A NOTE ON FRACTIONAL ORDER POINCARÉ'S INEQUALITIES

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Abstract. This note is devoted to the study of fractional order Poincaré's inequalities on different manifolds. Unlike compactness-uniqueness argument, by the spectrum theory and the frequency decomposition technique, we give the precise Poincaré constants, which depend on the geometric structure of the manifolds and symbols of the fractional order pseudodifferential operators.

Résumé. Cette note est consacrée à l'étude des inégalités de Poincaré généralisées d'ordre fractionnaire sur des variétés différentes. Contrairement à l'argument de compacité-unicité, par la théorie du spectre et la technique de la décomposition de fréquence, nous donnons des constantes de Poincaré précises, qui dépendent de la structure géométrique des variétés et des symboles des opérateurs pseudodifférentiels fractionnaires.

1. Introduction

Pseudodifferential operators, especially fractional order operators (also called Riesz fractional derivatives) are very important mathematical models which describe plenty of anomalous dynamic behaviors in our daily life, such as charge carrier transport in amorphous semiconductors, nuclear magnetic resonance diffusometry in percolative and porous media, transport on fractal geometries, diffusion of a scalar tracer in an array of convection rolls, dynamics of a bead in a polymeric network, transport in viscoelastic materials, etc. More interesting industrial applications and modeling process please refer to [8][9][10][14].

For $u \in H_0^1(\Omega)$, where Ω is a bounded Lipschitz domain in $\mathbb{R}^N$, classical Poincaré's inequality establishes a relationship between $\|u\|_{L^2(\Omega)}$ and $\|\nabla u\|_{L^2(\Omega)}$ by compactness-uniqueness argument. Due to this observation, in the bounded domain case, homogeneous Sobolev space is equivalent to the nonhomogeneous one. This inequality shows us that the L^2 norm of u is bounded above by the L^2 norm of its gradient. However, gradient operator is too restricted, what we are interested in is the estimate for general pseudodifferential operators, especially for the fractional order pseudodifferential operators on different manifolds. In Section 2, we introduce the precise definitions of pseudodifferential operators on different manifolds, namely, on $\mathbb{R}^N$, compact manifolds without boundary, for instance, torus $\mathbb{T}^N$ and unit sphere $\mathbb{S}^N$, and also smooth manifolds with boundary. Besides, we present several typical operators which are extensively investigated by mathematicians. It is known that the classical Poincaré's inequality is proved by compactness-uniqueness argument, which ignores the geometric impact. Consequently, the structure of Poincaré constant is unclear. In Section 2.1-2.3, we prove the fractional order Poincaré's inequalities for abstract pseudodifferential operators on different manifolds. This part generalizes the classical inequality extraordinarily and gives the precise Poincaré constants, which depend on the symbols of the linear operators and geometric structure of the manifolds. In particular, the boundedness of domain can be dismissed for certain pseudodifferential operators, for instance, the harmonic oscillating operator $-\Delta + |x|^2$ in [21]. In reality, these inequalities are very important in the analysis of domain decomposition, multilevel iterative methods for elliptic problems, etc.

2. Fractional order Poincaré's inequalities on different manifolds

Let M be a smooth manifold. A continuous linear operator $P : C_0^\infty(M) \to \mathscr{D}'(M)$ is a pseudodifferential operator in $OPS^m(M)$ provided its Schwartz kernel K is C^∞ off the diagonal in $M \times M$, and there exists an open cover Ω_j of M, a subordinate partition of unity ψ_j, and diffeomorphisms $F_j : \Omega_j \to \mathscr{O}_j \subset \mathbb{R}^N$ that transform the operators $\psi_k P \psi_j : C^\infty(\Omega_j) \to \mathscr{E}'(\Omega_k)$ into pseudodifferential operators in $OPS^m(\mathbb{R}^N)$. If M is noncompact, it is often of interest to place specific restrictions on K near infinity. For instance, since $\mathbb{R}^N$ is noncompact, we insist that the kernel be properly supported to avoid the negative consequence that pseudodifferential operators defined above cannot always be composed. And under this assumption, $P : C_0^\infty(\mathbb{R}^N) \to C_0^\infty(\mathbb{R}^N)$. If M is compact, this problem does not arise. In the following, we give the precise definitions of PDO on the compact smooth manifold with or without boundary and the

This work belongs to the following projects: 246775 - NUMERIWAVES funded by European Research Council (ERC), MTM2008-03541, funded by MICIN, ESF Research Networking Programme OPTPDE, PI2010-04, funded by Basque Government.

Work was supported by the Key Laboratory of Systems and Control, Chinese Academy of Sciences.

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AMS 2010 subject classification: 35S05, 35S11, 58J50.

Key words and phrases: Fractional pseudodifferential operators, Poincaré's inequality, spectrum theory, decomposition of frequency.

Mots-clés: Opérateurs pseudo-différentiels fractionnaires, inégalités de Poincaré, théorie du spectre, décomposition de fréquence.

whole space $\mathbb{R}^N$. On top of this, fractional order Poincaré's inequalities with explicit Poincaré constants on different manifolds are given.

2.1. **Discussion on the compact smooth manifold without boundary.** In this section we mainly consider the pseudodifferential operators (PDO) defined on the torus $\mathbb{T}^N$, which is a typical compact smooth manifold without boundary. The discussion can be applied to general Laplace-Beltrami operators successfully, such as on the sphere $\mathbb{S}^N$, etc. First and foremost, we give the precise definition of PDO on the torus.

