\mu>0}$ is relatively compact in $C([\tau,T],L^1(\mathbb{R}))$. □

A slight modification of the proof of the previous theorem gives as the necessary estimate to identify the initial data, stated in the following proposition.

**Proposition 3.3.** *For every test function $\varphi\in C_c^\infty(\mathbb{R})$, there exists $C>0$, independent of $\mu$, such that*

$$(3.24)\qquad \left|\int_{\mathbb{R}}u^\mu(t,x)\varphi(x)dx-\int_{\mathbb{R}}u_0^\mu(x)\varphi(x)dx\right|\le C(t+\sqrt{t}).$$

*Proof.* It is enough to multiply (3.2) by $\varphi\in C_c^\infty(\mathbb{R})$ and integrate it over $(0,t)\times\mathbb{R}$. Then, integrating by parts and repeating arguments similar to the ones in the second step of the proof for Theorem 3.1, we deduce (3.24). □

3.3. **Passing to the limit.** Finally, we have everything that we need to prove our main result, stated in Theorem 1.2, regarding the large-time behavior of the approximations to the solution of problem (1.3).

*Proof of Theorem 1.2.* By Theorem 3.1, we know that for every $0 < \tau < T < \infty$, the family $\{u^\mu\}_{\mu>0}$ is relatively compact in $C([\tau, T], L^1(\mathbb{R}))$. Consequently, there exists a subsequence of it (which we will not relabel) and a function $\bar{u} \in C((0,\infty), L^1(\mathbb{R}))$ such that

$$u^\mu \longrightarrow \bar{u} \in C([\tau, T], L^1(\mathbb{R})), \quad \text{as } \mu \to \infty. \tag{3.25}$$

We can also assume that $u^\mu(t,x) \to \bar{u}(t,x)$ almost everywhere in $(0,\infty) \times \mathbb{R}$ as $\mu \to \infty$.

Now, we multiply equation (3.2) by a test function $\phi \in C_c^\infty((0,\infty)\times\mathbb{R})$ and integrate it over $(0,\infty)\times\mathbb{R}$. We have:

(3.26)

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}} u_t^\mu(t,x)\phi(t,x)dxdt &= \frac{1}{4}\int_0^\infty \int_{\mathbb{R}} \Big(d^+_{\Delta x/\mu}\big(u^\mu(t,x)^2\big) + d^-_{\Delta x/\mu}\big(u^\mu(t,x)^2\big)\Big)\phi(t,x)dxdt \\ &+ \Delta x \int_0^\infty \int_{\mathbb{R}} d^+_{\Delta x/\mu} R\big(u^\mu(t, x - \frac{\Delta x}{\mu}), u^\mu(t,x)\big)\phi(t,x)dxdt \\ &+ \int_0^\infty \int_{\mathbb{R}} d^-_{\Delta x/\mu}\left(d^+_{\Delta x/\mu} u^\mu(t,x)\right)\phi(t,x)dxdt \\ &+ \int_0^\infty \int_{\mathbb{R}} \left(\mu^2 \sum_{m=1}^N \omega_m u^\mu(t, x - m\frac{\Delta x}{\mu}) - \mu^2 F_0^\Delta u^\mu(t,x) + \mu F_1^\Delta d^+_{\Delta x/\mu} u^\mu(t,x)\right)\phi(t,x)dxdt\end{aligned}$$

Our claim is that, passing to the limit $\mu \to \infty$, we obtain that $\bar{u}$ is a weak solution of the equation:

$$\begin{cases} \bar{u}_t = \bar{u}\bar{u}_x + (1 + F_2^\Delta)\bar{u}_{xx}, & (t,x) \in (0,\infty)\times\mathbb{R}, \\ \bar{u}(0) = M\delta_0. \end{cases} \tag{3.27}$$

All the limits in (3.26) are known (see [9]) except the last term. It is sufficient to check that we can take the limit $\mu \to \infty$ in

$$\mathcal{L}^\mu(t) = \int_{\mathbb{R}} \left(\mu^2 \sum_{m=1}^N \omega_m u^\mu(t, x - m\frac{\Delta x}{\mu}) - \mu^2 F_0^\Delta u^\mu(t,x) + \mu F_1^\Delta d^+_{\Delta x/\mu} u^\mu(t,x)\right)\phi(t,x)dx.$$

First, we reorder $\mathcal{L}_\mu$:

$$\mathcal{L}^\mu(t) = \mu^2 \int_{\mathbb{R}} u^\mu(t,x) \sum_{m=1}^N \omega_m \left(\phi(t, x + m\frac{\Delta x}{\mu}) - \phi(t,x) - m\frac{\Delta x}{\mu} d^-_{\Delta x/\mu}(\phi(t,x))\right) dx. \tag{3.28}$$

Now, due to Taylor's Theorem, for each $m \in \{1, \dots, N\}$

$$\phi(t, x + m\frac{\Delta x}{\mu}) - \phi(t,x) = m\frac{\Delta x}{\mu}\phi'(t,x) + \frac{1}{2}m^2\frac{\Delta x^2}{\mu^2}\phi''(t,x) + \frac{1}{\mu^3}O(\|\phi'''(t)\|_\infty).$$

In the same way,

$$d^-_{\Delta x/\mu}(\phi(t,x)) = \phi'(t,x) - \frac{1}{2}\frac{\Delta x}{\mu}\phi''(t,x) + \frac{1}{\mu^2}O(\|\phi'''(t)\|_\infty).$$

We combine this into (3.28) and get

$$\mathcal{L}^\mu(t) = F_2^\Delta \int_{\mathbb{R}} u^\mu(t,x)\phi''(t,x)dx + O(\|\phi'''(t)\|_\infty)\frac{1}{\mu}\int_{\mathbb{R}} u^\mu(t,x)dx. \tag{3.29}$$

