On the theory of Weak Turbulence for the Nonlinear Schrödinger Equation.

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Contents

Abstract

We study the Cauchy problem for a kinetic equation arising in the weak turbulence theory for the cubic nonlinear Schrödinger equation. We define suitable concepts of weak and mild solutions and prove local and global well posedness results. Several qualitative properties of the solutions, including long time asymptotics, blow up results and condensation in finite time are obtained. We also prove the existence of a family of solutions that exhibit pulsating behavior.

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CHAPTER 1

Introduction

The name weak turbulence is often used in the physical literature to describe the transfer of energy between different frequencies which takes place in several nonlinear wave equations with weak nonlinearities.

The theory of weak turbulence has been extensively developed in the last decades and many applications are available today. From the mathematical point of view, the starting point of all the problems which can be studied using the weak turbulence approach is a set of nonlinear wave equations with weak nonlinearities. We will denote as ε a small number which measures the strength of the nonlinear interactions. If ε is set to zero, the problem becomes a linear system of wave equations which will be termed in the following as the linearized problem. In order to simplify the presentation we will restrict this introductory description of weak turbulence theory to the cases in which the set of nonlinear equations is solved in the whole space $x \in \mathbb{R}^N$, for $t \in \mathbb{R}$ and where the equations are invariant under space and time translations. This allows to solve the linearized problem using standard Fourier transforms, but in principle the same ideas could be applied to nonhomogeneous systems. Suppose that the set of magnitudes by the wave equations is denoted as $u = u(t,x)$, where $u \in \mathbb{C}^L$ or $u \in \mathbb{R}^L$. Then, the linearized problem admits solutions proportional to $e^{i(kx+\omega t)}$ with $\omega = \Omega(k)$, where Ω is a function, perhaps multivalued, which is often referred as dispersion relation. In conservative (vs. dissipative) problems, the function $\Omega(\cdot)$ is real. A large class of initial data for the linearized problem can be decomposed in Fourier modes e^{ikx} and then, the solution of the linear equation is given by the form:

$$u(t,x) = \int a(t,k)\, e^{i(kx-\omega t)} d^N k \quad \text{with} \quad u(x,0) = u_0(x) = \int a(0,k)\, e^{ikx} d^N k$$

A crucial quantity in weak turbulence theories is the density in the wavenumber space $f(t,k) = |a(t,k)|^2$. Since $\Omega(\cdot)$ is real, the function $f(t,k)$ is constant in time for the solutions of the linearized problem. However, the dynamics of $f(t,k)$ becomes nontrivial if the nonlinear terms in the original system of wave equations are taken into account. Typically, due to the effect of resonances between some specific wavenumbers k, the function f changes in time with a rate that usually is a power law of the strength of the nonlinearities.

In principle, it is not possible to write a closed evolution equation for the function $f(t,k)$ because the dynamics of the function $a(t,\cdot)$ does not depend only on $|a(t,k)|$ but also in the phase of $a(t,\cdot)$. However, one of the key hypothesis in weak turbulence theory is that for a suitably chosen class of initial data u_0, it is possible to approximate the evolution of f by means of a kinetic equation. Moreover, in the limit of weak nonlinear interactions, it is possible to give an interpretation of the evolution of $f(t,k)$ by means of a particle model. The evolution of f is driven to the leading order in ε by resonances between linear modes with different values of k. This resonance condition can be given the interpretation of a collision between a number of particles, which results in another group of particles. The numbers of particles involved in these fictitious collisions depend on the dispersion relation for the linearized problem as well as in the form of the nonlinear terms. The resonance condition between modes can be understood as a condition for the conservation of the moment and energy of the particles in the collision process, assuming that k and ω are given the interpretation of moment and energy of the particles respectively. This makes this particle interpretation for the effect of the nonlinearities particularly appealing.

The precise conditions that allow to approximate the dynamics of wave equations by the kinetic models of weak turbulence have not been obtained in a fully rigorous manner. However, the physical derivations of the kinetic models of weak turbulence assume the statistical independence of the phases of the modes $a(0,k)$. From this point of view the derivation of the kinetic models of weak turbulence starting from wave equations have several analogies with the formal derivations of the Boltzmann equation starting from the dynamics of a particle system which can be found in the physical literature. It is also worth mentioning that the theory of weak turbulence assumes that the solutions of the underlying wave equation can be approximated to the leading order by means of solutions of the linearized problem. However, it is well known that effects induced by the nonlinear terms in the equation, which can become relevant for some ranges of k, can have a strong influence in the distribution function f (cf. [**12**], [**35**], [**39**], [**54**]).

The collision kernels arising in the kinetic equations of weak turbulence theory depend strongly on the details of the problem under consideration, as well as the number of particles involved in the collisions. Nevertheless this approach has been shown to be very fruitful in several physical problems, including water surface and capillary water waves (cf. [**20**], [**51**], [**52**]), internal waves on density stratifications (cf. [**7**], [**30**]), nonlinear optics (cf. [**12**]) and waves in Bose-Einstein condensates, planetary Rossby waves (cf. [**3**], [**31**]), and vibrating elastic plates (cf. [**11**]) among others. Many more applications as well as an extensive references list can be found in [**36**].

The first derivation of a kinetic model of weak turbulence was obtained, to our knowledge, in [**40**] in the context of the study of phonon interactions in anharmonic crystals. Derivations which take as starting point wave equations arising in a large variety of physical contexts and yielding analogous kinetic models were obtained in the 1960's in [**5**], [**17**], [**18**], [**19**], [**37**], [**38**], [**43**]. There has been a large increase in the number of applications of weak turbulence theory in the last fifteen years. References about these more recent developments can also be found in [**36**].

One of the most relevant mathematical results for the kinetic models of weak turbulence was the discovery by V. E. Zakharov of a class of stationary power law solutions for many models of weak turbulence. The earliest solutions of this class can be found in [**48**], [**49**]. Some of the solutions found in [**48**], [**49**] are just thermodynamic equilibria. These equilibria, take the form of a power law and they are usually termed as Rayleigh Jeans equilibria. However, some of the solutions found by V. E. Zakharov are characterized by the presence of fluxes of some physical magnitude (typically number of particles of energy) between different regions of the space k. From this point of view they have very strong analogies with the Kolmogorov solutions for the theory of turbulence in fluids, although in this last case the nonlinearities of the underlying problem (namely Euler's equations) are much stronger than in the case of weak turbulence. Due to this analogy, power law stationary solutions of kinetic models of weak turbulence which describe fluxes between different regions of the phase space are usually termed as Kolmogorov-Zakharov solutions. Some of the earliest examples of such type of solutions can be found in [**22**], [**52**]. Several other examples can be found in [**52**]. Methods to study linear stability for the Kolmogorov-Zakharov solutions in several models of weak turbulence can be found in [**4**].

One of the simplest, and most widely studied models of weak turbulence, is the one in which the underlying nonlinear wave equation is the nonlinear Schrödinger equation (cf. [**12**], [**36**], [**53**] and references therein). More precisely, the function $u = u(x,t) \in \mathbb{C}$ satisfies:

$$i\partial_t u = -\Delta u + \varepsilon |u|^2 u \quad , \quad u(0,\cdot) = u_0 \tag{1.1}$$

We will assume by definiteness that this problem is considered in $(t,x) \in \mathbb{R}^3 \times \mathbb{R}$. If $\varepsilon = 0$, equation (1.1) becomes the linear Schrödinger equation whose solutions are given by integrals of the form $\int_{\mathbb{R}^3} e^{i(kx-\omega t)} d\mu(k)$ for a large class of measures $\mu \in \mathcal{M}_+(\mathbb{R}^3)$ with $\omega = k^2$. Weak turbulence theory suggests that, for a choice of initial data u_0 according to a suitable class of probability measures homogeneous in space, the dynamics of the solutions of (1.1) for small ε can be obtained by means of the kinetic equation (cf. [**12**]):

$$\begin{aligned} \partial_t F_1 \quad &= \quad \frac{\varepsilon^2}{\pi} \iiint_{(\mathbb{R}^3)^3} \delta(k_1 + k_2 - k_3 - k_4)\, \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) \cdot \\ &\qquad \cdot \left[F_3 F_4 (F_1 + F_2) - F_1 F_2 (F_3 + F_4)\right] dk_2 dk_3 dk_4 \end{aligned} \tag{1.2}$$

with $\omega = \Omega(k) = k^2$, and where from now on we will use the notation $F_\ell = F(t, k_\ell)$, $\ell = 1,2,3,4$. Equation (1.2) is one of the most important examples of kinetic model arising in weak turbulence theory. It allows to understand some of the solutions of the nonlinear equation (1.1) in terms of particle collisions. Equation (1.2) has been extensively used to study problems in optical turbulence and Bose Einstein condensation (cf. [**12**], [**21**], [**26**], [**27, 28, 29**], [**36**], [**41**], [**44, 45**], [**46**], [**53**]).

In this paper we only consider the isotropic case of equation (1.2). The main reason for such a restriction is that it is not possible to give a meaning to the operator in the right-hand side of (1.2) in a unique way if F contains Dirac masses, as it was noticed in in [**32**] for an equation closely related, namely the Nordheim equation which will be discussed in Section 1.2 .

Suppose therefore that we look for solutions of (1.2) with the form: $F(t,k) = f(t,\omega)$, $\omega = k^2$. Then, after rescaling the time variable t in order to eliminate from the equation some constants we obtain that f solves:

$$\partial_t f_1 = \iint W \left[(f_1 + f_2) f_3 f_4 - (f_3 + f_4) f_1 f_2\right] d\omega_3 d\omega_4, \quad t > 0 \tag{1.3}$$

where $f_k = f(t, \omega_k)$, $k = 1,2,3,4$ and

$$W = \frac{\min\left\{\sqrt{\omega_1}, \sqrt{\omega_2}, \sqrt{\omega_3}, \sqrt{\omega_4}\right\}}{\sqrt{\omega_1}} \quad , \quad \omega_2 = \omega_3 + \omega_4 - \omega_1 \tag{1.4}$$

We are interested in the initial value problem associated to (1.3), (1.4). We will then assume that (1.3), (1.4) is solved with initial value $f_0(\omega)$, i.e.:

$$f(0,\omega) = f_0(\omega) \geq 0 \quad , \quad \omega \geq 0 \tag{1.5}$$

The function f is not a particle density in the space of frequencies ω, due to the presence of some nontrivial jacobians. A magnitude that is proportional to the density of particles in the space $\{\omega \geq 0\}$ is the function g defined by means of:

$$g = \sqrt{\omega}\, f \tag{1.6}$$

Then g solves:

$$\partial_t g_1 = \iint \Phi \left[\left(\frac{g_1}{\sqrt{\omega_1}} + \frac{g_2}{\sqrt{\omega_2}} \right) \frac{g_3 g_4}{\sqrt{\omega_3 \omega_4}} - \left(\frac{g_3}{\sqrt{\omega_3}} + \frac{g_4}{\sqrt{\omega_4}} \right) \frac{g_1 g_2}{\sqrt{\omega_1 \omega_2}} \right] d\omega_3 d\omega_4 \tag{1.7}$$

where:

$$\Phi = \min\left\{ \sqrt{\omega_1}, \sqrt{\omega_2}, \sqrt{\omega_3}, \sqrt{\omega_4} \right\} \tag{1.8}$$

$$g(0,\omega) = g_0(\omega) = \sqrt{\omega}\, f_0(\omega) \geq 0 \quad , \quad \omega \geq 0 \tag{1.9}$$

The integrations in (1.3), (1.7) are always restricted to the region

$$D(\omega_1) = \{\omega_3 \geq 0,\ \omega_4 \geq 0;\ \omega_3 + \omega_4 \geq \omega_1\},\ \ \omega_1 \geq 0.$$

In order to simplify the notation, we will assume in all the remainder of the paper that $\Phi = 0$ in $\mathbb{R}^4 \setminus [0,\infty)^4$.

As we already mentioned, the equations (1.1) and (1.2) have been used to study different questions related with the Bose Einstein condensation. In particular, on the basis of physical arguments, as well as formal asymptotics and numerical simulations, it has been accepted that, under some conditions, the solutions of equation (1.2) would contain, at least after some time, a Dirac mass at the origin (cf. [**12**], [**27, 28, 29**], [**41**], [**44, 45**]), a question that we will consider in some detail later. Let us only say here that, in the mentioned literature, this property is considered as reminiscent of the Bose Einstein condensation phenomena. Then, with some abuse of language, we will refer to the solutions of (1.2) that have a Dirac measure at the origin as solutions with a condensate.

1.1. Main results

The main goals of this paper are to study the Cauchy problem associated to (1.7)-(1.9) (or equivalently (1.3)-(1.5)), to obtain some of the qualitative behavior of the solutions and describe their long time asymptotic behavior. Although we prove several well-posedness results for initial data with infinite number of particles, i.e. $\int g_0 = \infty$, we restrict most of the analysis in this paper to the case where $\int g_0 < \infty$.

The stationary solutions of the equation (1.7)-(1.9) have been studied in the physics literature. On the other hand, the Cauchy problem for the Nordheim equation, (cf. Subsection 1.2), has been studied in [**13, 14**], [**15, 16**], [**32, 33, 34**].

The main results that are proved in this paper are the following:

1.1.1. Existence results & stationary solutions.

1.- Existence of bounded mild solutions of (1.3)-(1.5), i.e. solutions of the equation in the sense of the integral formulation which results from the Duhamel's formula. These solutions are locally defined in time for a large class of bounded initial data (cf. Theorem 2.16).

2.- Existence of global weak solutions of (1.3)-(1.5), i.e. solutions in the sense of distributions, globally defined in time for a large class of initial measures with total finite mass (cf. Theorem 2.18).

3. Characterisation of the weak stationary solutions of (1.7)-(1.8) with finite mass as the Dirac masses $g_{stat} = \delta_R$, with $R \geq 0$ (cf. Theorem 2.27).

1.1.2. Qualitative behavior of the solutions.

4. Characterization of the long time asymptotics of the weak solutions of (1.7)-(1.9) with finite mass $\int g(t, d\omega) < \infty$ in terms of the properties of the initial data g_0 (cf. Theorem 3.2). This result states that $g(t, \cdot) \rightharpoonup \delta_{R_*}$ with $R_* = \inf[\operatorname{supp} g_0]$.

5. Transport of the energy of the system towards $\omega \to \infty$. as $t \to \infty$, for a large class of weak solutions of (1.7)-(1.9) with finite energy the (cf. Corollary 3.9).

6. Optimal upper estimates for the rate of transport of the energy towards large values of ω for the solutions described in the point 8 (cf. Proposition 3.11).

7. If g is a weak solution of (1.7)-(1.9) globally defined in time and if we define $R_* = \inf[\operatorname{supp} g_0] = 0$, it is possible to show the following alternative: Either $\int_{\{0\}} g(t, d\omega) > 0$ for $t > t_*$, or the mass of g approaches towards $\omega = 0$ in a "pulsating manner" (cf. Theorem 3.13).

8. (Blow-up in finite time). Existence of solutions of (1.3)-(1.5) with initial data such that $\|f_0\|_{L^\infty([0,\infty))} < \infty$ for which $\limsup_{t\to T} \|f(t, \cdot)\|_{L^\infty([0,\infty))} = \infty$ for some $T < \infty$ (cf. Theorem 3.18).

9. There exist initial data g_0 such that the first alternative stated in the point 6 takes place (cf. Theorem 3.17). Moreover, there exist also initial data g_0 such that the second alternative stated in the point (7) takes place (cf. Theorem 4.1).

It may be useful to precise the meaning of the pulsating solutions mentioned in the point (7). Suppose that $R_* = 0$. Then, either there exists $t^* \in (0, \infty)$ such that $\int_{\{0\}} g(t, d\omega) > 0$ for $t > t_*$, or, alternatively, during most of the times, there exists $\rho = \rho(t) > 0$ such that $\frac{1}{\rho(t)} g\left(t, \frac{\cdot}{\rho(t)}\right)$ is close to the Dirac mass $M\delta_1(\cdot)$ in the weak topology. It turns out that the function $\rho(t)$ does not change its position continuously in general. On the contrary, we study in detail a class of initial data g_0 for which the function $\rho(t)$ can be shown to change by means of some jumps which take place at specific times (cf. Chapter 4). At those times g ceases being close to a Dirac mass and its mass spreads among a large set of values ω. After a transient time $g(t,\cdot)$ concentrates its mass again close to a Dirac mass whose position is closer to the origin than the previous one. This process is iterated infinitely many times as $g(t,\cdot)$ approaches to $M\delta_0(\cdot)$ as $t\to\infty$. During all this evolution the solution satisfies $\int_{\{0\}} g(t,d\omega)=0$.

The dynamics of the solutions of (1.7) has been extensively studied in the physics literature by means of physical simulations and formal asymptotic arguments. Two of the main questions that have been discussed are the finite time condensation and the asymptotic behaviour as $t\to+\infty$.

When considering the long time behavior for the solutions of kinetic equations of type (1.7) (or equivalently (1.3)) for gravity waves, it was seen in [**19**] that the Dirac masses where stationary solutions of the corresponding weak turbulence equation, but it was suggested that generic solutions should converge to the Rayleigh Jeans equilibria. In [**12**], using dimensional and scaling arguments, the authors indicate that as t tends to ∞, the solutions of (1.7), in presence of a condensate, converge towards a Dirac mass located at the origin containing the total number of particles. The same result is also described in [**41**] as well as in [**53**], where finite time condensation is also briefly described. Different scenarios of condensate formation in finite or infinite time have been discussed in [**23**],[**27, 28, 29**], [**47**]. It is now widely believed that a generic mechanism for the formation of a condensate is the one described in [**21**], [**26**], [**44**], [**45**]. In these papers, using numerical simulations and asymptotic arguments, it is shown how the condensate arises by means of a finite time blow up of the solutions of the equation (1.7). Near the blow up point the particle distribution f is given by a self similar solution of the second kind. Additional information about these issues may be found in See [**36**] Chapter 15.

Our results of points (3) and (4) above prove that all the weak solutions of (1.7)-(1.8), without flux at the origin and with finite mass, converge, in the weak sense of measures, to a Dirac delta containing all the mass of the solution and located at a suitable value of ω. This asymptotic behavior can take place either with the formation of condensate in finite time or without it. The results in points (7) and (9) show that both possibilities can take place.

It has been shown in [**50**] that the equation (1.3), (1.4) has two Kolmogorov-Zakharov solutions namely $f_1=\omega^{-7/6}$ and $f_2=\omega^{-3/2}$. The first, f_1, has a constant flux of particles towards the origin and a zero flux of energy. The second or f_2, has a constant flux of energy towards large values of ω and zero flux of particles, (cf [**12**] and [**50**]). However, since the integrals that define the fluxes for f_2 are divergent, we will only consider in this paper the solution f_1. It is nevertheless interesting to notice that the finite mass, zero flux weak solutions obtained in points (4) and (8) present both fluxes, in the directions predicted by these two Kolmogorov-Zakharov solutions. This behavior indicates a tendency of these solutions to transport particles towards small values of k. Since the energy is conserved in the particle collisions and the energy is reduced if $|k|$ is reduced, the inward particle flux must be compensated with an outward particle flux. The tendency to have these particle and energy fluxes will be made precise in this paper for isotropic solutions. We will derive in Section 3.2 estimates for the rate of transfer of energy towards infinity for particle distributions satisfying $\int g_0(d\omega)<\infty$. Some heuristic estimates about the characteristic time scales for the transfer of energy for arbitrary distributions g_0 are also discussed in Section 5.1.

It is interesting to compare the results concerning energy fluxes towards infinity with those obtained for the nonlinear Schrödinger equation obtained in the articles [**8**], [**24**] and [**25**]. The results in those papers show the existence of solutions of the NLS equation for which the energy can be transferred to large values of the frequency. The results in our paper concern just the kinetic approximation of the NLS equation, but prove rigorously the escape of the energy towards large values of $|k|$ as $t \to \infty$. In the absence of a precise rigorous results relating the solutions of the NLS equation and the corresponding kinetic theory of weak turbulence it is hard to precise the connections between both types of results.

1.2. Relation with the Nordheim equation

Several of the methods and results in this paper bear some analogies with those in [**15**] for the Nordheim equation. This equation, arises in the study of rarefied gases of quantum particles and takes the following form for homogeneous isotropic distributions:

$$\partial_t f_1 = \iint W \left[\left(1 + f_1 + f_2\right) f_3 f_4 - \left(1 + f_3 + f_4\right) f_1 f_2 \right] d\omega_3 d\omega_4 \tag{1.10}$$

$$W = \frac{\min\left\{\sqrt{\omega_1}, \sqrt{\omega_2}, \sqrt{\omega_3}, \sqrt{\omega_4}\right\}}{\sqrt{\omega_1}} \tag{1.11}$$

Equation (1.10), (1.11) differs from (1.3), (1.4) in the onset of the quadratic terms $f_3 f_4 - f_1 f_2$. These additional terms come from the use of the Bose Einstein statistics, instead of the classical, in the counting of the particles in the collisions.

The connection between the two equations (1.3), (1.10) has been already noticed by several authors, (cf. for example [**4**], [**26**], [**36**] and the references therein). It is suggested in particular that the cubic terms in (1.10) should be dominant in the limit of large occupation numbers. As a matter of fact, the results for the solutions of (1.3) (1.10) in points (1), (2) and (8) of Section (1.1) above, have also been obtained for the solutions of (1.10), (1.11) in [**15**] and their proofs are very similar.

1.3. Plan of the paper.

The plan of this paper is the following. Chapter 2 contains the definition of the different concepts of solutions of (1.7)-(1.9) which will be used in this paper, the relation between then and several well posedness results. Two main concepts of solutions will be used in this paper, namely mild solutions (i.e. solutions in the sense of the variation of constants formula), and weak solutions, which satisfy the equation in the sense of distributions. This chapter also contains a complete clasification of the stationary solutions with finite mass. We end Chapter 2 explaining how the Kolmogorov-Zakharov solutions and some related ones, fit into the framework of this paper. Chapter 3 describes several qualitative properties of the solutions of (1.7)-(1.9). We obtain a classification of all the possible long time asymptotics of one of the types of weak solutions that we have defined, namely those with interacting condensate and finite mass. We also derive some estimates for the rate of transfer of the energy towards large values of ω. We also prove a refined theorem concerning the long time asymptotic of the solutions which shows that if the mass concentrates at the origin, either a condensate appears in finite time or the solution exhibits a behavior that we will denote as pulsating. Finally we also prove in this chapter that solutions blow up in finite time. Chapter 4 contains a construction of a large family of initial data for which the solutions do not condensate in finite time but exhibit pulsating behavior as t goes to infinity. This is one of the most technical parts of the paper. Chapter 5 gives a description by means of heuristic arguments of how is the precise transfer of mass and energy for the different types of weak solutions considered in this paper. This chapter contains a list of open problems suggested by the results of this paper. Chapter 6 contains several results that are basically adaptation of previous results obtained in [**15**].

CHAPTER 2

Well-Posedness Results

We consider in this article different types of solutions for the equations (1.3), (1.4) and (1.7), (1.8). Some of them are measured valued solutions that do not solve the equations in classical form. Therefore, we need suitable concepts of generalized solutions for these equations.

An analysis of the physical literature shows that two different types of solutions of (1.3), (1.4) have been implicitly considered, depending on the interaction that is assumed between the condensate and the remaining particles of the system. For example, in [**26**], [**21**] it is assumed that there is no difference in the interactions between particles, whether they are or not in the condensate. On the contrary, for the Kolmogorov-Zakharov solutions, and related ones, it is implicitly assumed that the particles in the condensate do not interact with the remaining particles of the system. The difference between the two situations may be seen as reminiscent of the case of diffusive particles reaching a boundary, where either reflecting or absorbing boundary conditions can be imposed. Motivated by these two different situations we define two different types of weak solutions.

We will also use in this paper mild solutions of (1.3), (1.4) and (1.7), (1.8). They will be shown to be a subclass of weak solutions, and several of their properties will be studied later in this paper. Mild solutions of a regularized version of (1.7), (1.8) will be used as technical tool in one of the existence result for weak solutions.

We define:

$$\Phi_\sigma = \min\left\{\sqrt{(\omega_k-\sigma)_+}, k=1,2,3,4.\right\},\ \text{for}\ \sigma>0;\ \ \Phi_0=\Phi. \tag{2.1}$$

and introduce for further reference the analogous of equation (1.7) with the collision kernel Φ replaced by Φ_σ :

$$\partial_t g_1 = \iint \Phi_\sigma \left[\left(\frac{g_1}{\sqrt{\omega_1}}+\frac{g_2}{\sqrt{\omega_2}}\right)\frac{g_3g_4}{\sqrt{\omega_3\omega_4}} - \left(\frac{g_3}{\sqrt{\omega_3}}+\frac{g_4}{\sqrt{\omega_4}}\right)\frac{g_1g_2}{\sqrt{\omega_1\omega_2}}\right] d\omega_3 d\omega_4. \tag{2.2}$$

2.1. Weak solutions with interacting condensate.

The definition of weak solution that we introduce in this Section is similar to the one given for the Nordheim equation in [**32**].

