Self-similar planar curves related to modified Korteweg–de Vries equation

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Abstract

We exhibit a time reversible geometric flow of planar curves which can develop singularities in finite time within the uniform topology. The example is based on the construction of selfsimilar solutions of modified Korteweg–de Vries equation of a given (small) mean.

Keywords: Planar curves; Selfsimilarity; Modified Korteweg–de Vries equation

1. Introduction

The purpose of this paper is two fold. Firstly we will exhibit a time reversible geometric flow of planar curves which can develop singularities in finite time within the uniform topology. The singularities are corners and logarithmic spirals. And secondly we will prove existence of solutions of the modified KdV equation with initial conditions given by

\[ k(x,0) = a\delta + \mu \text{ p.v.} \frac{1}{x}, \quad a, \mu \in \mathbb{R}, \]  

with \( a, \mu \) small and \( \delta \) denoting Dirac’s \( \delta \) distribution.

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Consider the plane curve \((x(s,t), y(s,t))\) with \(t \in \mathbb{R}\) denoting time and \(s \in \mathbb{R}\) the arclength parameter and write
\[
z(s, t) = x(s, t) + i y(s, t).
\]
The geometric flow we are interested in is defined by
\[
\begin{cases}
zt = -z_{sss} + \frac{3}{2} z_s z_{ss}, \\
|z_s|^2 = 1.
\end{cases}
\]
(2)
Denoting \(k\) as the curvature of \(z\), i.e. \(z' = ikz'\), we get
\[
\begin{cases}
z_t = -(ik_s + \frac{1}{2} k^2) z_s, \\
|z_s|^2 = 1.
\end{cases}
\]
(3)
This flow was obtained by Goldstein and Petrich in [7]. Their motivation is the problem of the evolution of a vortex patch in the plane subject to Euler equations. The dynamics can be described in terms of the boundary of the patch which is assumed to be a simple closed curve, see [13]. One obtains in this way a geometric flow of planar curves which is non-local. This non-local flow is reduced in a completely formal way in [7] to an infinite sum of local flows. The first term of this sum is just a translation along the curve. The second one is given by (2)–(3). Goldstein and Petrich followed previous ideas by Hasimoto [8] in the setting of vortex filaments in \(\mathbb{R}^3\), ideas which were later extended by Lamb [12]. An immediate calculation in (3) gives that \(\frac{d}{dt}|z_s(s, t)|^2 = 0\), and that the curvature \(k\) satisfies the modified KdV equation,
\[
k_t + k_{sss} + \frac{3}{2} k^2 k_s = 0.
\]
(4)
As a consequence the initial data associated to (1) will be
\[
z(s, 0) = \begin{cases}
z_0 + \frac{s}{\sqrt{1 + \mu^2}} e^{i(\theta^+ - \mu \log s)}, & s \geq 0, \\
z_0 + \frac{s}{\sqrt{1 + \mu^2}} e^{i(\theta^- - \mu \log |s|)}, & s \leq 0
\end{cases}
\]
(5)
for some \(\theta^+, \theta^- \in [0, 2\pi)\).
In this paper we are interested in the existence of solutions of (2)–(3) for \(t > 0\) such that (5) is satisfied. Similarly we will exhibit solutions of (4) such that
\[
k(s, 0) = a \delta + \mu \text{ p.v.} \frac{1}{s}, \quad a, \mu \in \mathbb{R}.
\]
We will do so using a perturbation argument, and therefore \(a, \mu\) will be considered small. Although there is plenty of literature on (4) based on the complete integrability of the equation, see for example [9], we will use just ode’s techniques.
One motivation comes from the so-called binormal flow which is given by

$$X_t = X_s \wedge X_{ss},$$  \hspace{1cm} (6)

with $X(s, t)$ a curve in $\mathbb{R}^3$, $t$ the temporal variable and $s$ the arclength parameter. This flow was obtained by Da Rios [4] as an approximation to the evolution of a vortex filament in $\mathbb{R}^3$. In this context the flow is also known as the Localized Induction Approximation. Hasimoto established in [8] the connection of this flow with the 1d-focussing cubic non-linear Schrödinger equation:

$$i\psi_t + \psi_{ss} + \frac{1}{2}(|\psi|^2 + A(t))\psi = 0, \quad \psi(s, t) \in C, \ A(t) \in \mathbb{R}. \hspace{1cm} (7)$$

Considering the new function

$$\varphi(s, t) = e^{1/2 \int_0^t A(t') dt'} \psi(s, t)$$

we obtain for $\varphi$

$$i\varphi_t + \varphi_{ss} + \frac{1}{2}|\varphi|^2 \varphi = 0. \hspace{1cm} (8)$$

It was observed in [10] that if we impose the condition

$$\varphi(s, 0) = a\delta, \hspace{1cm} (9)$$

then the initial value problem either has more than one solution or has none. The reason is that if there is uniqueness the corresponding solution has to be invariant under Galilean transformations and therefore it should be

$$\varphi(s, t) = \frac{a}{\sqrt{t}} e^{i(\frac{s^2}{4t} + \frac{|a|^2}{2} \log t)}, \quad t > 0.$$  

Notice however that $\frac{a}{\sqrt{t}} e^{is^2/4t}$ is a solution of (7) with $A(t) = -\frac{|a|^2}{t}$ such that $\psi(s, 0) = ae^{i\pi/4} \delta$.

