RECENT PROGRESS ON OBSERVABILITY FOR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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We survey some recent progress in the observability estimate for stochastic partial differential equation, including the stochastic parabolic equation, the stochastic hyperbolic equation and the stochastic Schrödinger equation. All the observability estimates are obtained by global Carleman estimate.

Keywords: Observability estimate; global Carleman estimate; stochastic partial differential equations.

1. Introduction

Let $T > 0$, $G \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ be a given bounded domain with a C^2 boundary Γ. Put $Q \stackrel{\triangle}{=} (0,T) \times G$, $\Sigma \stackrel{\triangle}{=} (0,T) \times \Gamma$.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete filtered probability space on which a 1-dimensional standard Brownian motion $\{B(t)\}_{t\geq 0}$ is defined. Let H be a Banach space. We denote by $L^2_{\mathcal{F}}(0,T;H)$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{L^2(0,T;H)}) < \infty$; by $L^\infty_{\mathcal{F}}(0,T;H)$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted bounded processes; and by $L^2_{\mathcal{F}}(\Omega; C([0,T];H))$ the Fréchet space consisting of all H-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted continuous processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{C([0,T];H)}) < \infty$; (similarly, one can define $L^2_{\mathcal{F}}(\Omega; C^k([0,T];H))$ for any positive integer k), all of these spaces are endowed with the canonical norms respectively.

This paper is devoted to surveying some recent results for the observability estimate for stochastic partial differential equations(SPDEs for short). We only focus on the three typical SPDEs, that is, stochastic parabolic equations, stochastic hyperbolic equations and stochastic Schrödinger equations.

Observability estimate plays an important role in many application problems, such as controllability (see [3,13,24] for example), optimal control (see [12] for example), inverse problems (see [8] for example) and so on. There are a great many studies on observability estimate for deterministic partial differential equations (PDEs for short)(see [2,3,6,13,22,24] and the rich references cited therein). It would be natural to extend the observability estimate to the stochastic case. However, the study of such kind of problems is highly undeveloped. To our best knowledge, [1,15,16,21,23] are the only published papers concerning these problems. Lots of things should be done, and some of which seem to be very difficult.

People have introduced several powerful methods to establish the observability estimate for PDEs. For example, for the parabolic equations, we have the spectral method (see Ref. 11 for example) and the Carleman estimates (see Ref. 6 for example); for the hyperbolic equations, we have the spectral method (see Ref. 20 for example), the multiplier method (see Ref. 13 for example), the microlocal analysis method (see Ref. 2 for example), and the Carleman estimates (see Ref. 22 for example); for Schrödinger equations, we have the multiplier method (see Ref. 19 for example), the microlocal analysis method (see Ref. 10 for example), and the Carleman estimates(see Ref. 9 for example).

One will meet substantially new difficulties in the study of observability estimate for SPDEs. For instance, unlike the PDEs, the solution of a SPDE is usually non-differentiable with respect to the variable with noise (say, the time variable considered in this paper). Also, the usual compactness embedding result does not remain true for the solution spaces related to SPDEs. These new phenomenons lead that some useful methods for establishing observability estimate for PDEs cannot be applied to SPDEs. Especially, it seems that the multiplier method cannot be used to establish observability for stochastic hyperbolic equations or stochastic Schrödinger equations.

Until now, the most useful tool for studying the observability for SPDEs is the global Carleman estimate. Carleman estimate is simply a weighted energy method. There are two crucial steps in Carleman estimate. The first one is to obtain a suitable point-wise estimate related to the principal operator of the equation. The second one is to choose a proper weight function. This method has many advantages. For example, it is robust for a class of lower order terms, and it can give an explicit estimate on the observability constant with respect to suitable Sobolev space norms of the coefficients in the equations, which is crucial for some control problems.

The rest of this paper is organized as follows. Section 2 is devoted to introducing the observability estimate for stochastic parabolic equations. In Section 3, we review the observability estimate for stochastic hyperbolic equations. Section 4 is addressed to an introduction of the observability estimate for stochastic Schrödinger equations. As we have mentioned before, all the observability estimates presented in this paper are obtained by some global Carleman estimate and the crucial steps for the global Carleman estimate is to establish suitable point-wise estimate and to choose proper weight function, hence we organize the rest three sections in the following way:

We first present the equation. Next we introduce the result of observability estimate. Further, we give the pointwise estimate and the weight function. For the detailed proofs, please find them in the references.