We denote as $\mathcal{M}_+([0,\infty))$ the set of nonnegative Radon measures in $[0,\infty)$. Given $\rho\in\mathbb{R}$, we will denote as $\mathcal{M}_+([0,\infty):(1+\omega)^\rho)$ the set of measures $\mu\in\mathcal{M}_+([0,\infty))$ such that:

$$||\mu||_\rho = \sup_{R>0}\frac{1}{(1+R)^\rho}\frac{1}{R}\int_{\frac{R}{2}}^{R}\mu(d\omega) < +\infty.$$

We will use also the functional space $L_{+}^{\infty}\left(\mathbb{R}_{+}:\sqrt{\omega}\left(1+\omega\right)^{\rho-\frac{1}{2}}\right)$ which is the space of locally, nonnegative, bounded functions h such that:

$$||h||_{L_{+}^{\infty}\left(\mathbb{R}_{+}:\sqrt{\omega}(1+\omega)^{\rho-\frac{1}{2}}\right)}=\sup_{\omega>0}\frac{h\left(\omega\right)}{\sqrt{\omega}\left(1+\omega\right)^{\rho-\frac{1}{2}}}<\infty$$

REMARK 2.1. We will use also at several points in the arguments the fact that the space $\mathcal{M}_{+}\left([0,\infty):(1+\omega)^{\rho}\right)$ endowed with the weak topology is metrizable. We will denote the corresponding metric as $dist_{*}$. The dependence of this distance in ρ will not be written explicitly, since it will not play any role in the arguments.

We will assume in several of the results below that $\rho<-\frac{1}{2}$. This exponents corresponds to the slowest rate of decay which allows to define the integrals appearing in the definitions of the solutions in classical form. The typical behaviour of a function in $L_{+}^{\infty}\left(\mathbb{R}_{+}:\sqrt{\omega}\left(1+\omega\right)^{\rho-\frac{1}{2}}\right)$ with $\rho=-\frac{1}{2}$ is $g\left(\omega\right)\sim\frac{1}{\sqrt{\omega}}$ as $\omega\to\infty$ or equivalently $f\left(\omega\right)\sim\frac{1}{\omega}$ as $\omega\to\infty$. This corresponds to thermal equilibrium.

Notice that the range of powers $\rho<-\frac{1}{2}$ includes some functions g_{in} such that $\int_{0}^{\infty}g_{in}\left(d\omega\right)=\infty$. We will impose additional constraints on ρ if we need to consider solutions either with finite number of particles or finite energy.

DEFINITION 2.2. Given $\sigma\geq0$, and $\rho<-\frac{1}{2}$ we will say that the measure valued function $g\in C\left([0,T):\mathcal{M}_{+}\left([0,\infty):(1+\omega)^{\rho}\right)\right)$ is a weak solution of (2.2) with interacting condensate and with initial datum $g_{0}\in\mathcal{M}_{+}\left([0,\infty):(1+\omega)^{\rho}\right)$ if the following identity holds for any test function $\varphi\in C_{0}^{2}\left([0,T)\times[0,\infty)\right):$

$$\begin{aligned}\text{(2.3)}\quad &\int_{[0,\infty)}g\left(t_{*},\omega\right)\varphi\left(t_{*},\omega\right)d\omega-\int_{[0,\infty)}g_{0}\varphi\left(0,\omega\right)d\omega=\int_{0}^{t_{*}}\int_{[0,\infty)}g\partial_{t}\varphi d\omega dt+\\ &+\int_{0}^{t_{*}}\iiint_{[0,\infty)^{3}}\frac{g_{1}g_{2}g_{3}\Phi_{\sigma}}{\sqrt{\omega_{1}\omega_{2}\omega_{3}}}\times\\ &\qquad\times\left[\varphi\left(\omega_{1}+\omega_{2}-\omega_{3}\right)+\varphi\left(\omega_{3}\right)-\varphi\left(\omega_{1}\right)-\varphi\left(\omega_{2}\right)\right]d\omega_{1}d\omega_{2}d\omega_{3}dt\end{aligned}$$

for any $t_{*}\in[0,T)$.

REMARK 2.3. The reason to assume the condition $\rho<-\frac{1}{2}$, is to guarantee that the integrals on the right-hand side of (2.3) converge for large values of ω.

It is important to prove that the nonlinear operator in the last term of (2.3) is well defined for $g\in C\left([0,T):\mathcal{M}_{+}\left([0,\infty):(1+\omega)^{\rho}\right)\right),\ \rho<-1/2$. This is a consequence of the following Lemma.

LEMMA 2.4. *Suppose that* $\varphi\in C_{0}^{2}\left([0,\infty)\right)$. *Then, for all* $\sigma\in[0,1]$, *the functions defined by means of:*

$$\text{(2.4)}\quad \Delta_{\varphi,\sigma}\left(\omega_{1},\omega_{2},\omega_{3}\right)=\frac{\Phi_{\sigma}}{\sqrt{\omega_{1}\omega_{2}\omega_{3}}}\left[\varphi\left(\omega_{1}+\omega_{2}-\omega_{3}\right)+\varphi\left(\omega_{3}\right)-\varphi\left(\omega_{1}\right)-\varphi\left(\omega_{2}\right)\right]$$

with Φ_{σ} *as in (2.1), are uniformly continuous on compact subsets of* $(\omega_{1},\omega_{2},\omega_{3})\in[0,\infty)^{3},\ \omega_{4}=\omega_{1}+\omega_{2}-\omega_{3}$.

PROOF. We just need to prove uniform continuity of $\Delta_{\varphi,\sigma}$ near the boundary of $[0,\infty)^{3}$. We first derive an uniform estimate for $\Delta_{\varphi,\sigma}$ near the lines $\Gamma_{k,j}=\{\omega_{k}=\omega_{j}=0\ \ ,\ \ k\neq j\}$ for $k,j=1,2,3$. Using that $\Phi_{\sigma}\leq\Phi_{0}$ we obtain:

$$\text{(2.5)}\qquad \Delta_{\varphi,\sigma}\left(\omega_{1},\omega_{2},\omega_{3}\right)\leq\Delta_{\varphi,0}\left(\omega_{1},\omega_{2},\omega_{3}\right)$$

Notice that the line $\Gamma_{1,2}$ is contained in the set $\{\omega_3 \geq (\omega_1+\omega_2)\}$ where $\Phi_\sigma \leq \Phi_0 = 0$. Then $\Delta_{\varphi,\sigma}$ vanish in the set $\{\omega_3 \geq (\omega_1+\omega_2)\}$ and then, they are uniformly continuous there. We now examine the lines $\Gamma_{1,3},\ \Gamma_{2,3}$. Due to the symmetry of the functions $\Delta_{\varphi,\sigma}$ we can restrict to study to the line $\Gamma_{1,3}$. Suppose that $\omega_1 \leq \omega_2$, $\omega_3 \leq (\omega_1+\omega_2)$. We expand the term between brackets in (2.4) using Taylor at the point $(\omega_1,\omega_2,\omega_3) = (0,\omega_2,0)$. Then:

$$\Delta_{\varphi,0}(\omega_1,\omega_2,\omega_3) \leq \frac{\Phi_0}{\sqrt{\omega_1\omega_2\omega_3}} \times \left(\left|\varphi'(\omega_2)(\omega_1-\omega_3) - \varphi'(0)(\omega_1-\omega_3)\right| + C\left[(\omega_1)^2+(\omega_3)^2\right] \right) \tag{2.6}$$

for some constant $C>0$ depending only on φ and its derivatives, whence:

$$\Delta_{\varphi,0}(\omega_1,\omega_2,\omega_3) \leq \frac{C\Phi_0}{\sqrt{\omega_1\omega_2\omega_3}} \left(\omega_2\left|\omega_1-\omega_3\right| + (\omega_1)^2 + (\omega_3)^2 \right)$$

We now estimate Φ_0 by $\min\left\{\sqrt{\omega_1},\sqrt{\omega_3}\right\}$. Then:

$$\Delta_{\varphi,0}(\omega_1,\omega_2,\omega_3) \leq \frac{C}{\max\left\{\sqrt{\omega_1},\sqrt{\omega_3}\right\}\sqrt{\omega_2}} \left[\omega_2\left|\omega_1-\omega_3\right| + \left(\max\{\omega_1,\omega_3\}\right)^2 \right] \tag{2.7}$$

Combining (2.5), (2.7) we obtain the desired uniform convergence of the functions $\Delta_{\varphi,\sigma}$ in a neighbourhood of the line $\Gamma_{1,3}$, including the origin $(\omega_1,\omega_2,\omega_3) = (0,0,0)$. It only remains to obtain uniform continuity of the functions $\Delta_{\varphi,\sigma}$ in a neighbourhood of the planes $\Pi_k = \{\omega_k = 0\}$, $k=1,2,3$. This follows from the fact that after removing the neighbourhoods of the lines $\Gamma_{k,j},\ k,j = 1,2,3$ indicated above, we can restrict the analysis to points where at least two of the coordinates $\omega_j,\ j=1,2,3$, are bounded from below. Suppose that the remaining coordinate is ω_ℓ. The function $\frac{\Phi_\sigma}{\sqrt{\omega_\ell}}$ is then uniformly continuous in a neighbourhood of Π_k and the result follows. □

2.2. Weak solutions with non interacting condensate.

Although, most of the results that we will obtain in this paper are for weak solutions with interacting condensate, we wish to have a precise functional framework which allows to treat solutions that behave like the Kolmogorov-Zakharov solutions for small values of ω. We recall that the Kolmogorov-Zakharov solution $f_s(\omega) = K\omega^{-7/6}$, $g_s(\omega) = K\omega^{-2/3}$.

Some general properties of the solutions defined in this Section will be discussed in Section 2.7.

Definition 2.5. Given $\sigma \geq 0$, and $\rho < -\frac{1}{2}$ we will say that the measure valued function $g \in C\left([0,T) : \mathcal{M}_+\left([0,\infty) : (1+\omega)^\rho\right)\right)$ is a weak solution of (2.2) with non interacting condensate and with initial datum $g_0 \in \mathcal{M}_+\left([0,\infty) : (1+\omega)^\rho\right)$ if the following identity holds for any test function $\varphi \in C_0^2\left([0,T)\times[0,\infty)\right)$:

$$\begin{aligned} &\int_{[0,\infty)} g(t_*,\omega)\varphi(t_*,\omega)\,d\omega - \int_{[0,\infty)} g_0\varphi(0,\omega)\,d\omega = \int_0^{t_*}\int_{[0,\infty)} g\partial_t\varphi d\omega dt + \\ &+\int_0^{t_*}\iiint_{(0,\infty)^3} \frac{g_1g_2g_3\Phi_\sigma}{\sqrt{\omega_1\omega_2\omega_3}} \times \\ &\qquad\times\left[\varphi(\omega_1+\omega_2-\omega_3) + \varphi(\omega_3) - \varphi(\omega_1) - \varphi(\omega_2)\right] d\omega_1 d\omega_2 d\omega_3 dt \end{aligned} \tag{2.8}$$

for any $t_* \in [0,T)$.

Remark 2.6. The difference between Definitions 2.2 and 2.5 is extremely subtle. The domain of integration in the triple integral in the right hand side of (2.3) is $[0,+\infty)^3$, while the corresponding domain of integration in (2.8) is $(0,+\infty)^3$. The reason for this difference is to avoid, in the second case, any interaction between the particles in any possible condensate and the remaining particles. Notice also that as long as $\int_{\{0\}} g(t,d\omega)=0$, both Definitions are equivalent

2.3. Mild solutions.

We will use two different concepts of mild solutions, namely, measured valued mild solutions and bounded mild solutions. The idea behind these definitions is that they satisfy the equations in the sense of Duhamel's formula.

The reason to introduce the bounded mild solutions is due to our interest to prove finite time blow up in L^∞ norm for some of these solutions. Our interest in the measured valued mild solutions is twofold. First, we use them as technical tools in order to obtain global existence of weak solutions with interacting condensate in the sense of Definition (2.2). On the other hand, we construct a family of measured valued mild solutions with some peculiar asymptotic behavior (pulsating solutions) in Chapter 4.

We need to define some auxiliary functions.

Lemma 2.7. *Let $\rho<-\frac{1}{2}$. Suppose that either $g\in\mathcal{M}_+\left([0,\infty):(1+\omega)^\rho\right)$ and $\sigma>0$ or $g\in L_+^\infty\left(\mathbb{R}_+:\sqrt{\omega}\,(1+\omega)^{\rho-\frac{1}{2}}\right)$ and $\sigma\geq 0$. Then $A_\sigma(\omega_1)$ defined by means of*

$$A_\sigma(\omega_1)=-\iint\Phi_\sigma\left[\frac{2g_2g_3}{\sqrt{\omega_1\omega_2\omega_3}}-\frac{g_3g_4}{\sqrt{\omega_1\omega_3\omega_4}}\right]d\omega_3d\omega_4 \tag{2.9}$$

where $\omega_2=\omega_3+\omega_4-\omega_1$ defines a continuous function in $[0,\infty)$. Moreover, we have:

$$A_\sigma(\omega_1)\geq 0\quad,\quad\omega_1\in[0,\infty) \tag{2.10}$$

Remark 2.8. Given $g\in\mathcal{M}_+\left((1+\omega)^\rho\right)$, we define the measure:

$$\iint\Phi_\sigma\left[\frac{2g_2g_3}{\sqrt{\omega_1\omega_2\omega_3}}\right]d\omega_3d\omega_4$$

by means of its action over a test function $\varphi\in C_0^2([0,\infty))$, namely:

$$\begin{aligned}\int_0^\infty\varphi(\omega_1)\iint\Phi_\sigma\left[\frac{2g_2g_3}{\sqrt{\omega_1\omega_2\omega_3}}\right]d\omega_3d\omega_4d\omega_1=&\\ =\int_0^\infty\iint\Phi_\sigma\left[\frac{2g_2g_3\varphi(\omega_3+\omega_4-\omega_2)}{\sqrt{(\omega_3+\omega_4-\omega_2)\omega_2\omega_3}}\right]d\omega_3d\omega_4d\omega_2&\end{aligned} \tag{2.11}$$

Proof. Suppose first that $g\in\mathcal{M}_+\left([0,\infty):(1+\omega)^\rho\right)$ and $\sigma>0$. The function $\frac{\Phi}{\sqrt{\omega_1}}$ is continuous for $\omega_1>0$ and $(\omega_1,\omega_2)\in\mathbb{R}_+^2$. Moreover, $\Phi_\sigma=0$ if $\min\{\omega_1,\omega_2,\omega_3\}\leq\sigma$. Therefore, each of the terms $\frac{2\Phi_\sigma}{\sqrt{\omega_1\omega_2\omega_3}}$, $\frac{\Phi_\sigma}{\sqrt{\omega_1\omega_3\omega_4}}$ are bounded. Then, $\frac{2\Phi_\sigma g_2g_3}{\sqrt{\omega_1\omega_2\omega_3}}$ and $\frac{\Phi_\sigma g_3g_4}{\sqrt{\omega_1\omega_3\omega_4}}$ Radon measures in $\mathbb{R}_+^2$. Therefore, (2.9) defines a continuous function in $\{\omega_1>0\}$. Notice that the continuity at $\omega_1=0$ follows from the fact that $\Phi_\sigma=0$ if $\omega_1\leq\sigma$. Convergence of the integrals for large values of ω_3,ω_4 are a consequence of the fact that $g\in\mathcal{M}_+\left([0,\infty):(1+\omega)^\rho\right)$ with $\rho<-\frac{1}{2}$.

In order to prove (2.10) we rewrite $A(\omega_1)$ using that:

$$\iint \Phi \frac{2g_2g_3}{\sqrt{\omega_1\omega_2\omega_3}} d\omega_3 d\omega_4 = \iint \Phi \frac{g_2g_3}{\sqrt{\omega_1\omega_2\omega_3}} d\omega_3 d\omega_4 + \\ + \iint \Phi \frac{g_2g_4}{\sqrt{\omega_1\omega_2\omega_4}} d\omega_3 d\omega_4 \tag{2.12}$$

We now use the change of variables $\omega_2 = \omega_3 + \omega_4 - \omega_1$, $d\omega_2 = d\omega_4$ in the first integral and $\omega_2 = \omega_3 + \omega_4 - \omega_1$, $d\omega_2 = d\omega_3$ in the second one. Then, replacing the variable ω_2 by ω_4 in the first resulting integral and ω_2 by ω_3 in the second, we obtain that the integral in (2.12) becomes

$$\iint \frac{g_3g_4}{\sqrt{\omega_1\omega_3\omega_4}} \Psi d\omega_3 d\omega_4, \quad \Psi = \Psi_1 + \Psi_2$$

where:

$$\Psi_1 = \chi_{\{\omega_3 \geq \omega_4\}} \chi_{\{\omega_3 \geq \omega_1\}} \sqrt{(\omega_1 + \omega_4 - \omega_3)_+} + \chi_{\{\omega_3 \leq \omega_4\}} \chi_{\{\omega_3 \geq \omega_1\}} \sqrt{\omega_1} + \\ + \chi_{\{\omega_3 \geq \omega_4\}} \chi_{\{\omega_3 \leq \omega_1\}} \sqrt{\omega_4} + \chi_{\{\omega_3 \leq \omega_4\}} \chi_{\{\omega_3 \leq \omega_1\}} \sqrt{\omega_3}$$

$$\Psi_2 = \chi_{\{\omega_3 \leq \omega_4\}} \chi_{\{\omega_4 \geq \omega_1\}} \sqrt{(\omega_1 + \omega_3 - \omega_4)_+} + \chi_{\{\omega_3 \geq \omega_4\}} \chi_{\{\omega_4 \geq \omega_1\}} \sqrt{\omega_1} + \\ + \chi_{\{\omega_3 \leq \omega_4\}} \chi_{\{\omega_4 \leq \omega_1\}} \sqrt{\omega_3} + \chi_{\{\omega_3 \geq \omega_4\}} \chi_{\{\omega_4 \leq \omega_1\}} \sqrt{\omega_4}$$

Notice that $\Psi \geq \Phi$, whence (2.10) follows.

If $g \in L_+^\infty\left(\mathbb{R}_+ : \sqrt{\omega}(1+\omega)^{\rho - \frac{1}{2}}\right)$ we argue in a similar way. Notice that we can then assume that $\sigma = 0$ because the boundedness of g by $C\sqrt{\omega}$ for small ω implies the convergence of the integrals in (2.9). $\square$

LEMMA 2.9. *Let $\rho < -\frac{1}{2}$. Suppose that either $g \in \mathcal{M}_+([0,\infty) : (1+\omega)^\rho)$ and $\sigma > 0$ or $g \in L_+^\infty\left(\mathbb{R}_+ : \sqrt{\omega}(1+\omega)^{\rho-\frac{1}{2}}\right)$ and $\sigma \geq 0$. Then, the following formula defines mappings $\mathcal{O}_\sigma : \mathcal{M}_+([0,\infty) : (1+\omega)^\rho) \to \mathcal{M}_+([0,\infty) : (1+\omega)^\rho)$ and $\mathcal{O}_\sigma : L_+^\infty\left(\mathbb{R}_+ : \sqrt{\omega}(1+\omega)^{\rho-\frac{1}{2}}\right) \to L_+^\infty\left(\mathbb{R}_+ : \sqrt{\omega}(1+\omega)^{\rho-\frac{1}{2}}\right)$ respectively:*

$$\mathcal{O}_\sigma[g] = \iint \Phi_\sigma \frac{g_2g_3g_4}{\sqrt{\omega_2\omega_3\omega_4}} d\omega_3 d\omega_4 \ , \ \omega_2 = \omega_3 + \omega_4 - \omega_1 \tag{2.13}$$

where the action of the measure $\mathcal{O}_\sigma[g]$ acting over a test function $\varphi \in C_0([0,\infty))$ is given by:

$$\langle \mathcal{O}_\sigma[g], \varphi \rangle = \iiint \Phi_\sigma \frac{g_2g_3g_4}{\sqrt{\omega_2\omega_3\omega_4}} \varphi(\omega_1) d\omega_3 d\omega_4 d\omega_1 \tag{2.14}$$

If $g \in L_+^\infty\left(\mathbb{R}_+ : \sqrt{\omega}(1+\omega)^{\rho-\frac{1}{2}}\right)$ we can define directly $\mathcal{O}_\sigma[g]$ by means of the integration in (2.13) for any $\sigma \geq 0$.

PROOF. If $g \in \mathcal{M}_+([0,\infty) : (1+\omega)^\rho)$ and $\sigma > 0$ the function $\frac{\Phi_\sigma}{\sqrt{\omega_2\omega_3\omega_4}}$ is bounded and continuous in $[0,\infty)^3$. We can then compute the integral (2.14) which must be understood as:

$$\langle \mathcal{O}_\sigma[g], \varphi \rangle = \iiint \Phi_\sigma \frac{g_2g_3g_4}{\sqrt{\omega_2\omega_3\omega_4}} \varphi(\omega_3 + \omega_4 - \omega_2) d\omega_2 d\omega_3 d\omega_4 \tag{2.15}$$

Since the function $\varphi(\omega_3+\omega_4-\omega_1)$ is not compactly supported, we must examine carefully the convergence of the integral in (2.15). Not convergence problems arise for small $\omega_2,\ \omega_3,\ \omega_4$ due to the cutoff in Φ_σ. Since $g\in\mathcal{M}_+\left([0,\infty):(1+\omega)^\rho\right)$ with $\rho<-1$ and φ is bounded, we then obtain convergence of the integral in (2.15). It remains to prove that $\mathcal{O}_\sigma[g]\in\mathcal{M}_+\left([0,\infty):(1+\omega)^\rho\right)$. To this end we consider an increasing sequence of test functions $\{\varphi_n(\cdot)\}\subset C_0([0,\infty))$ and such that $\lim_{n\to\infty}\varphi_n(\omega)=(1+\omega)^\rho$ uniformly for ω in compact sets of $[0,\infty)$. Notice that, since the support of the functions $\varphi_n(\cdot)$ is contained in $\{\omega\geq 0\}$, we can restrict the integral in (2.15) to the set where $\omega_2\leq(\omega_3+\omega_4)$. Due to the symmetry under the permutation $\omega_3\longleftrightarrow\omega_4$ we can assume that $\omega_3\leq\omega_4$. Using the test functions $\varphi=\varphi_n$ as well as the fact that $\Phi_\sigma\leq\sqrt{\omega_2}$ we can estimate the integral on the right-hand side of (2.15) as:

$$C\int g_2d\omega_2\iint\frac{g_3g_4}{\sqrt{\omega_3\omega_4}}\left(1+(\omega_4)^{\rho-1}\right)d\omega_3d\omega_4<\infty$$

whence $\mathcal{O}_\sigma[g]\in\mathcal{M}_+\left([0,\infty):(1+\omega)^\rho\right)$ using Monotone Convergence.

