Some dispersive estimates for Schrödinger equations with repulsive potentials

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Abstract

We prove the local smoothing effect for Schrödinger equations with repulsive potentials for \(n \geq 3\). The estimates are global in time and are proved using a variation of Morawetz multipliers. As a consequence we give sharp constants to measure the attractive part of the potential and its rate of decay, which turns out to be different whether dimension 3 or higher are considered. Also a notion of zero resonance arises in a natural way. Our smoothing estimate allows us to use Sobolev inequalities and treat nonradial perturbations.

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1. Introduction

We consider Schrödinger Hamiltonians $\Delta + V(x)$, with $V$ a real potential $x \in \mathbb{R}^n$ and $n \geq 3$. We study a priori estimates for solutions of the evolution initial value problem

$$\begin{cases}
i \partial_t u + \Delta_x u + V(x)u = 0, & (x, t) \in \mathbb{R}^n \times \mathbb{R}, \ n \geq 3, \\
u(x, 0) = u_0(x).
\end{cases}$$

(1.1)

We suppose that $W(r)$ is a radial mayorant of the negative part of the radial derivative of $V(x)$, that is

$$(\partial_r V)_- (r) := \sup_{x \in \mathbb{R}^n: |x| = r} (\partial_r V)_- (x) \leq W(r).$$

(1.2)

There is a wide literature on the so-called repulsive potential setting, i.e. $(\partial_r V)_- (r) = 0$—see [19, vol. 4, Chapter XIII-7]. In this paper and for $n \geq 3$ we mainly revisit the work of M. Arai [3] with a double purpose. Firstly to obtain exact decay conditions on $W$, which turn out to be different either one considers dimension three or higher. And secondly to prove the so-called local smoothing effect of $1/2$ gain derivative—see [6,23,24]. At this respect we focus on estimates global in time so that Doi’s method [8] based on pseudo-differential operators does not seem to apply. One of the advantages of using just pure integration by parts is that we are able to give precise constants on the assumptions of the potential—see (1.8), (1.10) and (1.18). These constants are sharp at least for $n > 3$ as we shall prove in Section 3. As is usual when dealing with global estimates in time, one of the main difficulties is to rule out zero-resonances. The multipliers we use for the integration by parts suggest a natural definition for a zero-resonance. We also treat these questions in Section 3. In particular we prove that zero-resonances do not exist under our hypothesis.

It is important to notice that although we use a Morawetz type of multipliers we are able to obtain an estimate for the full gradient in a space introduced by Agmon and Hörmander [2] which is scaling invariant. As a bonus we allow typically nonradial perturbations using weighted Sobolev inequalities—see Theorem 3. From this point of view this work is an extension of [17]. In that paper the authors study Helmholtz equation and are able to extend their results to the wave equation. However their approach fails in the Schrödinger case because they use two type of multipliers, antisymmetric and symmetric. It is just the antisymmetric the one that works in the Schrödinger equation (Lemma 2.1). Therefore the results in this paper also apply to Helmholtz and wave equations. In [17] the potential becomes the refraction index and it is labelled as $n$ instead of $V$ as we do here. Then it is natural to consider $\Delta + n(x)$, and as a consequence we keep the expression $\Delta + V(x)$ instead of the standard $-\Delta + V$, to remark the comparison among the conditions on $n$ and $V$.

We assume that $V$ satisfies the following conditions:

**Hypothesis 1.** Problem (1.1) has a unique solution $u(x, t)$ for $u_0 \in H^{1/2}(\mathbb{R}^n)$ which satisfies the a priori estimate

$$\|u(\cdot, t)\|_{H^{1/2}(\mathbb{R}^n)} \leq C(V)\|u_0\|_{H^{1/2}(\mathbb{R}^n)}.$$

(1.3)

**Hypothesis 2.** There exists a class of data $u_0$, which is dense in $H^{1/2}(\mathbb{R}^n)$, such that the solution $u$ of (1.1) is in $C(\mathbb{R}; H^s(\mathbb{R}^n)) \cap C^1(\mathbb{R}; H^{s-2}(\mathbb{R}^n))$ for some $s \geq 3/2$ and $\int_{\mathbb{R}^n} |u|^2 |V| < \infty$. 
As examples of potentials $V$ satisfying the two above hypothesis, we can consider any real $V$ being $\Delta$-bounded with relative bound less than 1. This means that the a priori estimate

$$\|Vf\|_{L^2} \leq a \|\Delta f\|_{L^2} + b \|f\|_{L^2}$$

holds with some $a < 1$. This kind of assumptions is standard (see [7, Chapter 4]). In this case the dense class in Hypothesis 2 is the domain $H^s(\mathbb{R}^n)$ with $s = 2$ of the operator $\Delta + V(x)$.

All through the paper we could have assumed potentials which are time dependent taking also the supremum on $t$ in the left-hand side of (1.2). However there are no natural conditions so that Hypothesis 1 holds. Therefore for simplification of the exposition we will assume that $V$ is time independent.

Let us assume that

$$H(r) = r \int_r^\infty tW(t)\,dt < \infty, \quad (1.4)$$

and we suppose that

$$H(0) = \liminf_{r \to 0} r \int_r^\infty tW(t)\,dt < \infty, \quad (1.5)$$

$$H(\infty) = \limsup_{r \to \infty} r \int_r^\infty tW(t)\,dt < \infty. \quad (1.6)$$

We prove for $n > 3$ the following theorem.

**Theorem 1.** Let $V(x)$ be a potential in $\mathbb{R}^n$, $n > 3$, satisfying Hypotheses 1 and 2. Let $W(r)$ satisfy (1.2), (1.5), (1.6) and

$$\frac{n - 3}{2(n - 2)r^{n-2}} \int_0^r t^{n-2}W(t)\,dt \leq \frac{n - 1}{2nr^n} \int_0^r t^{n+1}W(t)\,dt$$

$$+ \frac{1}{n(n-2)} \int_r^\infty tW(t)\,dt. \quad (1.7)$$

Let $\epsilon > 0$ be such that

$$\epsilon + H(\infty) < \frac{(n - 1)(n - 3)}{2}. \quad (1.8)$$

Then the unique solution of (1.1) satisfies
\begin{align*}
\varepsilon \sup_{R>0} \frac{1}{R} \int_{B(0,R)} \int_{0}^{\infty} \left| \nabla u(x,t) \right|^2 \, dt \, dx + \varepsilon \sup_{R>0} \frac{1}{R^3} \int_{B(0,R)} \int_{0}^{\infty} \left| u(x,t) \right|^2 \, dt \, dx \\
+ H(0) \left( \int_{\mathbb{R}^n} \int_{0}^{\infty} \frac{|\partial_\tau u(x,t)|^2}{|x|} \, dt \, dx + \int_{\mathbb{R}^n} \int_{0}^{\infty} \left| u(x,t) \right|^2 (\partial_r V)_{+}(x) \, dt \, dx \right) \\
\leq C(n, V, W) \| u_0 \|_{H^{1/2}(\mathbb{R}^n)}^2,
\end{align*}

(1.9)

where \( \partial_\tau \) denotes the spherical tangential component of the gradient.

Condition (1.7) is equivalent to

\begin{align*}
r^2(n-3) \int_{0}^{r} t^{n-3} H(t) \, dt \leq (n-1) \int_{0}^{r} t^{n-1} H(t) \, dt,
\end{align*}

but we prefer to express it in terms of \( W \) in order to remark the role it plays in the proof of Theorem 1. The following corollary gives a pointwise condition on the radial derivative of \( V \).