Definition 2.1. *On the torus $\mathbb{T}^N$, let $u \in C^\infty(\mathbb{T}^N)$, then the sequence of Fourier series $\{\hat{u}(m)\}_{m\in\mathbb{Z}^N}$ defined by*

$$\hat{u}(m) \triangleq (2\pi)^{-N} \int_{\mathbb{T}^N} u(x)\exp(-i\langle m, x\rangle)dx, \tag{1}$$

is a rapidly decreasing sequence.([15]) By duality, we may produce an extension to the periodic distributions:

$$\mathcal{F} : \mathscr{D}'(\mathbb{T}^N) \to \mathscr{S}'(\mathbb{Z}^N), \quad \mathcal{F}^{-1} : \mathscr{S}'(\mathbb{Z}^N) \to \mathscr{D}'(\mathbb{T}^N).$$

With the Fourier transform, we define a generalized linear pseudodifferential operator

$$F(\sqrt{-\Delta}) : D(F(\sqrt{-\Delta})) \subset L^2(\mathbb{T}^N) \to L^2(\mathbb{T}^N)$$

as

$$F(\sqrt{-\Delta})u(x) \triangleq \sum_{m\in\mathbb{Z}^N} F(|m|)\mathcal{F}u(m)\exp(i\langle m, x\rangle). \tag{2}$$

The sequence $\{F(|m|) : m \in \mathbb{Z}^N\}$ is referred to as the torus symbol of $F(\sqrt{-\Delta})$, which is also a polynomially bounded sequence.

Theorem 2.2. *The RHS of (2) converges in the distributional sense. Moreover, when F is a real-valued functional, then the operator $F(\sqrt{-\Delta})$ in Definition 2.1 is a self-adjoint operator. We assume that the average of $u \in L^2(\mathbb{T}^N)$ on the torus $\mathbb{T}^N$ satifies*

$$\int_{\mathbb{T}^N} u(x)dx = 0. \tag{3}$$

And let F be a continuous, increasing and polynomially bounded real-valued functional on $[0,\infty)$, in particular, $F(x) > 0$ for $x > 0$. Then we have the following fractional order Poincaré's inequality,

$$\|u\|_{L^2(\mathbb{T}^N)} \le F^{-1}(1)\|F(\sqrt{-\Delta})u\|_{L^2(\mathbb{T}^N)}.$$

Proof. (I) Actually, in Definition 2.1, $F\Big(\sqrt{-\Delta}\Big)$ is defined in the distributional sense. Indeed, for $\forall\eta \in \mathscr{D}(\mathbb{T}^N)$, since $-\Delta(\mathscr{D}(\mathbb{T}^N)) = \mathscr{D}(\mathbb{T}^N)$, then there exists a unique $\eta_k \in \mathscr{D}(\mathbb{T}^N)$ such that $\underbrace{-\Delta\cdots-\Delta}_{k}\eta = \eta_k$ for each $k \in \mathbb{N}$. As a result,

$$\begin{aligned}
&\Big(\exp(i\langle m, x\rangle), \eta_k\Big)_{L^2(\mathbb{T}^N)} = \Big(\exp(i\langle m, x\rangle), \underbrace{-\Delta\cdots-\Delta}_{k}\eta\Big)_{L^2(\mathbb{T}^N)}\\
= \; & \Big(-\Delta\exp(i\langle m, x\rangle), \underbrace{-\Delta\cdots-\Delta}_{k-1}\eta\Big)_{L^2(\mathbb{T}^N)}\\
= \; & |m|^2\Big(\exp(i\langle m, x\rangle), \underbrace{-\Delta\cdots-\Delta}_{k-1}\eta\Big)_{L^2(\mathbb{T}^N)}\\
= \; & |m|^{2k}\Big(\exp(i\langle m, x\rangle), \eta\Big)_{L^2(\mathbb{T}^N)}.
\end{aligned}$$

Hölder's inequality tells that

$$|(\exp(i\langle m, x\rangle), \eta_k)_{L^2(\mathbb{T}^N)}| \le \|\exp(i\langle m, x\rangle)\|_{L^2(\mathbb{T}^N)}\|\eta_k\|_{L^2} = \sqrt{\int_{\mathbb{T}^N} 1dx}\|\eta_k\|_{L^2(\mathbb{T}^N)}.$$

As a result, $\{|(\exp(i\langle m, x\rangle), \eta)_{L^2(\mathbb{T}^N)}|\}_m$ is a rapidly decreasing sequence w. r. p. to m. On the other hand, $\{\hat{u}(m)\}_m$ is a polynomially bounded sequence w. r. p. to m. Since F is also a polynomially bounded function, consequently, the series on the RHS converges. i.e.

$$\sum_{m\in\mathbb{Z}^N} F(|m|)\mathcal{F}u(m)(\exp(i\langle m, x\rangle), \eta)_{L^2(\mathbb{T}^N)} < \infty.$$

(II) Let $u, v \in D(F(\sqrt{-\Delta}))$, then apply Definition 2.1, and one has

$$\begin{aligned}
(F(\sqrt{-\Delta})u, v)_{L^2(\mathbb{T}^N)} &= \Big(\sum_{m\in\mathbb{Z}^N} F(|m|)\mathcal{F}u(m)\exp(i\langle m, x\rangle), \sum_{n\in\mathbb{Z}^N} \mathcal{F}v(n)\exp(i\langle n, x\rangle)\Big)_{L^2(\mathbb{T}^N)} \\
&= \sum_{m\in\mathbb{Z}^N} \Big(F(|m|)\mathcal{F}u(m)\exp(i\langle m, x\rangle), \mathcal{F}v(m)\exp(i\langle m, x\rangle)\Big)_{L^2(\mathbb{T}^N)} \\
&= \Big(\sum_{m\in\mathbb{Z}^N} \mathcal{F}u(m)\exp(i\langle m, x\rangle), \sum_{n\in\mathbb{Z}^N} F(|n|)\mathcal{F}v(n)\exp(i\langle n, x\rangle)\Big)_{L^2(\mathbb{T}^N)} \\
&= (u, F(\sqrt{-\Delta})v)_{L^2(\mathbb{T}^N)}.
\end{aligned}$$

