# Convergence Properties of Overlapping Schwarz Domain Decomposition Algorithms

Minh-Binh TRAN
Basque Center for Applied Mathematics
Bizkaia Technology Park, Building 500, E-48160 Derio - Spain
Email: tbinh@bcamath.org

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**Abstract**

In this paper, we partially answer open questions about the convergence of overlapping Schwarz methods. We prove that overlapping Schwarz methods with Dirichlet transmission conditions for semilinear elliptic and parabolic equations always converge, while overlapping Schwarz methods with Robin transmission conditions only converge for semilinear parabolic equations, but the convergence is not guaranteed for semilinear elliptic ones. We then provide some conditions so that overlapping Schwarz methods with Robin transmission conditions converge for semilinear elliptic equations. Our new techniques can also be potentially applied to others kinds of partial differential equations.

**Keyword** Domain decomposition, Schwarz methods, semilinear parabolic equations, semilinear elliptic equations. Subject Class: 65M12.

## 1 Introduction

The Schwarz domain decomposition methods are procedures to solve partial differential equations in parallel, where each iteration involves the solutions of the original equation on smaller subdomains. The alternating method was originally proposed by H. A. Schwarz [24] in 1870 as a technique to prove the existence of a solution to the Laplace equation on a domain which is a combination of a rectangle and a circle. The idea was then used and extended by P. L. Lions [15], [16], [17] to parallel algorithms for solving partial differential equations. Since then, many kind of domain decomposition methods have been developed, to improve the performance of the classical domain decomposition method.

Many techniques have been developed to prove the convergence of classical Schwarz methods, or Schwarz methods with Dirichlet transmission conditions. One of the first techniques, used by P. L. Lions in [15], is the iterated projections for linear Laplace equation and linear Stokes equation. The idea is to prove that the classical Schwarz methods for these equations can be expressed in terms of sequences of projections in Hilbert Spaces. In the same paper, P. L. Lions also showed that the Schwarz sequences for nonlinear monotone elliptic equations are related to classical minimization methods over product spaces and proposed to use Schwarz methods for evolution equations. This idea was then used by L. Badea in [1] to prove the convergence of classical Schwarz methods for nonlinear monotone elliptic problems.
Following the pinoneering work of P. L. Lions, in the papers [6], [9], [7], E. Giladi, H. B. Keller, A. Stuart and M. Gander used Fourier and Laplace transforms, together with some explicit calculation to study classical Schwaz methods for some 1-dimensional evolution equations, with constant coefficients. Later, by using a maximum principle argument, M. Gander and H. Zhao proved that classical Schwarz method converges for the n-dimensional linear heat equation [8].
Another technique to study the convergence of classical Schwarz methods is to use the idea of upper-lower solutions methods, with initial guesses to be upper or lower solutions of the equations. This special class of domain decomposition methods with monotone iterations has been studied by S. H. Lui in [19], [20], [21]. Although many techniques have been developed to study the convergence problem of classical Schwarz methods, the problem with nonlinear equations in n-dimension and general multi-subdomains is still open.

A new class of Schwarz algorithms, in which Dirichlet transmission condition is replaced by Robin ones, has been studied recently in order to improve the performance of classical methods. The new algorithms are called optimized Schwarz methods since there are some parameters we can optimize to get faster algorithms. In 1989, P.L. Lions (see [16], [17]) established the convergence of nonoverlapping optimized Schwarz methods with Robin trans-

mission conditions by using an energy argument. Later, J. D. Benamou and B. Depres in [2] used this technique to study the convergence of nonoverlapping optimized Schwarz methods for Helmholtz equation. Energy estimates have then become a very powerful technique to prove the convergence of nonoverlapping optimized Schwarz methods with Robin transmission conditions (see [11]).
However, the convergence problem of overlapping optimized Schwarz methods, even for linear problems, still remains an open problem up to now. J.-H. Kimn [13], proved the convergence of an overlapping optimized Schwarz method for Poisson equation with Robin boundary data,

$$\begin{cases} -\Delta u = f \text{ in } \Omega, \\ \frac{\partial u}{\partial n} + pu = g \text{ on } \partial\Omega. \end{cases}$$

He proved that there is an $p_0 > 0$ such that the Schwarz iterations with Robin transmissions conditions converge for any Robin parameter $0 < p < p_0$. In [18], S. Loisel and D. B. Szyld extended the technique of J.-H. Kimn for the following equation

$$\begin{cases} -\nabla(a\nabla u) + cu = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $a$ is a $C^1$-function and $c$ is positive and belongs to $L^\infty(\Omega)$. The same constant $p$ is kept for all transmission operators and some conditions on the boundaries of the subdomains are then imposed.
A proof of convergence based on semi-classical analysis for overlapping optimized Schwarz methods with rectangular subdomains, linear advection diffusion equations on the half plane was given in [23].
Another technique is to use Fourier transform. This technique cannot be used to study the convergence of Schwarz methods for nonlinear problems and for general subdomains, but convergence rates can be obtained. Changing the boundary conditions will change the values of the convergence rates and then improve the performance of the algorithms, which proposes a new problem: the problem of optimizing the convergence rates. In [12], [3], [5] the authors showed that the problem of optimizing the convergence rates is in fact a new class of best approximation problems and suggested a new method to solve it.

In this paper, we present convergence proofs of overlapping classical and optimized Schwarz methods for elliptic and parabolic semilinear equations, in general forms, for general multi-subdomains. We prove that Schwarz methods with Dirichlet and Robin transmission conditions always converge for parabolic equations; since with parabolic equations the time variable can be controlled easily. However, the convergence of Schwarz methods with Robin transmission conditions is not guaranteed for elliptic equations and there are cases when the iterations diverge, while classical Schwarz methods always converge. We can see from Remark 3.1 that given a Schwarz algorithm with a specific Robin transmission condition, there exists a class of elliptic equations where the algorithm is unstable. A condition of convergence is then supplied: Schwarz methods with Robin transmission conditions for elliptic equations will converge if we multiply Robin parameters by a number large enough, and this can also be seen from Example 3.1. The techniques used in our proofs can also be used to prove the convergence of Schwarz methods for many other kinds of partial differential equations.

The paper is organized as follows.

- Section 2 is devoted to the convergence properties of Schwarz methods for semilinear parabolic equations. Section 2.1 gives the definition of the Schwarz algorithms for semilinear parabolic equations, and states the two theorems of convergence. Theorem 2.1 announces that classical Schwarz algorithms always converge with semilinear parabolic equations and its proof can be found in section 2.2. Theorem 2.2 is about the convergence of Schwarz algorithms with Robin transmission conditions and the proof is then given later in section 2.3.

- In section 3, we discuss the convergence properties of Schwarz methods for semilinear elliptic equations. Definitions of the algorithms, the two convergence theorems 3.1, 3.2, together with the counterexample 3.1 is announced in section 3.1. Section 3.2 and 3.3 contain the proofs of the two theorems.

## 2 Convergence for Semilinear Parabolic Equations

We introduce the abbreviation $\partial_{i,j} = \frac{\partial^2}{\partial x_i \partial x_j}$, $\partial_t = \frac{\partial}{\partial t}$ and $\partial_i = \frac{\partial}{\partial x_i}$ and consider a general semilinear parabolic equation

$$\begin{cases} \partial_t u - \sum_{i,j=1}^n \partial_j(a_{i,j}\partial_i u) + \sum_{i=1}^n b_i \partial_i u + cu = F(x,t,u) \text{ in } \Omega \times (0,\infty), \\ u(x,t) = h(x,t) \text{ on } \partial\Omega \times (0,\infty), \\ u(x,0) = h(x,0) \text{ on } \Omega, \end{cases} \tag{2.1}$$

where $\Omega$ is a bounded and $C^2$ domain in $\mathbb{R}^n$. The coefficients $a_{i,j}$, $b_i$, $c$ are functions of the space variable $x$, with the following properties
(A1) The functions $a_{i,j}$, $b_i$, $c$ are in $C^2(\mathbb{R}^n)$.
(A2) For all $i, j$ in $\{1, \ldots, I\}$, $a_{i,j}(x) = a_{j,i}(x)$. There exist strictly positive numbers $\lambda$, $\Lambda$ such that $A = (a_{i,j}(x)) \geq \lambda I$ in the sense of symmetric positive definite matrices and $|a_{i,j}(x)| < \Lambda$ in $\Omega$.
(A3) $h$ is in $C^2(\mathbb{R}^{n+1})$ and and $F$ is uniformly Lipschitz in the third variable, i.e. there exists $C > 0$, such that

$\forall\ t \in \mathbb{R}$, $\forall\ x \in \mathbb{R}^n$, $|F(x,t,z) - F(x,t,z')| \leq C|z - z'|$, $\forall\ z,\ z' \in \mathbb{R}$.

With Conditions $(A1)$, $(A2)$ and $(A3)$, Equation (2.1) has a unique bounded solution $u$ in $C^{2,1}(\overline{\Omega \times (0,\infty)})$, i.e $\partial_{i,j}\partial_t u$ belongs to $C(\overline{\Omega \times (0,\infty)})$ for all $i, j$ in $\{1, \ldots, n\}$. The proof of this result can be found in some classical books like [4], [14].

The domain $\Omega$ is divided into $I$ $C^2$ overlapping subdomains $\{\Omega_l\}_{l \in \{1,I\}}$, such that

$$\cup_{l=1}^n \Omega_l = \Omega;$$

$$(\partial\Omega_l \backslash \partial\Omega) \cap (\partial\Omega_{l'} \backslash \partial\Omega) = \emptyset, \quad \forall \quad l, l' \in \{1, \ldots, I\}, \quad l \neq l';$$

and

$$\forall l \in \{1, \ldots, I\}, \forall l', l'' \in J_l, l'' \neq l', \quad \Omega_{l'} \cap \Omega_{l''} = \emptyset,$$

where

$$J_l = \{l' | \Omega_{l'} \cap \Omega_l \neq \emptyset\}.$$

In short, we can say that there is no cross point in the decomposition of the domains. For any $l$ in $J$, for $l' \in J_l$, $\Gamma_{l,l'}$ is the set $(\partial\Omega_l \backslash \partial\Omega) \cap \overline{\Omega}_{l'}$.