2. Observability estimate for stochastic parabolic equation

Throughout this section, we make the following assumptions on the coefficients $a^{ij}: \ \Omega \times [0,T] \times \overline{G} \to \mathbb{R}^{n\times n} \ \ (i,j = 1,2,\cdots,n)$:

(H1) *$a^{ij} \in L^2_{\mathcal{F}}(\Omega; C^1([0,T]; W^{2,\infty}(G)))$ and $a^{ij} = a^{ji}$;*

(H2) *For any $\delta > 0$, there is $\rho > 0$ such that $|a^{ij}(\omega,t,x_1) - a^{ij}(\omega,t,x_2)| \leq \delta$ almost surely for any $t \in [0,T]$ and $x_1, x_2 \in \overline{G}$ which satisfy the following inequality: $|x_1 - x_2| \leq \rho$;*

(H3) *There is some constant $s_0 > 0$ such that*

$$\sum_{i,j} a^{ij}(\omega,t,x)\xi^i\xi^j \geq s_0|\xi|^2, \quad (\omega,t,x,\xi) \equiv (\omega,t,x,\xi^1,\cdots,\xi^n) \in \Omega\times Q\times\mathbb{R}^n. \tag{1}$$

Here and henceforth, we denote $\sum\limits_{i,j=1}^{n}$ simply by $\sum\limits_{i,j}$. For simplicity, we will use the notation $y_i \equiv y_i(x) = \partial y(x)/\partial x_i$, where x_i is the i-th coordinate of a generic point $x = (x_1,\cdots,x_n)$ in $\mathbb{R}^n$. In a similar manner, we use the notations z_i, v_i, etc. for the partial derivatives of z and v with respect to x_i. Also, we denote the scalar product in $\mathbb{R}^n$ by $\langle \cdot,\cdot \rangle$, and use C to denote a generic positive constant depending only on G, G_0 and $(a^{ij})_{n\times n}$, which may change from line to line.

Consider the following stochastic parabolic equation:

$$\begin{cases} dz - \sum_{i,j}(a^{ij}z_i)_j dt = [\langle\, a, \nabla z\,\rangle + bz]dt + czdB(t) & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(0) = z_0 & \text{in } G, \end{cases} \tag{2}$$

Here

$$a \in L^\infty_{\mathcal{F}}(0,T;L^\infty(G;\mathbb{R}^n)), b \in L^\infty_{\mathcal{F}}(0,T;L^{n^*}(G)), \text{ and } c \in L^\infty_{\mathcal{F}}(0,T;W^{1,\infty}(G)).$$

We begin with the following notion.

Definition 2.1. We call $z \in L^2_{\mathcal{F}}(\Omega; C([0,T];L^2(G))) \cap L^2_{\mathcal{F}}(0,T;H^1(G))$ a solution of equation (2) if the following hold:

1. $z(0) = z_0$ in G, P-a.s.,
2. For any $t \in [0,T]$ and any $\eta \in H^1_0(G)$, it holds

$$\begin{aligned} &\int_G z(t,x)\eta(x)dx - \int_G z(0,x)\eta(x)dx \\ =& \int_0^t \int_G \Big\{ -\sum_{i,j} a^{ij}(s,x)z_i(s,x)\eta_j(x) \\ &+[\langle\, a(s,x), \nabla z(s,x)\,\rangle + b(s,x)z(s,x)]\eta(x)\Big\}dxds \\ &+\int_0^t\int_G c(s,x)z(s,x)\eta(x)dxdB(s), \qquad P-\text{a.s.} \end{aligned} \tag{3}$$

We refer to Ref. 7 for the well-posedness of equation (2).

We have the following observability estimate for equation (2).

Theorem 2.1. *[21, Theorem 2.3] Let assumptions (H1), (H2) and (H3) be satisfied. Then there is a constant $C > 0$ such that all solutions z of equation (2) satisfy that*

$$|z(T)|_{L^2(\Omega,\mathcal{F}_T,P;L^2(G))} \le Ce^{C[T^{-4}(1+r_1^2)+Tr_1^2]}|z(\cdot)|_{L^2_{\mathcal{F}}(0,T;L^2(G_0))} \tag{4}$$

with

$$r_1 \triangleq |a|_{L^\infty_{\mathcal{F}}(0,T;L^\infty(G;\mathbb{R}^n))} + |b|_{L^\infty_{\mathcal{F}}(0,T;L^{n^*}(G))} + |c|_{L^\infty_{\mathcal{F}}(0,T;W^{1,\infty}(G))}.$$

A similar result was established in Ref. 1 if $(a^{ij})_{1\le i,j\le n}$ is the identity matrix.

The key tool for proving Theorem 2.1 is the following point-wise estimate.