If $g\in L_+^\infty\left(\mathbb{R}_+:\sqrt{\omega}\,(1+\omega)^{\rho-\frac{1}{2}}\right)$ and $\sigma\geq 0$ we obtain convergence for the integral in (2.13), even if $\sigma=0$, in the region where $\omega_2,\ \omega_3,\ \omega_4$ are smaller than one using the fact that $g(\omega)\leq C\sqrt{\omega}$ for $\omega\geq 0$. The convergence of the integrals for large values of ω follows from the fact that $g(\omega)\leq C\omega^{-\rho},\ \rho<-1$, if $\omega\geq 1$. In order to prove that $\mathcal{O}_\sigma[g]\in L_+^\infty\left(\mathbb{R}_+:\sqrt{\omega}\,(1+\omega)^{\rho-\frac{1}{2}}\right)$ we argue as follows. Since $\Phi_\sigma\leq\sqrt{\omega_1}$ and $g(\omega)\leq C\sqrt{\omega}$ for $\omega\geq 0$ we obtain:

$$\mathcal{O}_\sigma[g](\omega_1)=C\sqrt{\omega_1}\iint\frac{g_3g_4}{\sqrt{\omega_3\omega_4}}d\omega_3d\omega_4=C\sqrt{\omega_1}\left(\int\frac{g(\omega)}{\sqrt{\omega}}d\omega\right)^2$$

whence the estimate $\mathcal{O}_\sigma[g](\omega_1)\leq C\sqrt{\omega_1}$ for $\omega_1\leq 1$ follows. In order to obtain estimates for $\omega_1\geq 1$, notice that $\min\{\omega_3,\omega_4\}\geq\frac{\omega_1}{2}$. We can assume, without loss of generality that $\min\{\omega_3,\omega_4\}=\omega_3$ by means of a symmetrization argument. Then, using $\Phi_\sigma\leq\sqrt{\omega_2}$ in the region where $\omega_2\leq\omega_1$ and $\Phi_\sigma\leq\sqrt{\omega_1}$ if $\omega_2>\omega_1$ we would obtain:

$$\mathcal{O}_\sigma[g](\omega_1)\leq\iint_{\{\omega_2\leq\omega_1\}}\frac{g_2g_3g_4}{\sqrt{\omega_3\omega_4}}d\omega_3d\omega_4+\sqrt{\omega_1}\iint_{\{\omega_2>\omega_1\}}\frac{g_2g_3g_4}{\sqrt{\omega_2\omega_3\omega_4}}d\omega_3d\omega_4 \tag{2.16}$$

In order to estimate the first integral on the right we symmetrize the integral to have $\omega_4\geq\omega_3$. In this integral we have then ω_4 of order $\omega_1\geq 1$. Therefore:

$$\iint_{\{\omega_2\leq\omega_1\}}\frac{g_2g_3g_4}{\sqrt{\omega_3\omega_4}}d\omega_3d\omega_4\leq\frac{C}{(\omega_1)^{\frac{1}{2}+\rho}}\iint_{\{\omega_2\leq\omega_1\}}\frac{g_2g_3}{\sqrt{\omega_3}}d\omega_3d\omega_4$$

We can now change the variable ω_4 to ω_2 and integrate also ω_3 in the whole space. Both integrals are finite and we obtain the estimate:

$$\iint_{\{\omega_2\leq\omega_1\}}\frac{g_2g_3g_4}{\sqrt{\omega_3\omega_4}}d\omega_3d\omega_4\leq\frac{C}{(\omega_1)^{\frac{1}{2}+\rho}}\quad,\quad\omega_1\geq 1$$

We now estimate the second integral in (2.16). We can assume also by means of a symmetrization argument that $\omega_4\geq\omega_3$. Then $\omega_4\geq C\omega_1$. We also replace the integration in ω_4 by the integration in ω_2. Therefore

$$\begin{aligned}
&\sqrt{\omega_1}\iint_{\{\omega_2>\omega_1\}}\frac{g_2g_3g_4}{\sqrt{\omega_2\omega_3\omega_4}}d\omega_3 d\omega_4\\
&\leq \frac{C\sqrt{\omega_1}}{(\omega_1)^{\frac{1}{2}+\rho}}\iint_{\{\omega_2>\omega_1\}}\frac{g_2g_3}{\sqrt{\omega_2\omega_3}}d\omega_3 d\omega_2\\
&\leq \frac{C}{(\omega_1)^{\rho}}\frac{1}{(\omega_1)^{\rho-\frac{1}{2}}}\int\frac{g_3}{\sqrt{\omega_3}}d\omega_3 \leq \frac{C}{(\omega_1)^{2\rho-\frac{1}{2}}}
\end{aligned}$$

Therefore, we obtain, combining these estimates:

$$\mathcal{O}_\sigma[g](\omega_1)\leq C\min\left\{\sqrt{\omega_1},\frac{1}{(\omega_1)^{\frac{1}{2}+\rho}}\right\}$$

whence $\mathcal{O}_\sigma[g]\in L_+^\infty\left(\mathbb{R}_+:\sqrt{\omega}(1+\omega)^{\rho-\frac{1}{2}}\right)$. □

2.3.1. Measured valued mild solutions.

We can now define measured valued mild solutions for $\sigma>0$.

Definition 2.10. Let $\rho<-\frac{1}{2}$, $\sigma>0$ and $T\in(0,\infty]$. Given a measure $g_{in}\in\mathcal{M}_+([0,\infty):(1+\omega)^\rho)$ we will say that $g\in C([0,T]:\mathcal{M}_+([0,\infty):(1+\omega)^\rho))$ is a measured valued mild solution of (2.2) with initial value $g(\cdot,0)=g_{in}$ if the following identity holds in the sense of measures:

(2.17)

$$g(t,\cdot)=g_{in}(\cdot)\exp\left(-\int_0^t A_\sigma(s,\cdot)\,ds\right)+\int_0^t\exp\left(-\int_s^t A_\sigma(\xi,\cdot)\,d\xi\right)\mathcal{O}_\sigma[g](s,\cdot)\,ds$$

for $0\leq t<T$, where $A(\cdot,s)$ is defined as in Lemma 2.7 for each $g(\cdot,s)$ and $\mathcal{O}[g](\cdot,s)$ is defined as in Lemma 2.9 for each $g(\cdot,s)$.

Remark 2.11. The main reason to restrict the definition to $\sigma>0$ is because for $\sigma=0$ the operator $\mathcal{O}_\sigma[g]$ cannot be defined as a finite measure for arbitrary measures $g\in\mathcal{M}_+([0,\infty):(1+\omega)^\rho)$. The solutions defined here will be used as an auxiliary tool in order to construct global weak solutions of (2.2) for $\sigma=0$.

Remark 2.12. We understand by solutions in the sense of measures, solutions defined by means of their action over a test function i.e. (2.17) means:

$$\begin{aligned}
(2.18)\quad \int\varphi(\omega)\,g(t,d\omega)=&\int\varphi(\omega)\exp\left(-\int_0^t A_\sigma(s,\omega)\,ds\right)g_{in}(d\omega)+\\
&+\int_0^t\int\varphi(\omega)\exp\left(-\int_s^t A_\sigma(\xi,\omega)\,d\xi\right)\mathcal{O}_\sigma[g](s,d\omega)\,ds
\end{aligned}$$

for any $\varphi\in C_0([0,\infty))$. Notice that, since $A_\sigma(\cdot,s)$ is a continuous function all the terms in (2.18) are well defined.

2.3.2. Bounded mild solutions.

We now define solutions in the space of functions $C\left([0,T]:L_+^\infty\left(\mathbb{R}_+:\sqrt{\omega}(1+\omega)^{\rho-\frac{1}{2}}\right)\right)$.

Definition 2.13. Let $\rho<-\frac{1}{2}$, $\sigma\geq 0$ and $T\in(0,\infty]$. Given any $g_{in}\in L_+^\infty\left(\mathbb{R}_+:\sqrt{\omega}(1+\omega)^{\rho-\frac{1}{2}}\right)$ we say that $g\in C\left([0,T]:L_+^\infty\left(\mathbb{R}_+:\sqrt{\omega}(1+\omega)^{\rho-\frac{1}{2}}\right)\right)$

is a bounded mild solution of (2.2) with initial value $g(\cdot,0) = g_{in}$ if the following identity holds in the sense of measures:
(2.19)

$$g(t,\cdot) = g_{in}(\cdot)\exp\left(-\int_0^t A_\sigma(s,\cdot)\,ds\right) + \int_0^t \exp\left(-\int_s^t A_\sigma(\xi,\cdot)\,d\xi\right)\mathcal{O}_\sigma[g](s,\cdot)\,ds$$

for $0 \leq t < T$, where $A(s,\cdot)$ is defined as in Lemma 2.7 for each $g(\cdot,s)$ and $\mathcal{O}[g](s,\cdot)$ is defined as in Lemma 2.9 for each $g(s,\cdot)$.

Remark 2.14. Notice that, differently from Definition 2.10, in Definition 2.13 we allow σ to take the value 0.

2.3.3. Relation between the different concepts of solution. The relation between the different concepts of solution mentioned above is described in the following result.

Proposition 2.15. *(i) Suppose that $\sigma \geq 0$, $\rho < -\frac{1}{2}$ and that the function $g \in C\left([0,T] : L_+^\infty\left(\mathbb{R}_+ : \sqrt{\omega}(1+\omega)^{\rho-\frac{1}{2}}\right)\right)$ is a bounded mild solution of (2.2) in the sense of Definition 2.13. Then $g \in C([0,T] : \mathcal{M}_+([0,\infty) : (1+\omega)^\rho))$ and it is also a measured valued mild solution of (2.2) in the sense of Definition 2.10.*

(ii) Suppose that $\sigma > 0$, $\rho < -\frac{1}{2}$ and $g \in C([0,T] : \mathcal{M}_+([0,\infty) : (1+\omega)^\rho))$ is a measured valued mild solution of (2.2) in the sense of Definition 2.10. Then $g \in C([0,T) : \mathcal{M}_+([0,\infty)))$ and it is also a weak solution with interacting condensate of (2.2) in the sense of Definition 2.2.

Proof. The proof of (i) is immediate, because from the condition that $g \in C\left([0,T] : L_+^\infty\left(\mathbb{R}_+ : \sqrt{\omega}(1+\omega)^{\rho-\frac{1}{2}}\right)\right)$ it follows at once that one also has $g \in C([0,T] : \mathcal{M}_+([0,\infty) : (1+\omega)^\rho))$ and (2.19) implies (2.17) in the sense of measures (equivalently (2.18)).

In order to prove (ii), suppose now that $g \in C([0,T] : \mathcal{M}_+([0,\infty) : (1+\omega)^\rho))$ is a measured valued mild solution of (2.2). This implies the identity (2.18) for any $\varphi \in C_0([0,\infty))$. Using the regularity properties of g we can differentiate (2.18) for *a.e.* $t \in [0,T]$ and check that the following identity holds:

$$\partial_t\left(\int \varphi(t,\omega_1)\,g_1(t,d\omega_1)\right) = \int \partial_t\varphi(t,\omega_1)\,g_1(t,d\omega_1) +$$
$$+\iiint \Phi_\sigma\left[\left(\frac{g_1}{\sqrt{\omega_1}}+\frac{g_2}{\sqrt{\omega_2}}\right)\frac{g_3g_4}{\sqrt{\omega_3\omega_4}} - \left(\frac{g_3}{\sqrt{\omega_3}}+\frac{g_4}{\sqrt{\omega_4}}\right)\frac{g_1g_2}{\sqrt{\omega_1\omega_2}}\right]\varphi(\omega_1)\,d\omega_3 d\omega_4 d\omega_1$$

Symmetrizing the variables in the integrals of the right-hand side and integrating in the interval $[0,T]$ we obtain (2.3). □

2.4. Existence of bounded mild solutions.

We now prove the following result.

Theorem 2.16. *Let $\rho < -\frac{1}{2}$, $\sigma \geq 0$ be two given constants and uppose that $g_{in} \in L_+^\infty\left(\mathbb{R}^+; \sqrt{\omega}(1+\omega)^{\rho-\frac{1}{2}}\right)$. There exists $T > 0$, $T \leq \infty$, depending only on $\|g_0(\cdot)\|_{L_+^\infty\left(\mathbb{R}^+;\sqrt{\omega}(1+\omega)^{\rho-\frac{1}{2}}\right)}$ and σ, and a unique mild solution of (2.2) $g \in C\left([0,T) : L_+^\infty\left(\mathbb{R}_+ : \sqrt{\omega}(1+\omega)^{\rho-\frac{1}{2}}\right)\right)$ with initial value $g(\cdot,0) = g_{in}$ in the sense of Definition 2.13.*

If $\rho > 2$, the obtained solution g satisfies:

$$\int_0^\infty g_{in}(\omega)\,\omega d\omega = \int_0^\infty g(t,\omega)\,\omega d\omega\,, \quad t\in(0,T)\ . \tag{2.20}$$

If $\sigma > 0$ we have $T = \infty$.

If $\sigma = 0$, the function f is in the space $W^{1,\infty}\left((0,T);L^{\infty}(\mathbb{R}^+)\right)$ and it satisfies (1.3) a.e. $\omega\in\mathbb{R}^+$ for any $t\in(0,T_{\max})$. Moreover, f can be extended as a mild solution of (1.3)-(1.6) to a maximal time interval $(0,T_{\max})$ with $0<T_{\max}\leq\infty$. If $T_{\max}<\infty$ we have:

$$\lim\sup_{t\to T_{\max}^-}\|f(t,\cdot)\|_{L^\infty(\mathbb{R}^+)} = \infty.$$

Proof. The proof of this result can be obtained in the same manner a the Proof of Theorem 3.4 in [**15**]. The key idea is to interpret mild solutions as a fixed point for an operator $\mathcal{T}[g]$ which we define as the right-hand side of (2.19). Some of the main technical difficulties in the Proof of Theorem 3.4 in [**15**] are due to the need to control quadratic terms in f which appear in the Nordheim equation, but which are not present in $\mathcal{O}_\sigma[g]$. We only need to estimate then cubic terms and this allows to obtain well-posedness results for a larger range of exponents than the one obtained in [**15**] (namely $\rho<-\frac{1}{2}$).

The key estimates needed to implement the fixed point argument are the following ones:

$$\omega^\rho\mathcal{O}_\sigma[g](\omega) \leq \frac{C}{\omega^{\min\left\{\rho-\frac{1}{2},\frac{1}{2}\right\}}}\|g\|^3_{L^\infty_+\left(\mathbb{R}^+;\sqrt{\omega}(1+\omega)^{\rho-\frac{1}{2}}\right)} \quad,\quad \omega\geq 1 \tag{2.21}$$

$$0\leq A_\sigma(\omega)\leq C\|g\|^2_{L^\infty_+\left(\mathbb{R}^+;\sqrt{\omega}(1+\omega)^{\rho-\frac{1}{2}}\right)} \quad,\quad \omega\geq 0 \tag{2.22}$$

where $\mathcal{O}_\sigma[g]\,,\ A_\sigma$ are as in (2.9), (2.13) and C depends in ρ. In order to derive (2.22) we just notice that:

$$\begin{aligned} A_\sigma(\omega) &\leq \|g\|^2_{L^\infty_+\left(\mathbb{R}^+;\sqrt{\omega}(1+\omega)^{\rho-\frac{1}{2}}\right)}\int_0^\infty\int_0^\infty\frac{d\omega_3 d\omega_4}{(1+\omega_3)^{\rho+\frac{1}{2}}(1+\omega_4)^{\rho+\frac{1}{2}}} \\ &\leq C\|g\|^2_{L^\infty_+\left(\mathbb{R}^+;\sqrt{\omega}(1+\omega)^{\rho-\frac{1}{2}}\right)} \end{aligned}$$

To prove (2.21) we use the fact that we need to integrate only in the domain $\{\omega_3\geq 0,\ \omega_4\geq 0\ ,\ \omega_2\geq 0\}$, with ω_2 as in (2.13). We split the region in the sets $D_I = \{0\leq\max\{\omega_3,\omega_4\}\leq\omega_1\}\,,\ D_{II} = \{\omega_1\leq\omega_3<\infty,\ 0\leq\omega_4<\omega_1\}\,,\ D_{III} = \{\omega_1\leq\omega_4<\infty,\ 0\leq\omega_3<\omega_1\}$ and $D_{IV} = \{\omega_1\leq\omega_3,\ \omega_1\leq\omega_4\}$. We then split the integrals which define the operator $\mathcal{O}_\sigma[g]$ in the four domains. Notice that, since $g_2\leq\|g\|_{L^\infty_+\left(\mathbb{R}^+;\sqrt{\omega}(1+\omega)^{\rho-\frac{1}{2}}\right)}\frac{1}{\omega_1^\rho}$ in D_{IV} :

$$\iint_{D_{IV}}[\cdots]\leq\frac{\|g\|^3_{L^\infty_+\left(\mathbb{R}^+;\sqrt{\omega}(1+\omega)^{\rho-\frac{1}{2}}\right)}}{\omega_1^\rho}\int_{\omega_1}^\infty\int_{\omega_1}^\infty\frac{d\omega_3 d\omega_4}{(1+\omega_3)^{\rho+\frac{1}{2}}(1+\omega_4)^{\rho+\frac{1}{2}}}$$

and the term on the right-hand side can be estimated by the right-hand side of (2.21). On the other hand, using that $\omega_3\geq\omega_1$ in D_{II} we obtain:

$$\iint_{D_{II}}[\cdots]\leq\frac{\|g\|^3_{L^\infty_+\left(\mathbb{R}^+;\sqrt{\omega}(1+\omega)^{\rho-\frac{1}{2}}\right)}}{\omega_1^{\rho+\frac{1}{2}}}\int_{\omega_1}^\infty\int_0^{\omega_1}\frac{d\omega_3 d\omega_4}{(1+(\omega_3+\omega_4-\omega_1))^{\rho+\frac{1}{2}}(1+\omega_4)^{\rho+\frac{1}{2}}}$$

Using that $(\omega_3+\omega_4-\omega_1) \geq (\omega_3-\omega_1)$ we can estimate the integral on the right-hand side by the product of two integrals which can be bounded by the right-hand side of (2.21). The estimate of the integral in the domain D_{III} is similar.

Finally we estimate the integral in the domain D_I. Due to the symmetry of the integral it is enough to estimate the integrals in the region $\{\omega_3 \geq \omega_4\}$. We then have $\omega_3 \geq \frac{\omega_1}{2}$. We also change variables in order to replace the integration in ω_3 by integration in ω_2. Then:

$$\iint_{D_I} [\cdots] \leq \frac{\|g\|^3_{L^\infty_+\left(\mathbb{R}^+;\sqrt{\omega}(1+\omega)^{\rho-\frac{1}{2}}\right)}}{\omega_1^{\rho+\frac{1}{2}}} \int_0^{\omega_1}\int_0^{\omega_1} \frac{d\omega_2 d\omega_4}{(1+(\omega_2))^{\rho+\frac{1}{2}}(1+\omega_4)^{\rho+\frac{1}{2}}}$$

and this gives (2.21).

The rest of the fixed point argument can be made along the lines of the Proof of Theorem 3.4 in [**15**]. The fact that for $\sigma > 0$ the solutions can be extended to arbitrarily large values of t follows just from the boundedness of the function $\frac{\Phi_\sigma}{\sqrt{\omega_1\omega_2\omega_3}}$. □

2.5. Existence of global weak solutions with interacting condensate.

In order to prove well-posedness of measured valued weak solutions , we will restrict our analysis to integrable distributions g given that we will consider the long time asymptotics of the solutions only in this case. More precisely, we will assume that the initial data $g_{in} \in \mathcal{M}_+([0,\infty):(1+\omega)^\rho)$ with $\rho < -1$, and the resulting solutions $g(t,\cdot)$ will be shown to be in the same space for $t \geq 0$. The reason to assume that $\rho < -1$ is that we use in the argument yielding global existence the finiteness of $\int g(t,d\omega)$. It is likely that global weak solutions could be obtained just with the assumption $\rho < -\frac{1}{2}$ but the proof would require a careful study of the transfer of mass taking place at the region $\omega \to \infty$. Therefore, this case will not be considered in this paper. Notice that due to the cubic nonlinearity of the problem a simple Gronwall argument does not allow to obtain global existence, in spite of the fact that the operator $\mathcal{O}_\sigma[g]$ is defined for any $g \in \mathcal{M}_+([0,\infty):(1+\omega)^\rho)$, $\rho < -1$.

REMARK 2.17. On the other hand, we remark that the exponent $\rho = -\frac{1}{2}$ is in a suitable sense optimal. Indeed, the operator $\mathcal{O}_\sigma[g]$ cannot be defined in general for $g \in \mathcal{M}_+([0,\infty):(1+\omega)^\rho)$ with $\rho \geq -\frac{1}{2}$.

As a next step we prove a global existence Theorem of weak solutions for (1.7). We use an idea similar to the one in [**32**] for the Nordheim equation. We first regularize the problem using the kernels Φ_σ with $\sigma > 0$. It is possible to obtain global weak solutions in that case, just using the fact that mild solutions are weak solutions. Finally we take the limit $\sigma \to 0$. The available estimates for the solutions g_σ will allow to prove that the limit exists and it yields a global weak solution of (1.7). The main result that we prove in this Section is the following.

THEOREM 2.18. *Let $-2 < \rho < -1$ and $g_{in} \in \mathcal{M}_+([0,\infty):(1+\omega)^\rho)$. There exists a weak solution of (1.7) in the sense of Definition 2.2 with initial datum g_{in}.*

REMARK 2.19. Notice that if $\rho < -1$ taking a sequence of test functions φ_n which converge uniformly to one as $n \to \infty$, we can prove the following identity for any weak solution of (1.7) in the sense of Definition 2.2:

$$\int g_{in}(d\omega) = \int g(t,d\omega) \quad , \quad a.e.\ t \geq 0 \tag{2.23}$$

Moreover, if $g_{in} \in \mathcal{M}_+ ([0,\infty) : (1+\omega)^{\rho})$ with $\rho < -2$ we would obtain, using a similar argument that:

$$\int \omega g_{in} (d\omega) = \int \omega g (t, d\omega) \tag{2.24}$$

We split the proof of Theorem (2.18) in the different Subsections

2.5.1. Global measured valued weak solutions for the problem with regularized kernel $\Phi_\sigma,\ \sigma > 0$. As a first step we prove the existence of global weak solutions for the regularized problem (2.2) with $\sigma > 0$.

Lemma 2.20. *Let $\rho < -1,\ \sigma > 0$ and $g_{in} \in \mathcal{M}_+ ([0,\infty) : (1+\omega)^{\rho})$. Then, there exist $g_\sigma \in C([0,\infty) : \mathcal{M}_+ ([0,\infty) : (1+\omega)^{\rho}))$ which is a global weak solution of (2.2) in the sense of Definition 2.2 and satisfies $g(0,\cdot) = g_{in}$.*

Proof. We first construct a global mild solution in the sense of Definition 2.10. The main idea for this construction is to reformulate (2.19) as a fixed point Theorem. Given $T > 0$, we define the following operator

$$\mathcal{T}_\sigma : C([0,T] : \mathcal{M}_+ ([0,\infty) : (1+\omega)^{\rho})) \to C([0,T] : \mathcal{M}_+ ([0,\infty) : (1+\omega)^{\rho}))$$

by means of:

$$\mathcal{T}_\sigma [g] (t,\cdot) = g_{in} (\cdot) \exp\left(-\int_0^t A_\sigma (s,\cdot)\, ds \right) + \\ + \int_0^t \exp\left(-\int_s^t A_\sigma (\xi,\cdot)\, d\xi \right) \mathcal{O}_\sigma [g] (s,\cdot)\, ds$$

We now claim that the operators $g \to A_\sigma,\ g \to \mathcal{O}_\sigma [g]$ are continuous if we endow $C([0,T] : \mathcal{M}_+ ([0,\infty) : (1+\omega)^{\rho}))$ with the topology induced by the metric:

$$dist (g_1, g_2) = \sup_{0 \le t \le T} dist_* (g_1 (t,\cdot), g_2 (t,\cdot))$$

where $dist_*$ is as in Notation 2.1.

We then need to prove that, given any test function $\varphi \in C_0 ([0,T) \times [0,\infty))$, the following functions depend continuously on g in the weak topology:

$$I_1 [g] = \int g_{in} (\omega) \exp\left(-\int_0^t A_\sigma (s,\omega)\, ds \right) \varphi (\omega)\, d\omega$$

$$I_2 [g] = \iint_0^t \exp\left(-\int_s^t A_\sigma (\xi,\cdot)\, d\xi \right) \mathcal{O}_\sigma [g] (s,\cdot)\, \varphi (\omega)\, ds d\omega$$

We now notice that since $\sigma > 0$ the mapping $g(\cdot,t) \to A_\sigma (t,\cdot)$ defines a continuous map between $\mathcal{M}_+ ([0,\infty) : (1+\omega)^{\rho})$, endowed with the weak topology, and the set of continuous bounded functions $C_b ([0,\infty))$, endowed with the uniform convergence. This is due to the fact that, since $\sigma > 0$ the functions $\frac{\Phi_\sigma}{\sqrt{\omega_1 \omega_2 \omega_3}},\ \frac{\Phi_\sigma}{\sqrt{\omega_1 \omega_3 \omega_4}}$ are smooth, the values of $A_\sigma (t,\omega_1)$ depend only on g through integral quantities, and the decay of the measure g for large values implies that the contribution of the large values of ω can be made small.

It follows, that the operator $g \to A_\sigma$ from $C([0,T] : \mathcal{M}_+ ([0,\infty) : (1+\omega)^{\rho}))$ to $C([0,T] : C_b ([0,\infty)))$ is continuous. Moreover, since $A_\sigma \ge 0$ for $g \ge 0$ it then follows that the operator $I_1 [g]$, which maps $C([0,T] : \mathcal{M}_+ ([0,\infty) : (1+\omega)^{\rho}))$ to itself, is continuous.