Therefore, as $u^\mu \to \bar{u}$ in $C([\tau,T], L^1(\mathbb{R}))$, taking the limit $\mu \to \infty$ in (3.29), we obtain:

$$\lim_{\mu\to\infty} \int_0^\infty \mathcal{L}^\mu(t) = F_2^\Delta \int_0^\infty \int_{\mathbb{R}} \bar{u}(t,x)\phi''(t,x)dx.$$

It follows that $\bar{u}$ satisfies

$$-\int_0^\infty \int_{\mathbb{R}} \bar{u}\phi_t = -\frac{1}{2}\int_0^\infty \int_{\mathbb{R}} \bar{u}^2\phi_x + (1 + F_2^\Delta)\int_0^\infty \int_{\mathbb{R}} \bar{u}\phi_{xx},$$

so it is a weak solution of the equation in (3.27). It remains to identify the behavior of $\bar{u}$ as $t \to 0$. Due to Proposition 3.3, for any $\varphi \in C_c^\infty(\mathbb{R})$ we have

$$\left|\int_{\mathbb{R}} u^\mu(t,x)\varphi(x)dx - \int_{\mathbb{R}} u_0^\mu(x)\varphi(x)dx\right| \le C(t + \sqrt{t})$$

and from (3.25) we deduce

$$\left|\int_{\mathbb{R}} \bar{u}(t,x)\varphi(x)dx - M\varphi(0)\right| \le C(t + \sqrt{t})$$

by letting $\mu \to \infty$. Passing to the limit $t \to 0$ and using classical approximation arguments, we deduce that $\bar{u}(0) = M\delta_0$ in the sense of bounded measures. Thus, we conclude that $\bar{u}$ is the unique solution $u_M$ of equation (3.27), and that, in fact, the whole family $\{u^\mu\}_{\mu>0}$ converges to $u_M$ in $C((0,\infty), L^1(\mathbb{R}))$.

Therefore, by (3.25), we have:

$$\lim_{\mu\to\infty} \|u^\mu(1) - u_M(1)\|_1 = 0$$

and setting $\mu = \sqrt{t}$ and making use of the self-similar form of $u_M$ (see e.g. [6]) we obtain

$$\lim_{t\to\infty} \|u_\Delta(t) - u_M(t)\|_1 = 0. \tag{3.30}$$

Finally, the convergence in the $L^p$-norms for $p \in (1,\infty)$ follows from (3.30), the decay estimate of Proposition 3.1 for $p = \infty$ and the Hölder inequality. In fact, we have:

$$\|u_\Delta(t) - u_M(t)\|_p \le (\|u_\Delta(t)\|_\infty + \|u_M(t)\|_\infty)^{1-\frac{1}{p}} \|u_\Delta(t) - u_M(t)\|_1^{\frac{1}{p}} \le o(t^{-\frac{1}{2}(1-\frac{1}{p})}).$$

Using the piecewise constant interpolation of $u_M$, which we denote $S(u_M)$, and (A.2) from the Appendix, the case $p = \infty$ follows:

$$\begin{aligned}
\|u_\Delta(t) - u_M(t)\|_\infty &\le \|u_\Delta(t) - S(u_M(t))\|_\infty + \|S(u_M(t)) - u_M(t)\|_\infty \\
&\lesssim \|u_\Delta(t) - S(u_M(t))\|_2^{\frac{1}{2}} \|d^+_{\Delta x}(u_\Delta(t) - S(u_M(t)))\|_2^{\frac{1}{2}} + \Delta x \|u_{M,x}(t)\|_\infty \\
&\lesssim (\|u_\Delta(t) - u_M(t)\|_2 + \|u_M(t) - S(u_M(t))\|_2)^{\frac{1}{2}} (\|d^+_{\Delta x} u_\Delta(t)\|_2 + \|d^+_{\Delta x} S(u_M(t))\|_2)^{\frac{1}{2}} \\
&\qquad + \Delta x \|u_{M,x}(t)\|_\infty \\
&\le o(t^{-\frac{1}{2}} + t^{-\frac{3}{4}} + t^{-1}).
\end{aligned}$$

Now the proof is complete. □

3.4. **Convergence of the scheme.** To conclude this section, let us prove that $u_\Delta$ converges to the solution $u$ of (1.3) as $\Delta x \to 0$.

**Theorem 3.2.** *Let $u_0 \in L^1(\mathbb{R})$. The set of approximated solutions $\{u_\Delta\}_{\Delta x>0}$ given by (1.6) converges in $C((0,\infty), L^1(\mathbb{R}))$ to the solution $u$ of (1.3) as $\Delta x \to 0$.*

*Proof.* Following the same arguments as in Theorem 3.1, one shows that for every $0 < \tau < T < \infty$, the family $\{u_\Delta\}_{\Delta x>0} \subset C([\tau,T], L^1(\mathbb{R}))$ is relatively compact. Thus, there exists a subsequence of it (which we will not relabel) and a function $\bar{u} \in C((0,\infty), L^1(\mathbb{R}))$ such that

$$u_\Delta \longrightarrow \bar{u} \in C([\tau,T], L^1(\mathbb{R})), \quad \text{as } \Delta x \to 0. \tag{3.31}$$

We can also assume that $u_\Delta(t,x) \to \bar{u}(t,x)$ almost everywhere in $(0,\infty) \times \mathbb{R}$ as $\Delta x \to 0$.