**Corollary 1.** Let \( V(x) \) be a potential in \( \mathbb{R}^n, n > 3 \), satisfying Hypotheses 1 and 2. Assume that

\begin{align*}
(\partial_r V)_{-}(x) < c_0 \frac{(n-3)(n-1)}{|x|^3}
\end{align*}

(1.10)

with \( c_0 < 1/2 \) and \( \varepsilon > 0 \) satisfying

\begin{align*}
\varepsilon < \frac{1}{2} - c_0.
\end{align*}

(1.11)

Then the unique solution of (1.1) satisfies

\begin{align*}
\int_{\mathbb{R}^n} \int_{0}^{\infty} \frac{|\partial_\tau u(x,t)|^2}{|x|} \, dt \, dx + \varepsilon \sup_{R>0} \frac{1}{R} \int_{B(0,R)} \int_{0}^{\infty} \left| \nabla u(x,t) \right|^2 \, dt \, dx \\
+ \varepsilon \int_{\mathbb{R}^n} \int_{0}^{\infty} \frac{|u(x,t)|^2}{|x|^3} \, dt \, dx + \int_{\mathbb{R}^n} \int_{0}^{\infty} \left| u(x,t) \right|^2 (\partial_r V)_{+}(x) \, dt \, dx \\
\leq C(n, V, c_0) \| u_0(\cdot) \|_{H^{1/2}(\mathbb{R}^n)}^2.
\end{align*}

(1.12)

In Section 3 we study the sharpness of the above condition on \( V \).

**Remark.** 1. If we substitute \( H^{1/2} \) by the homogeneous \( \dot{H}^{1/2} \) in assumption (1.3), we can write (1.9) with homogeneous Sobolev norm in the right-hand side (this remark also applies to the 3D case, see Theorem 2). The appearance of the \( H^{1/2} \) norm on the right-hand side of (1.9) and
(1.12) is after using Lemma 2.1. In fact what naturally appears in the integration by parts is the usual radiation term

$$\mathcal{I}\left( \int_{\mathbb{R}^n} \tilde{u}(x, T) \nabla u(x, T) \nabla \Phi_R(x) \, dx \right),$$

for a family of test functions $\Phi_R$. If we know a priori that this term is bounded, then the left-hand side of (1.9) and (1.12) is also bounded. This will be used in Section 3 in connection with the existence of 0-resonances.

2. The standard function, see the corollary, satisfying (1.7) and (1.8) is

$$W(r) = \frac{2c_0}{r^3}$$

when

$$c_0 < \frac{(n - 1)(n - 3)}{4}.$$  \hspace{1cm} (1.14)

This radial majorant corresponds to the potential in Theorem 1:

$$V(x) = c_0 |x|^2.$$  \hspace{1cm} (1.15)

The multiplier, known as Morawetz multiplier, $\Phi(x) = |x|$ in Lemma 2.1 has been extensively used.

It corresponds to the case (1.13) and in this case (1.7) holds with equality. Notice that also for the free case $V = 0$ and taking $W$ as in (1.13) we recover the estimates in [6,23,24] for the Morrey norm of the full gradient and those estimates in [14] for the spherical tangential component of the gradient (the 3D case is also included); the control of the spherical gradient for some lower order perturbations has been treated in [4]. We also recover some estimates for perturbations treated in [9]. Recall that the estimate

$$\int_{\mathbb{R}^n} \int_0^\infty \frac{\left| \nabla e^{i\Delta t} u_0 \right|^2}{|x|} \, dt \, dx \leq C \|u_0\|_{L^2(\mathbb{R}^n)}^2,$$

is false, even in the free case $V = 0$, see [4].

Recently—see [5], global Strichartz estimates but not the local smoothing effect have been obtained for potentials as (1.15). Strictly speaking we cannot handle the above potential for the range

$$\frac{(n - 2)^2}{4} - 1 < c_0 < \frac{(n - 1)(n - 3)}{4}, \quad n \geq 4,$$
because it is unclear if Hypothesis 2 is satisfied in this range where just the usual quadratic form techniques for the construction of the solution \( u(t) \) are available [18, p. 172]. Hypothesis 1 still holds in this range. Therefore we could consider instead

\[
V(x) = \frac{c_0}{\delta + |x|^2}, \tag{1.16}
\]

for any \( \delta > 0 \). For those potentials and for fixed \( c_0 \) such that (1.14) holds we can apply Corollary 1 with constants which are uniform in \( \delta \). As we shall see in Section 3 these potentials will prove that (1.14) is sharp for Theorem 1 to hold. In any case notice that our condition (1.14) is worse than that one in [5].

3. Theorem 1 allows one to treat potentials with distributional derivatives. As an interesting example, we can consider \( V \) such that for small enough \( c_0 \) and \( a \in \mathbb{R}^+ \) \((\partial_r V)^-(r) = c_0 \delta_a \). These potentials can be attained by using in (1.2)

\[
W(r) = h \chi_{[a,d]}(r) + \frac{c}{r^3} \chi_{[b,\infty]}(r) \tag{1.17}
\]

with \( a < d \leq b \). Conditions (1.7) and (1.8) are satisfied under certain assumptions on the parameters; for instance when \( d = b \), \( h \) may be chosen of order \( 1/(d - a) \), and a limiting argument allows us to treat the Dirac \( \delta \) case. Similar arguments allow a superposition of a finite number of deltas

\[
(\partial_r V)^-(r) = \sum c_j \delta_{a_j}.
\]

Examples like (1.17) show that to attain the estimate for the full gradient in (1.9) the key point is the behavior of \( W \) at infinity.

4. It would be interesting to prove the gain of a half derivative for initial data in \( L^2 \) instead of \( H^{1/2} \) as we do here, see [21,22] for results in this direction.

If \( n = 3 \) the bilaplacian of a radial function \( \Phi \) has a very simple expression, \( \Phi^{iv} + \frac{4}{r} \Phi^{iii} \). This implies that the assumptions needed for the potential \( V \) in this case are different. We have the following result.

**Theorem 2.** Let \( V(x) \) be a potential in \( \mathbb{R}^3 \) satisfying Hypotheses 1 and 2. Let \( W(r) \) satisfy (1.2) and

\[
\eta + \int_0^\infty t^2 W(t) \, dt < 1 \tag{1.18}
\]

for \( \eta > 0 \).

Then the unique solution of (1.1) satisfies

\[
\int \int_{\mathbb{R}^3} \frac{|\partial_r u(x,t)|^2}{|x|} \, dt \, dx + \int \int_{\mathbb{R}^3} |u(x,t)|^2 (\partial_r V)^+ (x) \, dt \, dx
\]
\[ + \sup_{R>0} \frac{1}{R^3} \int_{B(0,R)} \int_0^{\infty} |u(x,t)|^2 \, dt \, dx + \sup_{R>0} \frac{1}{R^3} \int_{B(0,R)} \int_0^{\infty} |\partial_r u(x,t)|^2 \, dt \, dx \]

\[ \leq \frac{C(V,W)}{\eta} \|u_0\|^2_{H^{1/2}(\mathbb{R}^3)}. \quad (1.19) \]

The following theorem relaxes the radiality in the assumptions on the potential.