For any nonnegative and nonzero function $\psi \in C^3(\overline{G})$, any $k \ge 2$, and any (large) parameters $\lambda > 1$ and $\mu > 1$, put

$$\ell = \lambda\alpha, \quad \alpha(t,x) = \frac{e^{\mu\psi(x)} - e^{2\mu|\psi|_{C(\overline{G})}}}{t^k(T-t)^k}, \quad \varphi(t,x) = \frac{e^{\mu\psi(x)}}{t^k(T-t)^k}. \tag{5}$$

Theorem 2.2. *[21, Theorem 4.1] Let u be a $C^2(\overline{G})$-valued semimartingale. Set*

$$\theta = e^{\ell}, \quad v = \theta u, \quad \Psi = 2\sum_{i,j} a^{ij}\ell_{ij}. \tag{6}$$

Then for any $x \in G$ and $\omega \in \Omega$ (a.s. dP), it follows

$$\begin{aligned}
&2\int_0^T \theta\Big[-\sum_{i,j}(a^{ij}v_i)_j + Av\Big]\Big[du - \sum_{i,j}(a^{ij}u_i)_j dt\Big]\\
&\quad +2\int_0^T \sum_{i,j}(a^{ij}v_i dv)_j dt + \int_0^T \Big(\theta^2\sum_{i,j} a^{ij}\ell_i (du)^2\Big)_j dt\\
&\quad +2\int_0^T \sum_{i,j}\Big[\sum_{i',j'}\Big(2a^{ij}a^{i'j'}\ell_{i'}v_i v_{j'} - a^{ij}a^{i'j'}\ell_i v_{i'}v_{j'}\Big) + \Psi a^{ij}v_i v\\
&\qquad\qquad -a^{ij}\Big(A\ell_i + \frac{\Psi_i}{2}\Big)v^2\Big]_j dt\\
&\ge 2\sum_{i,j}\int_0^T c^{ij}v_i v_j dt + \int_0^T Bv^2 dt + \int_0^T \Big|-\sum_{i,j}(a^{ij}v_i)_j + Av\Big|^2 dt\\
&\quad -\int_0^T \theta^2\sum_{i,j} a^{ij}du_i du_j - \int_0^T \theta^2\Big[A - \sum_{i,j}\Big(a^{ij}\ell_i\ell_j + (a^{ij}\ell_i)_j\Big)\Big](du)^2,
\end{aligned} \tag{7}$$

where

$$A \triangleq -\sum_{i,j}\Big[a^{ij}\ell_i\ell_j - (a^{ij}\ell_i)_j\Big] - \Psi,$$

$$B \triangleq 2\Big[A\Psi - \sum_{i,j}(Aa^{ij}\ell_i)_j\Big] - A_t - \sum_{i,j}(a^{ij}\Psi_j)_i - \ell_t^2,$$

$$c^{ij} \triangleq \sum_{i',j'}\Big[2a^{ij'}(a^{i'j}\ell_{i'})_{j'} - (a^{ij}a^{i'j'}\ell_{i'})_{j'}\Big] - \frac{a^{ij}_t}{2} + \Psi b^{ij}.$$

Moreover, for λ and μ large enough, it holds

$$
\begin{aligned}
&A = -\lambda^2\mu^2\varphi^2\sum_{i,j} b^{ij}\psi_i\psi_j + \lambda\varphi O(\mu^2),\\
&B \geq 2s_0^2\lambda^3\mu^4\varphi^3|\nabla\psi|^4 + \lambda^3\varphi^3 O(\mu^3) + \lambda^2\varphi^2 O(\mu^4) + \lambda\varphi O(\mu^4)\\
&\qquad +\lambda^2\varphi^{2+2k^{-1}} O(e^{4\mu|\psi|_{C(\overline{G})}}) + \lambda^2\varphi^{2+k^{-1}} O(\mu^2) + \lambda\varphi^{1+k^{-1}} O(\mu^2),\\
&\sum_{i,j} c^{ij} v_i v_j \geq [s_0^2\lambda\mu^2\varphi|\nabla\psi|^2 + \lambda\varphi O(\mu)]|\nabla v|^2.
\end{aligned}
\tag{8}
$$

Once we have Theorem 2.2, we just need to choose a $\psi \in C^4(\overline{G})$ such that $\psi > 0$ in G and $\psi = 0$ on ∂G and $|\nabla\psi(x)| > 0$ for all $x \in \overline{G} \setminus G_1$ in inequality (7) and let $u = z$. Then Theorem 2.1 follows from inequality (7) and the standard energy estimate for equation (2).

