Abstract. We consider the problem of existence of heteroclinic solutions to the Hamiltonian system
\begin{equation}
\begin{cases}
U_{xx} = \nabla W(U), & U : \mathbb{R} \to \mathbb{R}^N, \\
U(-\infty) = a^{-}, & U(+\infty) = a^{+},
\end{cases}
\end{equation}
where $a^{\pm}$ are local minima of the potential $W \in C^2(\mathbb{R}^N)$ with $W(a^{\pm}) = 0$. (1) arises in the theory of phase transitions and has been considered before by Alikakos-Fusco [AF] and Sternberg [St]. Herein we give a new efficient proof of existence under assumptions different from those considered previously and we derive new a priori decay estimates, valid even when $a^{\pm}$ are degenerate. We establish existence by analyzing the loss of compactness: for any minimizing sequence of the Action functional $E(U) = \int_{\mathbb{R}} \left\{ \frac{1}{2} |U_{x}|^2 + W(U) \right\} dx$ in a suitable functional setup, there exist uniformly decaying designated translates, up to which, compactness is restored and passage to a solution of (1) is possible.

1. Introduction.

In this paper we consider the problem of existence of heteroclinic solutions to the Hamiltonian ODE system
\begin{equation}
\begin{cases}
U_{xx} = \nabla W(U), & U : \mathbb{R} \to \mathbb{R}^N, \\
U(-\infty) = a^{-}, & U(+\infty) = a^{+},
\end{cases}
\end{equation}
where $W \in C^2(\mathbb{R}^N)$ is a potential and $a^{\pm}$ are local minima of it with $W(a^{\pm}) = 0$. A typical $W$ for $N = 2$ is shown in Figures 1,2. Solutions to (1) are known as "heteroclinic connections", being standing waves of the gradient diffusion system
\begin{equation}
u_t = \nu_{xx} - \nabla W(u), \quad u : \mathbb{R} \times (0, +\infty) \to \mathbb{R}^N.
\end{equation}
(1) arises in the theory of phase transitions. For details we refer to Alikakos-Betelü-Chen [ABC] and to Alberti [Al]. Physically, (1) is the Newtonian law of motion with force $-\nabla(-W)$ induced by the potential $-W$ and $U$ the trajectory of a test particle which connects two maxima of $-W$. In the scalar case of $N = 1$, existence is textbook material by phase plane methods. For a variational approach we refer to Alberti [Al]. Even in this simple case the unboundedness of $\mathbb{R}$ implies that standard compactness and semicontinuity arguments fail when one tries to obtain solutions to $U_{xx} = W'(U)$ variationally as minimizers of the Action functional
\begin{equation}
E(U) = \int_{\mathbb{R}} \left\{ \frac{1}{2} |U_x|^2 + W(U) \right\} dx.
\end{equation}

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However, for $N = 1$ rearrangement methods do apply (Kawohl [Kaw]). When $N > 1$, (1) is much more difficult. It has first been considered by Sternberg in [St], as a problem arising in the study of the elliptic system $\Delta U = \nabla W(U)$. Noting the compactness problems, he utilizes the Jacobi Principle to obtain solutions by studying geodesics in the Riemannian manifold $(\mathbb{R}^N \setminus \{a^\pm\}, \sqrt{2W} \langle , \rangle)$.

Following a different approach, Alikakos-Fusco [AF] subsequently treated (1) utilizing the Least Action Principle. They derived their solutions as minimizers of (3). They introduced an artificial constraint in order to restore compactness and apply the Direct Method and obtained solutions to the (1) by eventually removing the constraint. The same approach has subsequently been applied by the author jointly with Alikakos [AK] to the respective travelling wave problem for (2), establishing existence of solution to $U_{xx} = \nabla W(U) - cU_x$ for $c \neq 0$. (1) has attracted some attention in connection with the study of system $\Delta U = \nabla W(U)$ and related material appears also in Alama-Bronsard-Gui [ABG], Bronsard-Gui-Schatzman [BGS], Alikakos [A, A2], Alikakos-Fusco [AF3] and Alikakos-Smirnelis [AS].