**Theorem 3.** Let \( V(x) \) be a potential in \( \mathbb{R}^n \), \( n \geq 3 \), satisfying Hypotheses 1 and 2. Let \( W(x) \) satisfy

\[ (\partial_r V)(x) \leq W(x) \quad (1.20) \]

for some nonnegative function \( W(x) \) which can be written as

\[ W(x) = \sum_{j=0}^{\infty} w_j(x) \quad (1.21) \]

with

\[ \text{supp } w_j \subset \{ x \in \mathbb{R}^n : 2^{j-2} < |x| \leq 2^{j-1} \} = \Omega_j, \quad j \geq 1; \]

\[ \text{supp } w_0 \subset \{ x \in \mathbb{R}^n : |x| \leq 2^{-1} \} = \Omega_0. \]

Assume that the a priori estimate

\[ \int_{\mathbb{R}^n} w_j(x) |u(x)|^2 \, dx \leq c(w_j) \int_{\mathbb{R}^n} |\nabla u|^2 \, dx \quad (1.22) \]

holds and that for \( 0 < \gamma < 1 \)

\[ \sum_{j=0}^{\infty} 2^j c(w_j) < \frac{(n-1)(n-3)}{4} (1 - \gamma), \quad n > 3, \quad (1.23) \]

\[ \sum_{j=0}^{\infty} 2^j c(w_j) < \frac{3}{7} (1 - \gamma), \quad n = 3. \quad (1.24) \]

Then the unique solution of (1.1) satisfies

\[ \gamma \sup_{R>0} \frac{1}{R} \int_{B(0,R)} \int_0^{\infty} |\nabla u(x,t)|^2 \, dt \, dx + \gamma(n-3) \int_{\mathbb{R}^n} \int_0^{\infty} \frac{|u(x,t)|^2}{|x|^3} \, dt \, dx \]

\[ + \gamma \sup_{R>0} \frac{1}{R^3} \int_{B(0,R)} \int_0^{\infty} |u(x,t)|^2 \, dt \, dx + \int_{\mathbb{R}^n} \int_0^{\infty} \frac{|\partial_r u(x,t)|^2}{|x|} \, dt \, dx \]
\[ + \int_0^\infty \int_{\mathbb{R}^n} \left| u(x,t) \right|^2 (\partial_r V)_+(x) \, dt \, dx \]
\[ \leq C(n, V, W) \| u_0 \|_{H^{1/2}(\mathbb{R}^n)}^2. \]  

(1.25)

**Remark.** 1. In this theorem the decay conditions on \( V \) are given by (1.23) and (1.24). We have to measure the size of \((\partial_r V)_-(x)\) on radial diadic sets with norms appropriated for weights in Sobolev inequalities, like Morrey–Campanato (see [16]) or Kato–Stummel (see [13]). As opposite to Theorems 1 and 2 we have a global nonradial norm on each diadic set \( \Omega_j \). We may take the Kato norm

\[ c(w_j) = \sup_{y \in \Omega_j, 0 \leq r \leq 2^j} \int_{B(y,r)} \frac{w_j(x)}{|x-y|^{n-2}} \, dx. \]

This integral makes sense for more general measures than \((d\mu)_j = w_j(x) \, dx\). For instance we may take \( w_j \epsilon(x) \) an \( \epsilon \) mollification of the singular measure \( \delta_{|x-x_j|=1} \) on a sphere centred at \( x_j \in \Omega_j \), then a superposition \( d\mu = \sum_{j=1}^{\infty} 2^{-js} \delta_{|x-x_j|=1} \) with \( s > 1 \) is a nonradial measure satisfying the hypothesis of Theorem 3 and with a Kato condition

\[ c(w_j) = \sup_{y,r} \int_{B(y,r)} \frac{(d\mu)_j}{|x-y|^{n-2}} \leq 2^{-js}. \]

If we consider the Morrey–Campanato norm, which is defined by

\[ \| w_j \|_{L^{2,p}} = \sup \left\{ r^2 \left( \int_{B(y,r)} |w_j(x)|^p \, dx \right)^{1/p} : y \in \mathbb{R}^n, \ 0 < r < 2^j \right\}, \]

then estimate (1.22) holds with \( c(w_j) = \| w_j \|_{L^{2,p}}^2 \) for some \( p > 1 \). We may take a majorant, associated to Coulomb type potentials,

\[ w_j(x) = 2^{-(1+\epsilon)j} \sum_{k=1}^{\ell} \frac{1}{|x-x_k|^2} \chi_{B(x_k,1)}(x), \]

where the \( x_k \) satisfy \( \|x_k\| \sim 2^j \) and they are in a lattice of resolution \( 2^{j/2}/n \).

2. It will be clear from the proof of Theorem 3 that we could allow a slightly more general situation. Namely to write \( W = W_1 + \delta W_2 \) with \( W_1 \) as either in Theorems 1 or 2, and \( W_2 \) as in Theorem 3, choosing \( \delta \) small enough depending either on (1.8) or on (1.18).

**2. The proof of the theorems**

We will need the following two lemmas.
Lemma 2.1. Let $V$ satisfy Hypotheses 1 and 2 and $\Phi(x) = \phi(|x|)$ be a smooth radial real-valued function, such that

$$\sup_{r > 0} \left\{ \phi'(r), r\phi''(r) \right\} \leq C.$$  \hspace{1cm} (2.1)

Then any solution

$$u \in C(\mathbb{R}; H^s(\mathbb{R}^n)) \cap C^1(\mathbb{R}; H^{s-2}(\mathbb{R}^n))$$

for some $s \geq 3/2$, \hspace{1cm} (2.2)

of the Schrödinger equation

$$i\partial_t u + \Delta_x u + V(x)u = 0, \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}, \ n \geq 3,$$ \hspace{1cm} (2.3)

satisfies for $T \in \mathbb{R}$

$$\int_0^T \int_{\mathbb{R}^n} \nabla \tilde{u}(x, t) D^2\Phi(x) \nabla u(x, t) \, dt \, dx - \frac{1}{4} \int_0^T \int_{\mathbb{R}^n} |u(x, t)|^2 \Delta^2\Phi(x) \, dt \, dx$$

$$+ \frac{1}{2} \int_0^T \int_{\mathbb{R}^n} |u(x, t)|^2 (\nabla V \cdot \nabla \Phi)_+(x) \, dt \, dx - \frac{1}{2} \int_0^T \int_{\mathbb{R}^n} |u(x, t)|^2 (\nabla V \cdot \nabla \Phi)_-(x) \, dt \, dx$$

$$= \frac{1}{2} \left( \int_{\mathbb{R}^n} \tilde{u}(x, T) \nabla u(x, T) \nabla \Phi(x) \, dx - \int_{\mathbb{R}^n} \tilde{u}(x, 0) \nabla u(x, 0) \nabla \Phi(x) \, dx \right),$$ \hspace{1cm} (2.4)

where

$$D^2\Phi(x) = \left( \frac{\partial^2\Phi}{\partial x_i x_j}(x) \right)_{1 \leq i, j \leq n}$$

and $\Delta^2$ is the bilaplacian.

Proof. Notice that, under the global assumption (2.2), all the integrals in identity (2.4) make sense either in $L^1([0, T] \times \mathbb{R}^n)$ or in $L^1(\mathbb{R}^n)$.

This identity follows by dual pairing, for fixed $t$, of Eq. (2.3) against the $H^{1/2}(\mathbb{R}^n)$ test function $\nabla \Phi(x) \cdot \nabla \tilde{u}(x, t) + \frac{1}{2} \Delta \Phi(x) \tilde{u}(x, t)$, then taking the real part and finally integrating in $t$.