Now, we take $\mu = 1$ in equation (3.2), multiply it by a test function $\phi \in C_c^\infty((0,\infty) \times \mathbb{R})$ and integrate it over $(0,\infty) \times \mathbb{R}$. We have:

$$\begin{aligned}
\int_0^\infty \int_{\mathbb{R}} u_{\Delta,t}(t,x)\phi(t,x)dxdt &= \frac{1}{4}\int_0^\infty \int_{\mathbb{R}} \Big(d^+_{\Delta x}\big(u_\Delta(t,x)^2\big) + d^-_{\Delta x}\big(u_\Delta(t,x)^2\big)\Big)\phi(t,x)dxdt \\
&+ \Delta x \int_0^\infty \int_{\mathbb{R}} d^+_{\Delta x} R\big(u_\Delta(t,x-\Delta x), u_\Delta(t,x)\big)\phi(t,x)dxdt \\
&+ \int_0^\infty \int_{\mathbb{R}} d^-_{\Delta x}\left(d^+_{\Delta x} u_\Delta(t,x)\right)\phi(t,x)dxdt \\
&+ \int_0^\infty \int_{\mathbb{R}} \left(\sum_{m=1}^N \omega_m u_\Delta(t,x-m\Delta x) - F_0^\Delta u_\Delta(t,x) + F_1^\Delta d^+_{\Delta x} u_\Delta(t,x)\right)\phi(t,x)dxdt
\end{aligned} \tag{3.32}$$

Our claim is that, passing to the limit $\Delta x \to 0$, we obtain that $\bar{u}$ is a weak solution of the equation (1.3). Thanks to (3.31) and to Propositions 3.1 and 3.2, we know that we can take all the limits in (3.32), except for the last term. Thus, it is sufficient to check that we can pass to the limit $\Delta x \to 0$ in

$$\mathcal{L}_\Delta(t) = \int_{\mathbb{R}} \left( \sum_{m=1}^{N} \omega_m u_\Delta(t, x - m\Delta x) - F_0^\Delta u_\Delta(t,x) + F_1^\Delta d_{\Delta x}^+ u_\Delta(t,x) \right) \phi(t,x) dx$$
$$= \int_{\mathbb{R}} u_\Delta(t,x) \left( \sum_{m=1}^{N} \omega_m \phi(t, x + m\Delta x) - F_0^\Delta \phi(t,x) - F_1^\Delta d_{\Delta x}^- \phi(t,x) \right) dx.$$

First, let us first observe that

$$F_0^\Delta = \sum_{m=1}^{N} e^{-m\Delta x}(e^{\Delta x} - 1) = 1 - e^{-N\Delta x} \to 1$$

and

$$F_1^\Delta = \Delta x(e^{\Delta x} - 1) \sum_{m=1}^{N} m e^{-m\Delta x} = \frac{\Delta x e^{-N\Delta x}(e^{(N+1)\Delta x} - e^{\Delta x}(N+1) + N)}{e^{\Delta x} - 1} \to 1,$$

as long as $N = N(\Delta x)$ is taken such that $N\Delta x \to \infty$ as $\Delta x \to 0$. Moreover, using (2.15) and that

$$(e^{\Delta x} - 1)\frac{1 - e^{-N\Delta x(1-i\xi)}}{e^{(1-i\xi)\Delta x} - 1} \to \frac{1}{1 - i\xi} = \frac{1 + i\xi}{1 + \xi^2}, \quad \text{as } \Delta x \to 0$$

we obtain

$$\left| \sum_{m=1}^{N} \omega_m \phi(t, x + m\Delta x) - \tilde{K} * \phi(t,x) \right| \le \int_{\mathbb{R}} |\widehat{\phi}(t,\xi)| \left| \sum_{m=1}^{N} \omega_m e^{im\Delta x \xi} - \widehat{K}(-\xi) \right| d\xi$$
$$= \int_{\mathbb{R}} |\widehat{\phi}(t,\xi)| \left| (e^{\Delta x} - 1)\frac{1 - e^{-N\Delta x(1-i\xi)}}{e^{(1-i\xi)\Delta x} - 1} - \frac{1 + i\xi}{1 + \xi^2} \right| d\xi \to 0,$$

where $\tilde{K}(z) = K(-z)$. Therefore

$$\lim_{\Delta x \to 0} \mathcal{L}_\Delta(t) = \int_{\mathbb{R}} \bar{u}(t,x) \left( \tilde{K} * \phi(t,x) - \phi(t,x) - \phi_x(t,x) \right) dx.$$

It follows that $\bar{u}$ satisfies

$$-\int_0^\infty \int_{\mathbb{R}} \bar{u}\phi_t = -\frac{1}{2}\int_0^\infty \int_{\mathbb{R}} \bar{u}^2 \phi_x + \int_0^\infty \int_{\mathbb{R}} \bar{u}\phi_{xx} + \int_0^\infty \int_{\mathbb{R}} \bar{u}(\tilde{K} * \phi - \phi - \phi_x).$$

so it is a weak solution of the equation in (1.3).