Problem (1) is nontrivial; except for the failure of the Direct Method for (3) due to the loss of compactness, an additional difficulty when $N > 1$ is that the Maximum Principle does not apply. In the papers [AF], [AK] and later in [AF2] were introduced substitutes of the Maximum Principle for minimizers. inspired by these results, the author later in [Ka] advanced these ideas to a general setting. A further difficulty of (1) is that additional minima of $W$ obstruct existence and suitable assumptions on $W$ must be imposed (see [ABC], [AF]).

In the present work, following [AF], we obtain solutions to (1) as minimizers of (3). We bypass their unilateral constraint method which is of independent interest, but requires a rather delicate analysis. We establish existence for (1) by an efficient method which analyzes the loss of compactness in minimizing sequences. Our motivation comes from the theory of Concentrated Compactness (see e.g. [Ev], [Str]). We introduce a functional space tailored for the study of (1) and show that given any minimizing sequence of (3), there exist designated uniformly decaying translates up to which compactness is restored and passage to a minimizer is available (Theorem 2.1). Our main ingredients are certain energy estimates and measure bounds which relate to those of [AF], [AK]. Herein however we utilize new ideas: we control the behavior of the minimizing sequence by the sup-level sets $\{W \geq \alpha\}$ and compactify the sequence by suitable translations.

Our basic assumption (A1) is slightly stronger than the respective of [AF], but we still allow for a certain degree of degeneracy. Under this assumption we obtain the new a priori decay estimates ($\ast$) by means of energy arguments, without linearizing the equation. The rest assumptions (A2'), (A2") allow for $W$’s with several minima and possibly unbounded from below, being similar to those of [AK].


**Hypotheses.** We assume $W \in C^2(\mathbb{R}^N)$ with $a^\pm$ local minima at zero: $W(a^\pm) = 0$. Moreover:

(A1) There exist $\alpha_0$, $w_0 > 0$ and $\gamma \geq 2$ such that for all $\alpha \in [0, \alpha_0]$ the sublevel sets $\{W \leq \alpha\}$ contains two convex components $\{W \leq \alpha\}^\pm$, each enclosing $a^\pm$ and

$$W(u) \geq w_0 |u - a^\pm|^\gamma, \quad u \in \{W \leq \alpha_0\}^\pm.$$
In addition, at least one of the following two properties is satisfied: either
(A2') \[ \{ W \leq \alpha_0 \} = \{ W \leq \alpha_0 \}^+ \cup \{ W \leq \alpha_0 \}^- , \]
(A2'') or there exists a convex (localization) set \( \Omega \subseteq \mathbb{R}^N \) and a \( w_{\text{max}} > \alpha_0 \) such that \( a^\pm \) are global minima for \( W|_\Omega \), while
\[ \Omega \subseteq \{ W \leq w_{\text{max}} \} \quad \text{and} \quad \partial \Omega \subseteq \{ W = w_{\text{max}} \} . \]

(A1) allows for \( C^{[\gamma]-1} \) flatness at the minima (but not \( C^\infty \) flatness as in [AF], [AK]). Assumption (A2') requires \( \{ W \leq \alpha \}^\pm \) to be the only components of the sublevel sets \( \{ W \leq \alpha \} \). Under (A2'), we immediately obtain \( \liminf_{|u| \to \infty} W(u) \geq \alpha_0 \).

(A2'') allows for \( W \)'s which may be unbounded from below, assuming nonnegativity of \( W \) only within \( \Omega \).

Under (A2'') the existence of a local minimizer \( U \) of (3) with \( E(U) > -\infty \) is a certain issue, but (A1) is more crucial. We shall refer to (A2') as the “coercive” and to (A2'') as the “non-coercive” assumption.

**Functional setup.** We derive solutions to (1) as minimizers of (3) in an affine Sobolev space which incorporates the boundary condition \( U(\pm \infty) = a^\pm \) and excludes the trivial solutions \( U = a^\pm \). Let \( W_{1,p}^{1,p}(\mathbb{R})^N \) denote the local Sobolev space of vector functions \( U : \mathbb{R} \to \mathbb{R}^N \). For \( p \in (1, \infty) \), the affine \( L^p \)-space, \( [L^p_{\text{aff}}(\mathbb{R})]^N := [L^p(\mathbb{R})]^N + U_{\text{aff}} \) is a complete metric space for the \( L^p \) distance. The following useful quantity

\[ U_{\text{aff}}(x) := \begin{cases} a^- , & x \leq -\varepsilon \\ \left( \frac{\varepsilon - x}{2\varepsilon} \right) a^- + \left( \frac{\varepsilon + x}{2\varepsilon} \right) a^+ , & -\varepsilon < x < \varepsilon \\ a^+ , & x \geq \varepsilon \end{cases} \]
measures the distance from \( U_{aff} \):

\[
\|U\|_{L^p_{aff}(\mathbb{R})} := \|U - U_{aff}\|_{L^p(\mathbb{R})}.
\]