In fact, writing this calculation as an integration by parts, in which some integrals have to be understood as distribution pairing, we have

$$0 = \Re \left( i \partial_t u + \Delta_x u + V(x)u \right) \left( \Phi(x) \cdot \nabla \tilde{u}(x, t) + \frac{1}{2} \Delta \Phi(x) \tilde{u}(x, t) \right)$$

$$= \frac{i}{2} \int \left\{ \partial_t \left( \nabla \Phi \nabla \tilde{u} + \frac{1}{2} \Delta \Phi \tilde{u} \right) - \partial_t \tilde{u} \left( \nabla \Phi \nabla u + \frac{1}{2} \Delta \Phi u \right) \right\}$$

$$+ \frac{1}{2} \int (\Delta u \nabla \Phi \cdot \nabla \tilde{u} + \Delta \tilde{u} \nabla \Phi \cdot \nabla u) + \frac{1}{4} \int (\Delta u \Delta \Phi \tilde{u} + \Delta_x \tilde{u} \Delta \Phi u).$$
\[ + \frac{1}{2} \int (V(x)u \nabla \Phi \cdot \nabla \tilde{u} + V(x)\tilde{u} \nabla \Phi \cdot \nabla u) \, dx + \frac{1}{2} \int V(x)|u|^2 \Delta \Phi \, dx \]
\[ = I_1 + I_2 + I_3 + I_4 + I_5. \tag{2.5} \]

Now by writing
\[ \int \partial_t u \nabla \Phi \nabla \tilde{u} = - \int \partial_t (\nabla u) \nabla \Phi \tilde{u} - \int \partial_t u \Delta \Phi \tilde{u}, \]
and since the expression is real, we have
\[ I_1 = -i \frac{d}{dt} \left( \int \tilde{u} \nabla u \cdot \nabla \Phi \, dx + \frac{1}{2} \int |u|^2 \Delta \Phi \, dx \right) = \frac{1}{2} \frac{d}{dt} \Im \int \tilde{u} \nabla u \cdot \nabla \Phi \, dx. \tag{2.6} \]

We have also
\[ I_3 = -\frac{1}{4} \int \nabla (|u|^2) \cdot \nabla (\Delta \Phi) \, dx - \frac{1}{2} \int |\nabla u|^2 (\Delta \Phi) \, dx, \tag{2.7} \]
\[ I_4 = \frac{1}{2} \int V(x) |u|^2 \cdot \nabla \Phi \, dx. \tag{2.8} \]

By the formula
\[ \frac{1}{2} \left( \nabla u \cdot \nabla (\nabla \Phi \cdot \nabla \tilde{u}) \right) + \nabla \tilde{u} \cdot \nabla (\nabla \Phi \cdot \nabla u) = \nabla \tilde{u} D^2 \Phi \nabla u + \frac{1}{2} \nabla \Phi \cdot \nabla (|\nabla u|^2), \]
\[ I_2 = -\int \nabla \tilde{u} D^2 \Phi \nabla u \, dx - \frac{1}{2} \int \nabla \Phi \cdot \nabla (|\nabla u|^2). \tag{2.9} \]

Now, by using the identities
\[ -\frac{1}{2} \int \nabla \Phi \cdot \nabla (|\nabla u|^2) = \frac{1}{2} \int |\nabla u|^2 \Delta \Phi \, dx, \tag{2.10} \]
\[ -\frac{1}{4} \int \nabla (|u|^2) \nabla (\Delta \Phi) \, dx = \frac{1}{4} \int |u|^2 \Delta^2 \Phi \, dx \tag{2.11} \]
and
\[ \frac{1}{2} \int V(x) \nabla (|u|^2) \cdot \nabla \Phi \, dx = -\frac{1}{2} \int |u|^2 \nabla V(x) \cdot \nabla \Phi \, dx - \frac{1}{2} \int |u|^2 V(x) \Delta \Phi \, dx, \tag{2.12} \]

together with (2.5)–(2.9), we obtain the desired identity (2.4). \square

**Lemma 2.2.** Let \( \Phi(x) = \phi(|x|) \) be a radial real-valued function such that \( \phi'(r) \) and \( r \phi''(r) \) are bounded, then we have
\[ \left| \int \tilde{u} (x) \nabla u(x) \nabla \Phi (x) \, dx \right| \leq \| u \|_{H^{1/2} (\mathbb{R}^n)}. \tag{2.13} \]
Proof. Let us denote
\[(Tf, g) = \int_{\mathbb{R}^n} f(x) \nabla g(x) \cdot \frac{x}{|x|} \phi'(|x|) \, dx.\]

Then \(|(Tf, g)| \leq C \|f\|_{L^2} \|g\|_{\dot{H}^1} \). This means
\[T : L^2 \rightarrow \dot{H}^{-1}.\]  

Integrating by parts, we may write for good \(f\) and \(g\):
\[\int_{\mathbb{R}^n} \phi''(|x|) f(x) \nabla g(x) \cdot \frac{x}{|x|} \, dx + \phi'(|x|) f(x) \nabla g(x) \cdot \frac{x}{|x|} \, dx + \phi'(|x|) f(x) \frac{n-1}{|x|} \, dx.\]

From our hypothesis
\[\left| \phi''(|x|) + \phi'(|x|) \frac{n-1}{|x|} \right| \leq C \frac{1}{|x|},\]
and hence we may use the uncertainty principle
\[\int_{\mathbb{R}^n} \left| f(x) \right|^2 \frac{1}{|x|^2} \, dx \leq C \|\nabla f\|_{L^2}^2\]

Interpolating between (2.14) and (2.15) gives
\[\left| (Tf, g) \right| \leq C \|g\|_{L^2} \|f\|_{\dot{H}^1}.\]

Which is to say
\[T : \dot{H}^1 \rightarrow L^2.\]  

Proof of Theorem 1. From Hypothesis 2 we might assume that \(u(t) \in H^s(\mathbb{R}^n), s \geq 3/2, \) for \(t \in \mathbb{R}.\)

Our goal is to insert in (2.4) a real function \(\Phi(x)\), depending on some fixed \(R > 0\), satisfying (2.1), such that the quadratic form determined by \(D^2 \Phi(x)\) is nonnegative and the bilaplacian \(\Delta^2 \Phi(x)\) is nonpositive.

Let us start by writing
\[\Delta^2 \Phi(x) = h(x).\]
This \( h(x) \) has to be nonpositive, notice that (2.16) “almost determines” \( \Phi(x) \), then we will need another condition in \( h(x) \) in order to have the nonnegativeness of \( D^2 \Phi(x) \).

If we suppose that \( h \) is a radial function, \( h(x) \equiv h(r) \) with \( r = |x| \), and we are looking for radial solutions \( \Phi(x) \equiv \phi(r) \), we can write (2.16) as

\[
\Delta^2 \Phi(x) = r^{-(n-1)} \partial_r \left( r^{n-1} \partial_r \left( r^{n-1} \partial_r \phi(r) \right) \right) \phi(r) = h(r). \tag{2.17}
\]

Our choice of \( h(r) \) will be

\[
\Delta^2 \Phi(x) \equiv h(r) = -m(r) - W(r), \tag{2.18}
\]

where

\[
m(r) = \frac{\epsilon}{R^3} \chi_{[0,R]}(r) + \frac{\eta}{r^3} \chi_{[R,\infty)}(r),
\]

for \( \epsilon \) as in (1.8), \( \eta > 0 \) with a condition that we will establish later and \( W(r) \) the function in (1.2).

