On conforming tetrahedralizations of prismatic partitions

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\textbf{Abstract:} We present an algorithm for conform (face-to-face) subdividing prismatic partitions into tetrahedra. This algorithm can be used in the finite element calculations and analysis.

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1 Introduction

Tetrahedral, prismatic, pyramidal, or block elements are usually used in finite element approximations of various engineering three-dimensional problems. Therefore, a natural question arises which of these elements are the most suitable for a particular problem in a given domain (cf. [1], [3], [7], [8]). The use of several types of elements enables us to compare the influence of the space discretization on the finite element solution.

However, among various types of elements, tetrahedral elements are the most popular for many reasons. For instance, if block trilinear elements are employed, the discrete maximum principle need not be satisfied (see [6], [5]). But if each block element is divided into 6 nonobtuse tetrahedra such that all of them contain the spatial diagonal, then the stiffness matrix associated to linear elements has the same size and the discrete maximum principle is fulfilled for a large class of nonlinear elliptic problems (see [4]). Moreover, the stiffness matrix associated with linear finite elements has less nonzero diagonals than
that one for trilinear block or triangular prism elements. Another important reason for
the use of tetrahedral elements is their flexibility to describe complicated boundaries.

In this work we show how to conformly (face-to-face) subdivide the given prismatic
partition into tetrahedra as local subdivisions of prisms into tetrahedra cannot be done
independently from each other in order to get a face-to-face tetrahedralization.

2 Subdivision of prismatic partitions into tetrahedra

By prism (or more precisely triangular prism) we shall mean a prism with two parallel
triangular faces and three rectangular faces.

Figure 1: Partition into prisms.

In what follows we consider only bounded polyhedra $\Omega \subset \mathbb{R}^3$ which can be decom-
posed into prisms. Let $P_h$ be a face-to-face partition of one such polyhedron into prisms
(see Figure 1). Here $h$ stands for the usual discretization parameter, i.e., the maximum
diameter of all prisms from $P_h$.

Throughout the paper we will always consider only face-to-face partitions, and there-
fore, the notion “face-to-face” will be sometimes omitted. It is evident from the definition
of a prism that any partition $P_h$ into prisms consists of parallel layers of prisms. In the
following we may suppose these layers to be horizontal and we call the bottom plane of a
layer its base and the triangular face of a prism we call the base triangle of the prism.

We shall consider such tetrahedralizations of $P_h$ that the triangular faces of the prisms
are not cut. Hence, the different layers of prisms can be subdivided independently into
tetrahedra and these altogether provide us with a conforming tetrahedral mesh over $\Omega$.

We shall subdivide each prism into three tetrahedra as marked in Figure 2 (left). We
see that its rectangular faces are divided by diagonals into triangles and these diagonals
determine three tetrahedra in the subdivision. However, these diagonals cannot be chosen
arbitrarily. In Figure 2 (right) we observe a division of three rectangular faces of a prism
which does not correspond to any partition of the prism into tetrahedra. Therefore, we
have to divide rectangular faces in the whole partition carefully.

In the next theorem we show how to practically construct from a given prismatic
partition $P_h$ a face-to-face tetrahedralization, thus avoiding the situation illustrated in
Figure 2 (right) (or its mirror image) when dividing rectangular faces by diagonals.

Theorem 2.1 For any conforming partition into prisms there exists a face-to-face sub-
division into tetrahedra.
Proof: From the beginning of this section we know that any partition of $\mathcal{P}_h$ into prisms consists of parallel layers which can be tetrahedralized independently (see Figure 1). Consider one of such layers supposed to be horizontal and let $\mathcal{T}_h$ be the triangulation of its base corresponding to the partition $\mathcal{P}_h$. Take an arbitrary vector $\vec{v} \neq 0$ in the plane containing the triangulation $\mathcal{T}_h$ (for instance $\vec{v} = (1, 0, 0)$). Now we define the orientation $\vec{e}$ of each edge $e$ of the triangulation $\mathcal{T}_h$ such that

$$(\vec{v}, \vec{e}) \geq 0.$$  \hfill (1)

If an edge $e$ is perpendicular to $\vec{v}$, we may take an arbitrary orientation of $\vec{e}$. In this way we get the planar digraph $G_h = (N, E)$, where $N$ is the set of nodes and $E$ is the set of the directed edges.

It is clear that $G_h$ does not contain a directed circle of which edges form a triangle of $\mathcal{T}_h$. Indeed, if, on the contrary, $\vec{e}_1, \vec{e}_2, \vec{e}_3$ form a circle then $\vec{e}_1 + \vec{e}_2 + \vec{e}_3 = \vec{0}$. Taking the scalar product of both sides by $\vec{v}$ and using (1), we get that $\vec{v}$ is perpendicular to the triangle with side vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$, which is a contradiction. $\blacksquare$

Remark 2.1 Non-allowed edge orientations are sketched in Figure 3. In Figure 4 we see a partition of the prism $ABCA'B'C'$ into three tetrahedra different from that in the left part of Figure 2.

We demonstrate now that the tetrahedral mesh generated as above is regular whenever the original prismatic mesh is regular as well.

Definition 2.1 A family of partitions $\mathcal{F} = \{\mathcal{T}_h\}_{h \to 0}$ of a polyhedron $\Omega$ into convex elements is said to be regular (strongly regular) if there exists a constant $\kappa > 0$ such that for any partition $\mathcal{T}_h \in \mathcal{F}$ and any element $T \in \mathcal{T}_h$ we have

$$\kappa h_T^3 \leq \text{meas} \, T$$

(2)

where $h_T = \text{diam} \, T$. 

Figure 2: Two subdivisions of rectangular faces of a prism.

Figure 3: Non-allowed edge orientations.
Remark 2.2. The above definition is equivalent to the inscribed ball condition (see, e.g., [2, Sec. 16]) which is more complicated. Note also that the regularity of a family of partitions into prisms is equivalent to Zlámal’s minimum angle condition [2, p. 128] applied to triangular bases of all prisms provided the height of all prisms is proportional to $h$.

Theorem 2.2. If a family of partitions $\{P_h\}_{h \to 0}$ of a polyhedron $\Omega$ into prisms is regular (strongly regular) then the associated family of partitions $\{T_h\}_{h \to 0}$ into tetrahedra is also regular (strongly regular).

Proof: It is evident that the volume of each of the three tetrahedra from Figure 2 (left) is equal to the one third of the volume of the prism. Therefore, if inequalities (2) hold for prisms, then the same relations hold also for tetrahedra with another constant $\kappa' = \kappa/3$.

Remark 2.3. Assume that a family of partitions of $\Omega$ into prisms is regular and that $\Omega$ has Lipschitz boundary. Then by Theorem 2.2 the optimal interpolation properties of tetrahedral elements in Sobolev space norms are satisfied.

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References