Also, by a proper choice of $A(t)$, solutions of (7) can be constructed such that—see [6]—

$$\psi(s, 0) = a\delta + \mu \text{ p.v.} \frac{1}{s}.$$  

Similarly the corresponding solutions of the binormal flow which are asymptotically either straight lines or 3d—logarithmic spirals are obtained—see [5,6].

If in mKdV equation we look for solutions in selfsimilar form

$$k(s, t) = \frac{2}{(3t)^{1/3}} u\left(\frac{s}{(3t)^{1/3}}\right), \quad t > 0, \hspace{1cm} (10)$$

we are lead to study the o.d.e.

$$u_{xx} - xu + 2u^3 = \mu, \quad x \in \mathbb{R}, \ \mu \in \mathbb{R}. \hspace{1cm} (11)$$
Similarly as in [6], in the case of (2) we consider

\[ z(s, t) = e^{-i \frac{\mu}{3} \lg t^{1/3}} \omega \left( \frac{s}{t^{1/3}} \right), \quad \mu \in \mathbb{R}, \quad t > 0, \]  

(12)

which gives us the corresponding complex o.d.e.

\[
\begin{cases}
-i \mu + \frac{1}{3} \omega - \frac{s}{3} \omega' = -\omega_{sss} + \frac{3}{2} \omega_s \omega_s^2, & s \in \mathbb{R}, \quad \mu \in \mathbb{R}, \\
|\omega_s|^2 = 1.
\end{cases}
\]

(13)

Our main results are the following ones.

**Theorem 1.1.** There is \( \epsilon_0 > 0 \) such that if \( a^2 + \mu^2 < \epsilon_0 \) then there exist \( \theta^\pm \in [0, 2\pi) \) and \( \omega \) an analytic solution of (13) such that if

\[ z(s, t) = e^{-i \frac{\mu}{3} \lg t^{1/3}} \omega \left( \frac{s}{t^{1/3}} \right), \quad t > 0, \]

then

(i) \( z \) solves (2)–(3) for \( t > 0 \) and

\[ |z(s, t) - \frac{1}{\sqrt{1 + \mu^2}} e^{-i \mu \lg |s|} e^{i \theta^+} \chi_{[0, \infty)}(s) - \frac{1}{\sqrt{1 + \mu^2}} e^{-i \mu \lg |s|} e^{i \theta^-} \chi_{(-\infty, 0]}(s)| \leq ct^{1/3}; \]

(ii) \( \theta^+ - \theta^- = 2a \);

(iii) The curvature \( k \) of \( z \) (i.e. \( z'' = ik' \)) satisfies (10)–(11).

In (i) \( \chi_I \) stands for the characteristic function: \( \chi_I(s) = 1 \) if \( s \in I \) and zero otherwise.

**Theorem 1.2.** There is \( \epsilon_0 > 0 \) such that given any \( a, \mu \) with \( a^2 + \mu^2 < \epsilon_0 \) there exists a bounded real analytic \( u(x, t) \) solution of (11) such that if

\[ k(s, t) = \frac{2}{(3t)^{1/3}} u \left( \frac{s}{(3t)^{1/3}} \right), \quad t > 0, \]

then

(i) \( k \) solves mKdV equation (4);

(ii) \( \int_{-\infty}^{\infty} u(x) \, dx = a; \)

and

(iii) \( \lim_{t \to 0} k(\cdot, t) = 2(a \delta - \mu \text{ p.v.} \frac{1}{x}) \)

in \( S' \), the space of tempered distributions.
Remarks.

(1) The change of coordinates \((s, t) \sim (−s, −t)\) proves that Eqs. (2) and (4) are reversible in time. Therefore the above results exhibit solutions with analytic initial conditions which develop a singularity in finite time. The curvature \(u\) is bounded, satisfies (ii) but it is not in \(L^2(\mathbb{R})\).

(2) The asymptotic behavior of \(u\) and therefore of \(ω_s\) is given in Propositions 2.1, 2.2 and 2.3 in Section 2. From that it is easy to prove that the Fourier transform of \(u\) is bounded and regular on \(\mathbb{R}\setminus\{0\}\). Therefore (iii) is achieved in a stronger sense.

(3) Numerical calculations [3] suggest that \(-\frac{\pi}{2} \leq a \leq \frac{\pi}{2}\) is a necessary condition. This question will be studied elsewhere.

(4) In the setting of vortex patches a natural assumption is to consider simple closed curves. In fact it is known that in that case if the curve is smooth remains smooth for later times [2]. It would be very interesting to know which is the evolution according to (3) of a simple closed curve with a corner.

Section 2 is devoted to the proof of Theorem 1.2 and the proof of Theorem 1.1 is given in Section 3. Although mKdV equation can be integrated using the inverse scattering transform [1,11], we will use just o.d.e. techniques in our proofs.

2. Proof of Theorem 1.2

Consider the real solutions of the equation

\[
u_{xx} − xu + 2u^3 = μ, \quad x \in \mathbb{R}.
\]

In order to prove Theorem 1.2 we shall need several propositions. Also let us introduce the notation \(\langle x \rangle = (1 + |x|^2)^{1/2}\).