3. Observability estimate for stochastic hyperbolic equations

Throughout this section, we make the following assumptions on the coefficients $b^{ij} \in C^1(G)$:

(H4). $b^{ij} = b^{ji}$ $(i, j = 1, 2, \cdots, n)$;

(H5). *For some constant* $s_0 > 0$,

$$
\sum_{i,j} b^{ij}\xi^i\xi^j \geq s_0|\xi|^2, \qquad \forall (x,\xi) \stackrel{\triangle}{=} (x, \xi^1, \cdots, \xi^n) \in G \times \mathbb{R}^n. \tag{9}
$$

Let us consider the following stochastic hyperbolic equation:

$$
\begin{cases}
dz_t - \displaystyle\sum_{i,j}(b^{ij}z_i)_j dt = \Big[b_1 z_t + \langle b_2, \nabla z\rangle + b_3 z + f\Big]dt + (b_4 z + g)dB(t) & \text{in } Q,\\
z = 0 & \text{on } \Sigma,\\
z(0) = z_0, z_t(0) = z_1 & \text{in } G.
\end{cases}
\tag{10}
$$

Here $(z_0, z_1) \in L^2(\Omega, \mathcal{F}_0, P; H_0^1(G) \times L^2(G))$,

$$
\begin{aligned}
&b_1 \in L^\infty_{\mathcal{F}}(0,T; L^\infty(G)), \qquad b_2 \in L^\infty_{\mathcal{F}}(0,T; L^\infty(G;\mathbb{R}^n)),\\
&b_3 \in L^\infty_{\mathcal{F}}(0,T; L^p(G))(p \geq n), \qquad b_4 \in L^\infty_{\mathcal{F}}(0,T; L^\infty(G)),
\end{aligned}
\tag{11}
$$

and

$$
f \in L^2_{\mathcal{F}}(0,T; L^2(G)), \qquad g \in L^2_{\mathcal{F}}(0,T; L^2(G)). \tag{12}
$$

We first give the definition of the solution to equation (10).

Definition 3.1. $z \in L^2_{\mathcal{F}}(\Omega; C([0,T]; H^1_0(G))) \cap L^2_{\mathcal{F}}(\Omega; C^1([0,T]; L^2(G)))$ is called a solution of equation (10) if the following hold:
1. $z(0) = z_0$ in G, P-a.s., and $z_t(0) = z_1$ in G, P-a.s.
2. For any $t \in [0,T]$ and any $\eta \in H^1_0(G)$, it holds that

$$\begin{aligned} & \int_G z_t(t,x)\eta(x)dx - \int_G z_t(0,x)\eta(x)dx \\ = & \int_0^t \int_G \Big\{ -\sum_{i,j} b^{ij}(x) z_i(s,x)\eta_j(x) + [b_1(s,x)z_t(s,x) + b_2(s,x)\cdot\nabla z(s,x) \\ & \qquad +b_3(s,x)z(s,x) + f(s,x)]\eta(x)\Big\} dxds \\ & \quad + \int_0^t \int_G (b_4(s,x)z(s,x) + g(s,x))\eta(x)dxdB(s), \quad \text{P-a.s.} \end{aligned} \tag{13}$$

For the well-posedness of equation (10), please see.[23]

For the further purpose, next we introduce two conditions.

Condition 3.1. There exists a positive function $d(\cdot) \in C^2(\overline{G})$ satisfying the following:
1. For some constant $\mu_0 > 0$, it holds

$$\begin{aligned} & \sum_{i,j}\Big\{\sum_{i',j'}\Big[2b^{ij'}(b^{i'j}d_{i'})_{j'} - b^{ij}_{j'}b^{i'j'}d_{i'}\Big]\Big\}\xi^i\xi^j \geq \mu_0 \sum_{i,j} b^{ij}\xi^i\xi^j, \\ & \qquad\qquad\qquad\qquad \forall (x,\xi^1,\cdots,\xi^n) \in \overline{G}\times\mathbb{R}^n. \end{aligned} \tag{14}$$

2. There is no critical point of $d(\cdot)$ in $\overline{G}$, i.e.,

$$\min_{x\in\overline{G}} |\nabla d(x)| > 0. \tag{15}$$

Remark 3.1. If $(b^{ij})_{1\leq i,j\leq n}$ is the identity matrix, then $d(x) = |x - x_0|^2$ satisfies Condition 3.1, where x_0 is any point which belongs to $\mathbb{R}^n \setminus \overline{G}$.