Now, it remains to identify the behavior of $\bar{u}$ as $t \to 0$. In the same way as in Proposition 3.3, we can prove that for every test function $\varphi \in C_c^\infty(\mathbb{R})$ and $\Delta x < 1$, there exists $C > 0$, independent of $\Delta x$, such that

$$\left| \int_{\mathbb{R}} u_\Delta(t,x)\varphi(x)dx - \int_{\mathbb{R}} u_\Delta^0(x)\varphi(x)dx \right| \le C(t + \sqrt{t}).$$

and from (3.31) and the definition of $u_\Delta^0$ in (1.5), we deduce

$$\left| \int_{\mathbb{R}} \bar{u}(t,x)\varphi(x)dx - \int_{\mathbb{R}} u_0(x)\varphi(x)dx \right| \le C(t + \sqrt{t})$$

by letting $\Delta x \to 0$. Passing to the limit $t \to 0$ and using classical approximation arguments, we deduce that $\bar{u}(0) = u_0$ in the sense of bounded measures. Thus, we conclude that $\bar{u}$ is the unique solution $u$ of equation (1.3), and that, in fact, the whole family $\{u_\Delta\}_{\Delta x > 0}$ converges to $u$ in $C((0,\infty), L^1(\mathbb{R}))$. Now the proof is complete. □

## 4. Numerical experiments

The aim of this last section is to support the necessity of using large-time behavior preserving schemes for the augmented Burgers equation. On the one hand, we show the importance of a numerical flux that does not destroy the N-wave shape at the early stages. On the other, we emphasize the role of the corrector factors $F_0^\Delta$ and $F_1^\Delta$ in the truncation of the convolution.

Regarding the time discretization, we opt for the explicit Euler for its simplicity. Even if there is no guarantee that the asymptotic behavior is preserved, numerical simulations exhibit a correct performance. Thus, we consider it enough to illustrate the key points enumerated above. We need to take into account that there is a stability condition that must be satisfied to ensure the convergence. It is easy to see (e.g. [3, 8]) that a sufficient condition is that

$$\frac{\Delta t}{\Delta x}\max_j\{u_j^0\} + 2\nu\frac{\Delta t}{\Delta x^2} + c\,\Delta t\sum_{m=1}^{N}(m+1)\omega_m \le 1 \tag{4.1}$$

Let us choose the following compactly supported initial data.

$$u_0(x) = \begin{cases} -\dfrac{1}{10}\sin\left(\dfrac{x}{2}\right), & x\in[-\pi,0],\\ -\dfrac{1}{20}\sin(2x), & x\in[0,\dfrac{\pi}{2}],\\ 0, & \text{elsewhere}\end{cases} \tag{4.2}$$

We take a mesh size $\Delta x = 0.1$. In order to avoid boundary issues, we choose a large enough spatial domain.

In Figure 1 we show the solution for $\nu = 10^{-2}$, $c = 2\times10^{-2}$ and $\theta = 1$ at time $t = 10^4$, as well as the corresponding asymptotic profile $u_M$, defined in (1.4). As we can observe, the solution given by (1.6) is already quite close to $u_M$. However, a non-suitable viscous numerical flux like, for instance, the modified Lax-Friedrichs (e.g. [8, Chapter 3]) can definitely modify the large-time behavior of the solution. In fact, in this case a viscosity proportional to $\Delta x^2/\Delta t$ is being added to the equation of the asymptotic profile (see [9]), producing a more diffused wave. Nevertheless, the discretization of the non-linear term is not the only one with the ability to perturb the dynamics of the model. Let us emphasize that an inappropriate discretization of the non-local term also leads to an incorrect asymptotic profile. Note that in Figure 1 we have the same scheme (1.6) but taking $F_0^\Delta = F_1^\Delta = 1$, which produces a translated solution.

The convergence rates, given in (1.9), are shown in Figure 2. The graphic highlights the different performances mentioned above. In fact, the solution given by (1.6) is the only one for which the norm is converging to zero with the corresponding rates.

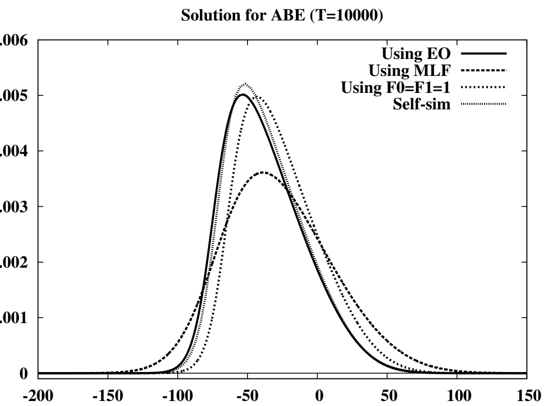


FIGURE 1. Solution of ABE with $\nu = 10^{-2}$, $c = 2\times10^{-2}$ and $\theta = 1$ at $t = 10^4$, using scheme (1.6) discretized explicitly. We use Engquist-Osher (solid) and modified Lax-Friedrichs (dashed) numerical fluxes for the nonlinearity, as well as no correcting factors (dotted), comparing the solutions to the asymptotic profile (gray).

To conclude, let us remark again the importance of taking a well-behaving numerical flux. In this paper we have proved that the asymptotic profile of (1.3) is a diffusive wave. Therefore, any sign-changing initial data will lose its positive or negative part, depending on the sign of its mass. As in the case of the viscous Burgers equation [13], simulations show that N-waves are intermediate states. Therefore, if the numerical viscosity is sufficiently large, the diffusion will become dominant much earlier than in the continuous model and destroy these profiles. For instance, let us consider the case $\nu = 10^{-4}$ and $c = 2\times10^{-4}$. In Figure 3, we can observe that at $t = 100$ the N-wave shape is not preserved if the modified Lax-Friedrichs flux is used, while Engquist-Osher is able to keep the continuous dynamics.