**Proposition 2.1.** Let \(u\) be a solution of (14). Then

(i) \(u, u_x\) are globally defined functions satisfying

\[|u| + \langle x \rangle^{-1/2}|u_x| \leq C \langle x \rangle^{1/2}, \quad x \in \mathbb{R},\]

(ii) There exist constants \(\rho_∞ \in \mathbb{R}, \varphi_∞ \in [0, 2\pi)\) such that as \(x \to -∞, u\) has the following asymptotic behavior

\[
u(x) = 2\rho_∞ (-x)^{-1/4} \cos \left(\frac{2}{3}(-x)^{3/2} + 3\rho_∞^2 \ln(-x) + \varphi_∞\right) - \frac{μ}{x} + O((-x)^{-7/4}),
\]

\[u_x(x) = 2\rho_∞ (-x)^{1/4} \sin \left(\frac{2}{3}(-x)^{3/2} + 3\rho_∞^2 \ln(-x) + \varphi_∞\right) - \frac{μ}{x} + O((-x)^{-5/4}).
\]

**Proof.** First, by multiplying the equation by \(u_x\) one checks that

\[
\frac{d}{dx}(u_x^2 - xu^2 + u^4 - 2μu) = -u^2.
\]
So the quantity \( u_x^2 - xu^2 + u^4 - 2\mu u \) satisfies

\[
E(x) \equiv u_x^2 - xu^2 + u^4 - 2\mu u = E(0) - \int_0^x dy u^2(y). \tag{15}
\]

As a consequence, we get that \( u, u_x \) are globally well defined and

\[
|u(x)| \leq C \langle x \rangle^{1/2}, \quad |u_x(x)| \leq C \langle x \rangle, \quad x \in \mathbb{R}. \tag{16}
\]

We next focus on the behavior of the solution as \( x \to -\infty \). Consider the auxiliary function \( f(x) \) defined by

\[
f(x) = \frac{1}{x} \int_0^x dy u^2(y) \geq 0. \tag{17}
\]

One has for the derivative

\[
f'(x) = \frac{1}{x^2} (u_x^2 + u^4 - 2\mu u - E(0)). \tag{18}
\]

Here we have made use of conservation law (15). It follows from (18) that

\[
\int_x^{-1} dy \frac{1}{y^2} (u_y^2 + u^4) = f(-1) - f(x) + \int_x^{-1} dy \frac{2\mu u + E(0)}{y^2}.
\]

By (16), (17), the right-hand side here is bounded by a constant. This means that the integral

\[
\int_{-\infty}^1 dy \frac{1}{y^2} (u_y^2 + u^4)
\]

converges, which in its turn implies the existence of the limit \( \lim_{x \to -\infty} f(x) \). Plugging this information into the conservation law one obtains that the following limit exists

\[
\lim_{x \to -\infty} \left( u^2 - \frac{u_x^2}{x} - \frac{u^4}{x} \right),
\]

and, in particular

\[
|u| \leq C, \quad |u_x| \leq C \langle x \rangle^{1/2}, \quad x \leq 0.
\]

The next step is to prove the same results for \((-x)^{1/2}u^2 + (-x)^{-1/2}u_x^2\). For this end we first estimate the integral

\[
\int_{-\infty}^{-1} dy \left( u^2 + \frac{u_x^2}{y} \right).
\]
Integrating by parts and taking into account the Eq. (14) one gets
\[
\int_{x}^{1} dy \left( u^2 + \frac{u_y^2}{y} \right) = \int_{x}^{1} dy \left( \frac{2u^4}{y} - \frac{\mu u}{y} + \frac{u_y u}{y^2} \right) + \left( \frac{u_y u}{y} \right) \bigg|_{x}^{1}.
\]

As a consequence,
\[
\left| \int_{x}^{1} dy \left( u^2 + \frac{u_y^2}{y} \right) \right| \leq C \ln(|x| + 2), \quad x \leq -1. \tag{19}
\]

On the other hand, the integration of the conservation law divided by \( x \) gives
\[
\int_{x}^{1} dy \left( \frac{u_y^2}{y} - u^2 \right) = \int_{x}^{1} dy \frac{1}{y} \int_{y}^{1} ds \ u^2(s) + \int_{x}^{1} dy \frac{2\mu u - u^4 + E(-1)}{y}
\]
\[
= \int_{x}^{1} dy \frac{1}{y} \int_{y}^{1} ds \ u^2(s) + \mathcal{O}(\ln |x|). \tag{20}
\]

Combining (19) and (20) one gets for the function \( g \)
\[
g(x) = \int_{x}^{1} dy \frac{1}{y} \int_{y}^{1} ds \ u^2(s)
\]

the following differential inequality
\[
|2xg' - g| \leq C \ln(|x| + 2), \quad x \leq -1. \tag{21}
\]

Integrating this inequality we obtain the existence of the limit \( \lim_{x \to -\infty} g(x)(-x)^{-1/2} \).