Remark 3.2. Condition 3.1 was first given in[5] for the purpose of obtaining an internal observability estimate for hyperbolic equations. In that paper, the authors also gave some explanation of Condition 3.1 and some interesting nontrivial examples satisfying it.

It is easy to check that if $d(\cdot)$ satisfies Condition 3.1, then for any given constants $a \geq 1$ and $b \in \mathbb{R}$, the function $\tilde{d} = ad + b$ still satisfies Condition 3.1 with μ_0 replaced by $a\mu_0$. Therefore we may choose d, μ_0, $c_0 > 0$, $c_1 > 0$ and T to guarantee that one of the following conditions holds:

Condition 3.2.

1. $$\frac{1}{4}\sum_{i,j} b^{ij}(x)d_i(x)d_j(x) \geq R_1^2 \stackrel{\triangle}{=} \max_{x\in\overline{G}} d(x) \geq R_0^2 \stackrel{\triangle}{=} \min_{x\in\overline{G}} d(x), \quad \forall x \in \overline{G}. \tag{16}$$
2. $T > T_0 \stackrel{\triangle}{=} 2\inf\{R_1 : d(\cdot)$ satisfies (16) $\}$.
3. $$\Big(\frac{2R_1}{T}\Big)^2 < c_1 < \frac{2R_1}{T}.$$
4. $\mu_0 - 4c_1 - c_0 > 0$

Throughout this section, we use C to denote a generic positive constant depending on G, T, Γ_0, $b^{ij}(i,j = 1,\cdots,n)$, δ, d, c_0 and c_1(unless otherwise stated), which may change from line to line.

Now we give the observability estimate. Here and henceforth, we denote by $\nu = (\nu_1,\cdots,\nu_n)$ the unit outer normal vector of Γ. Put

$$\Gamma_0 \stackrel{\triangle}{=} \Big\{x \in \Gamma\Big| \sum_{i,j} b^{ij} d_i(x)\nu^j(x) > 0\Big\} \tag{17}$$

and

$$\begin{cases} r_2 \stackrel{\triangle}{=} |b_2|_{L^\infty_{\mathcal{F}}(0,T;L^\infty(G;\mathbb{R}^n))} + |(b_1,b_4)|_{L^\infty_{\mathcal{F}}(0,T;(L^\infty(G))^2}, \\ r_3 \stackrel{\triangle}{=} |b_3|_{L^\infty_{\mathcal{F}}(0,T;L^p(G))}. \end{cases} \tag{18}$$

The first result is for the boundary observability estimate for equation (10).

Theorem 3.1. *Let Condition 3.1 and Condition 3.2 be satisfied. For any solution of equation* (10)*, we have that*

$$\begin{aligned} &|(z_0,z_1)|_{L^2(\Omega,\mathcal{F}_0,P;H_0^1(G)\times L^2(G))} \\ &\leq Ce^{C(r_2^2+r_3^{\frac{1}{3/2-n/p}}+1)}\Big(\Big|\frac{\partial z}{\partial\nu}\Big|_{L^2_{\mathcal{F}}(0,T;L^2(\Gamma_0))} + |f|_{L^2_{\mathcal{F}}(0,T;L^2(G))} \\ &\qquad +|g|_{L^2_{\mathcal{F}}(0,T;L^2(G))}\Big). \end{aligned} \tag{19}$$

An observability for the final state was established in Ref. 23 if $(b^{ij})_{1\leq i,j\leq n}$ is identity matrix.

A key tool for obtaining Theorem 3.1 is the following result.

Theorem 3.2. *[23, Theorem 4.1] Let* $p^{ij} \in C^1((0,T)\times\mathbb{R}^n)$ *satisfy*

$$p^{ij} = p^{ji}, \qquad i,j,= 1,2,\cdots,n,$$

$l, \Psi \in C^2((0,T)\times\mathbb{R}^n)$. Assume that u is an $H^2_{loc}(\mathbb{R}^n)$-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted process such that u_t is an $L^2(\mathbb{R}^n)$-valued semimartingale. Set $\theta = e^l$ and $v=\theta u$. Then, for a.e. $x\in\mathbb{R}^n$ and P-a.s. $\omega\in\Omega$,