If we insert (2.18) in (2.17) and we integrate, taking in account (1.6), we have

\[
\phi'(r) = \psi'(r) + \varphi'(r),
\]

where

\[
\psi'(r) = \frac{1}{r^{n-1}} \int_0^r u^{n-1} \int_0^\infty 1_s^{n-1} \int_0^s t^{n-1} W(t) \, dt \, ds \, du - \frac{c_1}{2(n-2)r^{n-3}} + \frac{c_2}{r^{n-1}} \tag{2.19}
\]

and

\[
\varphi'(r) = -\frac{1}{r^{n-1}} \int_0^r u^{n-1} \int_0^u 1_s^{n-1} \int_0^s t^{n-1} m(t) \, dt \, ds \, du + \frac{c_3 r}{n} \tag{2.20}
\]

with \( c_1, c_2 \) and \( c_3 \) constants.

Since

\[
\nabla \bar{u}(x,t) D^2 \Phi(x) \nabla u(x,t) = \phi''(|x|) \left| \partial_r u(x,t) \right|^2 + \frac{\phi'(|x|)}{|x|} \left| \partial_\tau u(x,t) \right|^2, \tag{2.21}
\]

in order to have the nonnegativeness of \( D^2 \Phi \) we need

\[
\inf_{r > 0} \{ \phi'(r), \phi''(r) \} \geq 0. \tag{2.22}
\]

We will prove (2.22) and also

\[
\phi'(r) \geq \frac{H(0)}{(n-1)(n-3)}, \tag{2.23}
\]

\[
\inf_{r \in (0,R)} \left\{ \phi'(r), \phi''(r) \right\} \geq \frac{C \epsilon}{R}, \tag{2.24}
\]
Now, by using
\[(\nabla V,\nabla \Phi)_+(x) = (\partial_r V)_+(x)\phi'(|x|)\] (2.25)
and (2.21)–(2.24), we have
\[
C \int_{\mathbb{R}^n} \int_0^T \nabla \bar{u}(x,t) D^2 \Phi(x) \nabla u(x,t) \, dt \, dx + C \int_{\mathbb{R}^n} \int_0^T |u(x,t)|^2 (\nabla V,\nabla \Phi)_+(x) \, dt \, dx
\geq H(0) \left( \int_{\mathbb{R}^n} \int_0^T \frac{\partial_r u(x,t)}{|x|} \, dt \, dx + \int_0^\infty \int_{\mathbb{R}^n} |u(x,t)|^2 (\partial_r V)_+(x) \, dt \, dx \right)
+ \frac{\epsilon}{R} \int_{B(0,R)} \int_0^T |\nabla u(x,t)|^2 \, dt \, dx. \tag{2.26}
\]

Now we would like the fourth term on the right-hand side of (2.4) to be absorbed by the second one, in order to obtain
\[
-\frac{1}{2} \int_{\mathbb{R}^n} \int_0^T |u(x,t)|^2 (\nabla V,\nabla \Phi)_-(x) \, dt \, dx - \frac{1}{4} \int_{\mathbb{R}^n} \int_0^T |u(x,t)|^2 \Delta^2 \Phi(x) \, dt \, dx
\geq \frac{\epsilon}{4R^3} \int_{B(0,R)} \int_0^T |u(x,t)|^2 \, dt \, dx. \tag{2.27}
\]
From (1.2) and (2.18) we have
\[
-\frac{1}{2} (\nabla V,\nabla \Phi)_-(x) - \frac{1}{4} \Delta^2 \Phi(x) \geq \frac{1}{2} W(|x|) \left( \frac{1}{2} - \phi'(|x|) \right) + \frac{\epsilon}{4R^3} \chi_{(0,R)}(|x|)
\]
and to obtain (2.27) we need
\[
\phi'(r) < \frac{1}{2}. \tag{2.28}
\]
We will also prove condition (2.1),
\[
\sup_{r > 0} \left\{ \phi'(r), r \phi''(r) \right\} \leq C, \tag{2.29}
\]
with C independent of R, which allows us to apply Lemma 2.2 and to obtain
\[
\frac{1}{2} \left| \int_0^T \frac{d}{dt} \left[ 3 \int_{\mathbb{R}^n} \bar{u}(x,t) \nabla u(x,t) \nabla \Phi(x) \, dx \right] \, dt \right| \leq C(V) \|u_0\|_{H^{1/2}(\mathbb{R}^n)}^2. \tag{2.30}
\]
Then (1.9) follows from (2.4), (2.26), (2.27) and (2.30).
Construction of the function $\Phi(x)$. By taking derivative in (2.19), we obtain

$$
\psi''(r) = -\frac{n-1}{r^n} \int_0^r w^{n-1} \int_0^u \int_0^{s^n-1} t^{n-1} W(t) \, dt \, ds \, du \\
+ \int_0^\infty \frac{1}{s^{n-1}} \int_0^r t^{n-1} W(t) \, dt \, ds + \frac{c_1(n-3)}{2(n-2)r^{n-2}} - \frac{c_2(n-1)}{r^n}.
$$

We use Fubini’s theorem in the first integral in the variable $u$ and $s$

$$
\psi''(r) = -\frac{n-1}{r^n} \int_0^r \int_0^s t^{n-1} W(t) \, dt \, ds + \frac{1}{n} \int_0^\infty \int_0^s t^{n-1} W(t) \, dt \, ds \\
+ \frac{c_1(n-3)}{2(n-2)r^{n-2}} - \frac{c_2(n-1)}{r^n},
$$

and now in the two integrals to obtain

$$
\psi''(r) = -\frac{n-3}{2(n-2)r^{n-2}} \int_0^r t^{n-1} W(t) \, dt + \frac{n-1}{2nr^n} \int_0^r t^{n+1} W(t) \, dt \\
+ \frac{1}{n(n-2)} \int_r^\infty t W(t) \, dt + \frac{c_1(n-3)}{2(n-2)r^{n-2}} - \frac{c_2(n-1)}{r^n}. \quad (2.31)
$$

A similar manipulation gives us

$$
\phi''(r) = -\frac{n-1}{2} \int_0^r \int_0^{s^{n+1}} t^{n+1} m(t) \, dt \, ds \\
+ \frac{n-3}{2} \int_0^r \int_0^{s^{n-1}} t^{n-1} m(t) \, dt \, ds + \frac{c_3}{n}. \quad (2.32)
$$

Since the terms

$$
-\frac{c_1}{2(n-2)r^{n-3}} + \frac{c_2}{r^{n-1}} \quad \text{in (2.19)} \quad \text{and} \quad \frac{c_1(n-3)}{2(n-2)r^{n-2}} - \frac{c_2(n-1)}{r^n} \quad \text{in (2.31)}
$$

have opposite sign, to obtain $\psi'$ and $\psi''$ nonnegative the only choice is $c_1 = c_2 = 0$.

We define

$$
\frac{c_3}{n} = \frac{n-1}{2} \int_0^\infty \int_0^s t^{n+1} m(t) \, dt \, ds - \frac{n-3}{2} \int_0^\infty \int_0^s t^{n-1} m(t) \, dt \, ds. \quad (2.33)
$$
Then

\[
\psi''(r) = -\frac{n-3}{2(n-2)r^{n-2}} \int_0^r t^{n-1} W(t) \, dt + \frac{n-1}{2nr^n} \int_0^r t^{n+1} W(t) \, dt \\
+ \frac{1}{n(n-2)} \int_r^\infty t W(t) \, dt,
\]

which from (1.7) gives \(\psi'' \geq 0\).