By (21), the same is true for
\[
(-x)^{1/2}g'(x) = (-x)^{-1/2} \int_{x}^{1} dy \ u^2.
\]

So, the limit \( \lim_{x \to -\infty} (-x)^{-1/2} \int_{x}^{1} dy \ u^2 \) does exist.

It follows then for conservation law (15) that the expression \( u_x^2(-x)^{-1/2} + u^2(-x)^{1/2} \) also has a limit as \( x \to -\infty \), and in particular, is bounded:
\[
u_x^2(-x)^{-1/2} + u^2(-x)^{1/2} \leq C, \quad x \leq 0. \tag{22}
\]
We are now able to prove part (ii) of Proposition 2.1. We start by performing a series of transformations of Eq. (14). First, we apply the following variation of parameter type of transformation:

\[
\begin{pmatrix}
  u \\
  u'
\end{pmatrix} = \begin{pmatrix}
  w \\
  w_x
\end{pmatrix} \begin{pmatrix}
  \zeta \\
  \tilde{\zeta}
\end{pmatrix}.
\] (23)

Here \( w \) is the solution of Airy equation

\[ w_{xx} - x w = 0, \]

with the following behavior as \( x \to -\infty \):

\[ w(x) = e^{i2/3(-x)^{3/2}} \chi(x), \quad \chi(x) = (-x)^{-1/4} \left( 1 + O((-x)^{-3/2}) \right), \]

\[ \chi^{(k)}(x) = O((-x)^{-1/4 + k}), \quad x \to -\infty. \] (24)

Note that \( \tilde{w}_x w - w_x \tilde{w} = 2i \).

Transformation (23) brings Eq. (14) to the form

\[ \zeta' = -3i |w|^4 |\zeta|^2 \zeta - i |w|^2 w^2 \zeta^3 - 3i |w|^2 \tilde{w}^2 |\zeta|^2 \tilde{\zeta} - i w^4 \tilde{\zeta}^3 + \frac{i \mu}{2} \tilde{w}. \] (26)

Notice that by (22), (24)–(26),

\[ |\zeta| \leq C, \quad |\zeta'| \leq C|x|^{-1/4}, \quad x \leq 0. \] (27)

The principal part of Eq. (26) is given by

\[ \zeta' = -3i |w|^4 |\zeta|^2 \zeta, \] (28)

the other terms in right-hand side of (26) are nonresonant and in principle can be removed by a suitable change of variables, see (31), (35) below. Model equation (28) can be integrated by means of the following substitution

\[ \zeta = ze^{-3i \int_0^x dy |w|^4 |z|^2}. \] (29)

Clearly, \( z \) satisfies the same estimate (27) as \( \zeta \). When applying this transformation to the full Eq. (26) one gets

\[ z' = I_1 + I_2 + I_3 + I_4, \] (30)

\[ I_1 = -i |w|^2 w^2 e^{-6i \int_0^x dy |w|^4 |z|^2} z^3, \]

\[ I_2 = -3i |w|^2 \tilde{w}^2 e^{6i \int_0^x dy |w|^4 |z|^2} \tilde{z}, \]

\[ I_3 = -i \tilde{w}^4 e^{12i \int_0^x dy |w|^4 |z|^2} \tilde{z}^3, \]

\[ I_4 = \frac{i \mu}{2} \tilde{w} e^{3i \int_0^x dy |w|^4 |z|^2}. \]
Next, we “remove” the nonresonant terms $I_j$, $j = 1, \ldots, 4$. Introduce the function $Z(0)$:

\[
Z(0) = z + \kappa_1 z^3 e^{-6i \int_0^x dy |w|^4 |z|^2} + \kappa_2 z^2 z e^{6i \int_0^x dy |w|^4 |z|^2} + \kappa_3 z^3 e^{12i \int_0^x dy |w|^4 |z|^2} + \mu \kappa e^{3i \int_0^x dy |w|^4 |z|^2},
\]

where the coefficients $\kappa_j(x)$ are chosen as follows

\[
\begin{align*}
\kappa_1 &= i \int_{-\infty}^x dy |w|^2 w^2 = -\frac{|w|^2 w^2}{2(-x)^{1/2}} + O((-x)^{-3}), \\
\kappa_2 &= 3i \int_{-\infty}^x dy |w|^2 \tilde{w}^2 = \frac{3}{2} \frac{|w|^2 \tilde{w}^2}{(-x)^{1/2}} + O((-x)^{-3}), \\
\kappa_3 &= i \int_{-\infty}^x dy \tilde{w}^4 = \frac{\tilde{w}^4}{4(-x)^{1/2}} + O((-x)^{-3}), \\
\kappa &= -\frac{i}{2} \int_{-\infty}^x dy \tilde{w} = -\frac{\tilde{w}}{2(-x)^{1/2}} + O((-x)^{-9/4}),
\end{align*}
\]

$Z(0)$ satisfies

\[
Z'(0) = \frac{3i}{2} \mu \kappa_1 \tilde{w} z e^{-3i \int_0^x dy |w|^4 |z|^2} + i \mu (3 \kappa |w|^4 - \kappa_2 w) |z|^2 e^{3i \int_0^x dy |w|^4 |z|^2}
\]

\[
+ \frac{i \mu}{2} (\kappa_2 \tilde{w} - 3 \kappa_3 w) \tilde{w}^2 e^{9i \int_0^x dy |w|^4 |z|^2} + F(0)(z, x),
\]

where the remainder $F_0$ admits the estimate

\[
|F(0)(z, x)| \leq C |w|^4 \left( \sum_{j=1}^3 |\kappa_j| \right) |z|^5 \leq C \frac{|z|^5}{(x)^{5/2}}.
\]