(III) According to Plancherel Theorem,

$$\begin{aligned}
\|u\|^2_{L^2(\mathbb{T}^N)} &= \| \sum_{m\in\mathbb{Z}^N} \mathcal{F}u(m)\exp(i\langle m, x\rangle)\|^2_{L^2(\mathbb{T}^N)} \\
&= \sum_{m\in\mathbb{Z}^N} |\mathcal{F}u(m)|^2 = \sum_{m\in\mathbb{Z}^N/0} |\mathcal{F}u(m)|^2 \\
&\le F^{-2}(1) \sum_{m\in\mathbb{Z}^N/0} F^2(|m|)|\mathcal{F}u(m)|^2 \\
&= F^{-2}(1)\| \sum_{m\in\mathbb{Z}^N/0} F(|m|)\mathcal{F}u(m)\exp(i\langle m, x\rangle)\|^2_{L^2(\mathbb{T}^N)} \\
&= F^{-2}(1)\|F(\sqrt{-\Delta})u\|^2_{L^2(\mathbb{T}^N)}.
\end{aligned}$$

Q. E. D. □

Remark 2.3. *As a matter of fact, the spectrum of Laplace-Beltrami operator Δ_T on the torus $\mathbb{T}^N$ is $\{0, -1^2, -2^2, -3^2, \cdots\}$. And the associated orthonormal basis for $L^2(\mathbb{T}^N)$ is*

$$\Big\{\Big(\int_{\mathbb{T}^N} 1dx\Big)^{-\frac{1}{2}} \exp(i\langle m, x\rangle) : |m| = 0, 1, 2, 3, \cdots\Big\}.$$

And the spectrum of Laplace-Beltrami operator Δ_S on the unit sphere $\mathbb{S}^N$ is

$$\{\lambda_k = -k(k + N - 1), k = 0, 1, 2, \cdots\}.$$

The pseudodifferential operators on $\mathbb{S}^N$ will be given in Section 2.2.

Remark 2.4. *We show an important application in the fractional order pseudodifferential problems. Let $u(x) = \exp(inx)$, $x \in [0, 2\pi]$, $n \in \mathbb{N}_+$, then by applying Definition 2.1, for each $\sigma \in [0, \infty)$, one has*

$$(-\frac{\partial^2}{\partial x^2})^\sigma u(x) = n^{2\sigma} u(x).$$

This property is essential in the construction of counter-examples for regularity discussions [9][10][11] *when the finite propagation speed fails.*

Remark 2.5. *Assumption (3) is important since it assures that $\hat{u}(0) = 0$. For instance, in the one dimensional case, when $u = \sum_{k=1}^{n} \exp(ikx)$, then u satisfies (3). Keep in mind the fact*

$$(-\frac{\partial^2}{\partial x^2})^\sigma u(x) = \sum_{k=1}^{n} k^{2\sigma} \exp(ikx).$$

Therefore, there is a uniform Poincaré constant 1 for such kind of finite combinations u, which means, the estimate is independent of n. Actually, when $F(0) > 0$, then assumption (3) can be dismissed and the Poincaré constant can be $F^{-1}(0)$. A typical example for this case is the positive operator $F(\sqrt{-\Delta}) = 1 - \Delta$.

2.2. **Discussion on a compact smooth manifold with boundary.** In the following we turn to the spectral theory in [16][17]. Assume that Ω is a compact smooth manifold. Let the linear differentiable operator A be self-adjoint, positive definite and with compact resolvent, then its spectrum $\Lambda(\Omega) = \{\lambda_i\}_{i\in\mathbb{N}^+}$ is discrete with finite multiplicity and

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \to \infty.$$

Moreover, there exists an orthonormal and complete system of eigenfunctions $\{\phi_\lambda(x)\}_{\lambda\in\Lambda(\Omega)}$ in $L^2(\Omega)$, namely, for each $\lambda \in \Lambda(\Omega)$,

$$\|\phi_\lambda\|_{L^2(\Omega)} = 1.$$

Actually, when the first eigenvalue is equal to 0, the following topics can be considered similarly. For instance, the Laplacian with Neumann boundary condition, Laplace-Beltrami operators defined on the torus $\mathbb{T}^N$ and unit sphere $\mathbb{S}^N$, which will be given in the form of corollaries. In this section, we mainly focus on the case with positive eigenvalues.

Definition 2.6. *With the above notations, one can define the generalized Fourier transform and Fourier series as follows:*

$$\mathcal{F}f(\lambda) = (f, \phi_\lambda)_{L^2(\Omega)},$$
$$f(x) = \sum_{\lambda\in\Lambda(\Omega)} \mathcal{F}f(\lambda)\phi_\lambda(x).$$

At the moment, we are ready to introduce the pseudodifferential operators on the smooth manifold Ω with boundary.