$$\begin{aligned}
&\theta\Big(-2l_tv_t+2\sum_{i,j}p^{ij}l_iv_j+\Psi v\Big)\Big[du_t-\sum_{i,j}(p^{ij}u_i)_jdt\Big]\\
&\quad+\sum_{i,j}\Big[\sum_{i'j'}\big(2p^{ij}p^{i'j'}l_{i'}v_iv_{j'}-p^{ij}p^{i'j'}l_iv_{i'}v_{j'}\big)-2p^{ij}l_tv_iv_t+p^{ij}l_iv_t^2\\
&\qquad\qquad+\Psi p^{ij}v_iv-\Big(Al_i+\frac{\Psi_i}{2}\Big)p^{ij}v^2\Big]_jdt \qquad (20)\\
&\quad+d\Big[\sum_{i,j}p^{ij}l_tv_iv_j-2\sum_{i,j}p^{ij}l_iv_jv_t+l_tv_t^2-\Psi v_tv+\Big(Al_t+\frac{\Psi_t}{2}\Big)v^2\Big]\\
&=\Big\{\Big[l_{tt}+\sum_{i,j}(p^{ij}l_i)_j-\Psi\Big]v_t^2-2\sum_{i,j}[(p^{ij}l_j)_t+p^{ij}l_{tj}]v_iv_t\\
&\quad+\sum_{i,j}\Big[(p^{ij}l_t)_t+\sum_{i',j'}\Big(2p^{ij'}(p^{i'j}l_{i'})_{j'}-(p^{ij}p^{i'j'}l_{i'})_{j'}\Big)+\Psi p^{ij}\Big]v_iv_j\\
&\quad+Bv^2+\Big(-2l_tv_t+2\sum_{i,j}p^{ij}l_iv_j+\Psi v\Big)^2\Big\}dt+\theta^2l_t(du_t)^2,
\end{aligned}$$

where $(du_t)^2$ denotes the quadratic variation process of u_t, A and B are stated as follows:

$$\begin{cases}
A\triangleq(l_t^2-l_{tt})-\sum_{i,j}(p^{ij}l_il_j-p^{ij}_jl_i-p^{ij}l_{ij})-\Psi,\\
B\triangleq A\Psi+(Al_t)_t-\sum_{i,j}(Ap^{ij}l_i)_j+\frac{1}{2}\Big[\Psi_{tt}-\sum_{i,j}(p^{ij}\Psi_i)_j\Big].
\end{cases}\qquad (21)$$

Once Theorem 3.2 is obtained, in order to prove Theorem 3.1, we choose $l=d(x)-c_2(t-\frac{T}{2})^2$.

4. Observability estimate for stochastic Schrödinger equations

Let us consider the following stochastic Schrödinger equation:

$$\begin{cases}
idz+\Delta zdt=(a_1\cdot\nabla z+a_2z+f)dt+(a_3z+g)dB & \text{in } Q,\\
z=0 & \text{on } \Sigma,\\
z(0)=z_0 & \text{in } G
\end{cases}\qquad (22)$$

with initial datum $y_0 \in L^2(\Omega, \mathcal{F}_0, P; H_0^1(G))$,

$$\begin{cases} ia_1 \in L_{\mathcal{F}}^{\infty}(0,T; W_0^{1,\infty}(G;\mathbb{R}^n)), \quad a_2 \in L_{\mathcal{F}}^{\infty}(0,T; W^{1,\infty}(G)), \\ a_3 \in L_{\mathcal{F}}^{\infty}(0,T; W^{1,\infty}(G)), \end{cases}$$

and

$$f \in L_{\mathcal{F}}^2(0,T; H_0^1(G)), \qquad g \in L_{\mathcal{F}}^2(0,T; H^1(G)). \tag{23}$$

We first recall the definition of the solution to equation (22).

Definition 4.1. We call $z \in L_{\mathcal{F}}^2(\Omega; C([0,T]; H_0^1(G)))$ a solution of equation (2) if the following hold:
1. $z(0) = z_0$ in G, P-a.s.;
2. For any $t \in [0,T]$ and any $\eta \in H_0^1(G)$, it holds that

$$\begin{aligned} & \int_G iz(t,x)\eta(x)dx - \int_G iz(0,x)\eta(x)dx \\ = & \int_0^t \int_G \Big[\nabla z(s,x)\cdot\nabla\eta(x) + \big(a_1\cdot\nabla z + a_2 z + f\big)\eta(x)\Big]dxds \\ & + \int_0^t \int_G (a_3 z + g)\eta(x)dxdB, \quad \text{P-a.s.} \end{aligned}$$

We refer to [4, Chapter 6] for the well-posedness of equation (2).