Function (4) will serve also as an a priori upper bound on the action (3) of the minimizer. For \( p, q \in (1, \infty) \), we introduce the affine anisotropic Sobolev space

\[
W^{1; p, q}_{aff}(\mathbb{R}) := \left\{ U \in [L^p_{aff}(\mathbb{R})]^N : U_x \in [L^q(\mathbb{R})]^N \right\}.
\]

(6) is a complete metric space, isometric to a reflexive Banach space for the distance

\[
d_{1; p, q}(U, V) := \|U - V\|_{L^p(\mathbb{R})} + \|U_x - V_x\|_{L^q(\mathbb{R})}.
\]

The purpose of this work is to establish the following:

**Theorem 2.1. (Existence - Compactness)** Assume that \( W \) satisfies (A1) and either (A2') or (A2''), with \( \alpha_0, \gamma, w_0 \), as in (A1), (A2'), (A2''). There exists a minimizing sequence \( (U_i)_{i=1}^\infty \) of the problem

\[
E(U) = \inf \left\{ E(V) : V \in [W^{1, \gamma, 2}_{aff}(\mathbb{R})]^N \right\}
\]

for (3) with \( E(U_i) \geq 0 \). For any such \( (U_i)_{i=1}^\infty \), there exist \( (x_i)_{i=1}^\infty \subseteq \mathbb{R} \) and translates \( \tilde{U}_i := U_i(\cdot - x_i) \) which have a subsequence converging weakly in \([W^{1, \gamma, 2}_{aff}(\mathbb{R})]^N\) to a minimizer \( U \) which solves (1):

\[
\begin{aligned}
U &= \nabla W(U), & U : \mathbb{R} &\to \mathbb{R}^N, \\
U(-\infty) &= a^-, & \lim_{x \to \infty} U &= a^+.
\end{aligned}
\]

In addition, all minimizing solutions \( U \) satisfy the decay estimates

\[
(*) \quad \begin{cases} 
|U(x) - a^\pm| \leq (Mw_0^{-1})^{\frac{1}{2}} |x|^{-\frac{1}{2}}, & |x| \geq M\alpha_0^{-1}, \\
|U_x(x)| \leq (2M)^{\frac{1}{2}} |x|^{-\frac{3}{2}}, & |x| \geq M\alpha_0^{-1}, 
\end{cases}
\]

as well as the bound \( E(U) \leq M \), with

\[
M = |a^+ - a^-| \max_{|x - \alpha|} \sqrt{2W}.
\]

**Corollary 2.2.** \((*)\) imply that the solution is nontrivial. In particular, \( U \neq a^\pm \).

Theorem 2.1 asserts that translation invariance of (1) and (3) causes the only possible loss of compactness to minimizing sequences. The space \([W^{1, \gamma, 2}_{aff}(\mathbb{R})]^N\) plays a special role to this description. Estimates \((*)\) is an essential property, satisfied uniformly by the compactified sequence of the translates and may not be satisfied by the initial \( (U_i)_{i=1}^\infty \). \((*)\) also guarantee that \( U(\pm \infty) = a^\pm \) and \( U_x(\pm \infty) = 0 \), both fully, not merely up to subsequences.

3. **Proof of the Main Result.**

**Control on the minimizing sequence.** Let \((U_i)_{i=1}^\infty\) be any minimizing sequence of (3). We will tacitly identify \( U_i \) with its precise representatives. Since

\[
|U(x'') - U(x')| \leq (x'' - x')^\frac{1}{2} \left( \int_{x'}^{x''} |U_x|^2 \, dx \right)^{\frac{1}{2}},
\]