Definition 2.7. *On the smooth manifold Ω, we define a generalized linear pseudodifferential operator as follows:*

$$F(\sqrt{A}) : D(F(\sqrt{A})) \subset L^2(\Omega) \to \mathscr{D}'(\Omega),$$

$$F(\sqrt{A})u(x) \triangleq \sum_{\lambda\in\Lambda(\Omega)} F(\sqrt{\lambda})\mathcal{F}u(\lambda)\phi_\lambda(x). \tag{4}$$

The sequence $\{F(\sqrt{\lambda}) : \lambda \in \Lambda(\Omega)\}$ is referred to as the symbol of $F(\sqrt{A})$, which is a polynomially bounded sequence w. r. p. to $\sqrt{\lambda}$.

In some literature, fractional Sobolev spaces are also called Gagliardo or Slobodeckij spaces. When we fix the fractional exponent $s \in (0, 1)$,one can give another definition as follows,

$$H^s(\Omega) \triangleq \{u \in L^2 : \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{2}+s}} \in L^2(\Omega \times \Omega)\},$$

which is endowed with the natural norm

$$\|u\|_{H^s} \triangleq \Big(\int_\Omega |u|^2dx + \int_\Omega\int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}}dxdy\Big)^{\frac{1}{2}}.$$

In particular,

$$[u]_{\dot{H}^s} \triangleq \Big(\int_\Omega\int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}}dxdy\Big)^{\frac{1}{2}},$$

is called Gagliardo(semi) norm.

Theorem 2.8. *The RHS of (4) converges in the distributional sense. Moreover, when F is a real-valued functional, then the operator $F(\sqrt{A})$ in Definition 2.7 is a self-adjoint operator. Let F be a continuous, increasing and polynomially bounded real-valued functional on $[0, \infty)$, in particular, $F(x) > 0$ for $x > 0$. Then we have the following fractional order Poincaré's inequality,*

$$\|u\|_{L^2} \le F^{-1}(\sqrt{\lambda_1})\|F(\sqrt{A})u\|_{L^2}.$$

Proof. (I) In (4), $F\Big(\sqrt{A}\Big)$ is defined in the distributional sense. Indeed, for $\forall\eta \in \mathscr{D}(\Omega)$, since $A(\mathscr{D}(\Omega)) = \mathscr{D}(\Omega)$, then there exists a unique $\eta_k \in \mathscr{D}(\Omega)$ such that $\underbrace{A \cdots A}_{k}\eta = \eta_k$ for each $k \in \mathbb{N}$. As a result,

$$\begin{aligned}
(\phi_\lambda, \eta_k)_{L^2} &= (\phi_\lambda, \underbrace{A \cdots A}_{k}\eta)_{L^2} \\
&= (A\phi_\lambda, \underbrace{A \cdots A}_{k-1}\eta)_{L^2} \\
&= \lambda(\phi_\lambda, \underbrace{A \cdots A}_{k-1}\eta)_{L^2} \\
&= \lambda^k(\phi_\lambda, \eta)_{L^2}.
\end{aligned}$$

Hölder's inequality tells that $|(\phi_\lambda, \eta_k)_{L^2}| \le \|\phi_\lambda\|_{L^2}\|\eta_k\|_{L^2} = \|\eta_k\|_{L^2}$. As a result, $\{|(\phi_\lambda, \eta)_{L^2}|\}_\lambda$ is a rapidly decreasing sequence w. r. p. to λ. On the other hand, by applying Hölder's inequality once more and one has

$$|\mathcal{F}u(\lambda)| = |(u, \phi_\lambda)_{L^2}| \le \|\phi_\lambda\|_{L^2}\|u\|_{L^2} = \|u\|_{L^2}.$$

This indicates, $\{|\mathcal{F}u(\lambda)|\}_\lambda$ is a polynomially bounded sequence w. r. p. to λ. Since F is also a polynomially bounded functional, consequently, the series on the RHS of (4) converges. i.e.

$$\Big\langle F\Big(\sqrt{A}\Big)u(x), \eta\Big\rangle_{\mathscr{D}',\mathscr{D}} = \sum_{\lambda\in\Lambda_{\mathscr{H}^2_{\mathbf{A}}}} F(\sqrt{\lambda})\mathcal{F}u(\lambda)(\phi_\lambda, \eta)_{L^2} < \infty.$$

Hereafter, we consider $u \in L^2$ with $F(\sqrt{A})u \in L^2$.
(II) Let $u, v \in D(F(\sqrt{A}))$, then apply Definition 2.7, and we have

$$\begin{aligned}
(F(\sqrt{A})u, v)_{L^2} &= \Big(\sum_{\lambda\in\Lambda(\Omega)} F(\sqrt{\lambda})\mathcal{F}u(\lambda)\phi_\lambda(x), \sum_{\eta\in\Lambda(\Omega)} \mathcal{F}v(\eta)\phi_\eta(x)\Big)_{L^2} \\
&= \sum_{\lambda\in\Lambda(\Omega)} \Big(\mathcal{F}u(\lambda)\phi_\lambda(x), F(\sqrt{\lambda})\mathcal{F}v(\lambda)\phi_\lambda(x)\Big)_{L^2} \\
&= \Big(\sum_{\lambda\in\Lambda(\Omega)} \mathcal{F}u(\lambda)\phi_\lambda(x), \sum_{\eta\in\Lambda(\Omega)} F(\sqrt{\eta})\mathcal{F}v(\eta)\phi_\eta(x)\Big)_{L^2} \\
&= (u, F(\sqrt{A})v)_{L^2}.
\end{aligned}$$

(III) In fact, since $\{\phi_\lambda(x)\}_{\lambda\in\Lambda(\Omega)}$ is an orthonormal basis in $L^2(\Omega)$, one can easily prove a similar result as Plancherel Theorem.