**Remark 2.1.** *Figure 1 gives an example which satisfies our assumptions about the way we divide $\Omega$ into several subdomains. In Figure 2, since there*

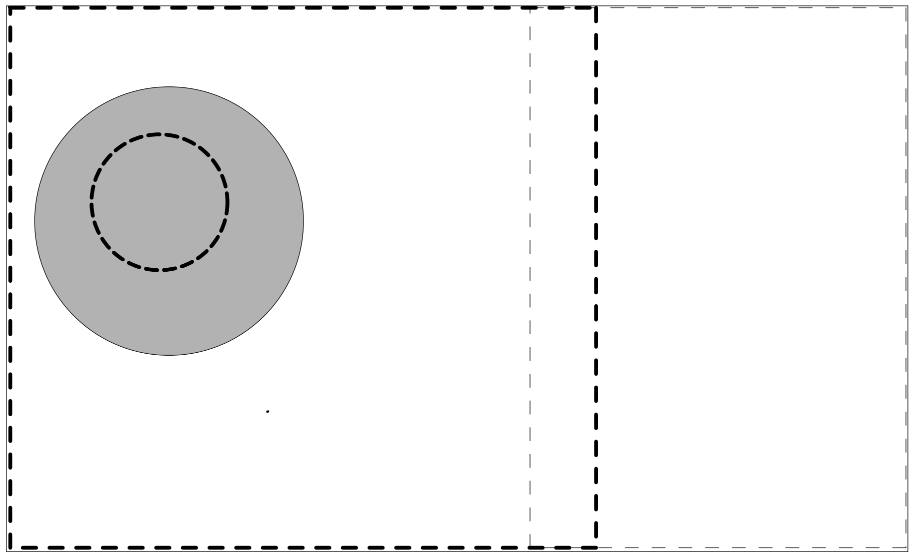

Figure 1: A good way of dividing $\Omega$

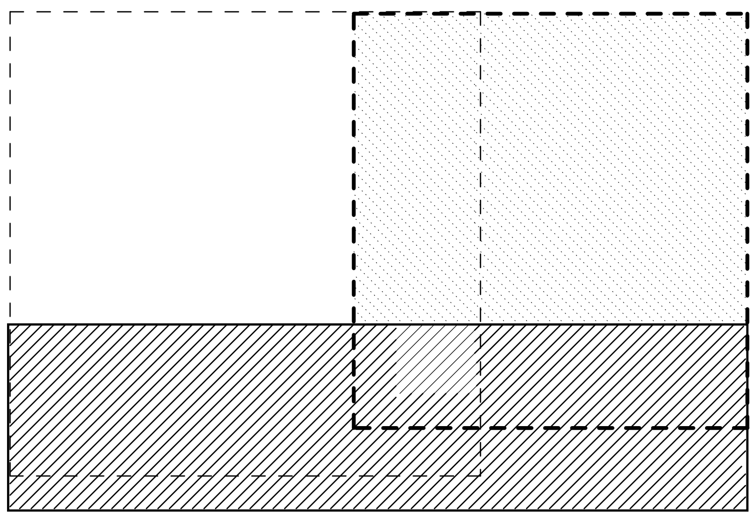

Figure 2: A bad way of dividing $\Omega$

*is an overlapping area between the three subdomains, this way of dividing* $\Omega$ *does not satisfy our conditions.*

The Schwarz waveform relaxation algorithm solves $I$ equations in $I$ subdomains instead of solving directly the main problem (2.1). The iterate $\#k$

in the $l$-th domain, denoted by $u_l^k$, is defined by

$$\begin{cases} \partial_t u_l^k - \sum_{i,j=1}^n \partial_j(a_{i,j}\partial_i u_l^k) + \sum_{i=1}^n b_i \partial_i u_l^k + c u_l^k = F(t,x,u_l^k), \text{ in } \Omega_l \times (0,\infty), \\ \mathfrak{B}_{l,l'} u_l^k = \mathfrak{B}_{l,l'} u_{l'}^{k-1}, \text{ on } \Gamma_{l,l'} \times (0,\infty), \forall l' \in J_l, \end{cases} \tag{2.2}$$

where the transmission operator $\mathfrak{B}_{l,l'}$ is either of the Dirichlet type or of the Robin type.
Each iterate inherits the boundary conditions and the initial values of $u$

$$u_l^k = h \text{ on } (\partial\Omega_l \cap \partial\Omega) \times (0,\infty), \quad u_l^k(.,0) = h(.,0) \text{ in } \Omega_l.$$

A bounded initial guess $u^0$ in $C^\infty(\overline{\Omega \times (0,\infty)})$ is provided, *i.e.* at step 1 Equations (2.2) are solved

$$\mathfrak{B}_{l,l'} u_l^1 = u^0 \quad \text{on } \Gamma_{l,l'} \times (0,\infty), \forall l' \in J_l.$$

We assume also the compatibility condition on $u^0$

$$\mathfrak{B}_{l,l'} h(.,0) = u^0(.,0) \quad \text{on } \Gamma_{l,l'}, \forall l' \in J_l.$$

Denote by $e_l^k$ the difference between $u_l^k$ and $u$, and subtract Equation (2.2) with the main equation (2.1) to obtain the following equations on $e_l^k$

$$\begin{cases} \partial_t e_l^k - \sum_{i,j=1}^n \partial_j(a_{i,j}\partial_i e_l^k) + \sum_{i=1}^n b_i \partial_t e_l^k + c e_l^k = F(t,x,u_l^k) - F(t,x,u) \text{ in } \Omega_l \times (0,\infty), \\ \mathfrak{B}_{l,l'} e_l^k = \mathfrak{B}_{l,l'} e_{l'}^{k-1} \text{ on } \Gamma_{l,l'} \times (0,\infty), \forall l' \in J_l. \end{cases} \tag{2.3}$$

Moreover,

$$e_l^k(.,.) = 0 \text{ on } (\partial\Omega_l \cap \partial\Omega) \times (0,\infty), \quad e_l^k(.,0) = 0 \text{ in } \Omega_l.$$

## 2.1 Classical Schwarz Methods

Consider the classical Schwarz waveform relaxation algorithm with Dirichlet transmission conditions $\mathfrak{B}_{l,l'} = Id$. By induction, each subproblem (2.2) in each iteration has a unique solution in $C^{2,1}(\overline{\Omega \times (0,\infty)})$ then in $L^2(0,\infty,H^1(\Omega)) \cap L^\infty(\Omega \times (0,\infty))$ also.

Consider (2.3) and let $g$, $f$ be bounded and strictly positive functions in $C^\infty(\mathbb{R}^n,\mathbb{R})$ and $C^\infty(\mathbb{R},\mathbb{R})$. Define

$$\Phi_l^k(x,t) := (e_l^k)^2 g(x) f(t).$$

Since $e_l^k$ belongs to $L^2(0,\infty,H^1(\Omega))\cup L^\infty(\Omega\times(0,\infty))$, $\Phi_l^k$ belongs to $L^2(0,\infty,H^1(\Omega))$. Let $c_i$ be $b_i+\sum_{j=1}^n 2a_{i,j}\partial_j g g^{-1}$, then $c_i\in L^\infty(\Omega_l\times(0,\infty))$, and define the following operator

$$\mathfrak{L}_{lD}(\Phi) \;=\; \partial_t\Phi-\sum_{i,j=1}^n \partial_j(a_{i,j}\partial_i\Phi)+\sum_{i=1}^n c_i(x,t)\partial_i\Phi. \qquad (2.4)$$

**Lemma 2.1.** *In each subdomain $\Omega_l$, for each iterate $k$*

$$\mathfrak{L}_{lD}(\Phi_l^k)\le 0,$$

*in the distributional sense, i.e. for all $\varphi$ in $H_0^1(\Omega)$ and $\varphi\ge 0$ a.e. on $\Omega$,*

$$\int_{\Omega_l}\mathfrak{L}_{lD}(\Phi_l^k)\varphi\le 0 \text{ a.e. in } (0,\infty).$$

*Proof.* Define the operator

$$\mathfrak{L}_{lD0} := \partial_t-\sum_{i,j=1}^n \partial_j(a_{i,j}\partial_i).$$

A lengthy but easy computation then implies that

$$\begin{aligned}\mathfrak{L}_{lD0}(\Phi_l^k) \;=\;& 2\left(\partial_t e_l^k-\sum_{i,j=1}^n \partial_j(a_{i,j}\partial_i e_l^k)\right)e_l^k g f-\sum_{i,j=1}^n 2a_{i,j}\partial_i e_l^k\partial_j e_l^k g f- \qquad (2.5)\\ &-\sum_{i,j=1}^n 4a_{i,j}\partial_i e_l^k e_l^k\partial_j g f+(e_l^k)^2\left(-\sum_{i,j=1}^n (a_{i,j}\partial_{i,j}g+\partial_i a_{i,j}\partial_j g)f+gf'\right).\end{aligned}$$

Thanks to Equation (2.3), and the lipschitzian property of $F$, the first term in (2.5) can be estimated in the distributional sense

$$2\left(\partial_t e_l^k-\sum_{i,j=1}^n \partial_j(a_{i,j}\partial_i e_l^k)\right)e_l^k g f \qquad (2.6)$$

$$
\begin{aligned}
&= 2gfe_l^k \left( F(t,x,u_l^k) - F(t,x,u) - \sum_{i=1}^n b_i \partial_i e_l^k - ce_l^k \right) \\
&\leq 2gf(e_l^k)^2 (C + ||c||_\infty) - \sum_{i=1}^n 2b_i \partial_i e_l^k gfe_l^k \\
&\leq 2gf(e_l^k)^2 (C + ||c||_\infty) - \sum_{i=1}^n b_i \left( \partial_i \Phi_l^k - (e_l^k)^2 \partial_i gf \right) \\
&\leq -\sum_{i=1}^n b_i \partial_i \Phi_l^k + (e_l^k)^2 \left( 2gf(C + ||c||_\infty) + \sum_{i=1}^n b_i \partial_i gf \right).
\end{aligned}
$$

Since $\mathfrak{L}_{lD0}$ is elliptic, the second term in (2.5) is negative.
Moreover, the third term in (2.5) can be transformed into

$$
-\sum_{i,j=1}^n 4a_{i,j} \partial_i e_l^k e_l^k \partial_j gf = -\sum_{i,j=1}^n 2a_{i,j} \partial_j g g^{-1} \partial_i \Phi_l^k + \sum_{i,j=1}^n 2a_{i,j} \partial_j g \partial_i g g^{-1} f (e_l^k)^2.
$$

Combine (2.5), (2.6) and (2.7) to get

$$
\mathfrak{L}_{lD0}(\Phi_l^k) + \sum_{i=1}^n \left( b_i + \sum_{j=1}^n 2a_{i,j} \partial_j g g^{-1} \right) \partial_i \Phi_l^k \leq (e_l^k)^2 \mathfrak{M}, \tag{2.7}
$$

where

$$
\mathfrak{M} = \left[ \sum_{i,j=1}^n \left(-a_{i,j} \frac{\partial_{i,j} g}{g} - \partial_j a_{i,j} \frac{\partial_i g}{g} + 2a_{i,j} \frac{\partial_j g}{g} \frac{\partial_i g}{g}\right) + \frac{f'}{f} + 2(C + ||c||_\infty) + \sum_{i=1}^n b_i \frac{\partial_i g}{g} \right] fg \tag{2.8}
$$

Notice that $\mathfrak{M}$ has the form of $fg(\frac{f'}{f} + G(g))$. Choosing $f$ such that $-\frac{f'}{f}$ large enough (for example, $f = \exp(-\alpha t)$, where $\alpha$ is a large positive constant), since the other terms are bounded in the bounded domain $\Omega_l \times (0, \infty)$, then $\mathfrak{M} \leq 0$. The nonlinear equation (2.3) has been transformed into the following linearized inequality of $\Phi_l^k$

$$
\partial_t \Phi_l^k - \sum_{i,j=1}^n \partial_j (a_{i,j} \partial_i \Phi_l^k) + \sum_{i=1}^n c_i \partial_i \Phi_l^k \leq 0. \tag{2.9}
$$

□

**Theorem 2.1.** *Consider the Schwarz algorithm with Dirichlet transmission condition, suppose that $f(t)$ is a strictly positive and continuous function satisfying $-\min_{t\in(0,\infty)}\frac{f'(t)}{f(t)}$ is sufficiently large, we get the geometrical convergence*

$$\lim_{k\to\infty}\max_{l\in\{1,\dots,I\}}||(u_l^k-u)^2 f(t)||_{L^\infty(\Omega_l\times(0,\infty))}=0.$$

**Remark 2.2.** *In the above theorem, if $f$ is chosen to be $\exp(-\alpha t)$, then when $\alpha$ is large enough,*

$$-\min_{t\in(0,\infty)}\frac{f'(t)}{f(t)}$$

*is large enough. In this case, the limit*

$$\lim_{k\to\infty}\max_{l\in\{1,\dots,I\}}||(u_l^k-u)^2 f(t)||_{L^\infty(\Omega_l\times(0,\infty))}=0$$

*implies the almost everywhere convergence of the sequence $\{u_l^k\}$ to $u$ on $\Omega_l\times(0,\infty)$.*

**Remark 2.3.** *In the proof, if $a_{ij}$, $b_i$, $c$ depend both on $t$ and $x$, the convergence result in the theorem remains true.*

*Proof.* The proof is divided into two steps.
**Step 1:** Construct estimates of the errors $e_l^k$ from Inequation (2.9).