On the other hand, the operator $\mathcal{O}_\sigma[g]$ defined in Lemma 2.9 is a continuous operator from $\mathcal{M}_+([0,\infty):(1+\omega)^\rho)$ to itself if this space is endowed with the weak topology if $\sigma>0$. The proof of this uses the fact that the integrals in (2.15) are well defined as it can be seen from the arguments in the Proof of Lemma 2.9. On the other hand the boundedness of $\frac{\Phi_\sigma}{\sqrt{\omega_2\omega_3\omega_4}}$ implies that the functional $\mathcal{O}_\sigma[g]$ depends continuously on convergent integrals of g. The continuity of the functional $g\to I_2[g]$ from $C([0,T):\mathcal{M}_+([0,\infty):(1+\omega)^\rho))$, follows then similarly. Therefore the transformation $\mathcal{T}_\sigma[g]$ defines a continuous mapping from $C([0,T):\mathcal{M}_+([0,\infty):(1+\omega)^\rho))$ to itself if this space is endowed with the weak topology. Moreover, this operator transforms the set

$$\mathcal{Y}_T=\left\{g\in C([0,T):\mathcal{M}_+([0,\infty):(1+\omega)^\rho));|||g|||_{\rho,T}\le 2\|g_{in}\|_\rho\right\} \tag{2.25}$$

$$|||g|||_{\rho,T}=\sup_{0\le t\le T}||g(t)||_\rho \tag{2.26}$$

into itself if T is sufficiently small.

Actually the operator $\mathcal{T}_\sigma[g]$ is compact in the set $\mathcal{Y}_T$. This a consequence of Arzela-Ascoli Theorem in metric spaces (cf. [**10**]), as well as the fact that the set $\left\{g\in\mathcal{M}_+([0,\infty):(1+\omega)^\rho):\sup_{0\le t\le T}\frac{1}{[1+R^\rho]}\frac{1}{R}\int_{\frac{R}{2}}^R g(t,d\omega)\le 2\|g_{in}\|\right\}$ is compact in $\mathcal{M}_+([0,\infty):(1+\omega)^\rho)$ endowed with the weak topology. The uniform continuity of $\mathcal{T}_\sigma[g]$ with respect to the time variable follows from the fact that the functions $t\to\Psi_\varphi(t)=\int\varphi(\omega)\mathcal{T}_\sigma[g](d\omega,t)$ is Lifschitz continuous for any test function φ, as it can be seen from the definition of $\mathcal{T}_\sigma[g]$.

Local existence of solutions then follows using Schauder's Theorem. Notice that, since $\sigma>0$ we can obtain that the corresponding fixed point, $g(t,\cdot)$ satisfies:

$$||g(t)||_\rho\le C_1+C_2\int_0^t||g(s)||_\rho ds$$

where C_1 only depends on σ and $||g_{in}||_\rho$ and C_2 only depends on σ and the total mass of $g(t)$ which is a constant and therefore depends on total mass of g_{in}. This is proved as follows. Integrating equation (2.19) in the interval $(R,2R)$ for any $R>0$, the first term is immediately estimated using $||g_{in}||_\rho$. The integral of the second is estimated by splitting the domain in the subdomains $\{\omega_3+\omega_4\ge 4R\}$ and $\{\omega_3+\omega_4\le 4R\}$. Since $\sigma>0$ the term $(\omega_2\omega_3\omega_4)^{-1/2}$ is bounded by a constant depending on σ. In the two resulting triple integrals, one of the integrations takes place in the interval $(R/4,4R)$, and the two others are estimated by the total mass of $g(t)$, that is constant. Using Gronwall's lemma we deduce, $|||g|||_{\rho,T}\le C(T)$ for any finite T. Iterating the construction it is then possible to prove that the solution is global in time.

In order to conclude the proof of the Lemma, we just notice that mild solutions of (2.2) in the sense of measures are weak solutions of (2.2) in the sense of Definition 2.2 due to Proposition 2.15. □

2.5.2. Monotonicity formula. The following result is analogous to one that has been proved in [**15**], [**34**].

Proposition 2.21. *Let $\sigma\ge 0$. Given $g\in\mathcal{M}_+([0,\infty):(1+\omega)^\rho)$ we define:*

$$q[g](\omega_1)=\left(\frac{g_1}{\sqrt{\omega_1}}+\frac{g_2}{\sqrt{\omega_2}}\right)\frac{g_3g_4}{\sqrt{\omega_3\omega_4}}-\left(\frac{g_3}{\sqrt{\omega_3}}+\frac{g_4}{\sqrt{\omega_4}}\right)\frac{g_1g_2}{\sqrt{\omega_1\omega_2}}$$

with $\omega_2 = \omega_3 + \omega_4 - \omega_1$. Let us denote as $\mathcal{S}^3$ the group of permutations of the three elements $\{1,2,3\}$. Suppose that $\varphi \in C_0^2([0,\infty))$ is a test function. The following identity holds:

$$\int_{[0,\infty)^3} d\omega_1 d\omega_3 d\omega_4 \ \Phi_\sigma q[g](\omega_1)\sqrt{\omega_1}\varphi(\omega_1) = \int_{[0,\infty)^3} d\omega_1 d\omega_2 d\omega_3 \ \frac{g_1\, g_2\, g_3}{\sqrt{\omega_1\omega_2\omega_3}}\mathcal{G}_{\sigma,\varphi} \tag{2.27}$$

where:

$$\mathcal{G}_{\sigma,\varphi} \equiv \mathcal{G}_{\sigma,\varphi}(\omega_1,\omega_2,\omega_3) = \frac{1}{6}\sum_{\sigma\in\mathcal{S}^3} H_\varphi\left(\omega_{\sigma(1)},\omega_{\sigma(2)},\omega_{\sigma(3)}\right)\Phi_\sigma\left(\omega_{\sigma(1)},\omega_{\sigma(2)};\omega_{\sigma(3)}\right)$$

$$H_\varphi(x,y,z) = \varphi(z) + \varphi(x+y-z) - \varphi(x) - \varphi(y)$$

with $\Phi_\sigma(\omega_1,\omega_2;\omega_3)$ given as:

$$\Phi_\sigma(\omega_1,\omega_2;\omega_3) = \min\left\{\sqrt{(\omega_1-\sigma)_+}, \sqrt{(\omega_2-\sigma)_+}, \sqrt{(\omega_3-\sigma)_+}, \sqrt{(\omega_1+\omega_2-\omega_3-\sigma)_+}\right\}$$

and:

$$\mathcal{G}_{\sigma,\varphi}(\omega_1,\omega_2,\omega_3) = \mathcal{G}_{\sigma,\varphi}\left(\omega_{\sigma(1)},\omega_{\sigma(2)},\omega_{\sigma(3)}\right) \quad \text{for any } \sigma\in\mathcal{S}^3 \tag{2.28}$$

Moreover, if the function φ is convex we have $\mathcal{G}_\varphi(\omega_1,\omega_2,\omega_3) \geq 0$ and if φ is concave we have $\mathcal{G}_\varphi(\omega_1,\omega_2,\omega_3) \leq 0$. For any test function φ the function $\mathcal{G}_\varphi(\omega_1,\omega_2,\omega_3)$ vanishes along the diagonal $\left\{(\omega_1,\omega_2,\omega_3)\in[0,\infty)^3 : \omega_1=\omega_2=\omega_3\right\}$.

Proof. It is essentially identical to the Proof of Proposition 4.1 of [**15**]. The only difference is that we use Φ_σ instead of Φ. However, using the fact that Φ_σ is invariant under permutations in their variables, we can argue exactly as in the Proof of Proposition 4.1 of [**15**] by means of a symmetrization argument. The only relevant difference with the result in [**15**] is that due to the fact that g are measures, we must check the continuity of the functions which are integrated against them. This follows from Lemma 2.4. ☐

We will need later a more detailed representation formula for the functions $\mathcal{G}_{\sigma,\varphi}$ in the case $\sigma=0$. To this end we define the following functions which have been used also in [**15**].

Definition 2.22. We define auxiliary functions $\omega_+,\ \omega_0,\ \omega_-$ from $[0,\infty)\times[0,\infty)\times[0,\infty)$ to $[0,\infty)$ as follows:

$$\omega_+(\omega_1,\omega_2,\omega_3) = \max\{\omega_1,\omega_2,\omega_3\},$$
$$\omega_-(\omega_1,\omega_2,\omega_3) = \min\{\omega_1,\omega_2,\omega_3\},$$
$$\omega_0(\omega_1,\omega_2,\omega_3) = \omega_k \in \{\omega_1,\omega_2,\omega_3\}\setminus\{\omega_+,\omega_-\}$$

with $k\in\{1,2,3\}$, where we will assume that the set $\{\omega_1,\omega_2,\omega_3\}$ has three different elements even if some of the values of the elements ω_j are identical.

Lemma 2.23. *The function $\mathcal{G}_{0,\varphi}$ defined in Proposition 2.21 can be written as:*

$$\mathcal{G}_{0,\varphi}(\omega_1,\omega_2,\omega_3) = \frac{1}{3}\left[\sqrt{\omega_-}H_\varphi^1(\omega_1,\omega_2,\omega_3) + \sqrt{(\omega_0+\omega_--\omega_+)_+}H_\varphi^2(\omega_1,\omega_2,\omega_3)\right]$$

$$H_\varphi^1(\omega_1,\omega_2,\omega_3) = \varphi(\omega_++\omega_--\omega_0) + \varphi(\omega_++\omega_0-\omega_-) - 2\varphi(\omega_+)$$
$$H_\varphi^2(\omega_1,\omega_2,\omega_3) = \varphi(\omega_+) + \varphi(\omega_0+\omega_--\omega_+) - \varphi(\omega_0) - \varphi(\omega_-)$$

If φ is concave both functions H_{φ}^{1}, H_{φ}^{2} are nonpositive.

Proof. This result has been proved in [**15**]. $\square$

Using Proposition 2.21 we can prove the following result.

Lemma 2.24. *For all $\sigma \geq 0$ let $g_{\sigma} \in C\left([0,\infty) : L_{+}^{\infty}\left(\mathbb{R}_{+} : \sqrt{\omega}\,(1+\omega)^{\rho-\frac{1}{2}}\right)\right)$ be a weak solution of (2.2) in the sense of Definition 2.2. Let $\varphi \in C([0,\infty))$ any convex function. Then:*

$$\frac{d}{dt}\left(\int_{0}^{\infty} g_{\sigma}(t,\omega)\,\varphi(\omega)\,d\omega\right) \geq 0, \quad a.e.\ t \in [0,\infty).$$

Proof. It is just a consequence of Proposition 2.21 as well as the identity:

$$\frac{d}{dt}\left(\int_{0}^{\infty} g(t,\omega)\,\varphi(\omega)\,d\omega\right) = \int_{0}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} \frac{g_1 g_2 g_3}{\sqrt{\omega_1\omega_2\omega_3}} \mathcal{G}_{0,\varphi}(\omega_1,\omega_2,\omega_3)\,d\omega_1 d\omega_2 d\omega_3 \tag{2.29}$$

a.e. $t \in [0,\infty)$. $\square$

2.5.3. Tightness of the measures $\{g_{\sigma}\}$. The following result will be used several times in the following in order to prove that the mass of the measures g_{σ} cannot escape too far away. In particular, since the Lemma provides uniform estimates in σ of the mass far away from the origin, it will play a crucial role taking the limit $\sigma \to 0$, in order to prove the existence of weak solutions of of (2.2) with $\sigma = 0$.

Lemma 2.25. *Suppose that $g_{\sigma} \in C\left([0,\infty) : \mathcal{M}_{+}\left([0,\infty) : (1+\omega)^{\rho}\right)\right),\ \rho < -\frac{1}{2}$ is a weak solution of (2.2) in the sense of Definition 2.2 for some $\sigma \geq 0$. Let $\eta > 0,\ R > 0$. Suppose that $g_{\sigma}(0,\cdot) = g_{in}(\cdot)$. Then:*

$$\int_{[0,L]} g_{\sigma}(t,d\omega) \geq (1-\eta)\int_{[0,R]} g_{in}(d\omega) \quad , \quad t \in [0,T] \tag{2.30}$$

where $L = \frac{R}{\eta}$.

Moreover, suppose that g_{in} satisfies $\int g_{in} = 1$ and $\int_{R}^{\infty} g_{in}(d\omega) \leq AR^{\rho+1}$, for some $-2 < \rho < -1$, $A > 0$ and any $R \geq 1$. Then:

$$\int_{0}^{R} g_{\sigma}(t,d\omega) \geq 1 - \frac{1}{R} - \frac{AR^{\rho+1}}{(\rho+2)} \quad , \quad R \geq 1 \tag{2.31}$$

Proof. We use the following test function:

$$\varphi(\omega) = (1-K\omega)_{+}$$

where $K > 0$ is a constant to be precised. The function φ is convex. Applying Proposition 2.21 and Lemma 2.24 it follows that:

$$\int_{0}^{\infty} g_{\sigma}(t,\omega)(1-K\omega)_{+}\,d\omega \geq \int_{0}^{\infty} g_{in}(\omega)(1-K\omega)_{+}\,d\omega \quad , \quad t \geq 0 \tag{2.32}$$

whence, assuming that $KR \leq 1$ and using that $(1-K\omega)_{+} \leq \chi_{\left(0,\frac{1}{K}\right)}$:

$$\int_{0}^{\frac{1}{K}} g_{\sigma}(t,d\omega) \geq (1-KR)\int_{0}^{R} g_{in}(\omega)\,d\omega.$$

Choosing then K by means of $KR = \eta$ and writing $L = \frac{1}{K}$ we obtain (2.30).

On the other hand, suppose that $\int g_{in} = 1$. We define $G_{in}(\omega) = \int_{[\omega,\infty)} g_{in}$. Using (2.32) with $K = \frac{1}{R}$ we obtain:

$$\int_0^R g_\sigma(t, d\omega) \geq -\int_0^\infty \frac{dG_{in}}{d\omega}\left(1 - \frac{\omega}{R}\right)_+ d\omega = 1 - \frac{1}{R}\int_0^R G_{in}(\omega)\, d\omega$$

Using that $G_{in}(\omega) \leq A\omega^{\rho+1}$ if $\omega \geq 1$ and $G_{in}(\omega) \leq 1$ for $\omega \geq 0$, it then follows that:

$$\int_0^R g_\sigma(t, d\omega) \geq 1 - \frac{1}{R} - \frac{AR^{\rho+1}}{(\rho+2)} \quad , \quad R \geq 1$$

if $\rho > -2$ whence (2.31) follows. □

REMARK 2.26. It is important to notice that Lemma (2.25) also holds for $\sigma = 0$. The proof of the existence of weak solutions for such a value of σ is concluded in the next Subsection.

2.5.4. Limit $\sigma \to 0$. Global existence of weak solutions. We can now prove Theorem 2.18:

PROOF OF THEOREM 2.18. We consider the solutions $\{g_\sigma : \sigma > 0\}$ of the problems (2.2) which have been found in Lemma 2.20. Our goal is to prove suitable compactness properties for these functions. The estimate (2.31) in Lemma 2.25 imply uniform tightness on σ for the measures $\{g_\sigma : \sigma > 0\}$. Moreover, this estimate yields also an uniform estimate of the measures g_σ in the space $\mathcal{M}_+([0,\infty) : (1+\omega)^\rho)$ with $-2 < \rho < -1$. Therefore, the limit of these functions will be in the same space. In order to prove the compactness of this family of measures in the space $C([0,\infty) : \mathcal{M}_+([0,\infty) : (1+\omega)^\rho))$ we need to obtain estimates for the increments of time. It is enough to estimate the differences:

$$\int_{[0,\infty)} g_\sigma(t_2, \omega)\varphi(\omega)\, d\omega - \int_{[0,\infty)} g_\sigma(t_1, \omega)\varphi(\omega)\, d\omega$$

for any $\varphi \in C^2([0,\infty))$, $t_1, t_2 \in [0,\infty)$. Using (2.3) and Lemma 2.4 we obtain:

$$\left|\int_{[0,\infty)} g_\sigma(t_2, \omega)\varphi(\omega)\, d\omega - \int_{[0,\infty)} g_\sigma(t_1, \omega)\varphi(\omega)\, d\omega\right| \leq C|t_2 - t_1|$$

where $C > 0$ is independent on σ. The compactness of the family $\{g_\sigma : \sigma > 0\}$ follows then from Arzela-Ascoli (cf. [**10**]). Taking a subsequence $\{\sigma_k\}$ we obtain that $g_{\sigma_k} \rightharpoonup g \in \mathcal{M}_+([0,\infty) : (1+\omega)^\rho)$, $-2 < \rho < -1$. Taking the limit in (2.3) and using also Lemma 2.4 we obtain that g is a weak solution of (1.7) in the sense of Definition 2.2 with initial datum g_{in} and the result follows. □

2.6. Stationary solutions.

In this Section we discuss the stationary solutions of (1.2). It turns out that in the isotropic case it is possible to obtain a complete classification of the equlibria.

2.6.1. Equilibria in the isotropic case. We first discuss the weak solutions in the sense of Definition 2.2 which do not depend on t. Such solutions will be termed as equilibria. In the isotropic case we can obtain a classification of all the equilibria.

THEOREM 2.27. *Suppose that* $g \in \mathcal{M}_+ ([0,\infty) : (1+\omega)^\rho)$, *with* $\rho < -\frac{1}{2}$ *has the property that the measure* $\bar{g} \in C([0,\infty) : \mathcal{M}_+ ([0,\infty) : (1+\omega)^\rho))$ *defined as* $\bar{g}(t,\cdot) = g(\cdot)$ *for any* $t \geq 0$ *is a weak solution of (2.2) in the sense of Definition 2.2 with* $\sigma = 0$. *We will assume also that* $\int_{[0,\infty)} g(d\omega) = m < \infty$. *Then there exists* $\omega_0 \geq 0$ *such that:*

$$g = m\delta_{\omega_0}$$

PROOF OF THEOREM 2.27. We can assume without loss of generality that $\int g(d\omega) = 1$. Let us assume first that $\int_{\{0\}} g = 0$. Using the fact that g is a weak solution of (2.2) in the sense of Definition 2.2 it follows from Proposition 2.21 that for any concave test function φ we have

$$\mathcal{G}_{0,\varphi} \leq 0 \tag{2.33}$$

Since g is an equilibrium it then follows that:

$$\int_{[0,\infty)^3} d\omega_1 d\omega_2 d\omega_3 \, \frac{g_1\, g_2\, g_3}{\sqrt{\omega_1\omega_2\omega_3}} \mathcal{G}_{0,\varphi}(\omega_1\, \omega_2\, \omega_3) = 0 \tag{2.34}$$

We then apply Lemma 6.2 to the function $\bar{g}(t,\cdot) = g(\cdot)$ which by assumption is a weak solution of (2.2). Then:

$$T \int_{\mathcal{S}_{R,\rho}} \left[\prod_{m=1}^{3} g_m(d\omega_m) \right] \leq \frac{2Bb^{\frac{7}{2}}R}{\rho^2 \left(\sqrt{b}-1\right)^2},$$

where $R > 0$, $b > 1$ can be chosen arbitrarily close to one. and ρ arbitrarily close to zero. The constant B is independent of $b,\ \rho,\ R, .T$. The set $\mathcal{S}_{R,\rho}$ is contained in $(0,\infty)^3$. Taking the limit $T \to \infty$ it then follows that:

$$\int_{\mathcal{S}_{R,\rho}} \left[\prod_{m=1}^{3} g_m(d\omega_m) \right] = 0$$

and taking the limit $\rho \to 0$ we obtain:

$$\int_{\{\omega_1=\omega_2=\omega_3:\omega_1>0\}} \left[\prod_{m=1}^{3} g_m(d\omega_m) \right] = 0$$

Therefore $g = \delta_{\omega_0}$ with $\omega_0 > 0$ and Theorem 2.27 would follow. Suppose then that $\int_{\{0\}} g > 0$. If $\int_{\{0\}} g = 1$ the conclusion of the Theorem follows with $\omega_0 = 0$. If $m = \int_{\{0\}} g \in (0,1)$ there exists a bounded set A such that $dist(A,\{0\}) > 0$ and $\int_A g > 0$. We then apply (2.34) with the concave test function $\varphi(\omega) = \frac{\omega}{1+\omega}$. Using (2.34), Lemma 2.23, as well as the fact that $\varphi''(\omega) \leq -c_1 < 0$ in bounded sets we obtain:

$$0 \leq -c_0 m \left(\int_A g \right)^2$$

with $c_0 > 0$. This gives a contradiction, whence $m \in \{0,1\}$. □

2.6.2. Equilibria in the nonisotropic case. The mathematical theory for the nonisotropic weak turbulence equation (1.2) is far less developed than in the isotropic case. The main reason for that is that the integral on the right-hand side of (1.2) does not define a measure for an arbitrary measure F. However, it is possible to obtain a huge class of measures F for which the right-hand side of (1.2) is well defined in the sense of measures which actually vanishes. The idea is construct measures F with the form $\sum_{\ell} \delta_{k_\ell}$ where the values k_ℓ do not interact with each other.

We first precise in which sense a measure $F \in \mathcal{M}_+\left(\mathbb{R}^3\right)$ is a stationary solution of (1.2).

Definition 2.28. We will say that $F \in \mathcal{M}_+\left(\mathbb{R}^3\right)$ is a stationary solution of (1.2) if for any $\varphi \in C_0\left(\mathbb{R}^3\right)$the following integrals are defined:

$$J_{k,\ell,m} = \iiint_{\left(\mathbb{R}^3\right)^3} \varphi\left(k_3 + k_4 - k_2\right) \delta\left(\omega_1 + \omega_2 - \omega_3 - \omega_4\right) F_k F_\ell F_m$$

$$\omega_1 = \left(k_3 + k_4 - k_2\right)^2 \quad , \quad \omega_j = k_j^2 \quad , \quad j = 2,3,4$$

with $(k,\ell,m) \in \{(3,4,1),(3,4,2),(1,2,3),(1,2,4)\}$ and, moreover, the following identity holds:

$$J_{3,4,1} + J_{3,4,2} = J_{1,2,3} + J_{1,2,4}$$

We can then construct infinitely many stationary solutions of (1.2) in the sense of Definition 2.28. The possibility of obtaining stationary solutions of weak turbulence equations by means of noninteracting particles was already pointed out in [**19**].

Theorem 2.29. *Given $L = 1,2,3,..,\infty$, it is possible to choose vectors $\{K_j\}_{j=1}^L$, $K_j \in \mathbb{R}^3$ in infinitely many ways, with the property that for any choice of numbers $\{m_j\}_{j=1}^L$, $m_j > 0$, the measure $F = \sum_{j=1}^L m_j \delta_{K_j}$ is a stationary solution of (1.2) in the sense of Definition 2.28.*

Proof. Given three arbitrary, different points $K_1, K_2, K_3 \in \mathbb{R}^3$ we choose a point $K_4 \in \mathbb{R}^3$ with the property that the functions

$$\Delta = \left(k_3 + k_4 - k_2\right)^2 + k_2^2 - k_3^2 - k_4^2$$

are different from zero for any choice of values $(k_1,k_2,k_3,k_4) \in \{K_1,K_2,K_3,K_4\}$. This choice of K_4 can be made in infinitely many different ways. It then follows that for any choice of numbers $\{m_j\}_{j=1}^4$, $m_j > 0$ we have:

$$\delta\left(\omega_1 + \omega_2 - \omega_3 - \omega_4\right) F_k F_\ell F_m = 0$$

with $F = \sum_{j=1}^4 m_j \delta_{K_j}$. Therefore $J_{k,\ell,m}$ for any choice of values of (k,ℓ,m). We can iterate the procedure in order to add an arbitrary number of particles. Actually it is possible to form countable sets of particles with the same property This proves the result. □

2.7. Weak solutions with non interacting condensate.

We now discuss the Kolmogorov-Zakharov solutions, in the framework used in this paper. These solutions have the form $f_s(\omega) = K\omega^{-7/6}$, whence $g_s(\omega) = K\omega^{-2/3}$ and have been extensively studied in the physical literature, where it has been seen that they yield a non-zero flux of particles from $\{\omega > 0\}$ to $\{\omega = 0\}$.