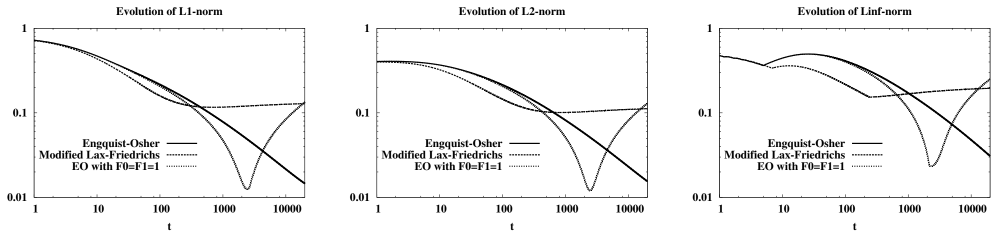


FIGURE 2. Evolution of the norms of the difference between the asymptotic profile and the solutions, multiplied by their corresponding rate $t^{\frac{1}{2}(1-\frac{1}{p})}$. From left to right, $L^1$, $L^2$ and $L^\infty$ norms. We compare (1.6) (solid), MLF numerical flux (dashed) and $F_0^\Delta = F_1^\Delta = 1$ (dotted).

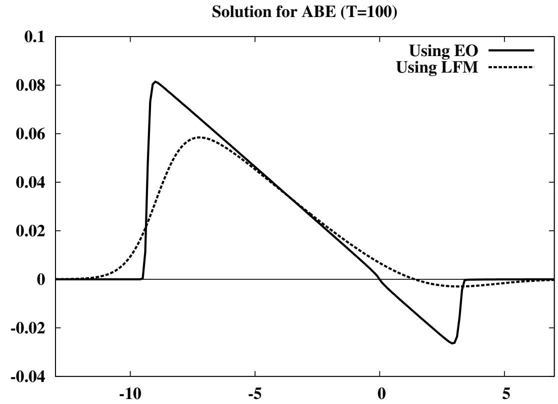


FIGURE 3. Solution of ABE with $\nu = 10^{-4}$, $c = 2\times 10^{-4}$ and $\theta = 1$ at $t = 100$, using scheme (1.6) discretized explicitly. We use Engquist-Osher (solid) and modified Lax-Friedrichs (dashed) numerical fluxes for the nonlinearity.

## APPENDIX A. AUXILIARY RESULTS

Here we prove some of the auxiliary results that we have used along the paper.

**Lemma A.1.** *For any piecewise constant function $w$ defined as in* (1.5) *and $\Delta x > 0$, the following holds:*

$$\|w\|_p^{p(p+1)/(p-1)} \le 4\|w\|_1^{2p/(p-1)}\|d^+_{\Delta x}|w|^{p/2}\|_2^2$$

*for all $p \in (1,\infty)$.*

*Proof.* First, let us define a piecewise linear function $v$ as follows:

$$v(x) := w_j\frac{x_{j+1}-x}{\Delta x} + w_{j+1}\frac{x-x_j}{\Delta x}, \quad x\in[x_j,x_{j+1}].$$

On the one hand, we know that

$$\|v\|_\infty^2 \le 2\|v\|_2\|v_x\|_2. \tag{A.1}$$

On the other hand, we have that:

$$\|v\|_2^2 = \Delta x\sum_{j\in\mathbb{Z}}\int_0^1 |w_j(1-x)+w_{j+1}x|^2\,dx \le \frac{1}{2}\Delta x\sum_{j\in\mathbb{Z}}\left(|w_j|^2+|w_{j+1}|^2\right) = \|w\|_2^2.$$

Moreover, it is easy to see that $\|v_x\|_2 = \|d^+_{\Delta x}w\|_2$. Therefore, we can obtain a similar inequality as (A.1) for $w$:

$$\|w\|_\infty^2 = \|v\|_\infty^2 \le 2\|v\|_2\|v_x\|_2 \le 2\|w\|_2\|d^+_{\Delta x}w\|_2. \tag{A.2}$$

Applying this inequality to $|w|^{p/2}$, we deduce:

$$\|w\|_\infty^{2p} = \||w|^{p/2}\|_\infty^4 \le 4\||w|^{p/2}\|_2^2\|d^+_{\Delta x}|w|^{p/2}\|_2^2 = 4\|w\|_p^p\|d^+_{\Delta x}|w|^{p/2}\|_2^2.$$

Thus, combining this with

$$\|w\|_p^{2p^2/(p-1)} \leq \|w\|_\infty^{2p} \|w\|_1^{2p/(p-1)},$$

we conclude

$$\|w\|_p^{p(p+1)/(p-1)} \leq 4\|w\|_1^{2p/(p-1)} \|d_{\Delta x}^+ |w|^{p/2}\|_2^2.$$

□

*Proof of Lemma 3.1.* For the first assertion, we simply integrate (3.2) over the whole space domain. We observe that all terms on the right hand side vanish, so

$$\frac{d}{dt}\int_{\mathbb{R}} u^\mu(t,x)dx = 0, \quad \forall t \geq 0,$$

for all $\mu > 0$ and, hence, the mass is conserved. Using the definition of $u^\mu$, we conclude

$$\int_{\mathbb{R}} u^\mu(t,x)dx = \int_{\mathbb{R}} u^\mu(0,x)dx = \int_{\mathbb{R}} \mu u_\Delta^0(\mu x)dx = \int_{\mathbb{R}} u_\Delta^0(x)dx.$$