Let us repeat this procedure once more. Introduce $Z(1)$,

\[
Z(1) = Z(0) + \mu \lambda_1 z^2 e^{-3i \int_0^x dy |w|^4 |z|^2} + \mu \lambda_2 z^2 e^{3i \int_0^x dy |w|^4 |z|^2} + \mu \lambda_3 z^2 e^{9i \int_0^x dy |w|^4 |z|^2},
\]

\[
\begin{align*}
\lambda_1 &= -\frac{3i}{2} \int_{-\infty}^x dy \kappa_1 \tilde{w}, \\
\lambda_2 &= -i \int_{-\infty}^x dy (3 \kappa |w|^4 - \kappa_2 w), \\
\lambda_3 &= -\frac{i}{2} \int_{-\infty}^x (\kappa_2 \tilde{w} - 3 \kappa_3 w).
\end{align*}
\]
It follows from asymptotics (24), (32) that
\[ \lambda_j = \mathcal{O}( (−x)^{-9/4} ), \quad j = 1, 2, 3. \]

\( Z_{(1)} \) obeys
\[ Z'_{(1)} = F_{(1)}(z, x), \quad F_{(1)}(z, x) = F_{(0)}(z, x) + \mu F_1(z, x) + \mu^2 F_2(z, x), \tag{36} \]
where \( F_1, F_2 \) satisfy
\[ |F_1(z, x)| \leq C|w|^4 \left( \sum_{j=1}^{3} |\lambda_j| \right) |z|^4 \leq C \frac{|z|^4}{(x)^{13/4}}, \]
\[ |F_2(z, x)| \leq C|w| \left( \sum_{j=1}^{3} |\lambda_j| \right) |z| \leq C \frac{|z|}{(x)^{5/2}}. \tag{37} \]

Then (36), (37) together with (27) mean that as \( x \to −\infty \), \( Z_{(1)}(x) \) has a limit \( z_\infty \in \mathbb{C} \), and
\[ |Z_{(1)}(x) − z_\infty| \leq C (x)^{-3/2}. \tag{38} \]

Returning to \( z \), one gets from (31), (35) and (38) the following representation
\[ z = z_\infty + \mu \frac{\bar{w}}{2} \int_0^x dy |w|^4 |z|^2 + \mathcal{O}( (−x)^{-3/2} ). \tag{39} \]

For the term \( \int_0^x dy |w|^4 |z|^2 \) this gives
\[ \int_0^x dy |w|^4 |z|^2 = -|z_\infty|^2 \ln(-x) + \varphi_* + \mathcal{O}( (−x)^{-3/2} ), \tag{40} \]
for some real constant \( \varphi_* \). In terms of \( \zeta \) (39), (40) yield
\[ \zeta = \zeta_\infty e^{3i|\zeta_\infty|^2 \ln(-x)} + \mu \frac{\bar{w}}{2} (-x)^{1/2} + \mathcal{O}( (−x)^{-3/2} ), \]
\[ \zeta_\infty = z_\infty e^{-3i\varphi_*}. \]

By (23) this leads directly to the desired asymptotics for \( u \) and \( u_x \):
\[ u = 2\rho_\infty (−x)^{-1/4} \cos \left( \frac{2}{3} (−x)^{3/2} + 3 \rho_\infty^2 \ln(-x) + \varphi_\infty \right) - \frac{\mu}{x} + \mathcal{O}( (−x)^{-7/4} ), \]
\[ u_x = 2\rho_\infty (−x)^{1/4} \sin \left( \frac{2}{3} (−x)^{3/2} + 3 \rho_\infty^2 \ln(-x) + \varphi_\infty \right) + \mathcal{O}( (−x)^{-5/4} ). \]

Here \( \rho_\infty e^{i\varphi_\infty} = \zeta_\infty \). This concludes the proof of Proposition 2.1 \( \square \)
In a special case of small \( \mu \) and small initial data one has in fact a uniform description of the solutions in terms of Airy functions \( w, \bar{w} \) for all \( x \leq 0 \). We formulate the corresponding statement in terms of Eq. (26), the connection between (14) and (26) being given by (23).