we have the inclusion \([W_{aff}^{1,\gamma,2}(\mathbb{R})]^N \subseteq [C_{loc}^0(\mathbb{R})]^N\). By (4), we obtain
\[
E(U_{aff}^\varepsilon) = \int_{-\varepsilon}^{\varepsilon} \left\{ \frac{|a^+ - a^-|^2}{8\varepsilon^2} + W \left( \frac{\varepsilon - x}{2\varepsilon} a^- + \left( \frac{\varepsilon + x}{2\varepsilon} \right) a^- \right) \right\} dx
\]
and hence the explicit bounds
\[
\frac{|a^+ - a^-|^2}{4\varepsilon} \leq E(U_{aff}^\varepsilon) \leq \frac{|a^+ - a^-|^2}{4\varepsilon} + 2\varepsilon \max_{[a^-,a^+]} W. \tag{8}
\]
We immediately get
\[
\inf_{\varepsilon > 0} E_{[W_{aff}^{1,\gamma,2}(\mathbb{R})]^N} \leq \inf_{\varepsilon > 0} E(U_{aff}^\varepsilon) \leq \max_{[a^-,a^+]} \sqrt{2W} = M < \infty.
\]
\(M\) is necessarily a strict upper bound since all \(U_{aff}^\varepsilon\) are merely Lipschitz while minimizing solutions to (1) must be smooth. Further, for \(i\) large we have
\[
\int_{\mathbb{R}} \frac{1}{2} |(U_i)_x|^2 dx + \int_{\mathbb{R}} W(U_i) dx \leq M. \tag{9}
\]
We now derive the \([L^\infty(\mathbb{R})]^N\) bounds. They are obtained in two different ways, depending on whether \((A2')\) of \((A2'')\) is assumed. In the case of \((A2')\), it is a consequence of the next energy estimate. For \(\alpha \in [0,\alpha_0]\) and \(i = 1, 2, ...\), we define the control set
\[
\Lambda_i^\alpha := \{ x \in \mathbb{R} : W(U_i) \geq \alpha \}. \tag{10}
\]
Let \(| \cdot |\) denote the Lebesgue measure on \(\mathbb{R}\) and \(M\) the constant in estimates (\(*\)).

**Lemma 3.1. (Energy Estimate I)** Assume \(W\) satisfies \((A2')\). Then we have
\[
M \geq \alpha \left| \Lambda_i^\alpha \right| + \frac{1}{2} \int_{\mathbb{R}} |(U_i)_x|^2 dx, \tag{11}
\]
\[
\sup_{i \geq 1} \left\| U_i \right\|_{[L^\infty(\mathbb{R})]^N} \leq \left| \Lambda_i^\alpha \right|^\frac{1}{2} \left( \int_{\mathbb{R}} |(U_i)_x|^2 dx \right)^\frac{1}{2} + \max_{u \in \{W \leq \alpha\}^\pm} |u|. \tag{12}
\]

**Proof of Lemma 3.1.** By (9) and (10), we have
\[
M \geq E(U_i) = \int_{\mathbb{R}} W(U_i) dx + \frac{1}{2} \int_{\mathbb{R}} |(U_i)_x|^2 dx
\]
\[
\geq \int_{\Lambda_i^\alpha} W(U_i) dx + \frac{1}{2} \int_{\mathbb{R}} |(U_i)_x|^2 dx
\]
\[
\geq \alpha \left| \Lambda_i^\alpha \right| + \frac{1}{2} \int_{\mathbb{R}} |(U_i)_x|^2 dx.
\]
This proves (11). Let now \(t, t_0\) be the endpoints of an interval \(I_i^\alpha \subseteq \Lambda_i^\alpha\) which is either \([t_0, t]\) or \([t, t_0]\) for which either \(U_i(t_0) \in \{W = \alpha\}^-\) or \(U_i(t_0) \in \{W = \alpha\}^+\) respectively. This means \(W(U_i) \geq \alpha\) on \(I_i^\alpha = [\min\{t, t_0\}, \max\{t, t_0\}]\). Since
\[
|U_i(t) - U_i(t_0)| \leq \left| I_i^\alpha \right|^\frac{1}{2} \left( \int_{I_i^\alpha} |(U_i)_x|^2 dx \right)^\frac{1}{2} \leq \left| \Lambda_i^\alpha \right|^\frac{1}{2} \left( \int_{\mathbb{R}} |(U_i)_x|^2 dx \right)^\frac{1}{2},
\]
by using that \(U_i(t_0) \in \{W = \alpha\}^\pm\), we deduce
\[
|U_i(t) - U_i(t_0)| \geq |U_i(t)| - |U_i(t_0)| \geq |U_i(t)| - \max_{u \in \{W \leq \alpha\}^\pm} |u|.
\]
This establishes estimate (12), proving Lemma 3.1. □