Also

\[
\phi''(r) = \frac{n-1}{2} \int_r^\infty \frac{1}{s^{n+1}} \int_0^s t^{n+1} m(t) \, dt \, ds - \frac{n-3}{2} \int_r^\infty \frac{1}{s^{n-1}} \int_0^s t^{n-1} m(t) \, dt \, ds
\]

(2.35)

(in the case \(n = 3\), this choice guarantees that \(\phi''(r)\) is nonnegative).

It is easy to check that \(\phi'(0) = 0\) and therefore

\[
\phi'(r) = \int_0^r \phi''(u) \, du.
\]

(2.36)

Using integration by part in (2.35) we obtain

\[
\phi''(r) = \frac{1}{n(n-2)} \int_r^\infty t m(t) \, dt + \frac{n-1}{2nr^n} \int_0^r t^{n+1} m(t) \, dt \\
- \frac{n-3}{2(n-2)r^{n-2}} \int_0^r t^{n-1} m(t) \, dt.
\]

(2.37)

Let \(p > 2\) and

\[
m(r) = \frac{\epsilon}{R^3} \chi_{[0,R]}(r) + \frac{\eta}{r^\frac{p}{2}} \chi_{[R,\infty]}(r).
\]

A calculation gives

\[
\int_0^r t^p m(t) \, dt = \begin{cases} 
\frac{\epsilon r^{p+1}}{R^{3(p+1)}}, & r \leq R, \\
\left(\frac{\epsilon}{p+1} - \frac{\eta}{p-2}\right) R^{p-2} + \eta r^{p-2}, & r > R,
\end{cases}
\]

(2.38)

and

\[
\int_r^\infty t m(t) \, dt = \begin{cases} 
\frac{\epsilon R^2 - \eta^2}{2R^2} + \frac{1}{R}, & r \leq R, \\
\frac{\eta}{r^\frac{p}{2}}, & r > R.
\end{cases}
\]

(2.39)
If we use (2.37)–(2.39) we have for $r \leq R$

$$\varphi''(r) = \frac{\epsilon}{2n(n-2)R} + \frac{\eta}{n(n-2)R} - \frac{3\epsilon}{2n(n+2)R} \left(\frac{r}{R}\right)^2,$$

and for $r \geq R$

$$\varphi''(r) = \frac{n-1}{2nR} \left(\frac{R}{r}\right)^n \left(\frac{\epsilon}{n+2} - \frac{\eta}{n-1}\right) + \frac{n-3}{2(n-2)R} \left(\frac{R}{r}\right)^{n-2} \left(\frac{\eta}{n-3} - \frac{\epsilon}{n}\right).$$

If we take $\eta$ such that

$$\frac{n+2}{n-1} \leq \frac{\epsilon}{\eta} \leq \frac{n}{n-3},$$

then

$$\varphi''(r) \geq \begin{cases} \frac{1}{n(n-2)R} \left[ \eta - \frac{\epsilon(n-4)}{n+2} \right], & r \leq R, \\ \frac{n-3}{2(n-2)R} \left(\frac{\eta}{n-3} - \frac{\epsilon}{n}\right) \left(\frac{R}{r}\right)^{n-2}, & r > R. \end{cases}$$

By using the above expression, (1.7), (2.34) and (2.36), we have (2.22) and (2.24):

$$\inf_{r \in (0,R)} \left\{ \frac{\varphi'(r)}{r}, \varphi''(r) \right\} \geq \frac{3\epsilon}{n^2(n+2)R}.$$ 

From (2.36)–(2.42) we have

$$\varphi'(r) \leq \frac{\epsilon(15+n)}{n(n-1)(n-2)}, \quad r > 0,$$

and the choice $c_1 = c_2 = 0$ with a calculation on (2.19) gives us

$$\psi'(r) = \frac{1}{2r^{n-3}} \int_0^r t^{n-3} \int_t^\infty s W(s) \, ds \, dt - \frac{1}{2r^{n-1}} \int_0^r t^{n-1} \int_t^\infty s W(s) \, ds \, dt.$$ 

Since $\phi'$ is nondecreasing and from (2.36)–(2.42) we have that $\phi' \geq 0$, then from (1.5), (1.6) and (2.43) we obtain

$$\frac{H(0)}{(n-1)(n-3)} \leq \phi' \leq \frac{H(\infty)}{(n-1)(n-3)} + \epsilon C(n).$$

Then we obtain (2.23) and hence (2.26). To obtain (2.28) and (2.29) and hence (2.27), we use the above inequality and the expressions for $\phi'$ and $\phi''$. □
Proof of Corollary 1. This corollary follows from Theorem 1. The proof can be simplified if we take in (2.18)

\[ m(r) = \frac{\eta}{r^3} \chi_{(R, \infty)}(r) \]

for some \( \eta > 0 \) and

\[ W(x) = c_0 \frac{(n-1)(n-3)}{|x|^3} \]

the radial function that allow us to recuperate the Morawetz multiplier.

In this case

\[ \phi'(r) = c_0 + \varphi'(r), \quad \phi''(r) = \varphi''(r). \]

From (2.36), (2.40) and (2.41) we have that (2.22) is true and

\[ \frac{\psi'(r)}{r} = \varphi''(r) = \frac{\eta}{n(n-2)R}, \quad r \in (0, R), \]

\[ \varphi'(r) \leq \int_0^\infty \varphi''(u) \, du = \frac{\eta}{(n-1)(n-3)}. \]

Now, by using (1.11), with \( \epsilon = 3\eta/((n-1)(n-3)) \), we have

\[ -\frac{1}{2} \langle \nabla V, \nabla \Phi \rangle_x - \frac{1}{4} \Delta^2 \Phi(x) \geq \frac{c_0 \eta}{|x|^3} \]

and the proof follows as in Theorem 1. \( \square \)

Proof of Theorem 2. We follow the proof of Theorem 1 with a few modifications.

We take in (2.18)

\[ m(r) = \frac{\epsilon}{R^3} \chi_{[0, R]}(r), \quad (2.45) \]

for some \( \epsilon > 0 \), \( W(r) \) the function in (1.2), \( \psi' \) as in (2.19) with \( c_2 = 0 \) and \( c_1 = -2\alpha \)

\[ \psi'(r) = \frac{1}{r^2} \int_0^r u^2 \int_u^\infty \frac{1}{s^2} \int_0^s t^2 W(t) \, dt \, ds \, du + \alpha \quad (2.46) \]

with \( \alpha \geq 0 \) and \( \varphi' \) as in (2.20) with \( c_3 / n \) as (2.33) for \( n = 3 \).

Since \( \psi'' \geq 0 \), see (2.34), we have

\[ \alpha \leq \psi'(r) \leq \alpha + \frac{1}{2} \int_0^\infty r^2 W(t) \, dt, \quad (2.47) \]
\( \phi \) satisfies (2.36) and

\[
\phi''(r) = \frac{1}{3r^3} \int_0^r t^4 m(t) \, dt + \frac{1}{3} \int_r^\infty t m(t) \, dt
\]

(2.48)

then \( \phi' \geq 0 \).