Define

$$M=\underset{\partial\Omega_l\times[0,\infty)\cup\Omega_l\times\{0\}}{esssup}\Phi_l^k(x,t),\tag{2.10}$$

we will prove that the maximum principle holds, i.e. $M\geq\Phi_l^k$ a.e. on $\Omega_l\times(0,\infty)$. Define the function

$$w=(\Phi_l^k-M)_+=\max\{\Phi_l^k-M,0\}.$$

Since $w\in L^2(0,\infty,H_0^1(\Omega_l))$, then

$$\partial_t w-\sum_{i,j=1}^n\partial_j(a_{i,j}\partial_i w)+\sum_{i=1}^n c_i\partial_i w\leq 0.\tag{2.11}$$

To prove that $M\geq\Phi_l^k$ a.e. on $\Omega_l\times(0,\infty)$, it suffices to prove that $w=0$ a.e. on $\Omega_l\times(0,\infty)$.

For $0 < p \leq \infty$ and $0 < q \leq \infty$, and for $0 \leq \tau_1 < \tau_2 \leq \infty$, define $||h||_{\tau_1,\tau_2,p,q} = ||h||_{L^q(\tau_1,\tau_2,L^p(\Omega_l))}$, for $h \in L^q(\tau_1, \tau_2, L^p(\Omega_l))$. If $\tau_1 = 0$, denote $||h||_{0,\tau_2,p,q}$ by $||h||_{\tau_2,p,q}$.

Let $\chi(\tau_1, \tau_2)$ be the characteristic function of the open interval $(\tau_1, \tau_2)$, where $0 < \tau_1 < \tau_2 \leq \infty$ and set $\varphi = w\chi$. Since $w \in L^2(0, T, H_0^1(\Omega_l)) \cap L^\infty(\Omega_l \times (0, \infty))$, it is evident that $\varphi \in L^2(0, \infty, H_0^1(\Omega_l)) \cap L^\infty(\Omega_l \times (0, \infty))$.

Use $\varphi$ as a test function for (2.11)

$$\int_{\tau_1}^{\tau_2}\int_{\Omega_l} \partial_t w w dxdt + \int_{\tau_1}^{\tau_2}\int_{\Omega_l} \sum_{i,j=1}^{n} a_{i,j}\partial_i w \partial_j w dxdt + \int_{\tau_1}^{\tau_2}\int_{\Omega_l} \sum_{i=1}^{n} c_i \partial_i w w dxdt \leq 0. \quad (2.12)$$

Equation (2.12) and Conditions $(A_1)$ and $(A_2)$ imply that

$$\int_{\Omega_l} \frac{w^2}{2} dx \Big|_{t=\tau_1}^{t=\tau_2} + \int_{\tau_1}^{\tau_2}\int_{\Omega_l} \lambda |\nabla w|^2 dxdt \leq \int_{\tau_1}^{\tau_2}\int_{\Omega_l} M_1 |\nabla w| w dxdt, \quad (2.13)$$

where $M_1$ is a positive constant. By the Cauchy inequality, the right hand side of (2.13) is bounded by

$$\int_{\tau_1}^{\tau_2}\int_{\Omega_l} \frac{M_1\epsilon}{2} |\nabla w|^2 dxdt + \int_{\tau_1}^{\tau_2}\int_{\Omega_l} \frac{M_1}{2\epsilon} w^2 dxdt, \quad (2.14)$$

where $\epsilon$ is a small positive constant.
For $\epsilon$ to be $\frac{\lambda}{M_1}$, Equality (2.14) implies

$$\int_{\Omega_l} \frac{w^2}{2} dx \Big|_{t=\tau_1}^{t=\tau_2} + \int_{\tau_1}^{\tau_2}\int_{\Omega_l} \frac{\lambda}{2} |\nabla w|^2 dxdt \leq \int_{\tau_1}^{\tau_2}\int_{\Omega_l} M_2 w^2 dxdt, \quad (2.15)$$

with $M_2 = \frac{M_1^2}{2}$.

Denote $X(t) = \int_{\Omega_l} w^2(x,t)dx$, and let $\tau_2$ be $t$ in the interval $\mathfrak{I} = (\tau_1, \tau_1+\delta)$, the previous estimate infers

$$X(t) + \lambda ||\nabla w||^2_{\tau_1,t,2,2} \leq M_2\delta\{\sup_{\mathfrak{I}} X(t)\} + X(\tau_1). \quad (2.16)$$

Choosing $\delta$ such that $M_2\delta = \frac{1}{2}$, the fact that $\sup_{\mathfrak{I}} X(t) \leq 2X(\tau_1)$ then follows. Since the inequality is true on any time interval with the length of $\delta$, and $X(0) = 0$, then $X(t) = 0$ for a.e. $t$ in $(0, \infty)$. Hence $w = 0$ for a.e. $t$

in $(0, \infty)$.

We have just proved that

$$\left(e_l^k(x,t)\right)^2 g(x)f(t) \leq \max_{l' \in J_l} \left( \underset{\Gamma_{l,l'} \times (0,\infty)}{esssup} \left(e_l^k(x,t)\right)^2 g(x)f(t) \right), \tag{2.17}$$

for all $l$ in $I$, for a.e. $(x,t)$ in $\Omega_l \times (0,\infty)$; for any strictly positive functions $g$, $f$ in $C^\infty(\mathbb{R}^n, \mathbb{R})$ and $C^\infty(\mathbb{R}, \mathbb{R})$.

**Step 2:** The convergence of the algorithm.

Denote

$$E^k = \max_{l \in I} \left( \underset{(x,t) \in (\Omega_l \times (0,\infty))}{esssup} \left(e_l^k\right)^2 f(t) \right). \tag{2.18}$$

From (2.17) comes that for every $l'$ in $J_l$ and for a.e. $(x,t)$ in $\Gamma_{l,l'} \times (0,\infty)$

$$\left(e_l^k(x,t)\right)^2 g(x)f(t) \leq \max_{l'' \in J_{l'}} \left( \underset{\Gamma_{l',l''} \times (0,\infty)}{esssup} \left(e_{l'}^{k-1}(x,t)\right)^2 g(x)f(t) \right), \tag{2.19}$$

that implies

$$\left(e_l^k(x,t)\right)^2 f(t) \leq \frac{1}{g(x)} \max_{l'' \in J_{l'}} \left( \underset{\Gamma_{l',l''} \times (0,\infty)}{esssup} \left(e_{l''}^{k-2}(x,t)\right)^2 g(x)f(t) \right). \tag{2.20}$$

Since $\Gamma_{l,l'}$ lies inside $\Omega_{l'}$, choose $g$ such that there exists a constant $M_{3,l}$ satisfying

$$1 > M_{3,l} > \frac{g(\zeta')}{g(\zeta)}, \quad \forall \zeta \in \Gamma_{l,l'} \text{ and } \forall \zeta' \in \cup_{l'' \in J_{l'}} \Gamma_{l',l''},$$

and this implies that for all $l'$ in $J_l$, for a.e. $(x,t)$ in $\Gamma_{l,l'} \times (0,\infty)$

$$\left(e_l^k(x,t)\right)^2 f(t) \leq M_{3,l} \max_{l'' \in J_{l'}} \left( \underset{\Gamma_{l',l''} \times (0,\infty)}{esssup} \left(e_{l''}^{k-2}(x,t)\right)^2 f(t) \right) \leq M_{3,l} E^{k-2}. \tag{2.21}$$

Choose $g$ to be the function 1, (2.17) yields

$$\left(e_l^k(x,t)\right)^2 f(t) \leq \max_{l' \in J_l} \left( \underset{\Gamma_{l,l'} \times (0,\infty)}{esssup} \left(e_l^k(x,t)\right)^2 f(t) \right), \quad \forall l' \in J_l, \text{ a.e. on } \Omega_l. \tag{2.22}$$

The estimates (2.21) and (2.22) imply the existence of a constant $M_4$ smaller than 1 and satisfy

$$E^k \leq M_4 E^{k-2}, \tag{2.23}$$

which shows that the errors converge geometrically

$$\lim_{k\to\infty} E^k = 0.$$

The theorem is proved. □

## 2.2 Optimized Schwarz Methods

The optimized Schwarz waveform relaxation algorithms are defined by replacing the Dirichlet by Robin transmission operators

$$\mathfrak{B}_{l,l'}v = \sum_{i,j=1}^{n} a_{i,j}\partial_i v n_{l,l',j} + p_{l,l'}v,$$

where $n_{l,l',j}$ is the $j$-th component of the outward unit normal vector of $\Gamma_{l,l'}$; $p_{l,l'}$ is positive and belongs to $L^\infty(\Gamma_{l,l'})$. By induction, each subproblem (2.2) in each iteration has a unique solution in $L^2(0,\infty,H^1(\Omega))$ and the algorithm is well-posed.

Let $f$ be a function in $L^2(0,\infty)$, define

$$\int_0^\infty f(x)\exp(-yx)dx.$$

Now, define for a fixed positive number $\alpha$

$$|f|_\alpha = \sup_{\alpha'>\alpha}\left[\int_{\alpha'}^{\alpha'+1}\left(\int_0^\infty f(x)\exp(-yx)dx\right)^2 dy\right]^{\frac{1}{2}},$$

and

$$\mathbb{L}^2_\alpha(0,\infty) = \{f \ : \ f\in L^2(0,\infty), |f|_\alpha < \infty\}.$$

Then $(\mathbb{L}^2_\alpha(0,\infty), |.|_\alpha)$ is a normed subspace of $L^2(0,\infty)$.

Consider Equation (2.3), let $g_l$ be a function bounded and greater than 1 in $C^\infty(\mathbb{R}^n,\mathbb{R})$, $\alpha$ be a positive constant, and define

$$\Phi_l^k(x) := \left(\int_0^\infty e_l^k \exp(-\alpha t)dt\right) g_l(x),$$

then $\Phi_l^k(x)$ belongs to $H^1(\Omega_l)$.

Let $B_i^l$ and $C^l$ be functions in $L^\infty(\mathbb{R}^n)$ defined in the following ways:

$$B_i^l := b_i + \sum_{j=1}^n \left( a_{i,j} \frac{\partial_j g_l}{g_l} \right),$$

$$C^l = \left[ \frac{\alpha}{2} + \sum_{i,j=1}^n \left( -a_{i,j} \frac{2\partial_i g_l \partial_j g_l}{(g_l)^2} - \partial_j a_{i,j} \frac{\partial_i g}{g} + a_{i,j} \frac{\partial_{i,j} g_l}{g_l} \right) - \sum_{i=1}^n b_i \frac{\partial_i g_l}{g_l} \right].$$

Define

$$\begin{aligned} \mathfrak{L}_{lR}(\Phi_l^k) &= -\sum_{i,j=1}^n \partial_j (a_{i,j} \partial_i \Phi_l^k) + \sum_{i=1}^n B_i^l \partial_i \Phi_l^k + C^l \Phi_l^k \\ &\quad + \left\{ \int_0^\infty \left[ \left( \frac{\alpha}{2} + c \right) e_l^k - F(u_l^k) + F(u) \right] \exp(-\alpha t) dt \right\} g_l. \end{aligned} \tag{2.24}$$

It is possible to suppose $\alpha$ to be large such that $C^l$ belongs to $(\frac{\alpha}{4}, \alpha)$.