For the contractivity we prove that for any $u_0, v_0 \in L^1(\mathbb{R})$, their corresponding solutions $u^\mu$ and $v^\mu$ satisfy

$$\|u^\mu - v^\mu\|_1 \leq \|u_0^\mu - v_0^\mu\|_1. \tag{A.3}$$

For the sake of clarity, let us define $w^\mu = u^\mu - v^\mu$. Clearly, $w^\mu$ verifies

$$\begin{aligned} w_t^\mu(t,x) &= \frac{1}{4}\Big(d_{\Delta x/\mu}^+\big(u^\mu(t,x)^2\big) + d_{\Delta x/\mu}^-\big(u^\mu(t,x)^2\big) - d_{\Delta x/\mu}^+\big(v^\mu(t,x)^2\big) - d_{\Delta x/\mu}^-\big(v^\mu(t,x)^2\big)\Big) \\ &+ \Delta x\, d_{\Delta x/\mu}^+ R\big(u^\mu(t,x-\frac{\Delta x}{\mu}), u^\mu(t,x)\big) - \Delta x\, d_{\Delta x/\mu}^+ R\big(v^\mu(t,x-\frac{\Delta x}{\mu}), v^\mu(t,x)\big) \\ &+ d_{\Delta x/\mu}^-\Big(d_{\Delta x/\mu}^+ w^\mu(t,x)\Big) \\ &+ \mu^2 \sum_{m=1}^N \omega_m w^\mu(t, x - m\frac{\Delta x}{\mu}) - \mu^2 F_0^\Delta w^\mu(t,x) + \mu F_1^\Delta d_{\Delta x/\mu}^+ w^\mu(t,x). \end{aligned}$$

We multiply it by $\operatorname{sign}(w^\mu)$ and integrate it on all $\mathbb{R}$. Using the definition of $R$ in (3.3) and reordering the terms we get

(A.4)

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}} |w^\mu(x)|dx \\ &= \frac{1}{4}\int_{\mathbb{R}} d_{\Delta x/\mu}^+\big(u^\mu(x)^2 + u^\mu(x-\frac{\Delta x}{\mu})^2 + u^\mu(x)|u^\mu(x)| - u^\mu(x-\frac{\Delta x}{\mu})|u^\mu(x-\frac{\Delta x}{\mu})|\big)\operatorname{sign}(w^\mu(x))dx \\ &- \frac{1}{4}\int_{\mathbb{R}} d_{\Delta x/\mu}^+\big(v^\mu(x)^2 + v^\mu(x-\frac{\Delta x}{\mu})^2 + v^\mu(x)|v^\mu(x)| - v^\mu(x-\frac{\Delta x}{\mu})|v^\mu(x-\frac{\Delta x}{\mu})|\big)\operatorname{sign}(w^\mu(x))dx \\ &+ \int_{\mathbb{R}} d_{\Delta x/\mu}^-(d_{\Delta x/\mu}^+(w^\mu(x)))\operatorname{sign}(w^\mu(x))dx \\ &+ \mu^2\sum_{m=1}^N \omega_m \int_{\mathbb{R}}\big(w^\mu(x-m\frac{\Delta x}{\mu}) - w^\mu(x)\big)\operatorname{sign}(w^\mu(x))dx + \mu F_1^\Delta \int_{\mathbb{R}} d_{\Delta x/\mu}^+ w^\mu(t,x)\operatorname{sign}(w^\mu(x))dx \\ &= I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned}$$

For $i = 0, 1$, let us denote $W_i^\pm = \{x \in \mathbb{R} : \pm w^\mu(x - i\Delta x) > 0\}$ and $W_i^0 = \{x \in \mathbb{R} : w^\mu(x - i\Delta x) = 0\}$. Now we can split the domains of the integrals into several parts, according to the sign of $w^\mu$. On the one hand, we have:

$$
\begin{aligned}
I_1+I_2 = &-\frac{1}{4}\int_{\mathbb{R}} \big(u^\mu(x)^2+u^\mu(x)|u^\mu(x)|\big) d^-_{\Delta x/\mu}(\operatorname{sign}(w^\mu(x)))dx\\
&-\frac{1}{4}\int_{\mathbb{R}} \big(u^\mu(x-\frac{\Delta x}{\mu})^2-u^\mu(x-\frac{\Delta x}{\mu})|u^\mu(x-\frac{\Delta x}{\mu})|\big) d^-_{\Delta x/\mu}(\operatorname{sign}(w^\mu(x)))dx\\
&+\frac{1}{4}\int_{\mathbb{R}} \big(v^\mu(x)^2+v^\mu(x)|v^\mu(x)|\big) d^-_{\Delta x/\mu}(\operatorname{sign}(w^\mu(x)))dx\\
&+\frac{1}{4}\int_{\mathbb{R}} \big(v^\mu(x-\frac{\Delta x}{\mu})^2-v^\mu(x-\frac{\Delta x}{\mu})|v^\mu(x-\frac{\Delta x}{\mu})|\big) d^-_{\Delta x/\mu}(\operatorname{sign}(w^\mu(x)))dx\\
=&-\frac{\mu}{2\Delta x}\int_{W_0^-\cap W_1^+} \big(u^\mu(x)^2+u^\mu(x)|u^\mu(x)|-v^\mu(x)^2-v^\mu(x)|v^\mu(x)|\big)\operatorname{sign}(w^\mu(x))dx\\
&-\frac{\mu}{2\Delta x}\int_{W_0^-\cap W_1^+} \big(v^\mu(x-\frac{\Delta x}{\mu})^2-v^\mu(x-\frac{\Delta x}{\mu})|v^\mu(x-\frac{\Delta x}{\mu})|\\
&\qquad -u^\mu(x-\frac{\Delta x}{\mu})^2+u^\mu(x-\frac{\Delta x}{\mu})|u^\mu(x-\frac{\Delta x}{\mu})|\big)\operatorname{sign}(w^\mu(x-\frac{\Delta x}{\mu}))dx\\
&-\frac{\mu}{2\Delta x}\int_{W_0^+\cap W_1^-} \big(u^\mu(x)^2+u^\mu(x)|u^\mu(x)|-v^\mu(x)^2-v^\mu(x)|v^\mu(x)|\big)\operatorname{sign}(w^\mu(x))dx\\
&-\frac{\mu}{2\Delta x}\int_{W_0^+\cap W_1^-} \big(v^\mu(x-\frac{\Delta x}{\mu})^2-v^\mu(x-\frac{\Delta x}{\mu})|v^\mu(x-\frac{\Delta x}{\mu})|\\
&\qquad -u^\mu(x-\frac{\Delta x}{\mu})^2+u^\mu(x-\frac{\Delta x}{\mu})|u^\mu(x-\frac{\Delta x}{\mu})|\big)\operatorname{sign}(w^\mu(x-\frac{\Delta x}{\mu}))dx\\
&-\frac{\mu}{4\Delta x}\int_{W_1^0} \big(u^\mu(x)^2+u^\mu(x)|u^\mu(x)|-v^\mu(x)^2-v^\mu(x)|v^\mu(x)|\big)\operatorname{sign}(w^\mu(x))dx\\
&-\frac{\mu}{4\Delta x}\int_{W_0^0} \big(v^\mu(x-\frac{\Delta x}{\mu})^2-v^\mu(x-\frac{\Delta x}{\mu})|v^\mu(x-\frac{\Delta x}{\mu})|\\
&\qquad -u^\mu(x-\frac{\Delta x}{\mu})^2+u^\mu(x-\frac{\Delta x}{\mu})|u^\mu(x-\frac{\Delta x}{\mu})|\big)\operatorname{sign}(w^\mu(x-\frac{\Delta x}{\mu}))dx.
\end{aligned}
$$