**Proposition 2.2.** Let \( \zeta \) be a solution of (26) with initial data \( \zeta(0) = \zeta_0 \). Then for \( |\zeta_0| + |\mu| \) sufficiently small, the solution \( \zeta \) admits the representation

\[
|\zeta(x) - \zeta_\infty e^{-3i|\zeta_\infty|^2 \int_0^x dy |w|^4} - \mu \kappa(x)| \leq \frac{(|\mu| + |\zeta_0|)^3}{(x)^{3/2}}, \quad x \leq 0
\]

(41)

for some complex constant \( \zeta_\infty \) satisfying

\[
|\zeta_\infty - \zeta_0 - \mu \kappa(0)| \leq C (|\mu| + |\zeta_0|)^3.
\]

Here

\[
\kappa(x) = -\frac{i}{2} \int_{-\infty}^x dy \bar{w}.
\]

**Proof.** This result follows directly from (36). Integrating this equation one gets

\[
Z_{(1)}(x) = Z_{(1)}(0) + \int_0^x dy F_{(1)},
\]

which by (34), (37) leads immediately to the following inequality for the function \( z(x) \):

\[
|z(x)| \leq C \left( |\zeta_0| + |\mu| + |z(x)|^3 + \int_x^0 dy \frac{|z|^5 + |z|^3 + |\mu|^3}{\langle y \rangle^{5/2}} \right).
\]

(42)

Inequality (42) means that

\[
|z(x)| \leq C \left( |\zeta_0| + |\mu| \right), \quad x \leq 0
\]

(43)

provided \( |\zeta_0| + |\mu| \) is sufficiently small. Plugging (43) into (36) one gets

\[
|Z'_{(1)}(x)| \leq C \left( |\zeta_0| + |\mu| \right)^3 \frac{1}{(x)^{5/2}}.
\]

As a consequence, the limit \( \lim_{x \to -\infty} Z_{(1)}(x) = z_\infty \) satisfies

\[
|z_\infty - \zeta(0) - \mu \kappa(0)| \leq C \left( |\zeta(0)|^3 + |\mu|^3 \right)
\]

and

\[
|Z_{(1)}(x) - z_\infty| \leq C \frac{|\zeta(0)|^3 + |\mu|^3}{\langle x \rangle^{3/2}}.
\]
Returning to $z$, one gets from (31),
\[
\left| z(x) + \mu \kappa(x)e^{3i\int_0^y dy |w|^4|z|^2} - z_{\infty}\right| \leq C \frac{(|\zeta(0)|^3 + |\mu|^3)}{\langle x \rangle^{3/2}},
\]
or in terms of $\zeta$,
\[
\left| \zeta(x) + \mu \kappa(x) - z_{\infty}e^{-3i|z_{\infty}|^2\int_0^y dy |w|^4 + i\varphi_{\infty}} \right| \leq C \frac{(|\zeta(0)|^3 + |\mu|^3)}{\langle x \rangle^{3/2}},
\] (44)
with some constant $\varphi_{\infty}$ satisfying
\[
|\varphi_{\infty}| \leq C (|\mu| + |\zeta(0)|)^2.
\]
Here we have made use of the obvious estimate
\[
\int_0^x dy |w|^4|z|^2 = |z_{\infty}|^2 \int_0^x dy |w|^4 - \varphi_{\infty}/3 + O\left(\frac{(|\mu| + |\zeta(0)|)^2}{\langle x \rangle^{3/2}}\right).
\]
Then (44) gives us (41) with $\zeta_{\infty} = z_{\infty}e^{i\varphi_{\infty}}$. \(\square\)

As $x \to +\infty$, there are two possible types of behavior: either the solution grows like $x^{1/2}$ or it decays. In this paper we will be concerned only with the decaying solution. For $\mu = 0$ they are described by the following proposition.

**Proposition 2.3.** Let $\mu = 0$. Then for any $a \in \mathbb{R}$ there exists a solution $u_a$ of (14) with the following behavior

\[
u_{\infty} = z_{\infty}e^{i\varphi_{\infty}}.
\]

\[
u(x) = x^{-1/4}e^{-\frac{(2/3)x}{x^{3/2}}} (1 + O(x^{-3/2})),
\]
\[
u_x(x) = -x^{1/4}e^{-\frac{(2/3)x}{x^{3/2}}} (1 + O(x^{-3/2})), \quad x \to +\infty.
\]

The asymptotics (45) is double: as $x \to +\infty$ and as $a \to 0$, $x \geq 0$.

**Proof.** To prove this result one rewrites (45) as an integral equation

\[
u(x) = a v(x) + \int_x^\infty dy k(x, y)u^3(y),
\] (46)
\[ k(x, y) = v^*(x)v(y) - v^*(y)v(x), \]
\[ v^* = \frac{e^{i\pi/4}w(x) + e^{-i\pi/4}\overline{w}(x)}{2} = e^{2/3x^{3/2}} \left( 1 + \mathcal{O}(x^{-3/2}) \right), \]
\[ v_x^* = x^{1/4}e^{2/3x^{3/2}} \left( 1 + \mathcal{O}(x^{-3/2}) \right), \]
\[ v^{*'}v - v^*v' = 2. \]

Clearly,

\[ |k(x, y)| \leq C e^{2/3(y^{3/2} - x^{3/2})/\langle y \rangle^{1/4} \langle x \rangle^{1/4}}, \quad y \geq x \geq 0, \]
\[ |k_x(x, y)| \leq C \langle x \rangle^{1/4} e^{2/3(y^{3/2} - x^{3/2})/\langle y \rangle^{1/4}}. \]

So, the desired result can be obtained by applying to (46) a standard fixed point argument associated to the space of continuous functions which are bounded with respect to the weight \( \langle x \rangle^{1/4} e^{(2/3)x^{3/2}} \). We leave the details to the reader.