Different, but equivalent, expressions for the fluxes have been obtained for instance in [**12**], [**14**], [**46**]. We will use the following formulas for the fluxes:

$$J_n[g](\omega) = J_{n,1}[g](\omega) + J_{n,2}[g](\omega) + J_{n,3}[g](\omega) + J_{n,4}[g](\omega) \tag{2.35}$$

with:

$$J_{n,1}[g](\omega) = \int_0^\infty d\omega_1 \int_0^\omega d\omega_2 \int_\omega^\infty d\omega_3 Q[g](\omega_1,\omega_2,\omega_3) \tag{2.36}$$

$$J_{n,2}[g](\omega) = \int_0^\omega d\omega_1 \int_{\omega-\omega_1}^\infty d\omega_2 \int_0^{\omega_1+\omega_2-\omega} d\omega_3 Q[g](\omega_1,\omega_2,\omega_3) \tag{2.37}$$

$$J_{n,3}[g](\omega) = -\int_0^\infty d\omega_1 \int_\omega^\infty d\omega_2 \int_0^\omega d\omega_3 Q[g](\omega_1,\omega_2,\omega_3) \tag{2.38}$$

$$J_{n,4}[g](\omega) = -\int_\omega^\infty d\omega_1 \int_0^\infty d\omega_2 \int_{\omega_1+\omega_2-\omega}^{\omega_1+\omega_2} d\omega_3 Q[g](\omega_1,\omega_2,\omega_3) \tag{2.39}$$

where $\int_a^b = \int_{(a,b)}$ and $Q[g](\omega_1,\omega_2,\omega_3) = \frac{\Phi g_1 g_2 g_3}{\sqrt{\omega_1\omega_2\omega_3}}$. We define also:

$$G_0(\omega) = \omega^{-\frac{2}{3}} \tag{2.40}$$

The methods developed in the papers [**13**], [**14**] for the Nordheim equation allow to obtain a large class of solutions of (2.2) which behave asymptotically as the Kolmogorov-Zakharov solutions for small values of ω. More precisely, the solutions described in the following Theorem have the asymptotics $f(t,\omega) \sim a(t)\,\omega^{-\frac{7}{6}}$ as $\omega \to 0$ for a suitable function $a(t)$.

THEOREM 2.30. *Given a function $f_0 \in C^1(0,\infty)$ satisfying $\left|\omega^{\frac{7}{6}} f_0(\omega) - A\right| + \left|\omega^{\frac{13}{6}} f_0(\omega) + \frac{7A}{6}\right| \leq C\omega^\delta$ for $0 < \omega \leq 1$ and $\left|\omega^{\frac{1}{2}+\rho} f_0(\omega)\right| \leq C$ for $\omega \geq 1$ and $\rho > \frac{1}{2}$, there exists $T > 0$ and functions $f \in C^{1,0}([0,T]\times[0,\infty))$, $a \in C([0,T])$, $|a(t)| \leq 2A$ such that f solves (1.3), (1.4) and $\left|\omega^{\frac{7}{6}} f(t,\omega) - a(t)\right| \leq 2C\omega^{\frac{\delta}{2}}$ for $0 < \omega \leq 1$, $t \in [0,T]$ and $\left|\omega^{\frac{1}{2}+\rho} f(t,\omega)\right| \leq 2C$ for $\omega \geq 1$, $t \in [0,T]$. Moreover, if $\rho > 1$ we have:*

$$\partial_t \left(\int f(t,\omega)\sqrt{\omega}\,d\omega \right) = J_n[G_0](1)\,(a(t))^3 \tag{2.41}$$

where $J_n[G_0](1)$ is obtained using (2.35), (2.40).

PROOF. The Proof of Theorem 2.30 is similar to the Proof of Theorem 2.1 in [**14**]. Its main idea is to linearize (1.3), (1.4) around the power law $\bar{f}(\omega) = \omega^{-\frac{7}{6}}$. The fundamental solution associated to this linearized problem can be computed explicitly using Wiener-Hopf methods and their properties can be described with great detail (cf. [**13**]). The quadratic terms in Nordheim's equation (1.10), (1.11) are lower order terms. Their contribution must be examined in detail for large values of ω, since their effect is the dominant one in that region. This detailed analysis of the effect of the quadratic terms in (1.10) has been made in [**14**], but in the analysis of (1.3), (1.4) we do not need to estimate the effect of the quadratic terms. This allows to assume initial data f_0 with less stringent decay conditions, since the contributions of the cubic terms for $\omega \to \infty$ can be estimated easily. $\square$

REMARK 2.31. Notice that the fluxes given in (2.41) cannot be prescribed for those solutions, but they arise naturally as a consequence of the evolution of the equation.

REMARK 2.32. The Kolmogorov-Zakharov solutions $f_s(\omega) = K\omega^{-7/6}$ are solutions of (1.3), (1.4) in the sense of Theorem 2.30 with initial datum $f_s(\omega) = K\omega^{-7/6}$. They are defined for arbitrarily large values of T.

Our next goal is to make precise how the solutions obtained in Theorem 2.30 can be set in the framework of weak solutions defined in Sections 2.1 and 2.2.We first remark that in both concepts of weak solutions there (cf. Definitions 2.2, 2.5) the resulting weak solutions must satisfy $\partial_t \left(\int_{[0,\infty)} g\left(t, d\omega\right) \right) = 0$ if the initial mass of the solutions is finite, as it can be readily seen using the test function $\varphi = 1$. It is mathematically simpler to work with this type of mass conserving weak solutions. However, due to (2.41) it would be impossible to have weak solutions of (2.2) in the sense of Definitions 2.2 or 2.5 unless an additional measure is added at the origin. Therefore, in our setting it is natural to state that the Kolmogorov-Zakharov solutions are:

$$g_{KZ}\left(t, d\omega\right) = -J_n\left[G_0\right](1) K^3 t \delta_{\omega=0} + \frac{K d\omega}{\omega^{\frac{2}{3}}} \tag{2.42}$$

It will be proved later that $J_n\left[G_0\right](1) < 0$, therefore, the mass at $\omega = 0$ increases.

In general, given any $f\left(t, \omega\right)$ which solves (1.3), (1.4) and has the properties in Theorem 2.30 we define:

$$g\left(t, d\omega\right) = m\left(t\right) \delta_{\omega=0} + \sqrt{\omega} f\left(t, \omega\right) d\omega \tag{2.43}$$

$$m\left(t\right) = -J_n\left[G_0\right](1) \int_0^t \left(a\left(s\right)\right)^3 ds \tag{2.44}$$

We then have the following result:

THEOREM 2.33. *Let $\sigma = 0$. The measure $g\left(t, \cdot\right)$ defined by means of (2.43), (2.44) solves (2.2) in the sense of Definition 2.5.*

The following result has some independent interest, because it states in which precise sense the Kolmogorov-Zakharov solutions solve (2.2).

COROLLARY 2.34. *Let $\sigma = 0$. The measure $g_{KZ}\left(t, d\omega\right)$ given by (2.42) solves (2.2) in the sense of Definition 2.5.*

We also have the following results which suggests that the correct definition of weak solutions for solutions with fluxes towards the origin is Definition 2.5.

THEOREM 2.35. *Let $\sigma = 0$. The measure $g\left(t, \cdot\right)$ defined by means of (2.43), (2.44) does not satisfy Definition 2.5.*

In order to prove Theorems 2.33, 2.35 it is convenient to prove the following Lemma.

LEMMA 2.36. *Suppose $\rho < -\frac{1}{2}$, $g \in \mathcal{M}_+\left([0,\infty) : (1+\omega)^\rho\right)$, $\varphi \in C_0^2\left([0,\infty)\right)$. Then:*

$$\begin{aligned} &\iiint_{(0,\infty)^3} \frac{\Phi g_1 g_2 g_3}{\sqrt{\omega_1 \omega_2 \omega_3}} \left[\varphi\left(\omega_1 + \omega_2 - \omega_3\right) + \varphi\left(\omega_3\right) - \right. \\ &\qquad \left. -\varphi\left(\omega_1\right) - \varphi\left(\omega_2\right)\right] d\omega_1 d\omega_2 d\omega_3 = \int_{(0,\infty)} J_n\left[g\right]\left(\omega\right) \varphi'\left(\omega\right) d\omega \end{aligned} \tag{2.45}$$

where $J_n[g]$ is as in (2.35).

PROOF. We rewrite the left hand side of (2.45) as:

$$\iiint_{(0,\infty)^3} Q[g]\left[\varphi(\omega_1+\omega_2-\omega_3)+\varphi(\omega_3)-\varphi(\omega_1)-\varphi(\omega_2)\right] d\omega_1 d\omega_2 d\omega_3 =$$
$$= K_1 + K_2$$
$$K_1 = \iiint_{(0,\infty)^3} Q[g]\left[\varphi(\omega_3)-\varphi(\omega_2)\right] d\omega_1 d\omega_2 d\omega_3$$
$$K_2 = \iiint_{(0,\infty)^3} Q[g]\left[\varphi(\omega_1+\omega_2-\omega_3)-\varphi(\omega_1)\right] d\omega_1 d\omega_2 d\omega_3$$

We now use the fact that $\iiint_{\{\omega_2=\omega_3\}} Q[g]\left[\varphi(\omega_3)-\varphi(\omega_2)\right] = 0$ to obtain:

$$K_1 = \iiint_{\{\omega_2<\omega_3\}} [\cdots] + \iiint_{\{\omega_2>\omega_3\}} [\cdots]$$

Using then that $\varphi(\omega_3)-\varphi(\omega_2) = \int_{\omega_2}^{\omega_3} \varphi'(\omega)\, d\omega$ for $\omega_2 < \omega_3$ and $\varphi(\omega_3) - \varphi(\omega_2) = -\int_{\omega_3}^{\omega_2} \varphi'(\omega)\, d\omega$ for $\omega_3 < \omega_2$ we obtain, applying Fubini's Theorem:

$$K_1 = \int_{(0,\infty)} J_{n,1}[g](\omega)\,\varphi'(\omega)\, d\omega - \int_{(0,\infty)} J_{n,3}[g](\omega)\,\varphi'(\omega)\, d\omega$$

On the other hand:

$$K_2 = \iiint_{\{\omega_2<\omega_3\}} [\cdots] + \iiint_{\{\omega_2>\omega_3\}} [\cdots]$$

Using then $\varphi(\omega_1+\omega_2-\omega_3)-\varphi(\omega_1) = \int_{\omega_1}^{\omega_1+\omega_2-\omega_3} \varphi'(\omega)\, d\omega$ for $\omega_2 > \omega_3$ and $\varphi(\omega_1+\omega_2-\omega_3)-\varphi(\omega_1) = \int_{\omega_1+\omega_2-\omega_3}^{\omega_1} \varphi'(\omega)\, d\omega$ for $\omega_3 > \omega_2$ we obtain, arguing in a similar manner:

$$K_2 = \int_{(0,\infty)} J_{n,2}[g](\omega)\,\varphi'(\omega)\, d\omega - \int_{(0,\infty)} J_{n,4}[g](\omega)\,\varphi'(\omega)\, d\omega$$

whence the result follows. □

PROOF OF THEOREM 2.33. By assumption f solves (1.3), (1.4). Multiplying (1.3) by $\sqrt{\omega_1}\tilde{\varphi}(\omega_1)$ with $\tilde{\varphi} \in C_0^1([0,\infty))$, where and $\tilde{\varphi}(0) = 0$ we obtain, after some changes of variables:

$$\partial_t\left(\int g\tilde{\varphi}\right) = \iiint_{(0,\infty)^3} Q[g]\left[\tilde{\varphi}(\omega_1+\omega_2-\omega_3)+\tilde{\varphi}(\omega_3)-\right.$$
$$\left.-\tilde{\varphi}(\omega_1)-\tilde{\varphi}(\omega_2)\right] d\omega_1 d\omega_2 d\omega_3 \tag{2.46}$$

where g is as in (2.43), (2.44).

Given $\varphi \in C_0^2([0,\infty))$ we split it as $\varphi = \phi_\varepsilon + \tilde{\varphi}_\varepsilon$ where $\phi_\varepsilon, \tilde{\varphi}_\varepsilon \in C_0^1([0,\infty))$, $\phi_\varepsilon(\omega) = \left(1 - \frac{\omega}{\varepsilon}\right)_+ \varphi$, $\tilde{\varphi}_\varepsilon(0) = 0$. Then:

$$\iiint_{(0,\infty)^3} Q[g]\left[\varphi(\omega_1+\omega_2-\omega_3) + \varphi(\omega_3) - \varphi(\omega_1) - \varphi(\omega_2)\right] d\omega_1 d\omega_2 d\omega_3$$
$$= K_3 + K_4$$
$$K_3 = \iiint_{(0,\infty)^3} Q[g]\left[\tilde{\varphi}_\varepsilon(\omega_1+\omega_2-\omega_3) + \tilde{\varphi}_\varepsilon(\omega_3) - \tilde{\varphi}_\varepsilon(\omega_1) - \tilde{\varphi}_\varepsilon(\omega_2)\right] d\omega_1 d\omega_2 d\omega_3$$
$$K_4 = \iiint_{(0,\infty)^3} Q[g]\left[\phi_\varepsilon(\omega_1+\omega_2-\omega_3) + \phi_\varepsilon(\omega_3) - \phi_\varepsilon(\omega_1) - \phi_\varepsilon(\omega_2)\right] d\omega_1 d\omega_2 d\omega_3$$

We now remark that, the asymptotics of f as $\omega \to 0$, stated in Theorem 2.30 implies:

$$\lim_{\omega\to 0}\left[J_n[g](\omega)\right] = (a(t))^3 J_n[G_0](1), \text{ with } G_0(\omega) = (\omega)^{-\frac{2}{3}} \tag{2.47}$$

The proof of (2.47) just requires to see that the asymptotics of $J_n[g](\omega)$ depends only on the local behaviour of g as $\omega \to 0$. The arguments requires for the proof are rather similar to the ones in the computations of the fluxes in [**46**].

Applying Lemma 2.36 to compute K_4 we obtain:

$$K_4 = \int_{(0,\infty)} J_n[g](\omega)\,\phi_\varepsilon'(\omega)\,d\omega$$

and taking the limit $\varepsilon \to 0$ we obtain:

$$K_4 = \varphi(0)\lim_{\omega\to 0}\left[J_n[g](\omega)\right] = -\varphi(0)(a(t))^3 J_n[G_0](1) \tag{2.48}$$

On the other hand, we compute $\partial_t\left(\int_{[0,\infty)} \varphi g(t,d\omega)\right)$ as:

$$\partial_t\left(\int_{[0,\infty)} \varphi g(t,d\omega)\right) = \partial_t\left(\int_{[0,\infty)} \phi_\varepsilon(\omega)\,g(t,d\omega)\right) + \partial_t\left(\int_{[0,\infty)} \tilde{\varphi}(\omega)\,g(t,d\omega)\right) \tag{2.49}$$

We compute the difference $\partial_t\left(\int_{[0,\infty)} \varphi g(t,d\omega)\right) - (K_3 + K_4)$. Using (2.46), (2.49) we obtain that this difference is:

$$\partial_t\left(\int_{[0,\infty)} \phi_\varepsilon(\omega)\,g(t,d\omega)\right) - K_4$$

The integrated version of this equation is:

$$\int_{[0,\infty)} \phi_\varepsilon(\omega)\,g(t,d\omega) - \int_{[0,\infty)} \phi_\varepsilon(\omega)\,g_0(d\omega) - \int_0^t K_4 ds$$

Taking the limit $\varepsilon \to 0$ and using (2.48) we obtain:

$$\left[m(t) + \int_0^t (a(s))^3 J_n[G_0](1)\,ds\right]\varphi(0)$$

Using (2.44) we obtain the cancellation of this quantity, and the result follows. $\square$

The Proof of Theorem 2.35 is now elementary.

Proof of Theorem 2.35. In order to prove that g is a solution of (2.2) in the sense of Definition 2.5 we need to compute $\partial_t \left(\int_{[0,\infty)} \varphi g(t, d\omega)\right) - \iiint_{[0,\infty)^3} [\cdots]$. Since, as we have seen in the Proof of Theorem 2.33, $\partial_t \left(\int_{[0,\infty)} \varphi g(t, d\omega)\right) - \iiint_{(0,\infty)^3} [\cdots] = 0$, we have:

$$\partial_t \left(\int_{[0,\infty)} \varphi g(t, d\omega)\right) - \iiint_{[0,\infty)^3} [\cdots] = -m(t) \iiint_{\mathcal{Z}} [\cdots] \tag{2.50}$$

where $\mathcal{Z} = \left(\{0\} \times [0,\infty)^2\right) \cup ([0,\infty) \times \{0\} \times [0,\infty)) \cup \left([0,\infty)^2 \times \{0\}\right)$. We now use the fact that Lemma 2.4 implies that the lines $\Gamma_{i,j} = \{\omega_i = \omega_j = 0\}$ do not contribute to the integral. We can then replace the set $\mathcal{Z}$ by $\mathcal{Y} = \left(\{0\} \times (0,\infty)^2\right) \cup ((0,\infty) \times \{0\} \times (0,\infty)) \cup \left((0,\infty)^2 \times \{0\}\right)$. Then the integral in right-hand side of (2.50) becomes:

$$\begin{aligned}
&\iint_{(0,\infty)^2 \cap \{\omega_2 > \omega_3\}} \frac{g_2 g_3}{\sqrt{\omega_2 \omega_3}} \left[\varphi(\omega_2 - \omega_3) + \varphi(\omega_3) - \varphi(0) - \varphi(\omega_2)\right] d\omega_2 d\omega_3 + \\
&+ \iint_{(0,\infty)^2 \cap \{\omega_1 > \omega_3\}} \frac{g_1 g_3}{\sqrt{\omega_1 \omega_3}} \left[\varphi(\omega_1 - \omega_3) + \varphi(\omega_3) - \varphi(\omega_1) - \varphi(0)\right] d\omega_1 d\omega_3 + \\
&\qquad + \iint_{(0,\infty)^2} \frac{g_1 g_2}{\sqrt{\omega_1 \omega_2}} \left[\varphi(\omega_1 + \omega_2) + \varphi(0) - \varphi(\omega_1) - \varphi(\omega_2)\right] d\omega_1 d\omega_2
\end{aligned}$$

We will assume that φ is compactly supported in $(0,\infty)$ in order to have convergence of the integrals. Therefore $\varphi(0) = 0$. Relabelling ω_3 to ω_2 in the second integral we obtain:

$$\begin{aligned}
&2 \iint_{(0,\infty)^2 \cap \{\omega_1 > \omega_2\}} \frac{g_1 g_2}{\sqrt{\omega_1 \omega_2}} \left[\varphi(\omega_1 - \omega_2) + \varphi(\omega_2) - \varphi(\omega_1)\right] d\omega_1 d\omega_2 + \\
&+ \iint_{(0,\infty)^2} \frac{g_1 g_2}{\sqrt{\omega_1 \omega_2}} \left[\varphi(\omega_1 + \omega_2) - \varphi(\omega_1) - \varphi(\omega_2)\right] d\omega_1 d\omega_2
\end{aligned}$$

A symmetrization argument and rearrangement of the different terms yields:

$$\begin{aligned}
\iint_{(0,\infty)^2} \frac{g_1 g_2}{\sqrt{\omega_1 \omega_2}} &\left[\varphi(\omega_1 + \omega_2) + \varphi(|\omega_1 - \omega_2|) + \varphi(\min\{\omega_1, \omega_2\}) - \right. \\
&\left. -\varphi(\max\{\omega_1, \omega_2\}) - \varphi(\omega_1) - \varphi(\omega_2)\right] d\omega_1 d\omega_2
\end{aligned}$$

Since the integrand is symmetric under the transformation $\omega_1 \longleftrightarrow \omega_2$ we can transform this integral in:

$$\iint_{\substack{(0,\infty)^2 \\ \{\omega_1 > \omega_2\}}} \frac{g_1 g_2}{\sqrt{\omega_1 \omega_2}} \left[\varphi(\omega_1 + \omega_2) + \varphi(\omega_1 - \omega_2) - 2\varphi(\omega_1)\right] d\omega_1 d\omega_2 \tag{2.51}$$

Suppose first that $g(\omega) = \omega^{-\frac{2}{3}}$. In this case this integral reduces to:

$$2 \int_0^\infty \frac{d\omega_1}{(\omega_1)^{\frac{7}{6}}} \int_0^{\omega_1} \frac{d\omega_2}{(\omega_2)^{\frac{7}{6}}} \left[\varphi(\omega_1 + \omega_2) + \varphi(\omega_1 - \omega_2) - 2\varphi(\omega_1)\right]$$

Integrating by parts in the integral in ω_2 we obtain:

$$-12\int_0^\infty \frac{d\omega_1}{(\omega_1)^{\frac{4}{3}}}\left[\varphi(2\omega_1)+\varphi(0)-2\varphi(\omega_1)\right]+$$
$$+12\int_0^\infty \frac{d\omega_1}{(\omega_1)^{\frac{7}{6}}}\int_0^{\omega_1} d\omega_2\,(\omega_2)^{-\frac{1}{6}}\left[\varphi'(\omega_1+\omega_2)-\varphi'(\omega_1-\omega_2)\right]$$

Applying Fubini in the second integral this integral becomes:

$$-12\int_0^\infty \frac{d\omega_1}{(\omega_1)^{\frac{4}{3}}}\left[\varphi(2\omega_1)+\varphi(0)-2\varphi(\omega_1)\right]+$$
$$+12\int_0^\infty (\omega_2)^{-\frac{1}{6}}\,d\omega_2\int_{\omega_2}^\infty \left[\varphi'(\omega_1+\omega_2)-\varphi'(\omega_1-\omega_2)\right]\frac{d\omega_1}{(\omega_1)^{\frac{7}{6}}}$$

and integrating by parts in the integral with respect to ω_1 the integral becomes:

$$-24\int_0^\infty \frac{d\omega_1}{(\omega_1)^{\frac{4}{3}}}\left[\varphi(2\omega_1)-\varphi(\omega_1)\right]+$$
$$+14\int_0^\infty (\omega_2)^{-\frac{1}{6}}\,d\omega_2\int_{\omega_2}^\infty \left[\varphi(\omega_1+\omega_2)-\varphi(\omega_1-\omega_2)\right]\frac{d\omega_1}{(\omega_1)^{\frac{13}{6}}}$$

Using the changes of variables $\xi=2\omega_1$ in the integral containing $\varphi(2\omega_1)$, $\xi=\omega_1+\omega_2$ in the integral containing $\varphi(\omega_1+\omega_2)$ and $\xi=\omega_1-\omega_2$ in the integral containing $\varphi(\omega_1-\omega_2)$ we obtain, applying Fubini in the second term:

$$24\left(2^{\frac{1}{3}}-1\right)\int_0^\infty \frac{\varphi(\xi)\,d\xi}{(\xi)^{\frac{4}{3}}}-14\int_0^\infty \varphi(\xi)\,d\xi\int_0^{\frac{\xi}{2}}(\omega_2)^{-\frac{1}{6}}(\xi-\omega_2)^{-\frac{13}{6}}\,d\omega_2+$$
$$+14\int_0^\infty \varphi(\xi)\,d\xi\int_0^\infty(\omega_2)^{-\frac{1}{6}}(\xi+\omega_2)^{-\frac{13}{6}}\,d\omega_2=\int_0^\infty\varphi(\xi)\,F(\xi)\,d\xi$$

$$F(\xi)=\frac{24\left(2^{\frac{1}{3}}-1\right)}{(\xi)^{\frac{4}{3}}}-14\int_0^{\frac{\xi}{2}}(\omega_2)^{-\frac{1}{6}}(\xi-\omega_2)^{-\frac{13}{6}}\,d\omega_2+$$
$$+14\int_0^\infty(\omega_2)^{-\frac{1}{6}}(\xi+\omega_2)^{-\frac{13}{6}}\,d\omega_2$$

A rescaling argument yields:

$$F(\xi)=\frac{c_*}{(\xi)^{\frac{4}{3}}}\tag{2.52}$$

where:

$$c_*=24\left(2^{\frac{1}{3}}-1\right)-14\int_0^{\frac{1}{2}}(x)^{-\frac{1}{6}}(1-x)^{-\frac{13}{6}}\,dx+14\int_0^\infty(x)^{-\frac{1}{6}}(1+x)^{-\frac{13}{6}}\,dx$$

Using the change of variables $y=(1+x)^{-1}$ in the last integral we can transform it in a Beta function. The second integral can be written in terms of the incomplete Beta function $B_z\left(\frac{5}{6},-\frac{7}{6}\right)$ (cf. [**1**]). Then:

$$c_*=24\left(2^{\frac{1}{3}}-1\right)-14B_{z=\frac{1}{2}}\left(\frac{5}{6},-\frac{7}{6}\right)+14B\left(\frac{4}{3},\frac{5}{6}\right)$$

and its numerical value is $c_* = 0.32964... \neq 0$. Therefore, if $g = \omega^{-\frac{2}{3}}$ we obtain that the right-hand side of (2.50) defines a nonzero functional. In particular there exist functions φ for which the integral $\iiint_{\mathcal{Z}} [\cdots]$ is different from zero.