Using that

$$\big(b(b+|b|)-a(a+|a|)\big)\operatorname{sign}(b-a)\geq 0,\quad \forall a,b\in\mathbb{R},$$

and that

$$\big(a(a-|a|)-b(b-|b|)\big)\operatorname{sign}(b-a)\geq 0,\quad \forall a,b\in\mathbb{R},$$

we conclude that $I_1+I_2\leq 0$. On the other hand, since

$$\int_{\mathbb{R}} w^\mu(x-m\frac{\Delta x}{\mu})\operatorname{sign}(w^\mu(x))dx\leq \int_{\mathbb{R}}|w^\mu(x)|dx,\quad \forall m\in\mathbb{Z},$$

it is immediate that

$$I_3=\frac{\mu^2}{\Delta x^2}\int_{\mathbb{R}} w^\mu(x-\frac{\Delta x}{\mu})\operatorname{sign}(w^\mu(x))dx+\frac{\mu^2}{\Delta x^2}\int_{\mathbb{R}} w^\mu(x+\frac{\Delta x}{\mu})\operatorname{sign}(w^\mu(x))dx-\frac{2\mu^2}{\Delta x^2}\int_{\mathbb{R}}|w^\mu(x)|dx\leq 0.$$

Moreover, for the same reason, we deduce that $I_4\leq$ and $I_5\leq 0$. Therefore, from (A.4) we get that

$$\frac{d}{dt}\int_{\mathbb{R}}|w^\mu(x)|dx\leq 0, \tag{A.5}$$

This guarantees the contractive property (A.3). □

*Proof of Lemma 3.2.* Let us consider the Fourier transform of $w$ as

$$\widehat{w}(\xi)=\int_{\mathbb{R}} e^{-ix\xi}w(x)dx,\quad \xi\in\mathbb{R}$$

and the discrete Fourier transform of the sequence $\{w_j\}_{j\in\mathbb{Z}}$ as

$$\overline{w}(\xi)=\Delta x\sum_{j\in\mathbb{Z}} w_j e^{-ij\Delta x\xi},\quad \xi\in[-\frac{\pi}{\Delta x},\frac{\pi}{\Delta x}].$$

It is also clear that for a piecewise constant function $w$ defined as in (1.5)

$$\widehat{w}(\xi) = \frac{2\sin(\frac{\xi\Delta x}{2})}{\xi\Delta x}\,\overline{w}(\xi) \quad \text{and} \quad \widehat{d^+_{\Delta x} w}(\xi) = \frac{e^{i\xi\Delta x}-1}{\Delta x}\widehat{w}(\xi).$$

Now, we know that

$$\|D^s w\|_2^2 = \int_{\mathbb{R}} |\xi|^{2s}|\widehat{w}(\xi)|^2 d\xi = \int_{-\pi/\Delta x}^{\pi/\Delta x} |\xi|^{2s}|\widehat{w}(\xi)|^2 d\xi + \sum_{j\neq 0}\int_{(2j-1)\pi/\Delta x}^{(2j+1)\pi/\Delta x} |\xi|^{2s}|\widehat{w}(\xi)|^2 d\xi. \tag{A.6}$$

For each $j \neq 0$, we have

$$\begin{aligned}
\int_{(2j-1)\pi/\Delta x}^{(2j+1)\pi/\Delta x} |\xi|^{2s}|\widehat{w}(\xi)|^2 d\xi &= \int_{(2j-1)\pi/\Delta x}^{(2j+1)\pi/\Delta x} |\xi|^{2s}|\,\overline{w}(\xi)|^2 \left|\frac{2\sin(\frac{\xi\Delta x}{2})}{\xi\Delta x}\right|^2 d\xi \\
&= \int_{-\pi/\Delta x}^{\pi/\Delta x} \left|\xi + 2j\frac{\pi}{\Delta x}\right|^{2s} |\,\overline{w}(\xi)|^2 \left|\frac{2\sin(\frac{\xi\Delta x}{2}+j\pi)}{(\xi+2j\frac{\pi}{\Delta x})\Delta x}\right|^2 d\xi \\
&= \int_{-\pi/\Delta x}^{\pi/\Delta x} \left|\frac{2}{\Delta x}\sin(\frac{\xi\Delta x}{2})\right|^{2s} |\,\overline{w}(\xi)|^2 \left|\frac{2}{\Delta x}\sin(\frac{\xi\Delta x}{2})\right|^{2-2s} \left|\xi+2j\frac{\pi}{\Delta x}\right|^{2s-2} d\xi \\
&\leq \int_{-\pi/\Delta x}^{\pi/\Delta x} |\xi|^{2s}|\,\overline{w}(\xi)|^2 \frac{|\xi|^{2-2s}}{\left|\xi+2j\frac{\pi}{\Delta x}\right|^{2-2s}} d\xi \\
&\leq \frac{1}{|2|j|-1|^{2-2s}}\int_{-\pi/\Delta x}^{\pi/\Delta x} |\xi|^{2s}|\,\overline{w}(\xi)|^2 d\xi
\end{aligned}$$