**Lemma 2.2.** *Choose $g_l$, $g_{l'}$ such that $\nabla g_l = \nabla g_{l'} = 0$ on $\Gamma_{l,l'}$ and $\frac{g_{l'}}{g_l} > 1$ on $\Gamma_{l,l'}$, for all $l'$ in $J_l$. $\Phi_l^k$ is then a solution of the following equation*

$$\begin{cases} \mathfrak{L}_{lR}(\Phi_l^k) = 0, & \text{in } \Omega_l \times (0,\infty), \\ \beta_l \mathfrak{B}_{l,l'}(\Phi_l^k) = \mathfrak{B}_{l,l'}(\Phi_{l'}^{k-1}) & \text{on } \Gamma_{l,l'} \times (0,\infty), \forall l' \in J_l. \end{cases} \tag{2.25}$$

*where $\beta_l = \frac{g_{l'}}{g_l}$ on $\Gamma_{l,l'}$, for all $l'$ in $J_l$.*

*Proof.* A complicated but easy computation leads to

$$\begin{aligned} & -\sum_{i,j=1}^n \partial_j (a_{i,j} \partial_i \Phi_l^k) + \alpha \Phi_l^k \\ = & \left[ \int_0^\infty \left( \partial_t e_l^k - \sum_{i,j=1}^n \partial_j (a_{i,j} \partial_i e_l^k) \right) \exp(-\alpha t) dt \right] g_l \\ & - \left( \int_0^\infty e_l^k \exp(-\alpha t) dt \right) \left( \sum_{i,j=1}^n a_{i,j} (\partial_{i,j} g_l + \partial_j a_{i,j} \partial_i g_l) \right) \\ & - \sum_{i,j=1}^n a_{i,j} \left[ \int_0^\infty e_l^k \left( \partial_i e_l^k \partial_j g_l + \partial_j e_l^k \partial_i g_l \right) \exp(-\alpha t) dt \right]. \end{aligned} \tag{2.26}$$

That implies

$$\begin{aligned}
& -\sum_{i,j=1}^{n} \partial_j(a_{i,j}\partial_i \Phi_l^k) + \alpha \Phi_l^k \qquad (2.27)\\
& + \left(\int_0^\infty e_l^k \exp(-\alpha t)dt\right)\left(\sum_{i,j=1}^{n}(a_{i,j}\partial_{i,j} g_l + \partial_j a_{i,j}\partial_i g_l)\right)\\
= & \left[\int_0^\infty \left(-\sum_{i=1}^{n} b_i \partial_i e_l^k - c e_l^k + F(u_l^k) - F(u)\right)\exp(-\alpha t)dt\right] g_l\\
& -\sum_{i,j=1}^{n} a_{i,j}\left[\int_0^\infty e_l^k \left(\partial_i e_l^k \partial_j g_l + \partial_j e_l^k \partial_i g_l\right)\exp(-\alpha t)dt\right].
\end{aligned}$$

Therefore

$$\begin{aligned}
0 = & -\sum_{i,j=1}^{n} \partial_j(a_{i,j}\partial_i \Phi_l^k) + \sum_{i=1}^{n} b_i \partial_i \Phi_l^k + \sum_{i,j=1}^{n} a_{i,j}\left(\partial_i \Phi_l^k \frac{\partial_j g_l}{g_l} + \partial_j \Phi_l^k \frac{\partial_i g_l}{g_l}\right)\\
& + \left(\frac{\alpha}{2} + \sum_{i,j=1}^{n} a_{i,j}(-\frac{2\partial_i g_l \partial_j g_l}{(g_l)^2} + a_{i,j}\frac{\partial_{i,j} g_l}{g_l} - \partial_j a_{i,j}\frac{\partial_i g_l}{g_l}) - \sum_{i=1}^{n} b_i \frac{\partial_i g_l}{g_l}\right)\Phi_l^k +\\
& + \left\{\int_0^\infty \left[\left(\frac{\alpha}{2} + c\right) e_l^k - F(u_l^k) + F(u)\right]\exp(-\alpha t)dt\right\} g_l.
\end{aligned}$$

Now, consider Robin transmission conditions on the boundary $\Gamma_{l,l'}$ and notice that $\nabla g_l = \nabla g_{l'} = 0$ on $\Gamma_{l,l'}$, the transmission conditions can be reported on $\Phi_l^k$

$$\begin{aligned}
\mathfrak{B}_{l,l'}(\Phi_l^k) = & \sum_{i,j=1}^{n} a_{i,j}\partial_i \Phi_l^k n_{l,l',j} + p_{l,l'}\Phi_l^k \qquad (2.28)\\
= & \int_0^\infty \left(\sum_{i,j=1}^{n} a_{i,j} n_{l,l',j}\partial_i e_l^k + p_{l,l'} e_l^k\right)\exp(-\alpha t) g_l + \sum_{i,j=1}^{n} a_{i,j} n_{l,l',j}\partial_i g_l e_l^k\\
= & \int_0^\infty \left(\sum_{i,j=1}^{n} a_{i,j} n_{l,l',j}\partial_i e_{l'}^{k-1} + p_{l,l'} e_{l'}^{k-1}\right)\exp(-\alpha t) g_l\\
= & \left(\sum_{i,j=1}^{n} a_{i,j}\partial_i \Phi_{l'}^{k-1} n_{l,l',j} + p_{l,l'}\Phi_{l'}^{k-1}\right)\frac{g_l}{g_{l'}} = \mathfrak{B}_{l,l'}(\Phi_{l'}^{k-1})\frac{g_l}{g_{l'}}.
\end{aligned}$$

□

**Theorem 2.2.** *Consider Schwarz algorithms with Robin transmission conditions. There exists a constant* $\alpha_0$ *such that for* $\alpha$ *to be greater than* $\alpha_0$

$$\lim_{k\to\infty}\sum_{l=1}^{I}\int_{\Omega_l}|e_l^k|_\alpha^2dx=0,$$

*Proof.* For all $l$ in $\{1,I\}$, denote by $\tilde{\Omega}_l$ to be the open set $\Omega_l\backslash\overline{\cup_{l'\in J_l}\Omega_{l'}}$. Let $\varphi_l^k$ be functions in $H^1(\tilde{\Omega}_l)$ and $\varphi_l^{k+1}$ be functions in $H^1(\Omega_l)$ for all $l$ in $I$ such that $\varphi_l^{k+1}=\varphi_{l'}^k$ on $\Gamma_{l,l'}$ for all $l'$ in $J_l$. Now, use $\varphi_l^{k+1}$ and $\varphi_l^k$ as test functions for (2.25), and take the sum with respect to $l$ in $\{1,I\}$ the integrals $\int_{\tilde{\Omega}_l}\mathfrak{L}_{lR}(\Phi_l^k)\varphi_l^k$ and $\int_{\tilde{\Omega}_l}\mathfrak{L}_{lR}(\Phi_l^{k+1})\varphi_l^{k+1}$, then

$$\begin{aligned}
&-\sum_{l=1}^{I}\Bigg\{\int_{\tilde{\Omega}_l}\sum_{i,j=1}^{n}a_{i,j}\partial_i\Phi_l^k\partial_j\varphi_l^kdx+\sum_{i=1}^{n}\int_{\tilde{\Omega}_l}B_i^l\partial_i\Phi_l^k\varphi_l^kdx+\int_{\tilde{\Omega}_l}C^l\Phi_l^k\varphi_l^kdx\\
&-\sum_{l'\in J_l}\int_{\Gamma_{l',l}}p_{l',l}\Phi_l^k\varphi_l^kd\sigma\\
&+\int_{\tilde{\Omega}_l}\left\{\int_0^\infty\left[\left(\frac{\alpha}{2}+c\right)e_l^k-F(u_l^k)+F(u)\right]\exp(-\alpha t)dt\right\}g_l\varphi_l^kdx\Bigg\} \qquad (2.29)\\
=&\sum_{l=1}^{I}\beta_l\Bigg\{\int_{\Omega_l}\sum_{i,j=1}^{n}a_{i,j}\partial_i\Phi_l^{k+1}\partial_j\varphi_l^{k+1}dx+\int_{\Omega_l}\sum_{i=1}^{n}B_i^l\partial_i\Phi_l^{k+1}\varphi_l^{k+1}dx\\
&+\int_{\Omega_l}C^l\Phi_l^{k+1}\varphi_l^{k+1}dx+\sum_{l'\in J_l}\int_{\Gamma_{l,l'}}p_{l,l'}\Phi_l^{k+1}\varphi_l^{k+1}d\sigma\\
&+\int_{\Omega_l}\left\{\int_0^\infty\left[\left(\frac{\alpha}{2}+c\right)e_l^{k+1}-F(u_l^{k+1})+F(u)\right]\exp(-\alpha t)dt\right\}g_l\varphi_l^{k+1}dx\Bigg\}.
\end{aligned}$$

In the above equality, choose $\varphi_l^{k+1}$ to be $\Phi_l^{k+1}$, then there exists $\varphi_l^k$, such that $\varphi_l^k=\varphi_{l'}^{k+1}$ on $\Gamma_{l,l'}$ for all $l'$ in $J_l$; moreover,

$$||\varphi_l^k||_{H^1(\Omega_l)}\leq C\sum_{l'\in J_l}||\varphi_{l'}^{k+1}||_{H^1(\Omega_{l'})}\text{ and }||\varphi_l^k||_{L^2(\Omega_l)}\leq C\sum_{l'\in J_l}||\varphi_{l'}^{k+1}||_{L^2(\Omega_{l'})},$$

where $C$ is a positive constant.
With these test functions, the right hand side of (2.29) is greater than or

equal to

$$
\begin{aligned}
& \sum_{l=1}^{I} \beta_l \left\{ \int_{\Omega_l} \lambda |\nabla \Phi_l^{k+1}|^2 dx - \sum_{i=1}^{n} \int_{\Omega_l} ||B_i^l||_{L^\infty(\Omega_l)} \left| \partial_i \Phi_l^{k+1} \right| |\Phi_l^{k+1}| dx \right. \\
& + \frac{\alpha}{4} \int_{\Omega_l} |\Phi_l^{k+1}|^2 dx + \sum_{l' \in J_l} \int_{\Gamma_{l,l'}} p_{l,l'} |\Phi_l^{k+1}|^2 d\sigma \\
& \left. + \int_{\Omega_l} \left\{ \int_0^\infty \left[ \left( \frac{\alpha}{2} + c \right) e_l^{k+1} - F(u_l^{k+1}) + F(u) \right] \exp(-\alpha t) dt \right\} g_l \varphi_l^{k+1} dx \right\} \\
\geq \; & \sum_{l=1}^{I} \beta_l \left\{ \int_{\Omega_l} \lambda |\nabla \Phi_l^{k+1}|^2 dx - \sum_{i=1}^{n} \int_{\Omega_l} ||B_i^l||_{L^\infty(\Omega_l)} \left| \partial_i \Phi_l^{k+1} \right| |\Phi_l^{k+1}| dx \right. \\
& + \frac{\alpha}{4} \int_{\Omega_l} |\Phi_l^{k+1}|^2 \qquad\qquad (2.30) \\
& \left. + \int_{\Omega_l} \left\{ \int_0^\infty \left[ \left( \frac{\alpha}{2} + c \right) e_l^{k+1} - F(u_l^{k+1}) + F(u) \right] \exp(-\alpha t) dt \right\} g_l \varphi_l^{k+1} dx \right\} \\
\geq \; & \sum_{l=1}^{I} \beta_l \left\{ \int_{\Omega_l} \frac{\lambda}{2} |\nabla \Phi_l^{k+1}|^2 dx + \frac{\alpha}{8} \int_{\Omega_l} |\Phi_l^{k+1}|^2 \right. \\
& \left. + \int_{\Omega_l} \left\{ \int_0^\infty \left[ \left( \frac{\alpha}{2} + c \right) e_l^{k+1} - F(u_l^{k+1}) + F(u) \right] \exp(-\alpha t) dt \right\} g_l \varphi_l^{k+1} dx \right\} \\
\geq \; & \sum_{l=1}^{I} \beta_l \left\{ \int_{\Omega_l} \frac{\lambda}{2} |\nabla \Phi_l^{k+1}|^2 dx + \frac{\alpha}{8} \int_{\Omega_l} |\Phi_l^{k+1}|^2 \right. \\
& \left. + \int_{\Omega_l} \left[ \int_0^\infty \left( \frac{\alpha}{2} + c - C' \right) e_l^{k+1} \exp(-\alpha t) dt \right] \left[ \int_0^\infty e_l^{k+1} \exp(-\alpha t) dt \right] g_l^2 dx \right\} \\
\geq \; & \sum_{l=1}^{I} \beta_l \left\{ \int_{\Omega_l} \frac{\lambda}{2} |\nabla \Phi_l^{k+1}|^2 dx + \frac{\alpha}{8} \int_{\Omega_l} |\Phi_l^{k+1}|^2 \right\},
\end{aligned}
$$