Lemma 2.9. *F, A are defined above, and $u \in L^2(\Omega)$, then one has*

$$\|F(\sqrt{A})u\|_{L^2}^2 = \sum_{\lambda\in\Lambda(\Omega)} F^2(\sqrt{\lambda})|\mathcal{F}u(\lambda)|^2.$$

Applying the above lemma, one has

$$\begin{aligned}
\|u\|_{L^2}^2 &= \| \sum_{\lambda\in\Lambda(\Omega)} \mathcal{F}u(\lambda)\phi_\lambda(x)\|_{L^2}^2 \\
&= \sum_{\lambda\in\Lambda(\Omega)} |\mathcal{F}u(\lambda)|^2 \\
&\leq F^{-2}(\sqrt{\lambda_1}) \sum_{\lambda\in\Lambda(\Omega)} F^2(\sqrt{\lambda})|\mathcal{F}u(\lambda)|^2 \\
&= F^{-2}(\sqrt{\lambda_1})\| \sum_{\lambda\in\Lambda(\Omega)} F(\sqrt{\lambda})\mathcal{F}u(\lambda)\phi_\lambda(x)\|_{L^2}^2 \\
&= F^{-2}(\sqrt{\lambda_1})\|F(\sqrt{A})u\|_{L^2}^2.
\end{aligned}$$

Q. E. D. □

Remark 2.10. *When $\Omega = (0, \pi)$, we consider the Dirichlet Laplacian $-\Delta : L^2(\Omega) \to L^2(\Omega)$. With simple calculations, one is able to check that, $\{1, 2^2, 3^2, \cdots, n^2, \cdots\}$ is the set of eigenvalues which are bounded away from 0. And the associated orthonormal basis for $L^2(\Omega)$ which consists of eigenfunctions of the Dirichlet Laplacian is*

$$\Big\{\sqrt{\frac{2}{\pi}}\sin(x), \sqrt{\frac{2}{\pi}}\sin(2x), \sqrt{\frac{2}{\pi}}\sin(3x), \cdots, \sqrt{\frac{2}{\pi}}\sin(nx), \cdots\Big\}. \tag{5}$$

Actually, in a general sense, in any bounded Lipschitz domains $\Omega \in \mathbb{R}^N$, there is a countable orthonormal basis in $L^2(\Omega)$ which consists of eigenfunctions of the Dirichlet Laplacian. The eigenfunctions belong to $L^2(\Omega)$ and the associated eigenvalues are all positive and bounded away from zero. Please refer to [5] *for specific proof. The case of Dirichlet Laplacian $-\Delta : H_0^1(\Omega) \to L^2(\Omega)$ please refer to* [11].

Remark 2.11. *When $\Omega = (0, \pi)$, we consider the Dirichlet operator $1 - \Delta : L^2(\Omega) \to L^2(\Omega)$. In this case, $\{1 + 1, 1 + 2^2, 1 + 3^2, \cdots, 1 + n^2, \cdots\}$ is the set of eigenvalues which are bounded away from 0. And the associated orthonormal basis in $L^2(\Omega)$ which consists of eigenfunctions of the Dirichlet operator is also (5).*

Remark 2.12. *When $\Omega = (0, \pi)$, we consider the biharmonic operator $\Delta^2 : H^4(\Omega) \bigcap H_0^1(\Omega) \to L^2(\Omega)$ with the boundary condition $u = \Delta u = 0$. In this case, $\{1, 2^4, 3^4, \cdots, n^4, \cdots\}$ is the set of eigenvalues which are bounded away from 0. And the associated orthonormal basis for $L^2(\Omega)$ which consists of eigenfunctions of the biharmonic operator is also (5).*

Remark 2.13. *Let $\Omega = (0, \pi)$, for the Dirichlet magnetic operator* [11]

$$(i\frac{\partial}{\partial x} - 1)^2 : L^2(\Omega) \to L^2(\Omega),$$

it is easy to calculate that $\{1, 2^2, 3^2, \cdots, N^2, \cdots\}$ is the set of eigenvalues which are bounded away from 0. And the associated orthonormal basis (in the sense of L^2-norm) is

$$\Big\{\sqrt{\frac{2}{\pi}}\sin(x)e^{-ix}, \sqrt{\frac{2}{\pi}}\sin(2x)e^{-ix}, \sqrt{\frac{2}{\pi}}\sin(3x)e^{-ix}, \cdots, \sqrt{\frac{2}{\pi}}\sin(Nx)e^{-ix}, \cdots\Big\}.$$

Remark 2.14. *Assume that there exists a constant $C(\Omega) > 0$ such that the inverse inequality holds, namely,*

$$\|F(\sqrt{A})u\|_{L^2} \leq C(\Omega)\|u\|_{L^2}.$$

Furthermore, if $D(I + F(\sqrt{A}))$ is compactly embedded in L^2, then there exists a constant $m \in \mathbb{N}$, such that $\mathcal{F}u(\lambda) \equiv 0$ for $\forall\lambda \geq m$ according to Riesz Lemma.