Therefore, replacing this in (A.6) and using that $0 < s < \frac{1}{2}$, we get

$$\begin{aligned}
\|D^s w\|_2^2 &\leq \int_{-\pi/\Delta x}^{\pi/\Delta x} |\xi|^{2s}|\widehat{w}(\xi)|^2 d\xi + \sum_{j\neq 0}\frac{1}{|2|j|-1|^{2-2s}}\int_{-\pi/\Delta x}^{\pi/\Delta x} |\xi|^{2s}|\,\overline{w}(\xi)|^2 d\xi \\
&= \int_{-\pi/\Delta x}^{\pi/\Delta x} |\xi|^{2s}\left|\frac{2\sin(\frac{\xi\Delta x}{2})}{\xi\Delta x}\right|^2 |\,\overline{w}(\xi)|^2 d\xi + \sum_{j\neq 0}\frac{1}{|2|j|-1|^{2-2s}}\int_{-\pi/\Delta x}^{\pi/\Delta x} |\xi|^{2s}|\,\overline{w}(\xi)|^2 d\xi \\
&\lesssim \int_{-\pi/\Delta x}^{\pi/\Delta x} |\xi|^{2s}|\,\overline{w}(\xi)|^2 d\xi
\end{aligned}$$

On the other hand, using analogous arguments, we also have

$$\begin{aligned}
\|d^+_{\Delta x} w\|_2^2 &= \int_{\mathbb{R}} \left|\frac{e^{i\xi\Delta x}-1}{\Delta x}\right|^2 |\widehat{w}(\xi)|^2 d\xi = \int_{\mathbb{R}} \left|\frac{e^{i\xi\Delta x}-1}{\Delta x}\right|^2 \left|\frac{2\sin(\frac{\xi\Delta x}{2})}{\xi\Delta x}\right|^2 |\,\overline{w}(\xi)|^2 d\xi \\
&\gtrsim \int_{-\pi/\Delta x}^{\pi/\Delta x} \left|\frac{e^{i\xi\Delta x}-1}{\Delta x}\right|^2 |\,\overline{w}(\xi)|^2 d\xi \gtrsim \int_{-\pi/\Delta x}^{\pi/\Delta x} |\xi|^2\,|\,\overline{w}(\xi)|^2 d\xi
\end{aligned}$$

Finally, we conclude

$$\begin{aligned}
\|w\|_{H^s(\mathbb{R})}^2 &= \int_{\mathbb{R}} (1+|\xi|^{2s})|\widehat{w}(\xi)|^2 d\xi = \int_{\mathbb{R}} |\widehat{w}(\xi)|^2 d\xi + \int_{\mathbb{R}} |\xi|^{2s}|\widehat{w}(\xi)|^2 d\xi \\
&\lesssim \int_{-\pi/\Delta x}^{\pi/\Delta x} |\,\overline{w}(\xi)|^2 d\xi + \int_{-\pi/\Delta x}^{\pi/\Delta x} |\xi|^{2s}|\,\overline{w}(\xi)|^2 d\xi \lesssim \int_{-\pi/\Delta x}^{\pi/\Delta x} (1+|\xi|^{2s})|\,\overline{w}(\xi)|^2 d\xi \\
&\lesssim \int_{-\pi/\Delta x}^{\pi/\Delta x} (1+|\xi|^{2})|\,\overline{w}(\xi)|^2 d\xi \lesssim (\|w\|_2^2 + \|d^+_{\Delta x} w\|_2^2).
\end{aligned}$$

□

**Lemma A.2.** *Given any $a \in (0,1)$ and $b \in \mathbb{C}$ with $|b| = 1$, the following inequality holds:*

$$\left|\sum_{n=1}^{N} a^n(b^n-1) + \left(\sum_{n=1}^{N} na^n\right)\left(\frac{1}{b}-1\right)\right| \leq |b-1|^2\,\frac{a}{(1-a)^3}$$

*Proof.* Using that $|b| = 1$, we have:

$$\begin{aligned}\left|\sum_{n=1}^{N} a^n(b^n-1) + \left(\sum_{n=1}^{N} na^n\right)\left(\frac{1}{b}-1\right)\right| &= \left|\sum_{n=1}^{N} a^n b(b^n-1) - \left(\sum_{n=1}^{N} na^n\right)(b-1)\right| \\ &= |b-1|\left|\sum_{n=1}^{N} a^n \sum_{k=0}^{n-1}(b^{k+1}-1)\right| = |b-1|^2\left|\sum_{n=1}^{N} a^n \sum_{k=0}^{n-1}\sum_{j=0}^{k} b^j\right| \\ &\le \frac{1}{2}|b-1|^2\sum_{n=1}^{\infty} n(n+1)a^n = |b-1|^2\,\frac{a}{(1-a)^3}\end{aligned}$$

□

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