where

$$
\begin{cases}
C' = C \text{ if } \int_0^\infty e_l^{k+1} \exp(-\alpha t) dt \geq 0, \\
C' = -C \text{ if } \int_0^\infty e_l^{k+1} \exp(-\alpha t) dt < 0,
\end{cases}
$$

and notice that $\alpha$ is large enough.
Similarly, we can estimate the left hand side of (2.29), which is in fact less

than or equal to

$$\begin{aligned}
&\sum_{l=1}^{I}\Bigg\{\int_{\tilde\Omega_l}\Lambda|\nabla\Phi_l^k||\nabla\varphi_l^k|dx+\int_{\tilde\Omega_l}2\alpha|\Phi_l^k||\varphi_l^k|dx\\
&+\sum_{i=1}^{n}\int_{\tilde\Omega_l}||B_i^l||_{L^\infty(\tilde\Omega_l)}\left|\partial_i\Phi_l^k\right||\varphi_l^k|dx+\sum_{l'\in J_l}\int_{\Gamma_{l',l}}p_{l',l}|\Phi_l^k||\varphi_l^k|d\sigma\Bigg\}\\
\leq\ &\sum_{l=1}^{I}M_1\Bigg[\Lambda\left(||\nabla\Phi_l^k||^2_{L^2(\tilde\Omega_l)}+||\nabla\varphi_l^k||^2_{L^2(\tilde\Omega_l)}\right)+\frac{\alpha}{2}||\Phi_l^k||^2_{L^2(\tilde\Omega_l)}+\frac{\alpha}{2}||\varphi_l^k||^2_{L^2(\tilde\Omega_l)}\\
&+\frac{1}{2}\left(||\nabla\Phi_l^k||^2_{L^2(\tilde\Omega_l)}+(\max_{i\in\{1,I\}}||B_i^l||_{L^\infty(\tilde\Omega_l)})^2||\varphi_l^k||^2_{L^2(\tilde\Omega_l)}\right) \qquad (2.31)\\
&+\sum_{l'\in J_l}||p_{l',l}||_{L^\infty(\Gamma_{l',l})}\left(||\Phi_l^k||^2_{H^1(\tilde\Omega_l)}+||\varphi_l^k||^2_{H^1(\tilde\Omega_l)}\right)\Bigg],
\end{aligned}$$

where $M_1$ is a positive constant depending only on $\{\Omega_l\}_{l\in\{1,I\}}$ and the coefficients of (2.3). Since $\alpha$ can be chosen such that $\alpha>(\max_{i\in\{1,I\}}||B_i^l||_{L^\infty(\tilde\Omega_l)})^2$, there exists $M_2$ positive, depending only on $\{\Omega_l\}_{l\in\{1,I\}}$ and the coefficients of (2.3) such that the right hand side of (2.31) is less than

$$\begin{aligned}
&\sum_{l=1}^{I}M_2\left[\int_{\tilde\Omega_l}\left(\frac{\lambda}{2}|\nabla\Phi_l^k|^2dx+\frac{\alpha}{8}|\Phi_l^k|^2+\frac{\lambda}{2}|\nabla\Phi_l^{k+1}|^2+\frac{\alpha}{8}|\Phi_l^{k+1}|^2\right)dx\right] \qquad (2.32)\\
\leq\ &\sum_{l=1}^{I}M_2\left(\frac{\lambda}{2}||\nabla\Phi_l^k||^2_{L^2(\Omega_l)}+\frac{\alpha}{8}||\Phi_l^k||^2_{L^2(\Omega_l)}+\frac{\lambda}{2}||\nabla\Phi_l^{k+1}||^2_{L^2(\Omega_l)}+\frac{\alpha}{8}||\Phi_l^{k+1}||^2_{L^2(\Omega_l)}\right).
\end{aligned}$$

Define

$$E_k:=\sum_{l=1}^{I}\left(\frac{\lambda}{2}||\nabla\Phi_l^k||^2_{L^2(\Omega_l)}+\frac{\alpha}{8}||\Phi_l^k||^2_{L^2(\Omega_l)}\right), \qquad (2.33)$$

then from (2.30), (2.31) and (2.32),

$$(\beta-M_2)E_{k+1}\leq M_2E_k, \qquad (2.34)$$

where $\beta=\min\{\beta_1,\dots,\beta_I\}$.
Since $M_2$ depends only on $\{\Omega_l\}_{l\in\{1,I\}}$ and the coefficients of (2.3), $\beta$ can be chosen large enough, such that

$$M_3:=\frac{M_2}{\beta-M_2}<1.$$

We obtain

$$\begin{aligned} E_k &\leq M_3^k E_0 \\ &\leq M_3^k \sum_{l=1}^{I} \left( \frac{\lambda}{2} ||\nabla \Phi_l^0||^2_{L^2(\Omega_l)} + \frac{\alpha}{8} ||\Phi_l^0||^2_{L^2(\Omega_l)} \right). \end{aligned}$$

That implies

$$||\Phi_l^k||^2_{L^2(\Omega_l)} \leq M_3^k \sum_{l=1}^{I} \left( \frac{4\lambda}{\alpha} ||\nabla \Phi_l^0||^2_{L^2(\Omega_l)} + ||\Phi_l^0||^2_{L^2(\Omega_l)} \right). \tag{2.35}$$

Notice that (2.35) still holds if $M_3$ and $\lambda$ are fixed, and $\alpha$ is replaced by all $y$ which is larger than $\alpha$. This observation leads to

$$\begin{aligned} & \sum_{l=1}^{I} \int_{\Omega_l} \left( \int_0^\infty e_l^k \exp(-yt) dt g_l \right)^2 dx \qquad (2.36) \\ \leq \;& M_3^k \left[ \frac{4\lambda}{y} \sum_{l=1}^{I} \int_{\Omega_l} \left( \int_0^\infty |\nabla e_l^0| \exp(-yt) dt \right)^2 g_l^2 dx \right. \\ & + \frac{4\lambda}{y} \sum_{l=1}^{I} \int_{\Omega_l} \left( \int_0^\infty e_l^0 \exp(-yt) dt \right)^2 |\nabla g_l|^2 dx \\ & \left. + \sum_{l=1}^{I} \int_{\Omega_l} \left( \int_0^\infty e_l^0 \exp(-yt) dt \right)^2 g_l^2 dx \right]. \end{aligned}$$

Let $\alpha'$ be a constant larger than or equal to $\alpha$, we obtain from the previous inequality that

$$\begin{aligned} & \sum_{l=1}^{I} \int_{\Omega_l} \int_{\alpha'}^{\alpha'+1} \left( \int_0^\infty e_l^k \exp(-yt) dt \right)^2 g_l^2 dy dx \qquad (2.37) \\ \leq \;& M_3^k \left[ \sum_{l=1}^{I} \int_{\Omega_l} \int_{\alpha'}^{\alpha'+1} \frac{4\lambda}{y} \left( \int_0^\infty |\nabla e_l^0| \exp(-yt) dt \right)^2 g_l^2 dy dx \right. \\ & + \sum_{l=1}^{I} \int_{\Omega_l} \int_{\alpha'}^{\alpha'+1} \frac{4\lambda}{y} \left( \int_0^\infty e_l^0 \exp(-yt) dt \right)^2 |\nabla g_l|^2 dy dx \\ & \left. + \sum_{l=1}^{I} \int_{\Omega_l} \int_{\alpha'}^{\alpha'+1} \left( \int_0^\infty e_l^0 \exp(-yt) dt \right)^2 g_l^2 dy dx \right]. \end{aligned}$$

Using the fact that $u^0$ belongs to $C_c^\infty(\overline{\Omega \times (0,\infty)})$, we can infer that the right hand side of (2.37) is bounded by a constant $M_3^k M_4(\alpha)$. Since $g_l$ is greater than 1, then

$$\sum_{l=1}^{I} \int_{\Omega_l} \int_{\alpha'}^{\alpha'+1} \left( \int_0^\infty e_l^k \exp(-yt)dt \right)^2 dydx \leq M_3^k M_4(\alpha). \tag{2.38}$$

(2.38) infers

$$\lim_{k\to\infty} \sum_{l=1}^{I} \int_{\Omega_l} |e_l^k|_\alpha^2 dx = 0, \tag{2.39}$$

that concludes the proof.

□

# 3 Convergence for Semilinear Elliptic Equations

Consider the semilinear elliptic equation

$$\begin{cases} -\sum_{i,j=1}^n \partial_j(a_{i,j}\partial_i u) + \sum_{i=1}^n b_i \partial_i u + cu = F(x,u), \text{ in } \Omega, \\ u = g, \text{ on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\Omega$ is a bounded and $C^2$ domain in $\mathbb{R}^n$. We impose on the coefficients of (3.1) Conditions $(A1)$, $(A2)$ in the previous section and the following condition
$(A3')$ There exists $C > 0$, such that $C < c(x)$ on $\overline{\Omega}$ and for all $x$ in $\mathbb{R}^n$:

$$|F(x,z) - F(x,z')| \leq C|z - z'|, \forall z, z' \in \mathbb{R}.$$

With Conditions $(A1)$, $(A2)$ and $(A3')$, Equation (3.1) has a unique solution $u$ in $W^{1,2}(\Omega) \cap L^\infty(\Omega)$ (see [10], [22]).

We impose the same way of dividing the domain $\Omega$ and the same notations as in the previous section.
The Schwarz algorithm at the iterate $\#k$ in the $l$-th domain, denoted by $u_l^k$, is then defined by

$$\begin{cases} -\sum_{i,j=1}^n \partial_j(a_{i,j}\partial_i u_l^k) + \sum_{i=1}^n b_i \partial_i u_l^k + cu_l^k = F(x,u_l^k), \text{ in } \Omega_l, \\ \mathfrak{B}_{l,l'} u_l^k = \mathfrak{B}_{l,l'} u_{l'}^{k-1}, \text{ on } \Gamma_{l,l'}, \forall l' \in J_l, \end{cases} \tag{3.2}$$

where $\mathfrak{B}_{l,l'}$ are either Dirichlet or Robin transmission operators.
Each iterate also inherits the boundary conditions of $u$:

$$u_l^k = g \text{ on } \partial\Omega_l \cap \partial\Omega.$$

A bounded initial guess $u^0$ in $C^\infty(\overline{\Omega \times (0,\infty)})$ is also provided

$$\mathfrak{B}_{l,l'} u_l^1 = u^0 \quad \text{on } \Gamma_{l,l'}, \forall l' \in J_l.$$

The difference $e_l^k$ between $u_l^k$ and $u$ is a solution of

$$\begin{cases} -\sum_{i,j=1}^n \partial_j(a_{i,j}\partial_i e_l^k) + \sum_{i=1}^n b_i \partial_i e_l^k + c e_l^k = F(x, u_l^k) - F(x,u), \text{ in } \Omega_l, \\ \mathfrak{B}_{l,l'} e_l^k = \mathfrak{B}_{l,l'} e_{l'}^{k-1}, \text{ on } \Gamma_{l,l'}, \forall l' \in J_l. \end{cases} \tag{3.3}$$

Moreover,

$$e_l^k = 0 \text{ on } \partial\Omega_l.$$

## 3.1 Classical Schwarz Methods

By induction, each subproblem (3.2) in each iteration has a unique solution in $W^{1,2}(\Omega) \cap L^\infty(\Omega)$ for the Dirichlet transmission condition. The algorithm is well-posed.
Consider (3.3) and let $g$ be a bounded and strictly positive function in $C^2(\mathbb{R}^n, \mathbb{R})$. Define the following function

$$\Phi_l^k(x) := (e_l^k(x))^2 g(x).$$

Let $c_i$ be $b_i + \sum_{j=1}^n 2a_{i,j}\partial_j g g^{-1}$, then $c_i$ belongs to $L^\infty(\Omega_l)$, and define

$$\mathfrak{L}_{lD}(\Phi) \; = \; -\sum_{i,j=1}^n \partial_j(a_{i,j}\partial_i \Phi) + \sum_{i=1}^n c_i \partial_i \Phi. \tag{3.4}$$