Corollary 3.2. \((L^\infty \text{ bound under } (A2'))\) If \(W\) satisfies (A1), (A2'), then

\[
\sup_{i \geq 1} \|U_i\|_{L^\infty(\mathbb{R})}^N \leq \sqrt{\frac{2}{\alpha_0}} M + \max_{u \in \{W \leq \alpha_0\}} |u|.
\]

Now we turn to the case of \((A2'')\). We obtain existence of a minimizing sequence \((U_i)^\infty_{i=1}\) of (3) localized inside \(\overline{\Omega} \subseteq \mathbb{R}^N\) whereon \(W|_{\overline{\Omega}} \geq 0\).

Lemma 3.3. \((L^\infty \text{ bound under } (A2''))\) If \(W\) satisfies (A1), (A2''), there is a minimizing sequence \((U_i)^\infty_{i=1}\) for which \(\bigcup_{i=1}^\infty U_i(\mathbb{R}) \subseteq \overline{\Omega}\) and \(W(U_i) \geq 0\). Moreover,

\[
\sup_{i \geq 1} \|U_i\|_{L^\infty(\mathbb{R})}^N \leq \max_{u \in \partial \Omega} |u|.
\]

Proof of Lemma 3.3. We show the existence of a deformation of \(W\) to a new \(\overline{W}\) such that \(\overline{W} = W\) on \(\Omega\) and all the minimizing sequences of (3) relative to \(\overline{W}\) in \([W_{\text{aff}}^{1,\gamma,2}(\mathbb{R})]^N\) are localized inside \(\Omega\). By (A2''), \(W < w_{\text{max}}\) inside \(\Omega\) and \(W = w_{\text{max}}\) on \(\partial \Omega\). We define \(\overline{W} := \max\{\}\) is generated by the reflection the graph of \(W\) with respect to the hyperplane \(\{w = w_{\text{max}}\}\), by reflecting all the portions of \(W\) which lie in the halfspace \(\{w < w_{\text{max}}\}\), to \(\{w > w_{\text{max}}\}\).

![Figure 3: The deformed coercive potential \(\overline{W}\), for which \(w = w_{\text{max}}\) is a lower bound outside of \(\Omega\).](image)

By construction, \(\overline{W}(u) \geq w_{\text{max}}\), for \(u \in \mathbb{R}^N \setminus \Omega\). Suppose for the shake of contradiction that \(\overline{W}\) has a minimizing sequence \((U_i)^\infty_{i=1}\) such that for some \(U_i\) and \(a < b\), \(U_i([a, b]) \subseteq \mathbb{R}^N \setminus \Omega^1\). By replacing \(U_i([a, b])\) by the straight line segment with the same endpoints, i.e. defining

\[
U_i(x) := \begin{cases} U_i(x), & x < a, \quad x > b \\ (x - a) / (b - a) U_i(b) + (b - x) / (b - a) U_i(a), & x \in [a, b] \end{cases}
\]

we obtain by convexity of \(\Omega\) that \(\overline{U}_i(\mathbb{R}) \subseteq \Omega\). By pointwise comparison,

\[
\int_a^b \overline{W}(U_i(x)) \, dx \leq \int_a^b \overline{W}(U_i(x)) \, dx.
\]

In addition, \(\overline{U}_i|_{(a,b)}\) minimizes the Dirichlet integral since it is a straight line, thus

\[
\frac{|U_i(b) - U_i(a)|}{b - a} = \int_a^b |(U_i)_x|^2 \, dx < \int_a^b |(U_i)_x|^2 \, dx.
\]

(16) and (17) imply that all minimizing sequences of the Action (3) with the potential \(\overline{W}\) in the place of \(W\) lie inside \(\Omega\). Finally, \(W|_{\Omega} = \overline{W}|_{\Omega}\) by construction. □

\(^1\text{This is the only case that has to be excluded since by the definition of } [W_{\text{aff}}^{1,\gamma,2}(\mathbb{R})]^N \text{ the "tails" of each } U_i \text{ approach asymptotically } a^\infty \in \Omega, \text{ at least along a sequence.}\)
In the case that \((A2')\) is assumed, we fix a sequence valued inside \(\Omega\). Moreover,

\[
M \geq \liminf_{i \to \infty} E(U_i) =: \inf \left\{ E(V) : V \in [W^{1,\gamma,2}_a(\mathbb{R})]^N \right\} \geq 0.
\]

As the notation suggests, the right hand side will henceforth stand for \(\liminf E(U_i)\).