Now we turn to the unit sphere $\mathbb{S}^N$, $N \geq 2$, which is a compact smooth manifold without boundary. One mainly considers the operator

$$A \triangleq -\Delta_S + \frac{(N-1)^2}{4} : D(-\Delta_S) \subset L^2(\mathbb{S}^N) \to L^2(\mathbb{S}^N),$$

where Δ_S is the Laplace-Beltrami operator on $\mathbb{S}^N$. In this case, the eigenvalues of A are

$$\Big\{\nu_k = \Big(\frac{N-1}{2} + k\Big)^2 : k = 0, 1, 2, \cdots\Big\}.$$

The eigenspace V_k of A with eigenvalue ν_k is the space of harmonic polynomials, homogeneous of degree k, restriction to $\mathbb{S}^N \subset \mathbb{R}^{N+1}$. Furthermore,

$$\dim V_k = \begin{pmatrix} k+N-2 \\ k-1 \end{pmatrix} + \begin{pmatrix} k+N-1 \\ k \end{pmatrix}.$$

When $N = 1$, this is the torus $\mathbb{T}$. One can easily check that, for each eigenvalue $\nu_k \neq 0$, the associated eigenspace is of 2 dimensions, namely, $\sqrt{\frac{1}{2\pi}}\exp(ikx)$ and $\sqrt{\frac{1}{2\pi}}\exp(-ikx)$ form an orthonormal basis in this eigenspace. From Theorem 2.8, we have the following result on the unit sphere $\mathbb{S}^N$, $N \geq 2$.

Corollary 2.15. *On the unit sphere $\mathbb{S}^N$, $N \geq 2$, let $A \triangleq -\Delta_S + \frac{(N-1)^2}{4}$, F be a continuous, increasing and polynomially bounded real-valued functional on $[0, \infty)$, in particular, $F(x) > 0$ for $x > 0$. Then we have the following fractional order Poincaré's inequality,*

$$\|u\|_{L^2} \leq F^{-1}(\frac{N-1}{2})\|F(\sqrt{-\Delta})u\|_{L^2}.$$

Recall Remark 2.3, we consider the case $A = -\Delta_S$. Combining Theorem 2.2 and Theorem 2.8, one has

Corollary 2.16. *On the unit sphere $\mathbb{S}^N$, $N \geq 2$, let $A \triangleq -\Delta_S$, F be a continuous, increasing and polynomially bounded real-valued functional on $[0, \infty)$, in particular, $F(x) > 0$ for $x > 0$.If $(u, \phi_0)_{L^2(\mathbb{S}^N)} = 0$, then one has the following fractional order Poincaré's inequality,*

$$\|u\|_{L^2} \leq F^{-1}(\sqrt{N})\|F(\sqrt{-\Delta})u\|_{L^2}.$$

In the case of unbounded domains, for instance, we consider the harmonic oscillator $-\Delta + |x|^2 : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$. The eigenvalues are all positive, namely,

$$\{2k + N : k = 0, 1, 2, \cdots\}.$$

And an orthomormal basis of eigenvalue $2k + N$ is given by

$$c_{k_1} \cdots c_{k_N} H_{k_1}(x_1) \cdots H_{k_N}(x_N), k_1 + \cdots + k_N = k,$$

where $k_i \in \{0, \cdots, k\}$, the $H_{k_i}(x_i)$ are the Hermite polynomials given by

$$H_k(x) = (-1)^k \exp(x^2)(\frac{d}{dx})^k \exp(-x^2),$$

and c_{k_i} are given by

$$c_k = \frac{1}{\sqrt{\sqrt{\pi}2^k(k!)}}.$$

In particular, the dimension of this eigenspace is the same as the dimension of the space of homogeneous polynomials of degree k in N variables. Please refer to [17][21] for more detailed discussion.

Corollary 2.17. *Let A be the harmonic oscillator $-\Delta + |x|^2 : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$. And let F be a continuous, increasing and polynomially bounded real-valued functional on $[0, \infty)$, in particular, $F(x) > 0$ for $x > 0$. Then we have the following fractional order Poincaré's inequality,*

$$\|u\|_{L^2} \leq F^{-1}(\sqrt{N})\|F(\sqrt{A})u\|_{L^2}.$$

Assume that $\Omega \in \mathbb{R}^N (N \geq 2)$ is a bounded domain with smooth boundary Γ. Denote by ν the outward unit normal to Γ. Now we consider the problem with Neumann boundary condition

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma \end{cases} \tag{6}$$

Classical theory of eigenvalue problems assures that problem (6) has a sequence of non-negative eigenvalues which tends to infinity and a sequence of corresponding eigenfunctions which define a Hilbert basis in L^2. Furthermore, the first eigenvalue is $\lambda = 0$, which is isolated and simple. As a consequence, one has the following result.

Corollary 2.18. *For the Laplacian with Neumann boundary condition (6), assume that*

$$(u, \phi_0)_{L^2} = 0.$$

Let F be a continuous, increasing and polynomially bounded real-valued functional on $[0, \infty)$, in particular, $F(x) > 0$ for $x > 0$. Then we have the following fractional order Poincaré's inequality,

$$\|u\|_{L^2} \leq F^{-1}(\sqrt{\lambda_1})\|F(\sqrt{-\Delta})u\|_{L^2}.$$

2.3. **A refined fractional order Poincaré's inequality.** For the Dirichlet-Laplacian, from a variational point of view, the lowest eigenvalue λ_1 can be characterized as the minimum of the Reyleigh quotient that is,

$$\lambda_1 = \inf_{u \in H_0^1(\Omega)/\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx}.$$

From [2], we know that the geometry of a Riemannian manifold completely determines the spectrum. Actually, let (M, g) be a closed Riemannian manifold of dimension $N \geq 2$, if the Ricci tensor field $\mathrm{Ric}(X, X) \geq (N-1)k \geq 0$ for some nonnegative constant k and for all $X \in \mathcal{T}(M)$, then the first nonzero eigenvalue has a lower bound, $\lambda_1 \geq \frac{(N-1)k}{4} + \frac{\pi^2}{D^2(M)}$, where $D(M)$ is the diameter of M. As to the upper bound, if the Ricci curvature is greater or equal to $(N-1)(-k)$, $k > 0$, then $\lambda_1 \leq \frac{(N-1)^2 k}{4} + \frac{c_N}{D^2(M)}$, where c_N is a positive constant depending only on N. In Section 2, we describe the Poincaré constant in an abstract manner by virtue of the first nonzero eigenvalue. In this section, with the frequency decomposition method from microlocal analysis, we give an explicit form of the Poincaré constant, which unveils the profound geometric impact. First and foremost, we recall the classical definition of pseudodifferential operators in [18] defined on $\mathbb{R}^N$.