Let $x_0 \in \big(\mathbb{R}^n \setminus \overline{G}\big)$ and Γ_0 be given by

$$\Gamma_0 \stackrel{\triangle}{=} \big\{x \in \Gamma : \, (x - x_0)\cdot\nu(x) > 0\big\}. \tag{24}$$

Put

$$\begin{aligned} r_4 \stackrel{\triangle}{=} & |a_1|^2_{L_{\mathcal{F}}^{\infty}(0,T;W_0^{1,\infty}(G;\mathbb{R}^n))} + |a_2|^2_{L_{\mathcal{F}}^{\infty}(0,T;W^{1,\infty}(G))} \\ & + |a_3|^2_{L_{\mathcal{F}}^{\infty}(0,T;W^{1,\infty}(G))} + 1. \end{aligned} \tag{25}$$

In this section, we denote by C a generic positive constant depending only on T, G and x_0, which may change from line to line.

The observability estimate for equation (22) is as follows.

Theorem 4.1. *For any solution of equation* (22)*, it holds that*

$$\begin{aligned} & |z_0|_{L^2(\Omega,\mathcal{F}_0,P;H_0^1(G))} \\ \leq & \exp(Cr_4)\Big(\Big|\frac{\partial z}{\partial\nu}\Big|_{L_{\mathcal{F}}^2(0,T;L^2(\Gamma_0))} + |f|_{L_{\mathcal{F}}^2(0,T;H_0^1(G))} + |g|_{L_{\mathcal{F}}^2(0,T;H^1(G))}\Big). \end{aligned} \tag{26}$$

Let $\beta(t,x)\in C^2(\mathbb{R}^{1+m};\mathbb{R})$, and let $b^{jk}(t,x)\in C^{1,2}(\mathbb{R}^{1+m};\ \mathbb{R})$ satisfy

$$b^{jk}=b^{kj},\qquad j,k=1,2,\cdots,n. \tag{27}$$

We define a (formal) second order stochastic partial differential operator $\mathcal{P}$ as

$$\mathcal{P}z\stackrel{\triangle}{=}i\beta(t,x)dz+\sum_{j,k=1}^{m}(b^{jk}(t,x)z_j)_kdt,\quad i=\sqrt{-1}. \tag{28}$$

As before, the key instrument for obtaining Theorem 4.1 is the following identity concerning $\mathcal{P}$.

Theorem 4.2. *Let $\ell,\ \Psi\in C^2(\mathbb{R}^{1+m};\ \mathbb{R})$. Assume that z is an $H^2_{loc}(\mathbb{R}^n,\mathbb{C})$-valued $\{\mathcal{F}_t\}_{t\geq0}$-adapted process. Put $\theta=e^{\ell}$, $v=\theta z$. Then for a.e. $x\in\mathbb{R}^n$ and P-a.s. $\omega\in\Omega$, it holds that*

$$\begin{aligned}
&\theta(\mathcal{P}z\overline{I_1}+\overline{\mathcal{P}z}I_1)+dM+div V\\
&=2|I_1|^2dt+\sum_{j,k=1}^{m}c^{jk}(v_k\overline{v}_j+\overline{v}_kv_j)dt+D|v|^2dt\\
&\quad+i\sum_{j,k=1}^{m}\Big[(\beta b^{jk}\ell_j)_t+b^{jk}(\beta\ell_t)_j\Big](\overline{v}_kv-v_k\overline{v})dt\\
&\quad+i\Big[\beta\Psi+\sum_{j,k=1}^{m}(\beta b^{jk}\ell_j)_k\Big](\overline{v}dv-vd\overline{v})\\
&\quad+(\beta^2\ell_t)dvd\overline{v}+i\sum_{j,k=1}^{m}\beta b^{jk}\ell_j(dvd\overline{v}_k-dv_kd\overline{v}),
\end{aligned} \tag{29}$$

where

$$\begin{cases}
I_1\stackrel{\triangle}{=}-i\beta\ell_tv-2\displaystyle\sum_{j,k=1}^{m}b^{jk}\ell_jv_k+\Psi v,\\
A\stackrel{\triangle}{=}\displaystyle\sum_{j,k=1}^{m}b^{jk}\ell_j\ell_k-\sum_{j,k=1}^{m}(b^{jk}\ell_j)_k-\Psi,
\end{cases} \tag{30}$$