Now we employ \((A1)\) to show that \(\Lambda_i^\alpha\) is connected. For \(\alpha \in [0, \alpha_0], i = 1, 2, ...,\) set

\[
\lambda_i^{\alpha^-} := \inf \Lambda_i^\alpha, \quad \lambda_i^{\alpha^+} := \sup \Lambda_i^\alpha.
\]

**Lemma 3.4.** \((\text{Control on the \(\lambda^{\alpha\pm}\) times})\) Assume \(W\) satisfies \((A1)\) and \((A2')\) or \((A2'')\). Then, for \(\alpha \in [0, \alpha_0], i = 1, 2, ...,\) the sets \(\Lambda_i^\alpha\) are intervals and

\[
\Lambda_i^\alpha = [\lambda_i^{\alpha^-}, \lambda_i^{\alpha^+}].
\]

**Proof of Lemma 3.4.** By convexity of \(\{W \leq \alpha\}\) of \(\{W \leq \alpha\}\), we apply again the comparison argument of Lemma 3.3 to see that \(\Lambda_i^\alpha\) is connected. It suffices to exclude the existence of an \([a, b]\) for which \(U_i((a, b))\) inside \(\{W \leq \alpha\}\) and \(\lambda_i^{\alpha^-} < a < b < \lambda_i^{\alpha^+}\) (Figure 4). On the complement of \([a, b]\), by replacing the connected components of \(U_i\) which have endpoints on the same \(\{W = \alpha\}\) by \(\overline{U_i}\) (given by \((15)\)), we obtain by \((16)\) and \((17)\) that \(E(\overline{U_i}) < E(U_i)\). This contradicts minimality of \(E(U_i)\), establishing therefore the Lemma.

\[
\text{Figure 4: } \mathbb{R}^{N=2}, \text{ the level sets } \{W = \alpha\}^{\pm} \text{ and the control set } \Lambda_i^\alpha \text{ of a minimizing function } U_i.
\]

The following sharpens \((11)\), under the additional information that \(\Lambda_i^\alpha\) is connected.

**Lemma 3.5.** \((\text{Energy estimate II})\) If \(d_\alpha := \text{dist}(\{W = \alpha\}^-, \{W = \alpha\}^+)\) is the distance between the two components, then for \(\alpha \in [0, \alpha_0]\) and \(i \geq 1\), we have

\[
M \geq E(U_i) \geq \frac{d_\alpha^2}{2(\lambda_i^{\alpha^+} - \lambda_i^{\alpha^-})} + \alpha (\lambda_i^{\alpha^+} - \lambda_i^{\alpha^-}).
\]

**Proof of Lemma 3.5.** Proceeding as in Lemma 3.1, we recall \((9)\) to obtain

\[
M \geq E(U_i) \geq \alpha (\lambda_i^{\alpha^+} - \lambda_i^{\alpha^-}) + \frac{1}{2} \int_{\lambda_i^{\alpha^-}}^{\lambda_i^{\alpha^+}} \| (U_i)_x \|^2 dx,
\]

where we have also used Lemma 3.4. In addition,

\[
d_\alpha \leq |U_i(\lambda_i^{\alpha^-}) - U_i(\lambda_i^{\alpha^+})| \leq (\lambda_i^{\alpha^+} - \lambda_i^{\alpha^-})^{\frac{1}{2}} \left( \int_{\lambda_i^{\alpha^-}}^{\lambda_i^{\alpha^+}} \| (U_i)_x \|^2 dx \right)^{\frac{1}{2}}.
\]

The Lemma follows.