Definition 2.19. *$F(D_x) : D(F(D_x)) \subset \mathcal{S}'(\mathbb{R}^N) \to \mathcal{S}'(\mathbb{R}^N)$ is a generalized linear pseudodifferential operator defined by*

$$F(D_x)u = \mathcal{F}^{-1}(F(\xi)\mathcal{F}(u)).$$

And $F(\xi)$ is referred to as the symbol of $F(D_x)$. Moreover, when $D(F(D_x)) \subset \mathcal{S}(\mathbb{R}^N)$, one has the explicit expression

$$F(D_x)u(x) \triangleq (2\pi)^{-N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \exp(i\langle x - y, \xi\rangle)F(\xi)u(y)dyd\xi.$$

Remark 2.20. *Since $\mathcal{S}$ is dense in L^2, so $F(\sqrt{-\Delta})$ is also a self-adjoint operator in the sense of L^2-norm. In the distributional sense, $\mathcal{F}((2\pi)^{-N}) = \delta(\xi)$. Therefore, once $g \equiv 1$, then*

$$F(\sqrt{-\Delta})g = (2\pi)^N \mathcal{F}^{-1}(F(|\xi|)\delta(\xi)) = F(0).$$

More theories concerned with symbol calculus, pseudo-local property, wave front set, etc., please refer to [18].

Remark 2.21. *Assume that $u \in C_0^\infty(\Omega)$. Let*

$$\bar{u} \triangleq \begin{cases} u(x) & x \in \Omega; \\ 0 & x \in \mathbb{R}^N/\Omega. \end{cases}$$

Since $u \in L^2(\Omega)$, according to Section 2.2, $u = \sum_{\lambda \in \Lambda(\Omega)} \mathcal{F}u(\lambda)\phi_\lambda(x)$. Consequently, $\|u\|^2_{L^2(\Omega)} = \sum_{\lambda \in \Lambda(\Omega)} |\mathcal{F}u(\lambda)|^2$. Actually, when the Sobolev index is a nonnegative integer, Definition 2.7 can be deduced from Definition 2.19. Indeed, *one has*

$$\begin{aligned}
(-\Delta)_x^m \bar{u} &= (2\pi)^{-N} \int_{\mathbb{R}^N} \exp(i\langle x,\xi\rangle)|\xi|^{2m} \int_{\mathbb{R}^N} \exp(-i\langle y,\xi\rangle)\bar{u}(y)dyd\xi \\
&= (2\pi)^{-N} \int_{\mathbb{R}^N} \exp(i\langle x,\xi\rangle)|\xi|^{2m} \int_{\Omega} \exp(-i\langle y,\xi\rangle)u(y)dyd\xi \\
&= (2\pi)^{-N} \int_{\mathbb{R}^N} \exp(i\langle x,\xi\rangle) \int_{\Omega} (-\Delta)_y^m \exp(-i\langle y,\xi\rangle)u(y)dyd\xi \\
&= (2\pi)^{-N} \int_{\mathbb{R}^N} \exp(i\langle x,\xi\rangle) \int_{\Omega} \exp(-i\langle y,\xi\rangle)(-\Delta)_y^m u(y)dyd\xi \\
&= (2\pi)^{-N} \int_{\mathbb{R}^N} \exp(i\langle x,\xi\rangle) \int_{\Omega} \exp(-i\langle y,\xi\rangle)(-\Delta)_y^m \sum_{\lambda\in\Lambda(\Omega)} \mathcal{F}u(\lambda)\phi_\lambda(y)dyd\xi \\
&= \sum_{\lambda\in\Lambda(\Omega)} \mathcal{F}u(\lambda)(2\pi)^{-N} \int_{\mathbb{R}^N} \exp(i\langle x,\xi\rangle) \int_{\Omega} \exp(-i\langle y,\xi\rangle)(-\Delta)_y^m \phi_\lambda(y)dyd\xi \\
&= \sum_{\lambda\in\Lambda(\Omega)} \mathcal{F}u(\lambda)(2\pi)^{-N} \int_{\mathbb{R}^N} \exp(i\langle x,\xi\rangle) \int_{\Omega} \exp(-i\langle y,\xi\rangle)\lambda^m \phi_\lambda(y)dyd\xi \\
&= \sum_{\lambda\in\Lambda(\Omega)} \lambda^m \mathcal{F}u(\lambda)\phi_\lambda(x) \\
&= (-\Delta)_x^m u.
\end{aligned}$$

Theorem 2.22. *When F is a real-valued functional, then the operator $F(\sqrt{-\Delta}) : D(F(\sqrt{-\Delta})) \subset \mathcal{S}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$ is a self-adjoint operator. Assume that $u \in C_0^\infty(\mathbb{R}^N)$, $\mathrm{supp}u \in \Omega$, where Ω is a bounded Lipschitz domain. Let F be a continuous, increasing and polynomially bounded real-valued functional on $[0,\infty)$, in particular, $F(x) > 0$ for $x > 0$. Then we have the following fractional order Poincaré's inequality,*

$$\|u\|_{L^2} \le \sqrt{\frac{1}{\beta}F^{-1}\Big(\sqrt[N]{\frac{1-\beta}{\omega(1)\mathrm{Vol}(\Omega)}}\Big)}\|F(\sqrt{-\Delta})u\|_{L^2}, \quad \forall \beta \in (0,1).$$