From these expressions we obtain (2.22)

\[
\phi'(r) \geq \alpha
\]

and

\[
\inf_{r \in (0,R)} \left\{ \frac{\phi'(r)}{r}, \phi''(r) \right\} \geq \frac{C \epsilon}{R}
\]

(2.49)

that implies, by using (2.21)

\[
C \int_{\mathbb{R}^3} \int_0^T \nabla \bar{u}(x,t) D^2 \Phi(x) \nabla u(x,t) \, dt \, dx + C \int_{\mathbb{R}^3} \int_0^T \left| u(x,t) \right|^2 (\nabla V \cdot \nabla \Phi)_+(x) \, dx
\]

\[
\geq \alpha \left( \int_{\mathbb{R}^3} \int_0^T \frac{\left| \partial_u u(x,t) \right|^2}{|x|} \, dt \, dx + \int_{\mathbb{R}^3} \int_0^T \left| u(x,t) \right|^2 (\partial_r V)_+(x) \, dt \, dx \right)
\]

\[
+ \frac{\epsilon}{R} \int_{B(0,R)} \int_0^T \left| \nabla u(x,t) \right|^2 \, dt \, dx.
\]

(2.50)

To obtain (2.27), we observe that (2.47) implies

\[
\phi'(r) \leq \alpha + \frac{1}{2} \int_0^\infty t^2 W(t) \, dt + \phi'(r).
\]

Then (2.36) and (2.48) give us that \( \phi'(r) \leq \epsilon/6 \) and (2.28) is a consequence of our hypothesis (1.18) with \( \eta = 2\alpha + \epsilon/3 \). We have proved the analogous of (2.27).

Since (2.29) is also true for \( n = 3 \), estimate (1.19) follows from (2.50), (2.27) for \( n = 3 \) and Lemmas 2.1 and 2.2.

\( \square \)

**Proof of Theorem 3.** Again we follow partly the lines of the proof of Theorem 1.

We start with the case \( n > 3 \).

For \( R > 0 \) we take in (2.18)

\[
\Delta^2 \Phi(x) = h(r) = -\frac{1}{|x|^3} \chi_{[R,\infty)}(|x|) - \frac{1}{|x|^3}.
\]
In the same way as in the proof of Corollary 1, we get

\[
C \int_{\mathbb{R}^n} \int_0^T \nabla \bar{u}(x,t) D^2 \Phi(x) \nabla u(x,t) \, dt \, dx + C \int_{\mathbb{R}^n} \int_0^T |u(x,t)|^2 (\nabla V, \nabla \Phi)_+(x) \, dt \, dx
- \frac{1}{4} \int_{\mathbb{R}^n} \int_0^T |u(x,t)|^2 \Delta^2 \Phi(x) \, dt \, dx
\geq \int_{\mathbb{R}^n} \int_0^T \frac{|\partial_t u(x,t)|^2}{|x|} \, dt \, dx + \int_{\mathbb{R}^n} \int_0^T |u(x,t)|^2 (\partial_r V)_+(x) \, dt \, dx
+ \frac{1}{R} \int_{B(0,R)} \int_0^T |\nabla u(x,t)|^2 \, dt \, dx + \int_{\mathbb{R}^n} \int_0^T \frac{|u(x,t)|^2}{|x|^3} \, dt \, dx.
\] (2.51)

If we use Lemmas 2.1, 2.2 and we take supremum in $R$ and $T$ in (2.51) we have

\[
\sup_{R > 0} \frac{1}{R} \int_{B(0,R)} \int_0^\infty |\nabla u(x,t)|^2 \, dt \, dx + \int_{\mathbb{R}^n} \int_0^\infty \frac{|u(x,t)|^2}{|x|^3} \, dt \, dx
+ \int_{\mathbb{R}^n} \int_0^\infty \frac{|\partial_t u(x,t)|^2}{|x|} \, dt \, dx + \int_{\mathbb{R}^n} \int_0^\infty |u(x,t)|^2 (\partial_r V)_+(x) \, dt \, dx
\leq C(n, V) \|u_0\|_{H^{1/2}(\mathbb{R}^n)}^2 + \frac{1}{2} \int_{\mathbb{R}^n} \int_0^\infty |u(x,t)|^2 (\nabla V, \nabla \Phi)_-(x) \, dt \, dx.
\] (2.52)

Now we will hide the second term on the right-hand side of (2.52) by the first and the second on the left-hand side of (2.52).

Since

\[
\phi'(r) \leq \frac{2}{(n - 1)(n - 3)},
\]
if we use (1.20) and (1.21) we have

\[
\frac{1}{2} \int_{\mathbb{R}^n} \int_0^\infty |u(x,t)|^2 (\nabla V, \nabla \Phi)_-(x) \, dt \, dx \leq B(n) \sum_{j=0}^\infty \int_{\Omega_j} \int_0^\infty |u(x,t)|^2 w_j(x) \, dx \, dt,
\] (2.53)

where

\[
B(n) = \frac{1}{(n - 1)(n - 3)}.
\]
Now we use a cutoff on $\Omega_j$, $j \geq 0$. Let $\beta$ be with supp $\beta \subset (1/4, 4)$ and $\beta = 1$ on $(1/2, 2)$. We define $\beta_j(x) = \beta(|x|/2^j)$, $j \geq 1$, $\beta_0$ with supp $\beta_0 \subset (-2, 2)$, $\beta_0 = 1$ on $(-1, 1)$ and

$$
\Omega_0^* = \{ x \in \mathbb{R}^n : |x| \leq 2 \}, \quad \Omega_j^* = \{ x \in \mathbb{R}^n : 2^{j-1} \leq |x| < 2^{j+1} \}.
$$

By using (1.22) and (2.53), we have

$$
\frac{1}{2} \int_{\mathbb{R}^n} \int_0^\infty |u(x, t)|^2 (\nabla V \cdot \nabla \Phi)(x) \, dt \, dx \\
\leq B(n) \sum_{j=0}^\infty \int_{\Omega_j} |\beta_j(x)u(x, t)|^2 w_j(x) \, dx \, dt \\
\leq B(n) \sum_{j=0}^\infty \int_{\Omega_j^*} c(w_j) \int |\nabla (\beta_j(-\cdot, t)) (x)|^2 \, dx \, dt \\
\leq B(n) \sum_{j=0}^\infty 2c(w_j) \int_{\Omega_j^*} |\nabla u(x, t)|^2 \, dx \, dt \\
+ B(n) \sum_{j=0}^\infty \frac{2c(w_j)}{2^{2j}} \int_{\Omega_j^*} |u(x, t)|^2 \, dx \, dt \\
\leq 2B(n) \left( \sum_{j=0}^\infty 2^{j} c(w_j) \right) \sup_{R>0} \frac{1}{R} \int_{B(0, R)} \int_0^\infty |\nabla u(x, t)|^2 \, dt \, dx \\
+ 4B(n) \sup_{j \geq 0} \left\{ 2^j c(w_j) \right\} \int_{\mathbb{R}^n} \int_0^\infty \frac{|u(x, t)|^2}{|x|^2} \, dtdx.
$$

Then condition (1.23) allows us to hide these terms by the first and the second terms in (2.52) and we obtain (1.25).