**Corollary 3.6.** \((\text{Uniform bounds on } |\Lambda_i^\alpha|)\) For \(i = 1, 2, ..., \alpha \in [0, \alpha_0]\), we have

\[
\frac{d_\alpha^2}{2M} \leq |\Lambda_i^\alpha| = \lambda_i^{\alpha^+} - \lambda_i^{\alpha^-} \leq \frac{M}{\alpha}.
\]
**Corollary 3.7. (Uniform bounds for the compactified sequence)** For \( i = 1, 2, \ldots \) and \( \alpha \in [0, \alpha_0] \), (20) can be rewritten in view of (21), (22), (23) as

\[
\frac{d_\alpha^2}{2M} \leq |\tilde{\Lambda}_i^\alpha| = \tilde{\lambda}_i^\alpha - \tilde{\lambda}_i^\alpha \leq \frac{M}{\alpha}.
\]

In particular, since \( 0 \in \tilde{\Lambda}_i^\alpha \) for \( \alpha \in [0, \alpha_0] \) and \( i = 1, 2, \ldots \), we have

\[
\max\{|	ilde{\lambda}_i^{\alpha-}|, |	ilde{\lambda}_i^{\alpha+}|\} \leq \frac{M}{\alpha}.
\]

**Bounds and Decay Estimates for the Compactified Sequence.** The \([L^2(\mathbb{R})]^N\) bound on the derivatives \((\tilde{U}_i)_x\) is immediate by the kinetic energy term of (3). The more interesting uniform \([L^\infty(\mathbb{R})]^N\) bound is a consequence of the \([L^\infty(\mathbb{R})]^N\) and the assumption (A1) on the nonconvex potential term.

**Lemma 3.8. (Estimates for the compactified sequence)** Let \((\tilde{U}_i)_1^\infty\) be given by (21) and (22). If \( W \) satisfies (A1) and either (A2') or (A2''), then \((\tilde{U}_i)_1^\infty\) lies in a ball of \([W^{1,2}_{\text{aff}}(\mathbb{R})]^N \cap [L^\infty(\mathbb{R})]^N\) centered at \( U_{\text{aff}} \). Moreover,

\[
\sup_{i \geq 1} \|\tilde{U}_i\|_{[L^\infty(\mathbb{R})]^N} \leq M^{\frac{1}{2}} \left\{ \frac{1}{w_0} + \frac{2}{\alpha_0} \sup_{i \geq 1} \left\| \tilde{U}_i \right\|_{[L^\infty(\mathbb{R})]^N} \right\}^{\frac{1}{2}}
\]
(27) \[ \sup_{i \geq 1} \| \tilde{U}_i \|_{[L^\infty(\mathbb{R})]^N} \leq \begin{cases} \sqrt{2} M + \max_{u \in \{ W \leq \alpha_0 \}} |w|, & \text{under (A2')} \\ \max_{u \in \{ W \leq \alpha_0 \}} |u|, & \text{under (A2'')} \end{cases} \]

(28) \[ \sup_{i \geq 1} \| (\tilde{U}_i)_x \|_{[L^2(\mathbb{R})]^N} \leq \sqrt{2M}. \]

**Proof of Lemma 3.8.** (28) follows from translation invariance, while (27) follows by (13), (14) and translation invariance. Thus, we only need to prove (26). For, 

\[ M \geq \int_{\mathbb{R}} W(U_i) dx = \int_{\mathbb{R}} W(\tilde{U}_i) dx \geq \int_{-\infty}^{-M \alpha_0} W(\tilde{U}_i) dx + \int_{M \alpha_0}^{+\infty} W(\tilde{U}_i) dx. \]

Utilizing (25), we obtain \( W(\tilde{U}_i(x)) \leq \alpha \), for \( i = 1, 2, \ldots \) when \( |x| \geq M \alpha^{-1} \). Thus, for such \( x \) we are in the domain of validity of (A1). For \( \alpha = \alpha_0 \), we get 

\[ w_0 \left( \int_{-\infty}^{-M \alpha_0} |\tilde{U}_i - a^-|^\gamma dx + \int_{M \alpha_0}^{+\infty} |\tilde{U}_i - a^+|^\gamma dx \right) \leq M. \]

By restricting to smaller \( \alpha \leq \alpha_1 (< \alpha_0) \), we may assume that \( [-M \alpha_0^{-1}, +M \alpha_0^{-1}] \supset [-1, 1] \). Hence, \( U_{\text{aff}} \) is \( a^\pm \) for \( |x| \geq M \alpha_0^{-1} \). To conclude, we employ (27) to get 

\[ \int_{-\infty}^{\frac{M \alpha_0}{\alpha}} |\tilde{U}_i - U_{\text{aff}}|^\gamma dx \leq \frac{2M}{\alpha_0} \left\{ \| \tilde{U}_i \|_{[L^\infty(\mathbb{R})]^N} \right\}^\gamma. \]

Putting these estimates together, we see that (26) has been established.