Proof. (I) Let $u, v \in D(F(\sqrt{-\Delta}))$, then by applying Definition 2.17, we have

$$\begin{aligned}
\langle F(\sqrt{-\Delta})u, v\rangle_{\mathcal{S}(\mathbb{R}^N)} &= (2\pi)^{-N} \int_{\mathbb{R}^N} \overline{v(x)} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \exp(i\langle x-y,\xi\rangle)F(|\xi|)u(y)dyd\xi dx \\
&= (2\pi)^{-N} \int_{\mathbb{R}^N} u(y) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \overline{\exp(i\langle y-x,\xi\rangle)F(|\xi|)v(x)}dxd\xi dy \\
&= \langle u, F(\sqrt{-\Delta})v\rangle_{\mathcal{S}(\mathbb{R}^N)}.
\end{aligned}$$

In fact, we can also deduce this from the asymptotic expansions of symbols of $F(\sqrt{-\Delta})$ and $F^*(\sqrt{-\Delta})$:

$$\sigma_{F*} \sim \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha D_x^\alpha \overline{\sigma_F(t,x,\xi)} = \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha D_x^\alpha \sigma_F(t,x,\xi) \sim \sigma_F(t,x,\xi).$$

(II) Dividing the frequency into two parts and applying Hölder's inequality, one has

$$\begin{aligned}
\|u\|_{L^2}^2 &= \int_{|\xi|\le\epsilon} |\hat{u}(\xi)|^2 d\xi + \int_{|\xi|\ge\epsilon} |\hat{u}(\xi)|^2 d\xi \\
&= \int_{|\xi|\le\epsilon} |\hat{u}(\xi)|^2 d\xi + \int_{|\xi|\ge\epsilon} \tfrac{F^2(|\xi|)|\hat{u}(\xi)|^2}{F^2(|\xi|)} d\xi \\
&= \int_{|\xi|\le\epsilon} |\int_\Omega u(x)\exp(-ix\cdot\xi)dx|^2 d\xi + \int_{|\xi|\ge\epsilon} \tfrac{F^2(|\xi|)|\hat{u}(\xi)|^2}{F^2(|\xi|)} d\xi \\
&\le \|u\|_{L^2}^2 \omega(1)\mathrm{Vol}(\Omega)\epsilon^N + F^{-2}(\epsilon)\|F(\sqrt{-\Delta})u\|_{L^2}^2
\end{aligned}$$

where $\omega(1)$ is the volume of a unit ball. Let us choose

$$\epsilon = \sqrt[N]{\frac{1-\beta}{\omega(1)\mathrm{Vol}(\Omega)}}, \quad \beta \in (0,1),$$

then we have

$$\|u\|_{L^2}^2 \le \frac{1}{\beta}F^{-2}\Big(\sqrt[N]{\frac{1-\beta}{\omega(1)\mathrm{Vol}(\Omega)}}\Big)\|F(\sqrt{-\Delta})u\|_{L^2}^2.$$

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Remark 2.23. *For fractional order pseudodifferential operators, namely,* $F(\sqrt{-\Delta}) = (\sqrt{-\Delta})^\gamma$, $\gamma \in [0,\infty)$, *choose* $\beta = \frac{1}{2}$ *and one has*

$$\|u\|_{L^2}^2 \leq 2\Big(2\omega(1)\mathrm{Vol}(\Omega)\Big)^{\frac{2\gamma}{N}} \|(\sqrt{-\Delta})^\gamma u\|_{L^2}^2.$$

Particularly, when $\gamma = 1$, *this corresponds to the classical Poincaré's inequality since* $\|\sqrt{-\Delta}u\|_{L^2} = \|\nabla u\|_{L^2}$.

Remark 2.24. *In the proof, one can discover that, the Poincaré's inequality holds since* Supp $\hat{u}$ *does not concentrate around 0 in the frequency space. And* $\sqrt{\frac{1}{\beta}}F^{-1}\Big(\sqrt[N]{\frac{1-\beta}{\omega(1)\mathrm{Vol}(\Omega)}}\Big)$ *is a Poincaré constant. Next we show a typical example in the non-compact case. Given* $\rho > 0$, *let* $\chi_\rho(\xi)$ *be the cut-off function which equals to 1 when* $|\xi_j| \leq \rho_j, j = 1,\cdots,N$, *and equals to zero elsewhere. A function* $u \in L^2(\mathbb{R}^N)$ *satisfies* $\hat{u}(\xi) = 0$ *for a.e.* $|\xi_j| < \rho_j, j = 1,\cdots,N$ *if and only if* $u = v - H_\rho * v$ *a.e. for some* $v \in L^2(\mathbb{R}^N)$, *where the high frequency filter is defined as*

$$H_\rho(x) = \prod_{j=1}^N H_{\rho_j}(x_j) = \pi^{-N}\prod_{j=1}^N \frac{\sin(\rho_j x_j)}{x_j}.$$

This statement is quickly checked by noting the fact that $\mathcal{F}(H_\rho * u) = \chi_\rho \hat{u}$. *Obviously, in this case, the Poincaré's inequality also holds.*

Acknowledgements: The author thanks Prof. Enrique Zuazua for the fruitful discussions. In particular, the author thanks Key Laboratory of Systems and Control of Chinese Academy of Sciences for the financial support and hospitality during his visit in June, 2010.

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