The proof in the case $n = 3$ is similar to the case $n > 3$. We take in (2.45) $\epsilon = 1$ and in (2.46) $\alpha = 1$. \square

### 3. Zero-resonances and sharpness of the results

Several authors introduced the concept of 0-resonance, in order to study a priori estimates or asymptotics of the resolvent operator at zero energy, see [10–12,15,20]. In these works the potentials $V$ are assumed to satisfy the short range condition

$$
|V(x)| \leq C \left( 1 + |x| \right)^{-\sigma} \text{ for } \sigma > 2,
$$

(3.1)
and a 0-resonance is a solution $u$ of the stationary equation

$$\Delta u + Vu = 0,$$

(3.2)

which is not in $L^2$ but is in $H^1_{\text{loc}}$ and satisfies

$$\int_{\mathbb{R}^n} \frac{|u(x)|^2}{(1 + |x|)^\alpha} \, dx < \infty$$

(3.3)

for any $\alpha > 1$. It can be seen, by using inequality (3.1) and Eq. (3.3), that such a function has to satisfy the global condition

$$\int_{\mathbb{R}^n} \frac{|
abla u(x)|^2}{(1 + |x|)^\alpha} \, dx < \infty$$

(3.4)

for any $\alpha > 1$.

Notice that for a solution of (3.2) the right-hand side of (2.2) is zero.

Let us consider solutions $u$ of the stationary equation which are only in the local spaces: $u \in H^1_{\text{loc}}$ and $|V|^{1/2}u \in L^2_{\text{loc}}$. If we perform the integration by parts in the proof of Lemma 2.1, in the domain $B(0, R)$, completing with the corresponding boundary terms in identities (2.7), (2.9)–(2.12), we obtain

$$0 = \int_{B(0,R)} \nabla \bar{u}(x) D^2 \Phi(x) \nabla u(x) \, dx$$

$$- \frac{1}{4} \int_{B(0,R)} |u(x)|^2 \Delta^2 \Phi(x) \, dx + \frac{1}{2} \int_{B(0,R)} |u(x)|^2 (\nabla V \cdot \nabla \Phi)(x) \, dx$$

$$+ \frac{1}{2} \int_{S_R} \partial_r \Phi(x) |\nabla u(x)|^2 \, d\sigma(x) - \Re \int_{S_R} \partial_r u(x) \nabla \Phi(x) \cdot \nabla \bar{u}(x) \, d\sigma(x)$$

$$+ \frac{1}{4} \int_{S_R} \partial_r \Delta \Phi(x) |u(x)|^2 \, d\sigma(x) - \frac{1}{4} \int_{S_R} \partial_r (|u(x)|^2) \Delta \Phi(x) \, d\sigma(x)$$

$$+ \frac{1}{2} \int_{S_R} \partial_r \Phi(x) |u(x)|^2 V(x) \, d\sigma(x),$$

(3.5)

where $S_R$ denotes the sphere of radius $R$. This suggests to consider the following definition of a 0-resonance.

**Definition 1.** A function $u$ is a zero-resonance if $u$ is not in $L^2$ and satisfies

$$\Delta u + Vu = 0, \quad u \in H^1_{\text{loc}} \text{ and } |V|^{1/2}u \in L^2_{\text{loc}}$$

(3.6)
and

\[ \sup_{R > 1} \frac{1}{R} \int_{B(0,R)} V^*|u|^2 < \infty \quad \text{and} \quad \liminf_{R \to \infty} \frac{1}{R} \int_{B(0,R)} V^*|u|^2 = 0 \] (3.7)

for \( V^* = |V| + (1 + |x|^2)^{-1} \).

**Lemma 3.1.** If \( u \) satisfies (3.6) and (3.7), then

\[ \sup_{R > 1} \frac{1}{R} \int_{B(0,R)} |\nabla u|^2 < \infty \quad \text{and} \quad \liminf_{R \to \infty} \frac{1}{R} \int_{B(0,R)} |\nabla u|^2 = 0. \] (3.8)

**Proof.** We assume that \( u \) is real. Let us take the cutoff function \( \eta(x) = \eta_0(|x|/R) \) where \( \eta_0 \) is supported on \([-2,2]\) and \( \eta_0 \equiv 1 \) on \([-1,1]\).

Then, for \( u \) satisfying (3.6) and (3.7), we have

\[
\int_{B(0,R)} |\nabla u|^2 \leq \int_{B(0,R)} |\nabla u|^2 \eta \leq \left| \int_{B(0,2R)} u \nabla \eta \cdot \nabla u \right| + \int_{B(0,2R)} u \eta \Delta u \leq \int_{B(0,2R)} |\nabla \eta \cdot |\nabla u|^2| + \int_{B(0,2R)} |u|^2 |V| \leq \int_{B(0,2R)} |u|^2 |\Delta \eta| + \int_{B(0,2R)} |u|^2 |V| \leq C \int_{B(0,2R)} |u|^2 (1 + |x|^2)^{-1} + \int_{B(0,2R)} |u|^2 |V| \leq C \int_{B(0,2R)} |u|^2 V^* .
\]

From the above (3.8) follows easily. \( \Box \)

For potentials bounded by \( c/(1 + |x|^2)^2 \), our definition is close to those of [10–12,20], but notice that they assume the “short range” condition (3.1). Also let us remark that in this case there is no resonance for dimensions \( n > 4 \) [12]. Our condition is an end-point substitution of the powers in (3.3), in the same sense as Besov norm in [2] replaces weighted norm in [1] (end-point dual trace lemma).

From the equation \( \Delta u + Vu = 0 \), our solutions are in \( H^2_{\text{loc}} \). For \( H^{3/2} \) solutions \( u(x,t) \) of the evolution equation, conditions (3.6) and (3.7) for any \( t \), imply that the boundary terms in (3.5) vanish as \( R \to \infty \). Since we can substitute the right-hand side of (1.12) by

\[
\frac{1}{2} \left\{ \int_{\mathbb{R}^n} \bar{u}(x,T) \nabla u(x,T) \nabla \Phi(x) dx - \int_{\mathbb{R}^n} \bar{u}(x,0) \nabla u(x,0) \nabla \Phi(x) dx \right\} ,
\]

any 0-resonance would violate the estimate (1.12) in Corollary 1. We may state, from our hypothesis, the previous lemma and this observation, the following proposition.

**Proposition 3.1.** Assume \( V \) satisfies the hypothesis of Theorems 1, 2 or 3. Then there are no zero-resonances.
For $n \geq 4$ and any $c > 1/2$ we construct a potential $V$ with $(\partial_r V)_-(x) \sim c(n-1)(n-3)|x|^{-3}$ as $|x| \to \infty$ with a zero-resonance. This is to say that the constant in (1.10) is sharp. In fact take
\[ u(x) = (\delta + |x|)^{-\alpha} \quad \text{with} \quad \delta > 0 \quad \text{and} \quad \frac{n-3}{2} < \alpha < n-2; \]
then $\Delta u + Vu = 0$ for
\[ V(x) = \frac{\alpha(n-2-\alpha)}{(\delta + |x|)^2} + \frac{\delta \alpha(n-1)}{(\delta + |x|)^2|x|}. \]
Notice that
\[ V'(r) = \frac{-2\alpha(n-2-\alpha)}{(\delta + |x|)^3} + \delta O(1/r^4), \]
which by taking $\alpha$ close to $(n-3)/2$ has the appropriate asymptotics as $|x| \to \infty$.

The resonance conditions (3.6) and (3.7) can be easily checked.

In dimension $n = 3$ we exhibit a potential such that
\[ \int r^2(\partial_r V)_-(r) \, dr = (\pi/2)^2 \quad (3.9) \]
with a resonance. Namely take
\[ u(x) = \begin{cases} \sin r/r & \text{if } |x| \leq \pi/2, \\ 1/r & \text{if } |x| \geq \pi/2. \end{cases} \quad (3.10) \]

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References