On the other hand, for arbitrary functions g obtained as $g = \sqrt{\omega} f$ with f as in Theorem 2.30 we obtain a similar result due to the fact that $g(\omega)$ is asymptotically close to $K\omega^{-\frac{2}{3}}$ as $\omega \to 0$. Indeed, we can consider test functions $\varphi(\omega) = \psi\left(\frac{\omega}{\varepsilon}\right)$, with $\varepsilon > 0$ small and ψ compactly supported in $(0, \infty)$. We can approximate g by means of $K\omega^{-\frac{2}{3}}$ with an error of order $\delta_1 \omega^{-\frac{2}{3}}$ with δ_1 small, if $0 < \omega < \delta_2$. Due to (2.52) we obtain that the contribution to (2.51) of the leading term $K\omega^{-\frac{2}{3}}$ is of order $m(t) K^2 \varepsilon^{-\frac{1}{3}}$. The error term due to the remainder $\delta_1 \omega^{-\frac{2}{3}}$ in the region $0 < \omega < \delta_2$ can be estimated using (2.51) by:

$$m(t)\,\delta_1 \int_0^{\delta_2} \frac{d\omega_1}{(\omega_1)^{\frac{7}{6}}} \int_0^{\omega_1} \frac{d\omega_2}{(\omega_2)^{\frac{7}{6}}} \left|\varphi(\omega_1 + \omega_2) + \varphi(\omega_1 - \omega_2) - 2\varphi(\omega_1)\right|$$

Using the form of the function φ and a rescaling argument we estimate this term by $m(t)\delta_1 \varepsilon^{-\frac{1}{3}}$. This contribution is small compared with the leading term. On the other hand, the regions where some of the variables $\omega_1,\ \omega_2$ are larger than δ_2 can be estimated, due to the fact that the support of φ has size ε, as well as $\omega_1 > \omega_2$, by:

$$m(t) \iint_{(0,\infty)^2 \cap \left\{\omega_1 > \omega_2 \geq \frac{\delta}{2}\right\}} \frac{g_1 g_2}{\sqrt{\omega_1 \omega_2}} \left|\varphi(\omega_1 - \omega_2)\right| d\omega_1 d\omega_2$$

Since by assumption $g(\omega) \leq \frac{C}{(1+\omega)^\rho}$ for $\omega \geq 1$, with $\rho > \frac{1}{2}$ and the support of φ has size ε, we can estimate this term as:

$$C_{\delta_2} m(t)\,\varepsilon \iint_{(0,\infty)^2 \cap \left\{\omega_1 > \omega_2 \geq \frac{\delta_2}{2}\right\}} \frac{d\omega}{(1+\omega)^{2\rho+1}} \leq C_{\delta_2} m(t)\,\varepsilon$$

Therefore, the contribution of this term is negligible if ε is small enough. This concludes the proof of the result. $\square$

Remark 2.37. If we had not added the mass at $\omega = 0$ in (2.42) or (2.43) the resulting measures g would not be a weak solution of (2.2) in the sense of neither Definition 2.2 or 2.5. Due to the absence of condensate in g both definitions would be equivalent and then it is enough to check that Definition 2.5 fails. This follows from the fact that the term $\partial_t \left(\int_{[0,\infty)} \varphi g(t, d\omega)\right) - (K_3 + K_4)$ computed in the Proof of Theorem 2.33 yields $\varphi(0) \int_0^t (a(s))^3 J_n [G_0](1)\, ds \neq 0$.

Remark 2.38. We have proved that the choice of mass at $\omega = 0$ in (2.43) gives a measure g which does not solve (2.2) in the sense of Definition 2.2. It is natural to ask if other choice of the mass at $\omega = 0$ could give such a solution. In the mass conserving case this cannot happen because the choice of the mass made in (2.43) at $\omega = 0$ is the only one compatible with mass conservation for the measure g.

The existence of weak solutions with interacting condensate has been considered in [**34**] for the Nordheim equation.

2.7.1. Negativity of the fluxes. We can now prove that $J_n [G_0](1)$, the constant that characterizes the fluxes from $\{\omega > 0\}$ to $\{\omega = 0\}$, is strictly negative. This has been shown in [**12**] using a representation formula for the fluxes inspired in the computations of Zakharov which give the exponents characterizing the stationary power law solutions of the weak turbulence equations. In [**46**] the negativity of this constant has been obtained computing it numerically. We prove here that the negativity of this constant is a consequence of the Monotonicity Formula. We remark that the choice of signs in [**12**] is the reverse of the one used in [**46**] as well as in this paper. It is assumed in [**12**] that fluxes tranporting particles from larger values of ω to smaller values of ω are positive.

THEOREM 2.39. *The constant* $J_n[G_0](1)$ *has the following representation formula:*

$$J_n[G_0](1) = -\frac{1}{3}\iiint_{(0,\infty)^3} \frac{d\omega_1 d\omega_2 d\omega_3}{(\omega_1\omega_2\omega_3)^{\frac{7}{6}}} \times \times\left[\sqrt{\omega_-}H^1_\varphi(\omega_1,\omega_2,\omega_3) + \sqrt{(\omega_0+\omega_- - \omega_+)_+}H^2_\varphi(\omega_1,\omega_2,\omega_3)\right] \tag{2.53}$$

where the functions H^1_φ*,* H^2_φ *are as in Lemma 2.23 with* $\varphi(\omega) = (1-\omega)_+$*. Moreover* $J_n[G_0](1) < 0$*.*

PROOF. We take as starting point (2.45) with $g = G_0$. In this case $J_n[g](\omega)$ is constant. Using the test function $\varphi(\omega) = (1-\omega)_+$ the right-hand side of (2.45) reduces to $-J_n[G_0](1)$. On the other hand, using Proposition 2.21 and Lemma 2.23 we can rewrite the left-hand side of (2.45) as the right-hand side of (2.53) with the reverse sign. Due to Lemma 2.23 we have $H^1_\varphi \geq 0$, $H^2_\varphi \geq 0$. Moreover, these functions are strictly positive at least in some compact subsets of $(0,\infty)^3$, whence the result follows. □

2.7.2. Energy fluxes. We can obtain formulas for the energy fluxes analogous to (2.45). This formula will allow us to prove, using elementary dimensional analysis arguments, that for the Kolmogorov-Zakharov solutions the fluxes of energy vanish.

LEMMA 2.40. *Let Suppose that* $\rho < -\frac{1}{2}$*,* $g \in \mathcal{M}_+([0,\infty):(1+\omega)^\rho)$*,* $\varphi \in C^2_0([0,\infty))$*. Let* $\varphi(\omega) = \omega\psi(\omega)$*. Then:*

$$\iiint_{(0,\infty)^3} \frac{\Phi g_1g_2g_3}{\sqrt{\omega_1\omega_2\omega_3}}\left[\varphi(\omega_1+\omega_2-\omega_3) + \varphi(\omega_3) - \varphi(\omega_1) - \varphi(\omega_2)\right] d\omega_1 d\omega_2 d\omega_3 = \int_{(0,\infty)} J_e[g](\omega)\varphi'(\omega)\,d\omega \tag{2.54}$$

where:

$$J_e[g](\omega) = \iiint_{(0,\infty)^3} \frac{\Phi g_1g_2g_3}{\sqrt{\omega_1\omega_2\omega_3}} F(\omega_1,\omega_2,\omega_3;\omega)\,d\omega_1 d\omega_2 d\omega_3 \tag{2.55}$$

$$F(\omega_1,\omega_2,\omega_3;\omega) = (\omega_1+\omega_2-\omega_3)\,\Omega(\omega_1,\omega,\omega_1+\omega_2-\omega_3) + +\omega_3\Omega(\omega_1,\omega,\omega_3) - \omega_2\Omega(\omega_1,\omega,\omega_3)$$

where the function $\Omega(\xi,\zeta,\eta)$ *is defined as follows:*

$$\begin{aligned} \Omega(\xi,\zeta,\eta) &= 1 \text{ if } \xi < \zeta < \eta \\ \Omega(\xi,\zeta,\eta) &= -1 \text{ if } \eta < \zeta < \xi \\ \Omega(\xi,\zeta,\eta) &= 0 \quad \text{otherwise} \end{aligned}$$

PROOF. Using the definition of φ we can write:

$$\varphi(\omega_1+\omega_2-\omega_3)+\varphi(\omega_3)-\varphi(\omega_1)-\varphi(\omega_2)=$$
$$=(\omega_1+\omega_2-\omega_3)\int_{\omega_1}^{\omega_1+\omega_2-\omega_3}\psi'(\omega)\,d\omega+\omega_3\int_{\omega_1}^{\omega_3}\psi'(\omega)\,d\omega-\omega_2\int_{\omega_1}^{\omega_3}\psi'(\omega)\,d\omega$$

where some of the integrals must be understood with a negative sign if the limits of integration are not ordered. The Lemma then follows by Fubini's Theorem. □

2.7.3. Kolmogorov-Zakharov solutions and dimensional considerations. The weak formulation given in Definition 2.5 combined with (2.45), (2.54) imply the following equations in the sense of distributions:

$$\partial_t(g)+\partial_\omega(J_n) = 0 \ , \ \omega>0 \tag{2.56}$$
$$\partial_t(\omega g)+\partial_\omega(J_e) = 0 \ , \ \omega>0 \tag{2.57}$$

These equations describe the transport of mass and energy in the set $\{\omega>0\}$. The Kolmogorov-Zakharov solutions arise naturally from (2.56), (2.57). Indeed, $g_s(\omega)=K\omega^{-2/3}$ is a solution of (2.56) in which $J_n=-c_0=J_n[G_0](1)\,K^3$. On the other hand, it would be natural to define a second Kolmogorov-Zakharov solution using (2.57), and more precisely, assuming that g is a power law yielding $J_e=$ *constant*. Using the rescaling properties of J_e (cf. (2.55)) this would suggest $g_s(\omega)=c_1\,\omega^{-1}$ for some constant $c_1>0$. This solution is usually assumed to be an admissible Kolmogorov-Zakharov solution in the physical literature. It would be associated to a transport of energy from low values of ω to higher values. However, since many of the integrals defining the fluxes are non-convergent we decided not to discuss it. It is not clear in which sense the solution $g_s(\omega)=c_1\,\omega^{-1}$ is a solution of (2.2) and for that reason we prefer not to pursue its analysis for the moment.

A remarkable fact of the Kolmogorov-Zakharov solution g_s is that its energy fluxes vanish for any value of ω. This can be seen in [**12**] using a suitable representation formula for the energy fluxes. We provide a different proof of this fact here that only requires dimensional analysis considerations. Due to the homogeneity of the functional J_e with respect to g it is enough to prove that the energy fluxes vanish for $g=G_0$, with $G_0(\omega)=\omega^{-\frac{2}{3}}$.

PROPOSITION 2.41. *Let $J_e[g]$ be as in (2.55). Then $J_e[G_0](\omega)=0$ for any $\omega>0$.*

PROOF. Definition 2.5 combined with Lemma 2.40 imply that G_0 is a distributional solution of the equation:

$$\partial_t(\omega G_0)+\partial_\omega(J_e[G_0])=0 \ , \ \omega>0 \tag{2.58}$$

Since G_0 is a power law we have, due to the homogeneity properties of J_e that $J_e[G_0](\omega)=c_0\omega$ for a suitable constant $c_0>0$. Since G_0 is stationary we obtain from (2.58) that $\partial_\omega(J_e[G_0])=0$. Integrating in the interval $[1,2]$ we obtain $J_e[G_0](2)=J_e[G_0](1)$, whence $c_0=0$ and the result follows. To make this argument fully rigorous, the weak formulation of (2.58) with suitable test functions must be used, followed by a passage to the limit. Since this argument is classical and elementary we skip it. □

2.7.4. On more general concepts of weak solutions of (2.2). We have defined two concepts of weak solutions of (2.2), namely Definitions 2.2 and 2.5. The main difference between the two Definitions is the form in which the particles in the condensate interact with the remaining particles of the system. A consequence of this is that for measures g without condensate both definitions are identical. In the case of Definition 2.2 the particles in the condensate interact with the remaining ones with an interaction that is obtained taking the limit of the interactions with particles with small size $\omega << 1$. On the contrary, in the case of Definition 2.5 it is assumed that the particles in the condensate do not interact at all with the remaining particles of the system.

It is natural to ask if it would be possible to introduce more general types of interactions between the condensate and the rest of the system in order to define more general concepts of weak solution of (2.2). The answer is affirmative. The following definition shows how to introduce in the system a rather large class of interactions.

DEFINITION 2.42. Suppose that $\alpha, \beta \in C\left([0,\infty)^2\right)$ are nonnegative functions such that $\alpha(\omega_1,\omega_2) = \alpha(\omega_2,\omega_1)$ for $(\omega_1,\omega_2) \in [0,\infty)^2$. Let $\rho < -\frac{1}{2}$. We will say that $g \in C([0,T) : \mathcal{M}_+([0,\infty) : (1+\omega)^\rho))$ is a weak solution of (2.2) with initial datum $g_0 \in \mathcal{M}_+([0,\infty) : (1+\omega)^\rho)$ and condensate interaction (α,β) if the following identity holds for any test function $\varphi \in C_0^2([0,T) \times [0,\infty))$:

$$\int_{[0,\infty)} g(t_*,\omega)\varphi(t_*,\omega)\,d\omega - \int_{[0,\infty)} g_0\varphi(0,\omega)\,d\omega = \int_0^{t_*}\int_{[0,\infty)} g\partial_t\varphi d\omega dt +$$
$$+\int_0^{t_*}\iiint_{(0,\infty)^3} \frac{g_1g_2g_3\Phi}{\sqrt{\omega_1\omega_2\omega_3}}\left[\varphi(\omega_1+\omega_2-\omega_3)+\varphi(\omega_3)-\varphi(\omega_1)-\right.$$
$$\left.-\varphi(\omega_2)\right]d\omega_1 d\omega_2 d\omega_3 dt +$$
$$+\int_0^{t_*}\iiint_{\substack{\{0\}\times(0,\infty)^2\\ \{\omega_2>\omega_3\}}} \frac{\beta(\omega_2,\omega_3)g_1g_2g_3}{\sqrt{\omega_2\omega_3}}\left[\varphi(\omega_2-\omega_3)+\varphi(\omega_3)-\varphi(0)-\right.$$
$$\left.-\varphi(\omega_2)\right]d\omega_1 d\omega_2 d\omega_3 dt +$$
$$+\int_0^{t_*}\iiint_{\substack{\{0\}\times(0,\infty)^2\\ \{\omega_1>\omega_3\}}} \frac{\beta(\omega_1,\omega_3)g_1g_2g_3}{\sqrt{\omega_1\omega_3}}\left[\varphi(\omega_1-\omega_3)+\varphi(\omega_3)-\varphi(\omega_1)-\right.$$
$$\left.-\varphi(0)\right]d\omega_1 d\omega_2 d\omega_3 dt +$$
$$+\int_0^{t_*}\iiint_{\{0\}\times(0,\infty)^2} \frac{\alpha(\omega_1,\omega_2)g_1g_2g_3}{\sqrt{\omega_1\omega_2}}\left[\varphi(\omega_1+\omega_2)+\varphi(0)-\varphi(\omega_1)-\right.$$
$$\left.-\varphi(\omega_2)\right]d\omega_1 d\omega_2 d\omega_3 dt$$

for any $t_* \in [0,T)$.

REMARK 2.43. The function α describes the probability rate for the collision of two particles with energy (ω_1,ω_2) to produce a particle with energy $\omega_3 = 0$ and other with $\omega_4 = (\omega_1+\omega_2)$. The function β describes the probability rate for the collision of one particle with energy $\omega_1 > 0$ and a particle in the condensate with energy $\omega_2 = 0$ to produce a particle with energy $\omega_3 < \omega_1$ and other with energy $\omega_4 = (\omega_1 - \omega_3)$. Notice that Definition 2.42 reduces to Definition 2.2 if $\alpha = \beta = 1$ and to Definition 2.5 if $\alpha = \beta = 0$.

CHAPTER 3

Qualitative behaviors of the solutions.

In this chapter we study several properties of the solutions whose existence has been proved in Chapter 2. We are interested first in the behavior as $t \to +\infty$ of the global weak solutions with interacting condensate of (1.7), and that is the content of Theorem 3.2 below. We consider next, in the Corollary 3.9 and Proposition 3.11, the evolution of the energy density of the particles. Then, in the case where the Dirac mass towards which the weak solution converges is located at the origin, we consider whether it is formed in finite time or it is only asymptotically achieved. That is the object of Theorem 3.13 and Theorem 3.17. We also prove blow up of a family of initially bounded solutions.

3.1. Weak solutions with interacting condensate as $t \to +\infty$.

In order to describe the long time asymptotics of the weak solutions of (1.7) we need some notation that allows us to classify the initial data g_{in}.

We recall that the support of a (nonnegative) Radon measure μ is defined as follows:

$$\operatorname{supp}(\mu) = [0,\infty) \setminus \bigcup \{\mathcal{U} \text{ open in } \mathbb{R} : \mu(\mathcal{U}) = 0\} \tag{3.1}$$

where we assume that $\mu(-\infty, 0) = 0$. Notice that $x \in \operatorname{supp}(\mu)$ iff for any $\rho > 0$ we have $\mu(B_\rho(x)) > 0$.

Definition 3.1. Given a set $A \subset [0,\infty)$, we define an extended set $A^* \subset (0,\infty)$ as:

$$A^* = \bigcup_{n=1}^{\infty} A_n$$

where we define the sets A_n inductively by means of:

$$A_1 = A, \quad A_{n+1} = (A_n + A_n - A_n) \cap (0,\infty) \quad , \quad n = 1,2,3,...$$

It is important to notice that by definition $0 \notin A^*$. We then have the following result.

Theorem 3.2. *Let $\rho < -1$. Suppose $g \in C([0,\infty) \colon \mathcal{M}_+([0,\infty) \colon (1+\omega)^\rho))$ is a weak solution of (1.7) in the sense of Definition 2.2 with initial datum $g_{in} \in \mathcal{M}_+([0,\infty) : (1+\omega)^\rho)$. Let $m = \int g_{in} d\omega$. Suppose that $m > 0$. Let $A = \operatorname{supp}(g_{in})$ and A^* as in Definition 3.1. We define $R_* = \inf(A^*)$. Then:*

$$\lim_{t\to\infty} \int g(t, d\omega)\,\varphi(\omega) = m\varphi(R_*) \tag{3.2}$$

for any test function $\varphi \in C_0([0,\infty))$.

REMARK 3.3. Theorem 3.2 just states that $g(t,\cdot) \rightharpoonup m\delta_{R_*}$ as $t\to\infty$ in the weak topology for measures in $\mathcal{M}_+([0,\infty):(1+\omega)^{\rho})$. Notice that since $m>0$ we have that $A\neq\emptyset$ and R_* is well defined and it satisfies $0\leq R_*<\infty$.

REMARK 3.4. The convergence towards a single Dirac mass at the origin containing the total number of particles has been suggested in several papers of the physical literature. In particular in [**12**], [**27**], [**41**], [**53**].

We first prove the following auxiliary result which will be used to prove some type of diffusive properties for (1.7).

LEMMA 3.5. *Suppose that the assumptions of Theorem 3.2 hold. Then, for any* $x\in A^*$, *any* $t^*>0$ *and any* $r>0$, *we have* $\int_{B_r(x)} g(t^*,d\omega)>0$.

PROOF. Let us consider any ball $B\subset(0,\infty)$. Taking a sequence of test functions φ_n in (2.3) converging pointwise to the characteristic function of $\bar{B}$ we obtain the following:

$$\int_{\bar{B}} g(t,d\omega)-\int_{\bar{B}} g_0(d\omega)=\int_0^t\iiint_{([0,\infty))^3}\frac{g_1g_2g_3\Phi}{\sqrt{\omega_1\omega_2\omega_3}}\times$$
$$\times\left[\chi_{\bar{B}}(\omega_1+\omega_2-\omega_3)+\chi_{\bar{B}}(\omega_3)-\chi_{\bar{B}}(\omega_1)-\chi_{\bar{B}}(\omega_2)\right]d\omega_1d\omega_2d\omega_3ds$$

for any $t>0$. Notice that this implies that the function $t\to\int_{\bar{B}} g(t,d\omega)$ is Lipschitz continuous and:

$$\partial_t\left(\int_{\bar{B}} g(t,d\omega)\right)=\iiint_{([0,\infty))^3}\frac{g_1g_2g_3\Phi}{\sqrt{\omega_1\omega_2\omega_3}}\times \tag{3.3}$$
$$\times\left[\chi_{\bar{B}}(\omega_1+\omega_2-\omega_3)+\chi_{\bar{B}}(\omega_3)-\chi_{\bar{B}}(\omega_1)-\chi_{\bar{B}}(\omega_2)\right]d\omega_1d\omega_2d\omega_3$$

a.e. $t\geq0$. Since $x\in A^*$ we have $x\in A_N$ for some $N\geq1$. The definition of the sets A_n implies the existence of a family of points

$$\mathcal{F}_N(x)=\left\{x^{(n)}_{\mathbf{k}_{N-n}}\in A_n;\mathbf{k}_{N-n}=(k_1,k_2,\cdots,k_{N-n}),k_j\in\{1,2,3\},n=1,\cdots,N\right\}$$

where $x^{(N)}=x$. Notice that this particular element has an empty family of indexes k_j. The family of points in $\mathcal{F}_N(x)$ satisfies:

$$x^{(n)}_{\mathbf{k}_{N-n}}=x^{(n-1)}_{(\mathbf{k}_{N-n},1)}+x^{(n-1)}_{(\mathbf{k}_{N-n},2)}-x^{(n-1)}_{(\mathbf{k}_{N-n},3)}$$

for any $x^{(n)}_{\mathbf{k}_{N-n}}\in\mathcal{F}_N(x)$. Notice that the family $\mathcal{F}_N(x)$ is not necessarily unique, but its existence is guaranteed by the definition of the sets A_n and any such a family could be used in the proof. By continuity we can find a set or radii $r_n>0$ such that $r_N=r$ and:

$$B_{r_{n-1}}\left(x^{(n-1)}_{(\mathbf{k}_{N-n},1)}\right)+B_{r_{n-1}}\left(x^{(n-1)}_{(\mathbf{k}_{N-n},2)}\right)-B_{r_{n-1}}\left(x^{(n-1)}_{(\mathbf{k}_{N-n},3)}\right)\subset B_{r_n}\left(x^{(n)}_{\mathbf{k}_{N-n}}\right) \tag{3.4}$$

for any $x^{(n)}_{\mathbf{k}_{N-n}}\in\mathcal{F}_N(x)$. Moreover, we choose the numbers $\{r_n\}$ in order to have:

$$B_{r_n}\left(x^{(n)}_{\mathbf{k}_{N-n}}\right)\subset(0,\infty)\ \text{ for any } x^{(n)}_{\mathbf{k}_{N-n}}\in\mathcal{F}_N(x) \tag{3.5}$$

Notice that (3.4) implies:

$$\chi_{B_{r_n}\left(x^{(n)}_{\mathbf{k}_{N-n}}\right)}(\omega_1+\omega_2-\omega_3)\geq\prod_{\ell=1}^{3}\chi_{B_{r_{n-1}}\left(x^{(n-1)}_{(\mathbf{k}_{N-n},\ell)}\right)}(\omega_\ell) \tag{3.6}$$

for $n=1,2,....,N$. We understand that the right-hand side of (3.6) is zero if $n=1$.

Therefore, applying (3.3) with $\bar{B} = B_{r_n}\left(x^{(n)}_{\mathbf{k}_{N-n}}\right)$, using that $\chi_{\bar{B}} \geq 0$, as well as (3.5) we obtain:

$$\partial_t \left(\int_{B_{r_n}\left(x^{(n)}_{\mathbf{k}_{N-n}}\right)} g(t, d\omega) \right) \geq \geq C_1 \prod_{\ell=1}^{3} \int_{\chi_{B_{r_{n-1}}\left(x^{(n-1)}_{(\mathbf{k}_{N-n},\ell)}\right)}} g(t, d\omega) - C_2 \int_{B_{r_n}\left(x^{(n)}_{\mathbf{k}_{N-n}}\right)} g(t, d\omega) \tag{3.7}$$

where $C_1 > 0,\ C_2 > 0$ depend on the family $\mathcal{F}_N(x)$ and $n = 1, ...N$. Notice that C_1 could become very small if some of the points in $\mathcal{F}_N(x)$ becomes large and C_2 could become very large if some of the points in $\mathcal{F}_N(x)$ approaches zero. However, since the family $\mathcal{F}_N(x)$ is finite, both constants are finite. Notice also that the constant C_2 depends also in $\int g(t, d\omega) = \int g_0(d\omega)$.