**Lemma 3.1.** *Choose $g$ to be $\tilde{g}^{-1}$ where $\tilde{g}$ is a solution in $C^2(\Omega) \cap C(\overline{\Omega})$ of the following equation*

$$\begin{cases} \sum_{i,j=1}^n \partial_j(a_{i,j}\partial_i \tilde{g}) + 2(C + ||c||_\infty)\tilde{g} - \sum_{i=1}^n b_i \partial_i \tilde{g} \le 0, \text{ in } \Omega_l, \\ \tilde{g} \text{ is strictly positive and bounded on } \overline{\Omega}, \end{cases} \tag{3.5}$$

*then $\mathfrak{L}_{lD}(\Phi_l^k) \le 0$ in the distributional sense.*

*Proof.* Define the operator

$$\mathfrak{L}_{lD0} := -\sum_{i,j=1}^{n} \partial_j(a_{i,j}\partial_i).$$

A complicated but easy computation gives

$$\begin{aligned} \mathfrak{H}(\Phi_l^k) &= \left(-\sum_{i,j=1}^{n} 2\partial_j(a_{i,j}\partial_i e_l^k)\right) e_l^k g - \sum_{i,j=1}^{n} 2a_{i,j}\partial_i e_l^k \partial_j e_l^k g - \\ &\quad - \sum_{i,j=1}^{n} 4a_{i,j}\partial_i e_l^k e_l^k \partial_j g + (e_l^k)^2 \left(\sum_{i,j=1}^{n}(-a_{i,j}\partial_{i,j} g + \partial_j a_{i,j}\partial_i g)\right). \end{aligned} \tag{3.6}$$

That implies

$$\mathfrak{H}(\Phi_l^k) + \sum_{i=1}^{n}\left(b_i + \sum_{j=1}^{n} 2a_{i,j}\partial_j g g^{-1}\right)\partial_i \Phi_l^k \leq (e_l^k)^2 \mathfrak{M}, \tag{3.7}$$

where

$$\mathfrak{M} = \left(\sum_{i,j=1}^{n} \partial_j(a_{i,j}\partial_i(g^{-1})) + 2(C + ||c||_\infty)(g^{-1}) - \sum_{i=1}^{n} b_i \partial_i(g^{-1})\right) g^2.$$

Therefore, the nonlinear equation (3.3) has been transformed into the following linearized inequality of $\Phi_l^k$:

$$-\sum_{i,j=1}^{n} \partial_j(a_{i,j}\partial_i \Phi_l^k) + \sum_{i=1}^{n} c_i(x,t)\partial_i \Phi_l^k \leq 0. \tag{3.8}$$

□

**Theorem 3.1.** *Consider the Schwarz algorithm with Dirichlet transmission condition,*

$$\lim_{k\to\infty} \max_{l\in\{1,\ldots,I\}} ||u_l^k - u||_{L^\infty(\Omega_l)} = 0.$$

*Proof.* **Step 1:** Construct some estimates of the errors $\{e_l^k\}$.

Consider (3.8) and define

$$M = \underset{\partial\Omega_l}{esssup}\, \Phi_l^k(x). \tag{3.9}$$

By the weak maximum principle, Theorem 8.1, [10], $\Phi_l^k(x)$ is bounded by $M$ almost every where on $\Omega_l$, that implies the following estimates,

$$\left(e_l^k\right)^2 g \leq \max_{l' \in J_l} \left( \underset{\Gamma_{l,l'}}{esssup} \left(e_l^k\right)^2 g \right), \text{ a.e. in } \Omega_l, \forall l \in \{1, \ldots, I\}. \tag{3.10}$$

**Step 2:** Convergence of the Algorithm.

Denote

$$E^k = \max_{l \in I} \left( \underset{\Omega_l}{esssup} \left(e_l^k\right)^2 \right). \tag{3.11}$$

From (2.17), for all $l'$ in $J_l$, and for a.e. $x$ in $\Gamma_{l,l'}$

$$\left(e_l^k(x)\right)^2 g(x) \leq \max_{l'' \in J_{l'}} \left( \underset{\Gamma_{l',l''}}{esssup} \left(e_{l'}^{k-1}(x)^2 g(x)\right) \right), \tag{3.12}$$

or

$$\begin{aligned} \left(e_l^k(x)\right)^2 &\leq \frac{1}{g(x)} \max_{l'' \in J_{l'}} \left( \underset{\Gamma_{l',l''}}{esssup} \left(e_{l'}^{k-1}(x)^2 g(x)\right) \right) \\ &\leq \tilde{g}(x) \max_{l'' \in J_{l'}} \left( \underset{\Gamma_{l',l''}}{esssup} \left(e_{l'}^{k-1}(x)^2 \tilde{g}(x)^{-1}\right) \right). \end{aligned} \tag{3.13}$$

Fix $l$ in $\{1, \ldots, I\}$ and let $f$ be a function in $C^2(\bar{\Omega})$ such that

- $f > 0$ on $\Omega_{l'}$, $\forall l' \in J_l$;
- There exist two positive real numbers $\epsilon$ small and $M$ large such that for $|\nabla f(x)| < \epsilon$, $\sum_{i,j=1}^n \partial_j(a_{i,j}\partial_i f)(x) > M$;
- $f = 0$ on $\partial\Omega_{l'}$, $\forall l' \in J_l$;

- $||f||_\infty = 1$. (We can construct this function by constructing a function $g$ which satisfies the first three properties, and then take $f = g/||g||_\infty$).

Let $\rho$ be a constant and put

$$\tilde{g} = M_0 - \exp(\rho f) = \exp(2\rho) - \exp(\rho f),$$

then,

$$\begin{aligned}
& \sum_{i,j=1}^{n} \partial_j(a_{i,j}\partial_i\tilde{g}) + 2(C + ||c||_\infty)\tilde{g} - \sum_{i=1}^{n} b_i\partial_i\tilde{g} \\
= & -\sum_{i,j=1}^{n} \rho^2 \exp(\rho f)\partial_j(a_{i,j}\partial_i f) - \sum_{i,j=1}^{n} a_{i,j}\rho\exp(\rho f)\partial_{i,j} f \\
& +2(C + ||c||_\infty)(\exp(2\rho) - \exp(\rho f)) + \sum_{i=1}^{n} b_i \exp(\rho f)\rho\partial_i f \\
< & \exp(-\rho f)\left[-\rho^2 M + \epsilon\rho + 2(C + ||c||_\infty)\left(\frac{\exp(2\rho)}{\exp(\rho f)} - 1\right) + \max_i ||b_i||_\infty\epsilon\rho\right] \quad (3.14) \\
< & \exp(-\rho f)\left[-\rho^2 M + \epsilon\rho + 2(C + ||c||_\infty)\exp(2\rho) + \max_i ||b_i||_\infty\epsilon\rho\right] \\
< & 0,
\end{aligned}$$

when $\rho$ is large enough and $M > \exp(2\rho)$.
Inequality (3.13) then becomes

$$\left(e_l^k(x)\right)^2 \le (M_0 - \exp(\rho f(x))) \max_{l'' \in J_{l'}} \left( \underset{\Gamma_{l',l''}}{esssup} \left( \left(e_{l'}^{k-1}(x)\right)^2 (M_0 - 1)^{-1} \right) \right) \quad (3.15)$$

Since $\Gamma_{l,l'}$ lies inside $\Omega_{l'}$, $f(x)$ is strictly positive on $\Gamma_{l,l'}$. Hence, there exists $M_{1,l}$ strictly less than 1, such that

$$\left(e_l^k(x)\right)^2 \le M_{1,l} \max_{l'' \in J_{l'}} \left( \underset{\Gamma_{l',l''}}{esssup} \left(e_{l'}^{k-1}(x)\right)^2 \right) \le M_{1,l} E^{k-1}, \forall l' \in J_l, \text{ for a.e } x \text{ in } \Gamma_{l,l'}. \quad (3.16)$$

Similarly as in (3.26), let $f$ be a function in $C^2(\bar{\Omega})$ such that

- $f = 0$ on $\partial\Omega_l$;
- $f > 0$ on on $\Omega_l$;

- There exist two positive constants $\epsilon$ small enough and $M$ large enough such that for $|\nabla f_l(x)| < \epsilon$, we have that $\sum_{i,j=1}^{n} \partial_j (a_{i,j} \partial_i f)(x) > M$;
- $||f||_\infty = 1$.

and let $\rho$ be a constant large enough. Setting

$$\tilde{g} = M_0 - \exp(\rho f),$$

$$\begin{aligned}\left(e_l^k(x)\right)^2 &\leq \tilde{g}(x) \max_{l' \in J_l} \left( \sup_{\Gamma_{l,l'}} \left(e_l^k(x)\right)^2 (M_0 - 1) \right) \\ &\leq \max_{l' \in J_l} \left( \sup_{\Gamma_{l,l'}} \left(e_l^k(x)\right)^2 \right), \text{ for a.e } x \text{ in } \Omega_l. \end{aligned} \tag{3.17}$$

Combining (3.16) and (3.17), we can deduce that there exists $M_2$ strictly less than 1 satisfying

$$E^k \leq M_2 E^{k-1}, \tag{3.18}$$

that leads to

$$\lim_{k \to \infty} E^k = 0.$$

□

## 3.2 Optimized Schwarz Methods

The optimized Schwarz waveform relaxation algorithms are defined by replacing the Dirichlet transmission operators by Robin ones

$$\mathfrak{B}_{l,l'} v = \sum_{i,j=1}^{n} a_{i,j} \partial_i v n_{l,l',j} + p_{l,l'} v,$$

where $n_{l,l',j}$ is the $j$-th component of the outward unit normal vector of $\Gamma_{l,l'}$; $p_{l,l'}$ is positive and belongs to $L^\infty(\Gamma_{l,l'})$. By induction argument, each subproblem (2.2) in each iteration has a unique solution in $H^1(\Omega)$ and the algorithm is well-posed.

In general, optimized Schwarz algorithms do not always converge when applied to elliptic equations, as shown in the following example.