$$
\begin{cases}
M \triangleq \beta^2 \ell_t |v|^2 + i\beta \displaystyle\sum_{j,k=1}^{m} b^{jk}\ell_j(\overline{v}_k v - v_k \overline{v}),\\
V \triangleq [V^1, \cdots, V^k, \cdots, V^m],\\
V^k \triangleq -i\beta \displaystyle\sum_{j=1}^{m} \Big[b^{jk}\ell_j(vd\overline{v} - \overline{v}dv) + b^{jk}\ell_t(v_j\overline{v} - \overline{v}_j v)dt \Big] \\
\qquad -\Psi \displaystyle\sum_{j=1}^{m} b^{jk}(v_j\overline{v} + \overline{v}_j v)dt + \sum_{j=1}^{m} b^{jk}(2A\ell_j + \Psi_j)|v|^2 dt \\
\qquad + \displaystyle\sum_{j,j',k'=1}^{m} \Big(2b^{jk'}b^{j'k} - b^{jk}b^{j'k'} \Big)\ell_j(v_{j'}\overline{v}_{k'} + \overline{v}_{j'}v_{k'})dt,
\end{cases}
\tag{31}
$$

and

$$
\begin{cases}
c^{jk} \triangleq \displaystyle\sum_{j',k'=1}^{m} \Big[2(b^{j'k}\ell_{j'})_{k'}b^{jk'} - (b^{jk}b^{j'k'}\ell_{j'})_{k'} \Big] - b^{jk}\Psi,\\
D \triangleq (\beta^2\ell_t)_t + \displaystyle\sum_{j,k=1}^{m} (b^{jk}\Psi_k)_j + 2\Big[\sum_{j,k=1}^{m} (b^{jk}\ell_j A)_k + A\Psi \Big].
\end{cases}
\tag{32}
$$

The weight function θ is chosen as follows. Let

$$
\psi(x) = |x - x_0|^2 + \tau, \tag{33}
$$

where τ is a positive constant such that $\psi \geq \frac{5}{6}|\psi|_{L^\infty(G)}$.

Let $s > 0$ and $\lambda > 0$. Put

$$
\ell = s\frac{e^{4\lambda\psi} - e^{5\lambda|\psi|_{L^\infty(G)}}}{t^2(T-t)^2}, \qquad \varphi = \frac{e^{4\lambda\psi}}{t^2(T-t)^2}, \qquad \theta = e^{\ell}. \tag{34}
$$

5. Some open problems

There are a great many unsolved problems for observability estimate for SPDEs. We only list a few problems which are important in our opinion.

1. Compared with the observability estimate for SPDEs, it is more challenging to establish the observability estimate for backward SPDEs with one observation. For example, let $\{\mathcal{F}_t\}_{t\geq 0}$ be the natural filtration gener-

ated by $\{B(t)\}_{t\geq 0}$ and consider the following backward parabolic equation:

$$\begin{cases} dz + \Delta z dt = (az + bZ)dt + ZdB & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(T) = z_T & \text{in } G, \end{cases} \tag{35}$$

where $z_T \in L^2(\Omega, \mathcal{F}_T, P; L^2(G))$ and $a, b \in L^\infty_{\mathcal{F}}(0,T; L^\infty(G))$. We are looking forward to the following inequality:

$$|z(0)|_{L^2(\Omega,\mathcal{F}_T,P;L^2(G))} \leq C|z|_{L^2_{\mathcal{F}}(0,T;L^2(G_0))}.$$

As far as we know, Ref. 15 is the only published paper concerning this problem, in which the observability estimate for a very special backward stochastic parabolic equation was obtained.

2. Both inequality (19) and inequality (26) are boundary observability estimate. A natural question is to establish the internal observability estimate for equation (10) and (22). For example, for the solution of equation (10) with initial datum $(z_0, z_1) \in L^2(\Omega, \mathcal{F}_0, P; L^2(G) \times H^{-1}(G))$, we expect the following inequality:

$$\begin{aligned} & |(z_0, z_1)|_{L^2(\Omega,\mathcal{F}_0,P;L^2(G)\times H^{-1}(G))} \\ & \leq Ce^{C(r_1^2 + r_2^{\frac{1}{3/2-n/p}} + 1)}\Big(|z|_{L^2_{\mathcal{F}}(0,T;L^2(\mathcal{O}_\delta(\Gamma_0)))} + |f|_{L^2_{\mathcal{F}}(0,T;L^2(G))} \\ & \quad + |g|_{L^2_{\mathcal{F}}(0,T;L^2(G))}\Big). \end{aligned} \tag{36}$$

However, to our best knowledge, people do not know how to achieve this result now.

3. It is well known that a sharp sufficient condition for establishing observability estimate for deterministic hyperbolic equations and deterministic Schrödinger equations with time invariant lower order terms is that the triple $(G, \Gamma_0, T)((G, \mathcal{O}_\delta, T))$ satisfies the geometric optic condition introduced in Ref. 2. It would be quite interesting and challenging to extend this result to the stochastic setting. However, in order to achieve this task, it seems that one needs the theory of propagation of singularity for SPDEs, which is completely undeveloped until now.

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