To describe the decaying solutions in the case \( \mu \neq 0 \), we first consider the corresponding linear equation

\[ u_{xx} - xu = \mu. \quad (47) \]

Consider the function \( U_a(x) \) defined by

\[ U_a(x) = av(x) - \frac{\mu}{2}\int_x^\infty dy v^*(x)v(y) - \frac{\mu}{2}\int_0^x dy v(x)v^*(y). \]

Clearly, it satisfies (47) and as \( x \to +\infty \)

\[ U_a(x) = -\frac{\mu}{x} + \mathcal{O}(x^{-5/2}), \]
\[ (U_a)_x(x) = \mathcal{O}(x^{-2}). \quad (48) \]

Here \( a \) is an arbitrary real constant. \[ \square \]

Returning to nonlinear equation (14) we prove the following result.

**Proposition 2.4.** For any real \( a \) there exists a solution \( u_a \) of (14) such that

\[ u_a(x) = U_a(x) + \mathcal{O}(x^{-4}), \]
\[ (u_a)_x(x) = \mathcal{O}(x^{-2}), \quad x \to +\infty. \quad (49) \]

Moreover if \( |a| \) and \( |\mu| \) are sufficiently small then one can construct \( u_a \) in such a way that
\[ |u_a(x) - U_a(x)| \leq C \frac{(|\mu| + |a|)^3}{\langle x \rangle^4}, \]
\[ |(u_a)_x(x) - (U_a)_x(x)| \leq C \frac{(|\mu| + |a|)^3}{\langle x \rangle^5}, \quad x \geq 0. \] (50)

Proof. Write \( u_a \) as the sum: \( u_a = U_a + r \). Then \( r \) satisfies the equation
\[ r_{xx} - xr + 6U_a^2r + 6U_ar^2 + 2r^3 = -2U_a^3. \]

We rewrite this equation in the form
\[ r = r_0 + \mathcal{H}(r), \] (51)
where
\[ r_0(x) = \int_x^\infty dy v^*(x)v(y)U_a^3(y) + \int_0^x dy v(x)v^*(y)U_a^3(y), \]
\[ \mathcal{H}(r) = \int_x^\infty dy v^*(x)v(y)\mathcal{J}(y) + \int_0^x dy v(x)v^*(y)\mathcal{J}(y), \]
\[ \mathcal{J} = 3U_a^2r + 3U_ar^2 + r^3, \]
\( x_0 \) being some constant. It is easy to check that
\[ |r_0(x)| \leq C \frac{(|a| + |\mu|)^3}{\langle x \rangle^4}, \]
\[ |(r_0)_x(x)| \leq C \frac{(|a| + |\mu|)^3}{\langle x \rangle^5}, \quad x \geq 0. \] (52)

We view (51) as a mapping in \( C([x_0, \infty]) \) equipped with the norm \( \sup_{x \geq x_0} |r(x)|/\langle x \rangle^4 \).

It is not difficult to check that the mapping \( r \mapsto r_0 + \mathcal{H}(r) \) is a contraction of the ball \( \sup_{x \geq x_0} |r(x)|/\langle x \rangle^4 \leq \eta \) into itself provided \( \eta \) is suitably chosen and \( x_0 \) is sufficiently large. This gives the existence of \( u_a \) satisfying (49).

In the case where \( |\mu|, |a| \) are small, inequality (52) allows us to take \( x_0 = 0 \), and leads to (50), the estimate for the derivative \( (u_a)_x \) being obtained by differentiating (51). \( \square \)

From the asymptotics obtained in Propositions 2.1, 2.3, and 2.4 the following holds. Let
\[ k(x, t) = \frac{2}{(3t)^{1/3}}u_a((3t)^{-1/3}x), \] (53)
where \( u_a \) is a solution of (14) decaying as \( x \to +\infty \) (see Propositions 2.3, 2.4).
Proposition 2.5. Take $k$ as in (53). Then $k$ solves mKdV equation (4) and

$$\lim_{t \downarrow 0} k(x, t) = C_1 \delta(x) - 2\mu \text{ p.v.} \left( \frac{1}{x} \right) \quad \text{in } S'(\mathbb{R}),$$

where

$$C_1 = 2 \lim_{R \to \infty} \int_{-R}^{R} dx u_a(x).$$

Remark. For $\mu = 0$, $a$ sufficiently small, Propositions 2.2 and 2.3 yield

$$\int dx u_a(x) = a \int dx v(x) + O(a^3) = 2\sqrt{\pi} a + O(a^3).$$

Here we have used that

$$\int dx v(x) = 2\sqrt{\pi},$$

which is an immediate consequence of the well-known integral representations of the Airy functions:

$$v(x) = \pi^{-1/2} \int dt e^{i \frac{3}{\pi} t^3 + ixt}.$$

So in the case $\mu = 0$ the corresponding solution $k(x, t)$ as $t \to +0$ converges in $S'$ to $C_1 \delta(x)$, $C_1 = 4\sqrt{\pi} a (1 + O(a^2))$.