**Example 3.1.** *Consider the elliptic problem on the domain* $\Omega = (0, L)$

$$\begin{cases} u'' - 3u' - 4u = f, & \text{in } \Omega, \\ u(0) = u(L) = 0, \end{cases} \tag{3.19}$$

*where* $f$ *belongs to* $C^\infty([0, L])$*. Divide* $\Omega$ *into two subdomains* $\Omega_1 = (0, L_2)$ *and* $\Omega_2 = (L_1, L)$*, with* $0 < L_1 < L_2 < L$*, and consider the domain decomposition algorithm*

$$\begin{cases} (u_1^{k+1})'' - 3(u_1^{k+1})' - 4u_1^{k+1} = f, \ \text{in } (0, L_2), \\ u_1^{k+1}(0) = 0 \ \text{and} \ (u_1^{k+1})'(L_2) + pu_1^{k+1}(L_2) = (u_2^k)'(L_2) + pu_2^k(L_2), \end{cases}$$

$$\begin{cases} (u_2^{k+1})'' - 3(u_2^{k+1})' - 4u_2^{k+1} = f, \ \text{in } (L_1, L), \\ u_2^{k+1}(L) = 0 \ \text{and} \ (u_2^{k+1})'(L_1) - qu_2^{k+1}(L_1) = (u_1^k)'(L_1) - qu_1^k(L_1), \end{cases}$$

*where* $p$*,* $q$ *are positive numbers.*
*The errors* $e_1^k$ *and* $e_2^k$ *from the above equations can be obtained*

$$e_1^{k+1} = A_{k+1}(\exp(4x) - \exp(-x)),$$

$$e_2^{k+1} = B_{k+1}(\exp(4(x-L)) - \exp(-(x-L))),$$

*where*

$$\tau_1 = \frac{A_{k+1}}{B_k} = \frac{4\exp(4(L_2-L)) + \exp(-(L_2-L)) + p(\exp(4(L_2-L)) - \exp(-(L_2-L)))}{4\exp(4L_2) + \exp(-L_2) + p(\exp(4L_2) - \exp(-L_2))},$$

$$\tau_2 = \frac{B_{k+1}}{A_k} = \frac{4\exp(4L_1) + \exp(-L_1) - q(\exp(4L_1) - \exp(-L_1))}{4\exp(4(L_1-L)) + \exp(-(L_1-L)) - q(\exp(4(L_1-L)) - \exp(-(L_1-L)))}.$$

*Set*

$$\tau = \left| \frac{A_{k+1}B_{k+1}}{B_k A_k} \right|,$$

*then*

$$\begin{aligned} \tau \;=\; & \left| \frac{4\exp(5L_2) + \exp(5L) + p(\exp(5L_2) - \exp(5L))}{4\exp(5L_2) + 1 + p(\exp(5L_2) - 1)} \right| \\ & \times \left| \frac{4\exp(5L_1) + 1 - q(\exp(5L_1) - 1)}{4\exp(5L_1) + \exp(5L) - q(\exp(5L_1) - \exp(5L))} \right|. \end{aligned} \tag{3.20}$$

*The algorithm converges if and only if* $\tau$ *is smaller than* 1.

*For* $p=1$ *and* $q$ *large,*

$$\tau \backsim \frac{\exp(5L_1)-1}{-\exp(5L_1)+\exp(5L)}. \tag{3.21}$$

*Since the right hand side of* (3.21) *is greater than* 1 *for* $L_1 > L/\ln 2$ *and* $L$ *large, the algorithm does not converge for* $p=1$ *and* $q$ *large.*
*Note that*

$$\tau_1 = \left| \frac{4\exp(5L_2)+\exp(5L)+p(\exp(5L_2)-\exp(5L))}{4\exp(5L_2)+1+p(\exp(5L_2)-1)} \right|,$$

*can be made larger than 1 since*

$$\left| \frac{4\exp(5L_2)+\exp(5L)}{4\exp(5L_2)+1} \right|$$

*is larger than 1;*
*and the second term can be made larger than 1 since*

$$\tau_2 = \frac{\exp(5L_1)-1}{-\exp(5L_1)+\exp(5L)}$$

*is larger than* 1 *for* $L_1 > L/\ln 2$ *and* $L$ *large.*

**Remark 3.1.** *In the above example, the Schwarz algorithms converge if* $p$ *and* $q$ *are large. This observation leads to Theorem* 3.2 *below. Moreover, the convergence factor is still small if* $p$, $q$ *are chosen to be negative. At least, the behavior of* $\tau$ *is quite the same when* $q$ *tend to* $+\infty$ *and* $-\infty$. *This observation is different from what was seen from the convergence rate in the case where the subdomains are two half lines in [3] and [5].*

Introduce the modified Robin transmission operator, based on $(T_2)$

$$\mathfrak{B}^{\rho}_{l,l'} v = \sum_{i,j=1}^{n} a_{i,j}\partial_i v n_{l,l',j} + \rho p_{l,l'} v,$$

where $\rho$ is a positive parameter.

**Theorem 3.2.** *Consider Schwarz Algorithms with Robin transmission conditions, if we replace $\mathfrak{B}_{l,l'}$ by $\mathfrak{B}^{\rho}_{l,l'}$, then there exists $\rho_0$ such that for $\rho > \rho_0$, the algorithms converge in the following sense*

$$\lim_{k\to\infty} \max_{l\in\{1,\dots,I\}} ||u_l^k - u||_{L^2(\Omega_l)} = 0.$$

**Remark 3.2.** *Consider again Example 3.1, and a Schwarz algorithm which diverges with $p = 0$ and $q = q_0$, there exists $L_1$, $L_2$, $L$ such that the algorithm diverges*

$$\begin{cases} (u_1^k)'' - 3(u_1^k)' - 4u_1^k = f, \ \textit{in } (0, L_2), \\ u_1^k(0) = 0 \ \textit{and } (u_1^k)'(L_2) = (u_2^{k-1})'(L_2), \end{cases}$$

$$\begin{cases} (u_2^k)'' - 3(u_2^k)' - 4u_2^k = f, \ \textit{in } (L_1, L), \\ u_2^k(L) = 0 \ \textit{and } (u_2^k)'(L_1) - q_0 u_2^k(L_1) = (u_1^{k-1})'(L_1) - q_0 u_1^{k-1}(L_1), \end{cases}$$

*where $q_0$ is a large constant.*
*Let $P$ be a function in $C^1(\mathbb{R})$, and put $w = u\exp(P)$, Equation* (3.19) *can be transformed into*

$$\begin{cases} w'' - (3 + 2P')w' + (-4 - 3P' + (P')^2 - P'')w = f, \quad \textit{in } (0, L), \\ w(0) = w(L) = 0. \end{cases} \tag{3.22}$$

*The Schwarz algorithm then becomes*

$$\begin{cases} (w_1^k)'' - (3 + 2P')(w_1^k)' + (-4 - 3P' + (P')^2 - P'')w_1^k = f, \ \textit{in } (0, L_2), \\ w_1^k(0) = 0 \ \textit{and } ((w_1^k)' - P'w_1^k)(L_2) = ((w_2^{k-1})' - P'w_2^{k-1})(L_2), \end{cases}$$

$$\begin{cases} (w_2^k)'' - (3 + 2P')(w_2^k)' + (-4 - 3P' + (P')^2 - P'')w_2^k = f, \ \textit{in } (L_1, L), \\ w_2^k(L) = 0 \ \textit{and } (w_2^k)'(L_1) - (P' + q_0)w_2^k(L_1) = ((w_1^{k-1})'(L_1) - (P' + q_0)w_2^k(L_1), \end{cases}$$

*We can deduce that given a pair of numbers $(p, q)$, we can find a class of functions $P$ such that $-P'(L_2) = p$ and $-P'(L_1) = q + q_0$, and the Schwarz algorithm with the associated equation* (3.22) *and this Robin transmission condition does not converge. However, Theorem* 3.1 *announces that we can make the algorithms converge, even if they do not converge initially, by increasing the parameter $\rho$.*

*Proof.* **Step 1:** Linearize the equation (2.3).

Consider the equation (2.3) and let $g_l$ be a strictly positive bounded function in $C^2(\Omega_l, \mathbb{R})$. Define the following function

$$\Phi_l^k(x) := e_l^k(x) g_l(x).$$

A complicated but easy computation gives

$$\begin{aligned} 0 \;\; = \;\; & -\sum_{i,j=1}^{n} \partial_j (a_{i,j} \partial_i \Phi_l^k) + \sum_{i=1}^{n} b_i \partial_i \Phi_l^k + \sum_{i,j=1}^{n} a_{i,j} \left( \partial_i \Phi_l^k \frac{\partial_j g_l^k}{g_l^k} + \partial_j \Phi_l^k \frac{\partial_j g_l^k}{g_l^k} \right) \quad (3.23) \\ & + \left( \sum_{i,j=1}^{n} a_{i,j} \frac{\partial_{i,j} g_l^k}{g_l^k} - \sum_{i,j=1}^{n} a_{i,j} \frac{2\partial_i g_l^k \partial_j g_l^k}{(g_l^k)^2} - \sum_{i=1}^{n} b_i \frac{\partial_i g_l^k}{g_l^k} + (c - \overline{F})(g_l^k)^{-1} \right) \Phi_l^k, \end{aligned}$$

where

$$\begin{cases} \bar{F}(x) = 0 \text{ if } u_l^k(x) = u(x), x \in \Omega, \\ \bar{F}(x) = \frac{F(u_l^k(x)) - F(u(x))}{u_l^k(x) - u(x)} \text{ if } u_l^k(x) \neq u(x), x \in \Omega, \end{cases}$$

$\bar{F}$ is then bounded as $F$ is Lipschitz.
Similar as in (3.8) and in Step 2 of the proof of Theorem 3.1, we rewrite the last term on the right hand side of (3.23) into the following form

$$\left( -\sum_{i,j=1}^{n} \partial_i (a_{i,j} \partial_i ((g_l^k)^{-1})) + (c - \overline{F})((g_l^k)^{-1}) + \sum_{i=1}^{n} b_i \partial_i ((g_l^k)^{-1}) \right) \Phi_l^k, \quad (3.24)$$

and use the same argument as in (3.5): choose $g_l^k$ to be $\tilde{g}_l{}^{-1}$ where $\tilde{g}_l$ is a solution in $C^2(\Omega) \cap C(\overline{\Omega})$ of the following equation

$$\begin{cases} -\sum_{i,j=1}^{n} \partial_i (a_{i,j} \partial_i \tilde{g}_l) - K \tilde{g}_l + \sum_{i=1}^{n} b_i \partial_i \tilde{g}_l \geq 0, \text{ in } \Omega_l, \\ \tilde{g} \text{ is strictly positive and bounded on } \overline{\Omega}, \end{cases} \quad (3.25)$$

where $K$ is a large enough constant.

Since $p_{l,l'}$ is strictly positive for all $l$ in $\{1, \dots, I\}$ and $l'$ in $J_l$, there exist functions $f_l$, $l \in \{1, \dots, I\}$, in $C^2(\bar{\Omega})$ such that

- $\sum_{i,j=1}^{n} a_{i,j} n_{l,l',j} \partial_i f_l = p_{l,l'}$ on $\Gamma_{l,l'}$, $\forall l' \in J_l$.
- $\sum_{i,j=1}^{n} a_{i,j} n_{l',l,j} \partial_i f_l = p_{l',l}$ on $\Gamma_{l',l}$, $\forall l'$: $l \in J_{l'}$.

- There exist two positive constants $\epsilon$ small enough and $M$ large enough such that for $|\nabla f_l(x)| < \epsilon$, $\sum_{i,j=1}^n \partial_j(a_{i,j}\partial_i f)(x) > M$.
- $f_l = 0$, on $\Gamma_{l,l'}$, $\forall l' \in J_l$; and $f_{l'} = \alpha_l$, on $\Gamma_{l',l}$, $\forall l' : l \in J_{l'}$.
- $||f||_\infty = 1$. (We can construct this function by constructing a function $g$ which satisfies the first properties, and then take $f = g/||g||_\infty$).

Similar as in (3.26), let $\rho$ be a constant large enough and put $\tilde{g} = M_3 - \exp(-\rho f)$, where $M_3$ is a positive constant,

$$\begin{aligned}
&\sum_{i,j=1}^n \partial_j(a_{i,j}\partial_i\tilde{g}) + 2(C + ||c||_\infty)\tilde{g} - \sum_{i=1}^n b_i\partial_i\tilde{g} \\
= \;& -\sum_{i,j=1}^n \rho^2 \exp(\rho f)\partial_j(a_{i,j}\partial_i f) - \sum_{i,j=1}^n a_{i,j}\rho\exp(\rho f)\partial_{i,j}f \\
&+2(C + ||c||_\infty)(M_3 - \exp(\rho f)) + \sum_{i=1}^n b_i \exp(\rho f)\rho\partial_i f \\
= \;& \Bigg(-\lambda\rho^2 M - \sum_{i,j=1}^n a_{i,j}\partial_{i,j} f\rho \\
&+2(C + ||c||_\infty)\frac{M_3 - \exp(\rho f)}{\exp(\rho f)} + \sum_{i=1}^n b_i\rho\partial_i f\Bigg)\exp(\rho f) \\
< \;& 0,
\end{aligned}$$

when $\rho$ is large enough.