We now apply (3.7) iteratively, starting with $n = 1$. By assumption:

$$\min_{x^{(1)}_{\mathbf{k}_{N-1}}} \left(\int_{B_{r_n}\left(x^{(1)}_{\mathbf{k}_{N-1}}\right)} g_0(d\omega) \right) > 0.$$

Then:

$$\min_{0 \leq t \leq t^*} \min_{x^{(1)}_{\mathbf{k}_{N-1}}} \left(\int_{B_{r_n}\left(x^{(1)}_{\mathbf{k}_{N-1}}\right)} g(t, d\omega) \right) \geq c_1 > 0 \tag{3.8}$$

where c_0 depends in $\mathcal{F}_N(x)$ and t^*. Using (3.8) in (3.7) with $n = 2$ we obtain:

$$\min_{\frac{t^*}{N} \leq t \leq t^*} \min_{x^{(2)}_{\mathbf{k}_{N-1}}} \left(\int_{B_{r_n}\left(x^{(1)}_{\mathbf{k}_{N-1}}\right)} g(t, d\omega) \right) \geq c_2 > 0$$

Iterating, and using the nonnegativity of g, we obtain:

$$\min_{\frac{(n-1)t^*}{N} \leq t \leq t^*} \min_{x^{(n)}_{\mathbf{k}_{N-1}}} \left(\int_{B_{r_n}\left(x^{(1)}_{\mathbf{k}_{N-1}}\right)} g(t, d\omega) \right) \geq c_n > 0$$

for $n = 1, ...N$, whence the result follows. $\square$

The following Lemma will be used to prove that the dynamics of g can be reduced to a discrete problem if $R_* > 0$.

Lemma 3.6. *Let $\rho < -1$, $g_{in} \in \mathcal{M}_+\left([0,\infty) : (1+\omega)^\rho\right)$. Define $m = \int g_{in} d\omega$. Suppose that $m > 0$. Let $A = \operatorname{supp}(g_{in})$ and A^* as in Definition 3.1 and let be $R_* = \inf(A^*)$. If $R_* > 0$, there exists a finite set of positive real numbers $\{D_k\}_{k=1}^L$ such that*

$$A^* = \left\{ R_* + \sum_{k=1}^{L} n_k D_k : n_k \in \mathbb{N}_* \right\} \tag{3.9}$$

Moreover, for any $j, k \in \{1, 2, ..., L\}$ the quotients $\frac{D_j}{D_k}$ are rational numbers and we have:

$$\min\{D_k\}_{k=1}^L \geq R_* \tag{3.10}$$

Proof. Suppose that $x, y \in A,\ y > x$. Let $D_1 \equiv (y-x)$. Then $Q_1 \equiv [x + D_1\mathbb{Z}] \cap \mathbb{R}_+ \subset A^*$. This is just a consequence of the definition of A^*. Notice that $R_* \leq \min Q_1 \leq D_1$. Suppose that $[A^* \setminus Q_1] \neq \emptyset$. We choose $z_2 \in [A^* \setminus Q_1]$. Since $R_* > 0$ we have $z_2 > 0$. Moreover, $D_2 \equiv dist\,(z_2, Q_1) < D_1$. Then $Q_2 \equiv (Q_1 \cup [\min Q_1 + D_2\mathbb{Z}] \cup [z_2 + D_2\mathbb{Z}]) \cap \mathbb{R}_+ \subset A^*$. Notice that if $D_2 > \frac{D_1}{2}$ we have $z_2 \leq \frac{D_1}{2}$. Since $\min Q_2 \leq D_2$ it follows that $\min Q_2 \leq \frac{D_1}{2}$. Indeed, if $D_2 \leq \frac{D_1}{2}$ this follows immediately. Otherwise we have that $R_* \leq \min Q_2 \leq z_2 \leq \frac{D_1}{2}$, since $z_2 \in Q_2$. We then define sets Q_k in an iterative manner. More precisely, as long as $[A^* \setminus Q_{k-1}] \neq \emptyset,\ k = 2, 3, ...$ we can choose $z_k \in [A^* \setminus Q_{k-1}]$. We have $D_k \equiv dist\,(z_k, Q_{k-1}) < D_{k-1}$ and we then define:

$$Q_k \equiv (Q_{k-1} \cup [\min Q_{k-1} + D_k\mathbb{Z}] \cup [z_k + D_k\mathbb{Z}]) \cap \mathbb{R}_+ \subset A^* \tag{3.11}$$

Arguing as in the case $k = 2$ we obtain $R_* \leq D_k \leq \frac{D_{k-1}}{2}$. Then D_k decreases exponentially in k and since $R_* > 0$ the process must stop after a finite number of steps, say $M \geq 1$. More precisely, there exists $1 \leq M < \infty$ such that $A^* \subset Q_L$. Otherwise, if the iteration procedure could be iterated for arbitrarily large values of k we would arrive at a contradiction. On the other hand, since (3.11) holds for $k = M$ we have $Q_M \subset A^*$ whence $Q_M = A^*$. Then $R_* = \min(Q_M) \in A^*$. Moreover, we have proved the existence of points $x_k, y_k \in A^*$ such that $(y_k - x_k) = D_k,\ k = 1, ..., M$. Then, using the definition of A^* we obtain:

$$\mathcal{U}_M = \left\{ R_* + \sum_{k=1}^{M} n_k D_k : n_k \in \mathbb{N}_* \right\} \subset A^* \tag{3.12}$$

where:

$$\mathbb{N}_* = \{0, 1, 2, 3, ...\}$$

If the inclusion in (3.12) is strict we can find $z \in A \setminus \mathcal{U}_L$. We must have $z > R_*$ since, otherwise there would exist two points in A^* at a distance smaller than R_* and this is impossible as seen above. Otherwise we introduce additional distances $D_j,\ j = M+1, ...$ and extend iteratively the sets $\mathcal{U}_M$ to $\mathcal{U}_{M+1},\ \mathcal{U}_{M+2}, ...$ including in the set also the points $\{nD_{M+1} : n \in \mathbb{N}_*\},\ \{nD_{M+2} : n \in \mathbb{N}_*\},\ ...$ respectively. Since $D_{k+1} \leq \frac{D_k}{2}$ the process must stop in a finite number of steps. Therefore we obtain $\mathcal{U}_L = A^*$ for some L. This gives (3.9).

In order to prove that $\frac{D_j}{D_k}$ are rational numbers for any j, k we just notice that, if this quotient is irrational for any pair j, k we would have

$$dist\,(\{R_* + nD_j : n \in \mathbb{N}_*\}, \{R_* + nD_k : n \in \mathbb{N}_*\}) = 0 \tag{3.13}$$

whence $R_* = 0$ and the resulting contradiction yields the result. We just remark that the property (3.13) follows from the well known fact that the set of points $\{m\alpha \pmod 1 : m \in \mathbb{N}_*\}$ is dense in the interval $[0, 1]$ for any irrational α (cf. [**2**]). □

The following result about the set A^* follows easily.

Corollary 3.7. *Let $\rho < -1,\ g_{in} \in \mathcal{M}_+([0,\infty) : (1+\omega)^\rho)$. Define $m = \int g_{in} d\omega$. Suppose that $m > 0$. Let $A = \operatorname{supp}(g_{in})$ and A^* as in Definition 3.1. If $R_* > 0$, the set A^* has the form (3.9) in Lemma 3.6. If $R_* = 0$ we have $\overline{A^*} = [0, \infty)$.*

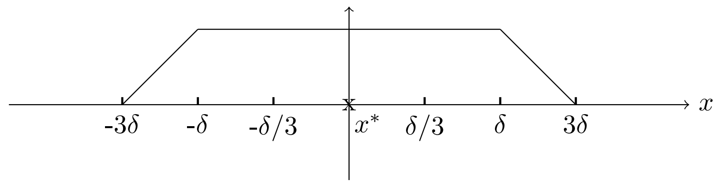

FIGURE 1. The function φ_0.

PROOF. If $R_* > 0$ we can apply Lemma 3.6. If $R_* = 0$ we have the following due to the definition of A^*. Given any $\varepsilon > 0$, there exist $x_1, x_2 \in A^*$ with $0 < x_1 < x_2 < \varepsilon$. This implies that $x_1 + \ell (x_2 - x_1) \in A^*$ for any $\ell = 0, 1, 2, 3, ...$ and since $(x_2 - x_1) < \varepsilon$ we then obtain that A^* is dense in $[0, \infty)$. $\square$

The following result combined with Lemma 3.5 will be used to characterize the support of the measures $g(t, \cdot)$. The main difficulty in the proof of Lemma 3.8 is due to the fact that equation (2.2) is singular at $\omega = 0$. Therefore we need to obtain detailed estimates for the measure of g supported in regions with ω small, because tiny amounts of the measure of g arriving to that region could have a huge effect in the dynamics of g.

LEMMA 3.8. *Suppose that the assumptions of Theorem 3.2 hold and* $R_* > 0$. *Then* $\operatorname{supp}(g(t, .)) \subset \overline{A^*}$ *for any* $t \geq 0$.

PROOF. We will assume, without loss of generality, that $\int g(t, d\omega) = 1$. We fix $\delta > 0$ small, in particular $\delta < \frac{R_*}{8}$. We define $N = N(\delta)$ as the smallest positive integer such that $3^N \delta > \frac{3R_*}{4}$. Notice that $3^{N-1}\delta \in \left[\frac{R_*}{4}, \frac{3R_*}{4}\right)$. We define the following sets:

$$Z_k = A_* + B_{3^k\delta}(0) \quad , \quad k = -1, 0, 1, 2, 3, ... (N - 1)$$
$$Z_N = Z_{N+1} = [0, \infty)$$

$$\mathcal{U}_0 = Z_0 \quad , \quad \mathcal{U}_k = Z_k \setminus Z_{k-1} \quad , \quad k = 1, ..., N$$

It is relevant to notice that, using the definition of N as well as (3.10) and the invariance of A_* under the transformations $(\omega_1, \omega_2, \omega_3) \in A_*^3 \to (\omega_1 + \omega_2 - \omega_3)$, we have:

$$Z_{k-1} + Z_{k-1} - Z_{k-1} = Z_k \quad , \quad k = 1, 2, 3, ... (N - 1) \tag{3.14}$$

We now define a set of nonnegative test functions $\varphi_k \in C_0^1([0, \infty))$, $k = 0, 1, ..., N$ as follows. We will assume that $0 \leq \varphi_k \leq 1$, $k = 0, 1, ..., N$ and we assume that:

$$\varphi_0(\omega) = 1 \text{ if } \omega \in Z_0; \quad \varphi_0(\omega) = 0 \text{ if } \omega \notin Z_1$$

and, for $k = 1, 2, 3, ...N$:

$$\varphi_k(\omega) = 1 \text{ if } \omega \in Z_k \setminus Z_{k-1} = \mathcal{U}_k \quad ; \quad \varphi_k(\omega) = 0 \text{ if } \omega \in ([0, \infty) \setminus Z_{k+1}) \cup Z_{k-2}$$

Moreover, we choose the functions φ_k satisfying the inequalities:

$$|\varphi_k'| \leq \frac{C}{3^k\delta} \quad , \quad |\varphi_k''| \leq \frac{C}{(3^k\delta)^2} \quad , \; k = 0, 1, 2, 3, ...N \tag{3.15}$$

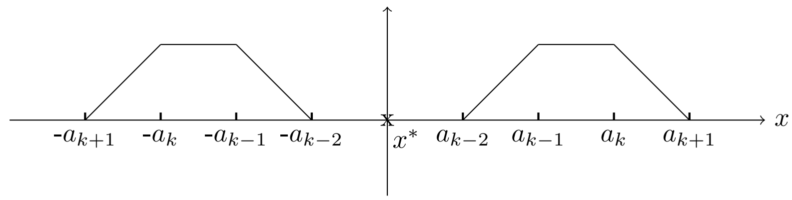

FIGURE 2. The function φ_k, $k \geq 1$ where $a_\ell = x^* + 3^\ell \delta$.

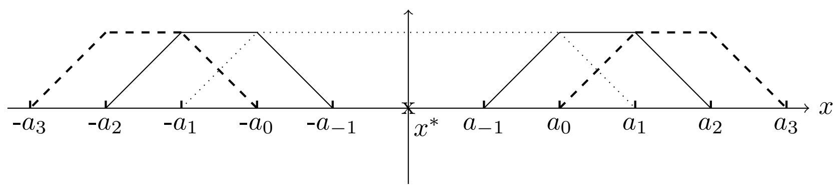

FIGURE 3. The functions φ_0 (dotted), φ_1 and φ_2 (dashed).

(See figures 1, 2 and 3.) Using (2.3) we obtain:

$$\partial_t \left(\int g(t, d\omega)\, \varphi_k \right) = \sum_{j_1=0}^{N} \sum_{j_2=0}^{N} \sum_{j_3=0}^{N} \int_{\mathcal{U}_{j_1}} \int_{\mathcal{U}_{j_2}} \int_{\mathcal{U}_{j_3}} \frac{g_1 g_2 g_3 \Phi}{\sqrt{\omega_1 \omega_2 \omega_3}} \times \tag{3.16}$$
$$\times \left[\varphi_k(\omega_1 + \omega_2 - \omega_3) + \varphi_k(\omega_3) - \varphi_k(\omega_1) - \varphi_k(\omega_2) \right] d\omega_1 d\omega_2 d\omega_3$$

a.e. $t \geq 0$. We now proceed to estimate the different terms on the right of (3.16). We will estimate in a different manner the terms containing at least two among the indexes j_1, j_2, j_3 equal to N and all the others choices of indexes. Let us denote the set of indexes (j_1, j_2, j_3) with at least two values equal to N as $\mathcal{S}$. If $(j_1, j_2, j_3) \notin \mathcal{S}$ we have that at least two among the values $\omega_1,\ \omega_2,\ \omega_3$ remain at a distance of $\omega = 0$ larger than $\frac{R_*}{4}$. Therefore $\frac{\Phi}{\sqrt{\omega_1 \omega_2 \omega_3}} \leq C$, with $C > 0$ independent on δ. Then:

$$\sum_{(j_1,j_2,j_3)\notin\mathcal{S}} \int_{\mathcal{U}_{j_1}} \int_{\mathcal{U}_{j_2}} \int_{\mathcal{U}_{j_3}} [\cdots] \leq$$
$$\leq C \sum_{(j_1,j_2,j_3)\notin\mathcal{S}} \int_{\mathcal{U}_{j_1}} \int_{\mathcal{U}_{j_2}} \int_{\mathcal{U}_{j_3}} g_1 g_2 g_3 \left[\varphi_k(\omega_1 + \omega_2 - \omega_3) + \varphi_k(\omega_3) \right]$$

Using the definition of the functions φ_k as well as the finiteness of the measure g we obtain:

$$\sum_{(j_1,j_2,j_3)\notin\mathcal{S}} \int_{\mathcal{U}_{j_1}} \int_{\mathcal{U}_{j_2}} \int_{\mathcal{U}_{j_3}} [\cdots] \leq C \int g(t, d\omega)\, \varphi_k + \tag{3.17}$$
$$+ C \sum_{(j_1,j_2,j_3)\notin\mathcal{S}} \int_{\mathcal{U}_{j_1}} \int_{\mathcal{U}_{j_2}} \int_{\mathcal{U}_{j_3}} g_1 g_2 g_3 \varphi_k(\omega_1 + \omega_2 - \omega_3)$$

Due to the definition of the test functions φ_k as well as the property (3.14) and the fact that $\mathcal{U}_k \subset Z_k$ it follows that, if $\max\{j_1, j_2, j_3\} \leq (k-2)$, we have $\varphi_k(\omega_1 + \omega_2 - \omega_3) = 0$ if $\omega_\ell \in \mathcal{U}_\ell$, $\ell \in \{j_1, j_2, j_3\}$. Using that $\varphi_k(\omega_1 + \omega_2 - \omega_3) \leq 1$, as well as the finiteness of the mass of g as well as the fact that $\varphi_\ell(\omega) =$

1 for $\omega \in \mathcal{U}_\ell$ it then follows that the last term in (3.17) can be estimated as $C\sum_{\ell=(k-2)_+}^{N} \int g(t,d\omega)\varphi_\ell$, with C depends on the total mass of g. Therefore it follows from (3.17) that:

$$\sum_{(j_1,j_2,j_3)\notin\mathcal{S}} \int_{\mathcal{U}_{j_1}}\int_{\mathcal{U}_{j_2}}\int_{\mathcal{U}_{j_3}} [\cdots] \leq C\sum_{\ell=(k-2)_+}^{N} \int g(t,d\omega)\varphi_\ell \tag{3.18}$$

We now estimate the contribution to the sum in (3.16) of the indexes satisfying $(j_1,j_2,j_3)\in\mathcal{S}$. Suppose first that $(j_1,j_2,j_3)\neq(N,N,N)$. We can then estimate one of the quotients $\frac{1}{\sqrt{\omega_k}}$, $k=1,2,3$ by a constant independent of δ since $\max\{\omega_1,\omega_2,\omega_3\}\geq\frac{R_*}{4}$. If $j_3<N$ we obtain a zero contribution to the integrals if $\max\{\omega_1,\omega_2\}<\frac{R_*}{8}$, since then $(\omega_1+\omega_2-\omega_3)<0$. Therefore, it is enough to consider the case in which $\max\{\omega_1,\omega_2\}>\frac{R_*}{8}$. However, in this case, using the definition of Φ we obtain:

$$\frac{\Phi}{\sqrt{\omega_1\omega_2\omega_3}}\leq C$$

with C independent of δ (but depending on R_*). We then obtain, arguing as in the estimate for the terms with the indexes in the complement of $\mathcal{S}$:

$$\begin{aligned}\sum_{\substack{(j_1,j_2,j_3)\in\mathcal{S}\setminus\{(N,N,N)\}\\ j_3<N}} \int_{\mathcal{U}_{j_1}}\int_{\mathcal{U}_{j_2}}\int_{\mathcal{U}_{j_3}} [\cdots] \leq \\ \leq C\sum_{\substack{(j_1,j_2,j_3)\in\mathcal{S}\\ j_3<N}} \int_{\mathcal{U}_{j_1}}\int_{\mathcal{U}_{j_2}}\int_{\mathcal{U}_{j_3}} g_1g_2g_3d\omega_1d\omega_2d\omega_3 C\left(\int g(t,d\omega)\varphi_N\right)^2\end{aligned} \tag{3.19}$$

Suppose now that $j_3=N$. Since $(j_1,j_2,j_3)\in\mathcal{S}\setminus\{(N,N,N)\}$ we have that exactly one of the indexes j_1 or j_2 is equal to N. We can assume, without loss of generality that $j_2=N$. Then:

$$\begin{aligned}\sum_{\substack{(j_1,j_2,j_3)\in\mathcal{S}\setminus\{(N,N,N)\}\\ j_3=N}} \int_{\mathcal{U}_{j_1}}\int_{\mathcal{U}_{j_2}}\int_{\mathcal{U}_{j_3}} [\cdots] \leq C\sum_{j_1\neq N}\int_{\mathcal{U}_{j_1}}\int_{\mathcal{U}_N}\int_{\mathcal{U}_N} \frac{g_1g_2g_3\Phi}{\sqrt{\omega_2\omega_3}}\times \\ \times\left[\left|\varphi_k(\omega_1+\omega_2-\omega_3)-\varphi_k(\omega_1)\right|+\left|\varphi_k(\omega_3)-\varphi_k(\omega_2)\right|\right]d\omega_1d\omega_2d\omega_3\end{aligned} \tag{3.20}$$

Using (3.15) we obtain the inequality:

$$\left|\varphi_k(\omega_1+\omega_2-\omega_3)-\varphi_k(\omega_1)\right|+\left|\varphi_k(\omega_3)-\varphi_k(\omega_2)\right|\leq\frac{C}{3^k\delta}\min\left\{\omega_2+\omega_3,3^k\delta\right\}$$

Then, the right-hand side of (3.20) can be estimated by:

$$\frac{C}{\sqrt{3^k\delta}}\sum_{j_1\neq N}\int_{\mathcal{U}_{j_1}}\int_{\mathcal{U}_N}\int_{\mathcal{U}_N} g_1g_2g_3d\omega_1d\omega_2d\omega_3\leq\frac{C}{\sqrt{3^k\delta}}\left(\int g(t,d\omega)\varphi_N\right)^2 \tag{3.21}$$

We finally estimate the case $(j_1,j_2,j_3)=(N,N,N)$. We derive from (3.15) the estimate:

$$\left|\varphi_k(\omega_1+\omega_2-\omega_3)+\varphi_k(\omega_3)-\varphi_k(\omega_1)-\varphi_k(\omega_2)\right|\leq\frac{C\min\left\{\omega_+\omega_0,\left(3^k\delta\right)^2\right\}}{\left(3^k\delta\right)^2} \tag{3.22}$$

where the functions $\omega_0,\ \omega_+$ are as in Definition 2.22. Notice that deriving (3.22) we first substract an affine function from φ_k whose contribution to the left-hand side of (3.22) vanishes. The resulting function to be estimated takes the value zero at $\omega = 0$ as well as its first derivative. Notice that for $k \leq (N-2)$ we should have $\max\{\omega_1, \omega_2\} \geq \frac{R_*}{8}$ in order to avoid a vanishing integral. In that case we can argue as in the derivation of (3.21) to obtain:

(3.23)
$$\left| \iiint_{(\mathcal{U}_N)^3} g_1 g_2 g_3 \Delta_{\varphi_k,0}(\omega_1,\omega_2,\omega_3)\, d\omega_1 d\omega_2 d\omega_3 \right| \leq \frac{C}{\sqrt{3^k\delta}} \left(\int g(t,d\omega)\, \varphi_N \right)^2$$

if $k = 0, 1, ..., (N-2)$, with $\Delta_{\varphi_k,0}(\omega_1,\omega_2,\omega_3)$ as defined in (2.1), (2.4). If $k > (N-2)$ we use (3.22) which combined with the definition of N implies that $\frac{1}{3^k\delta}$ is bounded and of order one. Then $\sqrt{3^k\delta}$ and $(3^k\delta)^2$ are comparable, whence we obtain again the estimate (3.23).

Combining (3.16), (3.18), (3.19), (3.21), (3.23) we arrive at:

$$\partial_t m_k(t) \leq C_0 \left[\sum_{\ell=(k-2)_+}^{N} m_\ell(t) + \frac{1}{\sqrt{3^k\delta}} (m_N(t))^2 \right] \quad , \quad a.e.\ t \geq 0 \tag{3.24}$$

for $k = 0, 1, 2, .., N$, where:

$$m_k(t) = \int g(t,d\omega)\, \varphi_k \tag{3.25}$$

and where C_0 is a positive constant. We notice also that the definition of the test functions φ_k implies:

$$m_k(0) = \delta_{k,0} \tag{3.26}$$

Since the support of the functions φ_k overlap at most a finite number of times, and the total mass of g_{in} has been normalized to one, we have:

$$\sum_{k=0}^{N} m_k(t) \leq 3 \quad , \quad t \geq 0 \tag{3.27}$$

We have also, since each $m_k(t)$ is smaller than the total mass of $g(t,\cdot)$:

$$m_k(t) \leq 1 \quad , \quad t \geq 0 \quad , \quad k = 0, 1, 2, .., N \tag{3.28}$$

We first derive some upper estimates for the asymptotics of $m_k(t)$ as $t \to 0$. Using (3.24) and (3.27) we obtain:

$$m_k(t) \leq C_0 \left(3 + \frac{1}{\sqrt{3^k\delta}} \right) t \quad , \quad k = 1, 2, .., N \tag{3.29}$$

Combining (3.28) and (3.29) we obtain the following estimate:

$$\max\{m_1(t), m_2(t)\} \leq C_0 t + O_\delta(t^2) \tag{3.30}$$

where from now on we denote as $O_\delta(f(t))$ a function $g(t)$, perhaps depending on δ, such that $\lim_{t\to 0} \frac{g(t)}{f(t)} = 0$. We emphasize the fact that this convergence is not uniform in δ in general. Using again (3.29), (3.28) for $k \geq 3$ we obtain:

$$m_k(t) = O_\delta(t^2) \quad , \quad k = 3, ..., N \tag{3.31}$$

Combining now (3.29), (3.30), (3.31) we arrive at:

$$\max\{m_3(t), m_4(t)\} \leq \frac{2C_0^2 t^2}{2!} + O_\delta(t^3)$$

$$m_k(t) = O_\delta\left(t^3\right) \quad , \quad k = 5, ..., N$$

Iterating the argument we obtain:

$$\max\left\{m_{2\ell-1}(t), m_{2\ell}(t)\right\} \leq \frac{1}{2}\frac{(2C_0 t)^\ell}{\ell!} + O_\delta\left(t^{\ell+1}\right) \quad , \quad \ell = 1, 2, ... \left[\frac{N+1}{2}\right] \tag{3.32}$$

where we will assume that $m_{N+1}(t) \equiv 0$. This term formally appears in (3.32) if N is an odd number.