More generally, if $\mu \neq 0$, one gets from Propositions 2.2 and 2.4

$$\lim_{R \to \infty} \int_{-R}^{R} dx u_a(x) = \lim_{R \to \infty} \int_{-R}^{R} dx U_a(x) + O\left((|a| + |\mu|)^3\right)$$

$$= a \int dx v(x) + c\mu + O\left((|a| + |\mu|)^3\right),$$

with some constant $c$ independent of $a$ and $\mu$.

So in this case $C_1 = 4\sqrt{\pi} a + 2c\mu + O\left((|a| + |\mu|)^3\right)$.

Since $C_1$ is clearly a continuous function of $a$ and $\mu$, the proof of Theorem 1.2 follows directly from Propositions 2.1–2.5 and the above remark.

3. Proof of Theorem 1.1

Take

$$z(s, t) = e^{-it \frac{4}{3} \log t^{1/3}} \omega \left( \frac{s}{t^{1/3}} \right), \quad t > 0.$$ (54)
Then \( z \) solves (2) if

\[
\begin{cases}
-i\mu + \frac{1}{3} - \omega - \frac{s}{3} \omega' = -\omega'' + \frac{3}{2} \omega' (\omega''), & s \in \mathbb{R}, \\
|\omega_s|^2 = 1.
\end{cases}
\]

If we write \( \omega'(s) = e^{\int_0^s k(s',1) ds'} \) then \( \omega'' = ik\omega' \) and \( \omega''' = i k_s \omega' - k^2 \omega' \). Therefore we have to take:

\[
\omega(0) = \frac{-3}{1 - i\mu} \left( ik_s(0) + \frac{1}{2} k^2(0) \right).
\]

Then if \( k(s, 1) \) solves

\[
k_{ss} - \frac{s}{3} k + \frac{k^3}{2} = \frac{\mu}{3},
\]

\( \omega \) solves (55).

Take

\[
k(s, 1) = \frac{2}{3^{1/3}} u \left( \frac{s}{3^{1/3}} \right).
\]

Therefore \( u \) has to solve

\[
u_{xx} - xu + 2u^3 = \frac{\mu}{2}, \quad x \in \mathbb{R}.
\]

From Propositions 2.1, 2.2 and 2.3 we know that if \( a^2 + \mu^2 < \epsilon_0 \) there exists a solution of (57) which is bounded and real analytic and such that

\[
\int_{-\infty}^{\infty} u(x) \, dx = a.
\]

Consider the function \( g \) defined for \( s \neq 0 \) as

\[
g(s) = e^{i\mu \lg |s|} \frac{\omega(s)}{s}.
\]

Then

\[
g'(s) = e^{i\mu \lg |s|} \frac{(i\mu - 1) \omega + s \omega'}{s^2} = \frac{3}{s^2} \left( ik_s + \frac{1}{2} k^2 \right) \omega' e^{i\mu \lg |s|}.
\]
From Propositions 2.1 and 2.3 we conclude that \( g' \) is integrable at \( \pm \infty \) and \( g(\pm \infty) \) can be defined. Then

\[
\begin{aligned}
g(s) &= g(\infty) - 3 \int_s^\infty \frac{1}{r^2} (i k_s + \frac{1}{2} k^2) \omega_s e^{i \mu \lg r} \, dr, \quad s > 0, \\
g(s) &= g(-\infty) + 3 \int_{-\infty}^s \frac{1}{r^2} (i k_s + \frac{1}{2} k^2) \omega_s e^{i \mu \lg |r|} \, dr, \quad s < 0.
\end{aligned}
\]  

(60)

Moreover, after integration by parts we get

\[
g(s) = g(\infty) + 3 \frac{k}{s^2} \omega_s - 9 \int_s^\infty \frac{k^2}{r^2} \omega_s e^{i \mu \lg r} \, dr - (6i + 3\mu) \int_s^\infty \frac{k}{r^3} \omega_s e^{i \mu \lg r} \, dr
\]

(61)

for \( s > 0 \), and a similar expression for \( s < 0 \).

From (54) and (59)

\[
ed^{i \mu \lg |s|} \omega' = (1 - i \mu) g + s g', \quad |\omega'| = 1.
\]

(62)

Hence \( |g(\pm \infty)| = \frac{1}{\sqrt{1 + \mu^2}} \) and there exist \( \theta^\pm \) such that

\[
g(\pm \infty) = \frac{1}{\sqrt{1 + \mu^2}} e^{i \theta^\pm}.
\]

(63)

For \( s > 0, \ t > 0 \) we have

\[
\left| z(s, t) - \frac{1}{\sqrt{1 + \mu^2}} e^{-i \mu \lg s} s e^{i \theta^+} \right| \leq 3 \left| s \int_{s/t^{1/3}}^\infty \frac{1}{r^2} \left( i k_s + \frac{k^2}{2} \right) \omega_s e^{i \mu \lg r} \, dr \right| \leq c t^{1/3},
\]

where the last inequality follows from (61) and Propositions 2.1 and 2.2 if \( s \geq t^{1/3} \), and similarly otherwise.

Finally from (62)

\[
ed^{i(\theta^+-\theta^-)} = \frac{g(\infty)}{g(-\infty)} = e^{\int_{-\infty}^\infty k(s, 1) \, ds},
\]

which proves (ii). The proof of Theorem 1.1 is over.

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References