Denote the right hand side of the equation (3.23) by $\mathfrak{L}_l(\Phi_l^k)$, then it can be rewritten in the following form

$$\mathfrak{L}_l(\Phi_l^k) \;=\; -\sum_{i,j=1}^n \partial_j(a_{i,j}\partial_i\Phi_l^k) + \sum_{i=1}^n B_i^l\partial_i\Phi_l^k + C^l\Phi_l^k, \tag{3.26}$$

where $B_i^l$ and $C^l$ are functions in $L^\infty(\mathbb{R}^n)$, $C^l$ is bounded from below by $K+c+\overline{F}$. $\rho$ can be chosen such that there exists $\alpha$ large enough, $2\alpha > C_l > \alpha$.

Now, consider the Robin transmission condition on the boundary $\Gamma_{l,l'}$

$$\mathfrak{B}_l(\Phi_l^k) \;=\; \sum_{i,j=1}^n a_{i,j}\partial_i\Phi_l^k n_{l,l',j} \tag{3.27}$$

$$
\begin{aligned}
&= \left(\sum_{i,j=1}^{n} a_{i,j} n_{l,l',j} \partial_i e_l^k\right) g_l + \sum_{i,j=1}^{n} a_{i,j} n_{l,l',j} \partial_i g_l e_l^k \\
&= \left(\sum_{i,j=1}^{n} a_{i,j} n_{l',l,j} \partial_i e_l^k + p_{l,l'} e_l^k\right) g_l \\
&= \left(\sum_{i,j=1}^{n} a_{i,j} n_{l',l,j} \partial_i e_{l'}^{k-1} + p_{l,l'} e_{l'}^k\right) g_l \\
&= \left(\sum_{i,j=1}^{n} a_{i,j} \partial_i \Phi_{l'}^{k-1} n_{l',l,j}\right) \frac{g_l}{g_l'} = \mathfrak{B}_l(\Phi_{l'}^{k-1}) \frac{g_l}{g_l'}.
\end{aligned}
$$

We can choose $f_l$ such that

$$
\frac{g_{l'}}{g_l} = \beta_l, \text{ on } \Gamma_{l,l'}, \forall l' \in J_l,
$$

where $\beta_l$ is a constant greater than 1.

From the previous calculation on $\mathfrak{B}_l(\Phi_l^k)$ and $\mathfrak{L}_l(\Phi_l^k)$, $\Phi_l^k$ is in fact a solution of the following equation

$$
\begin{cases}
\mathfrak{L}_l(\Phi_l^k) = 0, & \text{in } \Omega_l \times (0, \infty), \\
\beta_l \mathfrak{B}_{l,l'}(\Phi_l^k) = \mathfrak{B}_{l,l'}(\Phi_{l'}^{k-1}) & \text{on } \Gamma_{l,l'} \times (0, \infty), \forall l' \in J_l.
\end{cases}
\tag{3.28}
$$

**Step 2:** The Proof of Convergence.

Denote by $\tilde{\Omega}_l$ to be the open set $\Omega_l \backslash \overline{\cup_{l' \in J_l} \Omega_{l'}}$. For each $l$ in $\{1, \ldots, I\}$, let $\varphi_l^k$ to be a function in $H^1(\Omega_l)$ and $\varphi_l^{k+1}$ to be a function in $H^1(\Omega_l)$ such that $\varphi_l^{k+1} = \varphi_{l'}^k$ on $\Gamma_{l,l'}$ for all $l'$ in $J_l$. Now, using $\varphi_l^{k+1}$ and $\varphi_l^k$ as test functions for all subdomains, we obtain

$$
\begin{aligned}
& -\sum_{l=1}^{I} \Bigg\{ \int_{\tilde{\Omega}_l} \sum_{i,j=1}^{n} a_{i,j} \partial_i \Phi_l^k \partial_j \varphi_l^k dx + \int_{\tilde{\Omega}_l} \sum_{i=1}^{n} B_i^l \partial_i \Phi_l^k \varphi_l^k dx + \int_{\tilde{\Omega}_l} C^l \Phi_l^k \varphi_l^k dx \\
& - \sum_{l' \in J_l} \int_{\Gamma_{l',l}} p_{l',l} \Phi_l^k \varphi_l^k d\sigma \Bigg\} \\
= & \sum_{l=1}^{I} \beta_l \Bigg\{ \int_{\Omega_l} \sum_{i,j=1}^{n} a_{i,j} \partial_i \Phi_l^{k+1} \partial_j \varphi_l^{k+1} dx + \int_{\Omega_l} \sum_{i=1}^{n} B_i^l \partial_i \Phi_l^{k+1} \varphi_l^{k+1} dx
\end{aligned}
$$

$$+\int_{\Omega_l} C^l \Phi_l^{k+1} \varphi_l^{k+1} dx + \sum_{l' \in J_l} \int_{\Gamma_{l,l'}} p_{l,l'} \Phi_l^{k+1} \varphi_l^{k+1} d\sigma \Bigg\}. \tag{3.29}$$

In the above equality, choose $\varphi_l^{k+1}$ to be $\Phi_l^{k+1}$, then there exists $\varphi_l^k$ such that $\varphi_l^k = \varphi_{l'}^{k+1}$ on $\Gamma_{l,l'}$ for all $l'$ in $J_l$; and

$$||\varphi_l^k||_{H^1(\Omega_l)} \le C \sum_{l' \in J_l} ||\varphi_{l'}^{k+1}||_{H^1(\Omega_{l'})}; ||\varphi_l^k||_{L^2(\Omega_l)} \le C \sum_{l' \in J_l} ||\varphi_{l'}^{k+1}||_{L^2(\Omega_{l'})}.$$

With these test functions, the right hand side of (3.29) is greater than or equal to

$$\begin{aligned}
& \sum_{l=1}^{I} \beta_l \Bigg\{ \int_{\Omega_l} \lambda |\nabla \Phi_l^{k+1}|^2 dx - \sum_{i=1}^{n} \int_{\Omega_l} ||B_i^l||_{L^\infty(\Omega_l)} \left|\partial_i \Phi_l^{k+1}\right| |\Phi_l^{k+1}| dx \\
& +\alpha \int_{\Omega_l} |\Phi_l^{k+1}|^2 dx + \sum_{l' \in J_l} \int_{\Gamma_{l',l}} p_{l',l} |\Phi_l^{k+1}|^2 d\sigma \Bigg\} \\
\ge \ & \sum_{l=1}^{I} \beta_l \Bigg[ \int_{\Omega_l} \lambda |\nabla \Phi_l^{k+1}|^2 dx - \sum_{i=1}^{n} \int_{\Omega_l} ||B_i^l||_{L^\infty(\Omega_l)} \left|\partial_i \Phi_l^{k+1}\right| |\Phi_l^{k+1}| dx \\
& +\alpha \int_{\Omega_l} |\Phi_l^{k+1}|^2 \Bigg] \\
\ge \ & \sum_{l=1}^{I} \beta_l \left[ \int_{\Omega_l} \frac{\lambda}{2} |\nabla \Phi_l^{k+1}|^2 dx + \frac{\alpha}{2} \int_{\Omega_l} |\Phi_l^{k+1}|^2 \right],
\end{aligned}$$

with $\alpha$ being large enough.
Similarly, we estimate the left hand side of (3.29), which is in fact bounded by

$$\begin{aligned}
& \sum_{l=1}^{I} \Bigg\{ \int_{\tilde{\Omega}_l} \Lambda |\nabla \Phi_l^k| |\nabla \varphi_l^k| dx + \int_{\tilde{\Omega}_l} 2\alpha |\Phi_l^k| |\varphi_l^k| dx \\
& + \sum_{i=1}^{n} \int_{\tilde{\Omega}_l} ||B_i^l||_{L^\infty(\tilde{\Omega}_l)} \left|\partial_i \Phi_l^k\right| |\varphi_l^k| dx + \sum_{l' \in J_l} \int_{\Gamma_{l',l}} p_{l',l} |\Phi_l^k| |\varphi_l^k| d\sigma \Bigg\} \\
\le \ & \sum_{l=1}^{I} M_4 \Big[ \Lambda \left( ||\nabla \Phi_l^k||^2_{L^2(\tilde{\Omega}_l)} + ||\nabla \varphi_l^k||^2_{L^2(\tilde{\Omega}_l)} \right) + \alpha ||\Phi_l^k||^2_{L^2(\tilde{\Omega}_l)} + \alpha ||\varphi_l^k||^2_{L^2(\tilde{\Omega}_l)}
\end{aligned}$$

$$+\left(||\nabla\Phi_l^k||^2_{L^2(\tilde\Omega_l)} + (\max_{i\in\{1,I\}}||B_i^l||_{L^\infty(\tilde\Omega_l)})^2||\varphi_l^k||^2_{L^2(\tilde\Omega_l)}\right)$$

$$+\sum_{l'\in J_l}||p_{l',l}||_{L^\infty(\Gamma_{l',l})}\left(||\Phi_l^k||^2_{H^1(\tilde\Omega_l)} + ||\varphi_l^k||^2_{H^1(\tilde\Omega_l)}\right)\Bigg], \tag{3.30}$$

where $M_4$ is a positive constant which depends only on $\{\Omega_l\}_{l\in\{1,I\}}$ and the coefficients of (3.3). Since $\alpha$ can be chosen such that $\alpha > (\max_{i\in\{1,\dots,I\}}||B_i^l||_{L^\infty(\tilde\Omega_l)})^2$, there exists $M_5$ positive, depending only on $\{\Omega_l\}_{l\in\{1,\dots,I\}}$ and the coefficients of (3.3) such that the right hand side of (3.30) is less than

$$\sum_{l=1}^{I} M_5\left[\int_{\tilde\Omega_l}\left(\frac{\lambda}{2}|\nabla\Phi_l^k|^2 + \frac{\alpha}{2}|\Phi_l^k|^2 + \frac{\lambda}{2}|\nabla\Phi_l^{k+1}|^2 + \frac{\alpha}{2}|\Phi_l^{k+1}|^2\right)dx\right] \tag{3.31}$$

$$\leq \sum_{l=1}^{I} M_5\left(\frac{\lambda}{2}||\nabla\Phi_l^k||^2_{L^2(\Omega_l)} + \frac{\alpha}{2}||\Phi_l^k||^2_{L^2(\Omega_l)} + \frac{\lambda}{2}||\nabla\Phi_l^{k+1}||^2_{L^2(\Omega_l)} + \frac{\alpha}{2}||\Phi_l^{k+1}||^2_{L^2(\Omega_l)}\right).$$

Define

$$E_k := \sum_{l=1}^{I}\left(\frac{\lambda}{2}||\nabla\Phi_l^k||^2_{L^2(\Omega_l)} + \frac{\alpha}{2}||\Phi_l^k||^2_{L^2(\Omega_l)}\right), \tag{3.32}$$

then from (3.30), (3.31) and (3.32),

$$(\beta - M_5)E_{k+1} \leq M_5 E_k, \tag{3.33}$$

$\beta = \min\{\beta_1,\dots,\beta_I\}$. Since $M_5$ depends only on $\{\Omega_l\}_{l\in 1,I}$ and the coefficients of (2.3), $\beta$ can be chosen large enough, such that $\frac{M_5}{\beta - M_5} < 1$, then

$$E_k \leq \left(\frac{M_5}{\beta - M_5}\right)^{k-1} E_1, \tag{3.34}$$

which means $E_k$ tends to 0 as $k$ tends to infinity. ☐

# 4 Conclusions

We have introduced a new class of techniques to study the convergence of Schwarz methods. In particular, classical Schwarz methods are proved to converge when being applied to both parabolic and elliptic equations. On

the contrary, Schwarz methods with Robin transmission conditions only converge when we use them for parabolic equations, though they were proved to converge faster than classical ones in previous studies. For elliptic equations, we have given a counter example, where we can see that optimized Schwarz methods do not converge; and for each optimized Schwarz algorithm, there exists a class of elliptic equations which is not stable with this algorithm. A new way of stabilizing the algorithms has then been proposed.
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