**Lemma 3.9. (Uniform decay estimate)** If \( W \) satisfies (A1), the compactified sequence \( (\tilde{U}_i)_i \) satisfies \( |\tilde{U}_i(x) - a^\pm| \leq (Mw_0^{-1})^{\frac{1}{\gamma}} |x|^{-\frac{1}{\gamma}}, \) for \( |x| \geq M \alpha_0^{-1} \).

**Proof of Lemma 3.9.** We have already seen in Lemma 3.8 that (25) implies \( W(\tilde{U}_i(x)) \leq \alpha \), for \( i = 1, 2, \ldots \) when \( |x| \geq M \alpha^{-1} \). By (A1), \( w_0 \left( |\tilde{U}_i(x) - a^-|^\gamma \right) \leq W(\tilde{U}_i(x)) \leq \alpha \), for all such \( x \in \mathbb{R} \). Therefore, \( |\tilde{U}_i(x) - a^\pm|^\gamma \leq \alpha w_0^{-1} \), for all \( |x| \geq M \alpha^{-1} \) and all \( \alpha \leq \alpha_0 \). We fix an \( x \in \mathbb{R} \) for which \( |x| \geq M \alpha_0^{-1} \) and choose \( \alpha = \alpha(x) := M|x|^{-1} \). This is a legitimate choice since \( |x| = M \alpha(x)^{-1} \geq M \alpha_0^{-1} \). We thus obtain that \( |\tilde{U}_i(x) - a^\pm|^\gamma \leq \alpha(x) w_0^{-1} \leq M w_0^{-1} |x|^{-1} \) and letting \( x \) vary, the estimate follows.

**Corollary 3.10. (A priori decay estimates)** Assume \( W \) satisfies (A1). Then, if a solution \( U \) to (1) exists, it must satisfy estimates (*) of Theorem 2.1.

**Proof of Corollary 3.10.** We recall from [AF] the equipartition property \( |U_x|^2 = 2W(U) \) satisfied by solutions of (1). Equipartition implies \( |U_x|^2 = 2W(U) \leq 2\alpha, \) for \( |x| \geq M \alpha^{-1} \) and \( \alpha \leq \alpha_0 \). The rest follows closely the proof of Lemma 3.9.

**Passage to a minimizing solution.** We conclude by proving existence of minimizers. By (26), (27) and (28), the sequence of translates \( (\tilde{U}_i)_i \) converges to some \( U \) weakly in \( [W_{\text{aff}}^{1,\gamma^2}(\mathbb{R})]^N \). By denoting the subsequence again by \( (\tilde{U}_i)_i \), we have \( \tilde{U}_i - U \rightharpoonup 0 \) in \( [L^\gamma(\mathbb{R})]^N \) and \( (\tilde{U}_i - U)_x \rightharpoonup 0 \) in \( [L^2(\mathbb{R})]^N \) as \( i \to \infty \). By \( L^p \) interpolation and Rellich’s theorem, we obtain up to a further subsequence that
\( \tilde{U}_i \to U \) in \( [L^2_{loc}(\mathbb{R})]^N \) and a.e. on \( \mathbb{R} \) as \( i \to \infty \). By weak lower semicontinuity of the \( L^2 \) norm, a.e. convergence \( W(\tilde{U}_i) \to W(U) \) and Fatou’s Lemma, we obtain \( E(U) \leq \liminf_{i \to \infty} E(\tilde{U}_i) \). By (8), we also get \( 0 \leq E(U) \leq M \). Thus \( U \) is a local minimizer of the functional (3) in \( [W^{1,2}_{aff}(\mathbb{R})]^N \). Since \( [\mathcal{C}^\infty_0(\mathbb{R})]^N + U_{aff} \) is dense in \( [W^{1,2}_{aff}(\mathbb{R})]^N \), by standard arguments \( \tilde{U} \) solves (1) and satisfies estimates (\( \ast \)) of Theorem 2.1. The proof of Theorem 2.1 is complete.

\( \Box \)

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References


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