Our next goal is to prove the following estimate:

$$\max\left\{m_{2\ell-1}(t), m_{2\ell}(t)\right\} \leq \frac{(4C_0 t)^\ell}{\ell!} \quad , \quad \ell = 1, 2, ... \left[\frac{N+1}{2}\right] \tag{3.33}$$

for $0 \leq t \leq T$, with $T < \min\left\{\frac{e\sqrt{R_*}}{12C_0}, \frac{1}{16eC_0}\right\}$. Notice that due to (3.32) we have that (3.33) holds for $0 \leq t \leq t_\delta$, with $t_\delta > 0$. We define

$$t^* = \sup\left\{0 \leq \bar{t} \leq T : (3.33) \text{ holds in } 0 \leq t \leq \bar{t}\right\}$$

We already know that $t^* \geq t_\delta$. Our goal is to prove that $t^* = T$. Suppose that $t^* < T$. Using (3.24) we obtain:

$$\partial_t m_k(t) \leq C_0\left[\sum_{\ell=(k-2)_+}^{k-1} m_\ell(t) + \sum_{\ell=k}^{N} m_\ell(t) + \frac{1}{\sqrt{3^k\delta}}\left(m_N(t)\right)^2\right] \quad , \quad k = 1, 2, \cdots, N.$$

Using (3.33), which holds for $0 \leq t \leq t^*$, we arrive at:

$$\partial_t m_k(t) \leq C_0\left[\sum_{\ell=(k-2)_+}^{k-1} m_\ell(t) + 2\sum_{\ell=\left[\frac{k+1}{2}\right]}^{\left[\frac{N+1}{2}\right]} \frac{(4C_0 t)^\ell}{\ell!} + \frac{1}{\sqrt{3^k\delta}}\left(\frac{(4C_0 t)^{\left[\frac{N}{2}\right]}}{\left(\left[\frac{N}{2}\right]\right)!}\right)^2\right].$$

for $k = 1, 2, \cdots, N$. Notice that:

$$\begin{aligned}\sum_{\ell=\left[\frac{k+1}{2}\right]}^{\left[\frac{N+1}{2}\right]} \frac{(4C_0 t)^\ell}{\ell!} &= \frac{(4C_0 t)^{\left[\frac{k+1}{2}\right]}}{\left(\left[\frac{k+1}{2}\right]\right)!}\sum_{m=0}^{N}\frac{\left(\left[\frac{k+1}{2}\right]\right)!\,(4C_0 t)^m}{\left(\left[\frac{k+1}{2}\right]+m\right)!} \\ &< \frac{(4C_0 t)^{\left[\frac{k+1}{2}\right]}}{\left(\left[\frac{k+1}{2}\right]\right)!}\exp(4C_0 t)\end{aligned}$$

Assuming that $t \leq T < \frac{1}{16eC_0} < \frac{1}{4C_0}$ we obtain that this sum can be estimated by $\frac{(4C_0 t)^{\left[\frac{k+1}{2}\right]}}{\left(\left[\frac{k+1}{2}\right]\right)!}e$.

On the other hand, using that $3^{N-1}\delta \in \left[\frac{R_*}{4}, \frac{3R_*}{4}\right)$ we obtain:

$$\frac{1}{\sqrt{3^k\delta}}\left(\frac{(4C_0t)^{\left[\frac{N+1}{2}\right]}}{\left(\left[\frac{N+1}{2}\right]\right)!}\right)^2 \leq \frac{2}{\sqrt{R_*}}\frac{\sqrt{3^{N-k-1}}\,(4C_0t)^{2\left[\frac{N+1}{2}\right]}}{\left(\left(\left[\frac{N+1}{2}\right]\right)!\right)^2}$$

$$= 2\frac{(4C_0t)^{\left[\frac{k+1}{2}\right]}}{\left(\left[\frac{k+1}{2}\right]\right)!}e\times$$

$$\times\left[\frac{1}{e\sqrt{R_*}}\left(\sqrt{3}\right)^{N-2\left[\frac{N+1}{2}\right]}\left(\sqrt{3}\right)^{2\left[\frac{N+1}{2}\right]}(3)^{-\frac{k+1}{2}+\left[\frac{k+1}{2}\right]}(3)^{-\left[\frac{k+1}{2}\right]}\times\right.$$

$$\left.\times(4C_0t)^{2\left[\frac{N+1}{2}\right]-\left[\frac{k+1}{2}\right]}\frac{\left(\left[\frac{k+1}{2}\right]\right)!}{\left(\left(\left[\frac{N+1}{2}\right]\right)!\right)^2}\right]$$

Using that $\sqrt{3} < 3$, $\frac{\left(\left[\frac{k+1}{2}\right]\right)!}{\left(\left(\left[\frac{N+1}{2}\right]\right)!\right)^2} < 1$, $-\frac{k+1}{2} + \left[\frac{k+1}{2}\right] \leq 0$, $N - 2\left[\frac{N+1}{2}\right] \leq 0$, we can estimate the term between brackets as as:

$$\frac{1}{e\sqrt{R_*}}(12C_0t)^{2\left[\frac{N+1}{2}\right]-\left[\frac{k+1}{2}\right]}$$

Since $2\left[\frac{N+1}{2}\right] - \left[\frac{k+1}{2}\right] \geq 1$, it then follows that, since $t \leq T \leq \min\left\{\frac{1}{12C_0}, \frac{e\sqrt{R_*}}{12C_0}\right\}$ that this term is smaller than one. Then:

$$\frac{1}{\sqrt{3^k\delta}}\left(\frac{(4C_0t)^{\left[\frac{N+1}{2}\right]}}{\left(\left[\frac{N+1}{2}\right]\right)!}\right)^2 \leq 2\frac{(4C_0t)^{\left[\frac{k+1}{2}\right]}}{\left(\left[\frac{k+1}{2}\right]\right)!}e$$

and

$$\partial_t m_k(t) \leq C_0\left[\sum_{\ell=(k-2)_+}^{k-1} m_\ell(t) + 4e\frac{(4C_0t)^{\left[\frac{k+1}{2}\right]}}{\left(\left[\frac{k+1}{2}\right]\right)!}\right], \quad k = 1,2,..,N \tag{3.34}$$

We now derive estimates for the terms $m_k(t)$ iteratively, taking as starting point the fact that $m_0(t) \leq 1$ as well as (3.26). Arguing by induction in ℓ it follows that, for $0 \leq t \leq t^*$ we have:

$$\max\{m_{2\ell-1}(t), m_{2\ell}(t)\} \leq \frac{3}{4}\frac{(4C_0t)^\ell}{\ell!}, \quad \ell = 1,2,...\left[\frac{N+1}{2}\right] \tag{3.35}$$

Indeed, if $\ell = 1$, using (3.34) as well as the fact that

$$16eC_0t \leq 1 \text{ for } 0 \leq t \leq T \tag{3.36}$$

we obtain

$$\partial_t m_1(t) \leq C_0[1 + 16eC_0t] < 3C_0 \tag{3.37}$$

whence $m_1(t) \leq 3C_0t$. The definition of T then implies that $3C_0t \leq 1$ for $0 \leq t \leq T$. Then, using again (3.34) as well as (3.36):

$$\partial_t m_2(t) \leq C_0[2 + 16eC_0t] \leq 3C_0 \tag{3.38}$$

Integrating (3.37), (3.38) we obtain (3.35) for $\ell = 1$. Suppose now that $1 < \ell \leq \left[\frac{N+1}{2}\right]$. Then, using the induction hypothesis and (3.34) we obtain:

$$\partial_t m_{2\ell-1}(t) \leq C_0\left[2\frac{(4C_0t)^{\ell-1}}{(\ell-1)!} + 4e\frac{(4C_0t)^\ell}{(\ell)!}\right]$$

Using again (3.36) as well as the fact that $(\ell)! > (\ell-1)!$ we obtain $\partial_t m_{2\ell-1}(t) \leq 3C_0 \frac{(4C_0 t)^{\ell-1}}{(\ell-1)!}$ whence:

$$m_{2\ell-1}(t) \leq \frac{3}{4}\frac{(4C_0 t)^{\ell}}{\ell!} \tag{3.39}$$

Using again (3.34), combined with the induction hypothesis (3.35) as well as (3.39) and the fact that $\frac{(4C_0 t)^{\ell}}{\ell!} \leq \frac{(4C_0 t)^{\ell-1}}{(\ell-1)!}$ if $\frac{4C_0 t}{\ell!} < \frac{1}{(\ell-1)!}$ for $0 \leq t \leq T$, we obtain:

$$\partial_t m_{2\ell}(t) \leq C_0 \left[2\frac{3}{4}\frac{(4C_0 t)^{\ell-1}}{(\ell-1)!} + 4e\frac{(4C_0 t)^{\ell}}{(\ell)!} \right]$$

whence, using once more (3.36) we obtain: $\partial_t m_{2\ell}(t) \leq 3C_0 \frac{(4C_0 t)^{\ell-1}}{(\ell-1)!}$ thus $m_{2\ell}(t) \leq \frac{3}{4}\frac{(4C_0 t)^{\ell}}{\ell!}$. This concludes the proof of (3.35) for $0 \leq t \leq t_*$. Then the inequality (3.33) can be obtained, due to the continuity of the functions $m_k(t)$ to some interval $0 \leq t \leq t_* + \varepsilon_0$ with $\varepsilon_0 > 0$, but this contradicts the definition of t_* and implies that $t^* = T$.

In order to conclude the Proof of the Lemma we notice that the definition of $m_k(t)$ and (3.33) yield:

$$\int_{\mathcal{U}_k} g(t, d\omega) \leq \int g(t, d\omega)\varphi_k = m_k(t) \leq \frac{(4C_0 t)^{\left[\frac{k+1}{2}\right]}}{\left(\left[\frac{k+1}{2}\right]\right)!}$$

for $k = 1, ..., N$. We now choose $k_0 = \frac{N}{2}$. Then $3^{k_0}\delta \leq C\sqrt{\delta}$. Moreover $k_0 \to \infty$ as $\delta \to 0$. Then:

$$\int_{\bigcup_{k \geq k_0} \mathcal{U}_k} g(t, d\omega) \leq C\frac{(4C_0 t)^{\left[\frac{k_0+1}{2}\right]}}{\left(\left[\frac{k_0+1}{2}\right]\right)!} \to 0 \text{ as } \delta \to 0$$

Classical continuity results for Radon measures then imply that:

$$\int_{[0,\infty)\setminus A_*} g(t, d\omega) = 0 \quad , \quad 0 \leq t \leq t_*$$

and the Lemma follows iterating the argument in time intervals $t_* \leq t \leq 2t_*, ...$ $\square$

Proof of the Theorem 3.2. We assume without loss of generality that $m = \int g_0(d\omega) = 1$. Let $\varphi \in C_0([0,\infty))$. Suppose first that $R_* = 0$. We define $\bar{A} = A \setminus \{0\}$. Let $\bar{R}_* = \inf\left(\bar{A}^*\right)$, where we will use the notation $\inf(\emptyset) = \infty$. We will consider separately the cases $\bar{R}_* = 0$ and $\bar{R}_* > 0$. Suppose first that $\bar{R}_* = 0$. Our goal is to show (3.2). Let $\varepsilon > 0$ arbitrarily small. Since $\bar{R}_* = 0$ there exist points $z \in \bar{A}^*$ with z arbitrarily small and $B_r(z) \subset \left(0, \varepsilon^2\right]$ for some $r > 0$. Then, Lemma 3.5 with $t^* = 1$ yields $c_0 = \int_{(0,\varepsilon^2]} g(1, d\omega) \geq \int_{B_r(z)} g(1, d\omega) > 0$. Let $\eta = \frac{1}{2}$ and $R = \varepsilon^2$. Then Lemma 2.25 implies that:

$$\int_{[0,2\varepsilon^2]} g(t, d\omega) \geq \frac{c_0}{2} \quad , \quad t \geq 1 \tag{3.40}$$

Suppose now that $R_* = 0$ and $\bar{R}_* > 0$. Then $\int_{\left(0,\bar{R}_*\right)} g_0(d\omega) = 0$. However, since $R_* = 0$ we have $\int_{\left[0,\bar{R}_*\right)} g_0(d\omega) > 0$, whence $\int_{\{0\}} g_0(d\omega) > 0$. Then $\int_{[0,\varepsilon^2]} g_0(d\omega) \geq c_0$. Applying then Lemma 2.25 with $R = \varepsilon^2$ and $\eta = \frac{1}{2}$ we then obtain (3.40). Therefore we always have (3.40) if $R_* = 0$.

We now have the following alternative. Either there exists $\bar{t} \geq 1$ such that $\int_{[0,4\varepsilon^2)} g(\bar{t}, d\omega) \geq 1 - \frac{\varepsilon}{3}$, or otherwise we have

$$\int_{[0,4\varepsilon^2]} g(t, d\omega) \leq 1 - \frac{\varepsilon}{3} \quad \text{for any} \quad t \geq 1 \tag{3.41}$$

Our aim is to show that second case implies a contradiction. Suppose that (3.41) takes place. We choose now R sufficiently large to have $\int_{[0,R]} g_0(d\omega) \geq 1 - \frac{\varepsilon}{12}$. Applying Lemma 2.25 it then follows that

$$\int_{\left[0, \frac{12R}{\varepsilon}\right]} g(t, d\omega) \geq 1 - \frac{\varepsilon}{6} \quad , \quad t \geq 0 \tag{3.42}$$

Combining (3.41), (3.42):

$$\int_{\left(4\varepsilon^2, \frac{12R}{\varepsilon}\right]} g(t, d\omega) \geq \frac{\varepsilon}{6} \quad , \quad t \geq 1 \tag{3.43}$$

Let $\varphi \in C^2([0,\infty))$ a test function satisfying the following properties:

$$\text{for any } \omega \in [0,\infty): \ \varphi(\omega) \geq 0, \quad \varphi'(\omega) > 0, \quad \varphi''(\omega) < 0;$$
$$\lim_{\omega \to \infty} \varphi(\omega) = 1 \quad , \quad \varphi(0) = 0$$

Let the function H_φ^1 as in Lemma 2.23. Suppose that $\omega_- \in \left[0, 2\varepsilon^2\right]$ and $\omega_0, \omega_+ \in \left(4\varepsilon^2, \frac{12R}{\varepsilon}\right]$. Then $(\omega_+ - \omega_-) \geq (\omega_0 - \omega_-) \geq 2\varepsilon^2$. Using the strict concavity of φ in bounded regions as well as Taylor Theorem we obtain that for such values of $(\omega_1, \omega_2, \omega_3)$:

$$H_\varphi^1(\omega_1, \omega_2, \omega_3) \leq -\kappa\varepsilon^2$$

where $\kappa > 0$ depends on the values of the second derivative of φ in $\left[0, \frac{12R}{\varepsilon}\right]$. (Therefore it depends on ε). Using Proposition 2.21 as well as Lemma 2.23 it then follows that:

$$\partial_t \left(\int \varphi(\omega) g(t, d\omega) \right) \leq -\bar{\kappa}\varepsilon^2 \left(\int_{\left(4\varepsilon^2, \frac{12R}{\varepsilon}\right]} g(t, d\omega) \right)^2 \int_{[0,2\varepsilon^2]} g(t, d\omega) \quad , \quad t \geq 1$$

Using (3.40) and (3.43) we then obtain:

$$\partial_t \left(\int \varphi(\omega) g(t, d\omega) \right) \leq -\frac{\bar{\kappa} c_0}{72} \varepsilon^4 \quad , \quad t \geq 1$$

but this contradicts the boundedness of $\int \varphi(\omega) g(t, d\omega)$. Therefore (3.41) cannot be satisfied and we then obtain that there exists $\bar{t} = \bar{t}(\varepsilon) \geq 1$ such that $\int_{[0,4\varepsilon^2)} g(\bar{t}, d\omega) \geq 1 - \frac{\varepsilon}{3}$.

We then apply again Lemma 2.25 with $\eta = \frac{\varepsilon}{3}$ to prove that $\int_{\left[0, \frac{12\varepsilon^2}{\varepsilon}\right)} g(t, d\omega) = \int_{[0,12\varepsilon^2)} g(t, d\omega) \geq 1 - \frac{2\varepsilon}{3}$ for any $t \geq \bar{t}$. Then $\int_{[12\varepsilon^2,\infty)} g(t, d\omega) < \frac{2\varepsilon}{3}$. Using the continuity of φ we then obtain that:

$$\left| \int g(t, d\omega) \varphi(\omega) - \varphi(0) \right| \leq \sup_{\omega \in [0,12\varepsilon^2]} |\varphi(\omega) - \varphi(0)| + \frac{4\varepsilon}{3} \sup_\omega |\varphi(\omega)| \quad \text{if } t \geq \bar{t}(\varepsilon)$$

Since ε can be made arbitrarily small we obtain (3.2) if $R_* = 0$.

Let us assume now that $R_* > 0$. In this case, due to Lemma 3.6 the set A^* has the form (3.9). Lemma 3.5 with $t^* = 1$ implies that $\int_{\{R_*\}} g(1, d\omega) > 0$. Moreover, Lemma 3.8 implies that $\text{supp}(g(t, .)) \subset A^*$ for any $t \geq 0$. We define the test function

$\varphi(\omega) = \frac{(\bar{\omega}-\omega)_+}{(\bar{\omega}-R_*)}$ with $\bar{\omega} = \frac{1}{2}\min_{k=1,\dots L}\{R_* + D_k\}$. Since φ is convex, Lemma 2.24 implies that $\partial_t \left(\int_{\{R_*\}} g(t, d\omega)\right) = \partial_t \left(\int g(t, d\omega)\varphi(\omega)\right) \geq 0$. Moreover, using the form of the function $\mathcal{G}_{0,\varphi}$ in Lemma 2.23 as well as (2.29) and the fact that $H_\varphi^2 \geq 0$ we obtain:

$$\frac{d}{dt}\left(\int_{\{R_*\}} g(t, d\omega)\right) \geq \frac{1}{3}\int_0^\infty \int_0^\infty \int_0^\infty \frac{g_1 g_2 g_3}{\sqrt{\omega_0\omega_+}} H_\varphi^1(\omega_1, \omega_2, \omega_3)\, d\omega_1 d\omega_2 d\omega_3 \tag{3.44}$$

Notice that we can restrict the integration in the right of (3.44) to the set $\omega_+ \in A^* \setminus \{R_*\}$, due to the fact that $\omega_+ \in \{R_*\}$ implies $\omega_+ = \omega_0 = \omega_- = R_*$ and $H_\varphi^1 = 0$. Our choice of the function φ implies that $\varphi(\omega_+ + \omega_0 - \omega_-) = \varphi(\omega_+) = 0$ if $\omega_+ \in A^* \setminus \{R_*\}$. Then, using the form of H_φ^1 we obtain:

$$\frac{d}{dt}\left(\int_{\{R_*\}} g(t, d\omega)\right) \geq \frac{1}{3}\iiint_{\{\omega_+ \in A^* \setminus \{R_*\}\}} \frac{g_1 g_2 g_3}{\sqrt{\omega_0\omega_+}} \varphi(\omega_+ + \omega_- - \omega_0)\, d\omega_1 d\omega_2 d\omega_3$$

and since $\varphi \geq 0$ we obtain:

$$\begin{aligned}\frac{d}{dt}\left(\int_{\{R_*\}} g(t, d\omega)\right) &\geq \iiint_{\left\{\substack{\omega_+,\omega_0 \in A^* \setminus \{R_*\} \\ \omega_- = R_*}\right\}} \frac{g_1 g_2 g_3}{\sqrt{\omega_0\omega_+}} \varphi(\omega_+ + \omega_- - \omega_0)\, d\omega_1 d\omega_2 d\omega_3 \\ &= \frac{1}{3}\iiint_{\left\{\substack{\omega_+,\omega_0 \in A^* \setminus \{R_*\} \\ \omega_- = R_*}\right\}} \frac{g_1 g_2 g_3}{\sqrt{\omega_0\omega_+}} d\omega_1 d\omega_2 d\omega_3 \\ &= K_1 \int_{\{R_*\}} g(t, d\omega) \left[\int_{A^* \setminus \{R_*\}} g(t, d\omega)\right]^2\end{aligned}$$

where $K_1 > 0$ contains combinatorial factor whose precise value is not relevant. Due to the monotonicity of $\int_{\{R_*\}} g(t, d\omega)$ we have:

$$\begin{aligned}\frac{d}{dt}\left(\int_{\{R_*\}} g(t, d\omega)\right) &\geq K_1 \int_{\{R_*\}} g(1, d\omega) \left[\int_{A^* \setminus \{R_*\}} g(t, d\omega)\right]^2 \\ &\geq K_2 \left[\int_{A^* \setminus \{R_*\}} g(t, d\omega)\right]^2 = K_2 \left[1 - \int_{\{R_*\}} g(t, d\omega)\right]^2\end{aligned}$$

for $t \geq 1$. Then $\int_{\{R_*\}} g(t, d\omega) \to 1$ as $t \to \infty$ and the Theorem follows. $\square$

The results in this Section provide detailed information on the asymptotics of the weak solutions for arbitrary initial data g_{in}. It is important to remark that, that for any $R_* > 0$ there exist initial data g_{in} such that the corresponding solution $g(t, \cdot)$ converges to $m\delta_{R_*}$ as $t \to \infty$. Indeed, any initial distribution g_{in} supported in a set A such that A^* is one of the sets (3.9) yields this asymptotics for $g(t, \cdot)$.

It is interesting to remark that the aggregation of the particles towards $\omega = 0$ does not imply that the energy of the system becomes concentrated in the region where ω is close to zero. Indeed, the particles with $\omega = 0$ have zero energy. Since $g\chi_{(0,\infty)}(\omega) \rightharpoonup 0$ as $t \to \infty$ the only alternative left, due to the conservation of the energy, is the flux of the remaining energy towards large values of ω. The fact that the energy tends to move towards large values of k has been noticed repeatedly in the physical literature (cf. for instance [**12**], [**41**]). The precise result that we prove is the following:

COROLLARY 3.9. *Let $\rho < -2$ and $g \in C([0,\infty) : \mathcal{M}_+([0,\infty) : (1+\omega)^\rho))$ be a weak solution of (1.7) in the sense of Definition 2.2 with initial datum $g_{in} \in \mathcal{M}_+([0,\infty) : (1+\omega)^\rho)$. Let $m = \int g_{in} d\omega$ and $e = \int \omega g_{in} d\omega$. Suppose that $m > 0$ and let R_* as in Theorem 3.2. Suppose that $e > mR_*$. Then, there exists an increasing function $R(t)$ such that $\lim_{t\to\infty} R(t) = \infty$ and:*

$$\lim_{t\to\infty} \int_{R(t)}^{\infty} \omega g(t, d\omega) = (e - mR_*)$$

REMARK 3.10. This Corollary just means that all the excess of energy which does not accumulate at the point $\omega = R_*$ escapes to infinity. Notice that the inequality $e > mR_*$ is satisfied for any initial distribution g_{in} except the ones given by $g_{in} = m\delta_{R_*}$.

PROOF. It is just a consequence of Theorem 3.2 as well as the conservation of energy for these distributions (cf. (2.24)). □

3.2. Energy transfer towards large values of k.

Notice that Theorem 3.2 implies that the mass of the distribution g tends to concentrate towards the smallest value of ω compatible with the collision mechanism. Suppose that $R_* = \inf(A^*)$ On the other hand, if we assume that the total energy of the initial distribution is bounded, i.e. $\int g_0(\omega)\omega d\omega < \infty$, we would have, due to the conservation of the energy that $MR_* < \int g_0(\omega)\omega d\omega$, with $M = \int g_0(\omega) d\omega$. Then:

$$\int (g(t,\omega) - M\delta_{R_*})\omega d\omega \to \left[\int g_0(\omega)\omega d\omega - MR_*\right]$$

as $t \to \infty$. Therefore, since $\int_{(R_*,\infty)} g(t,\omega) \rightharpoonup 0$ as $t \to \infty$, it follows that a fraction of the energy of the system should move towards large values of ω. Actually, if $R_* = 0$, all the energy of the system moves towards large values of ω as $t \to \infty$.

It turns out that the rate of transfer of the energy towards larger values of k is given by the interaction between particles of a given size, say R, with smaller particles. This is made more precise in the following result which provides a characteristic time scale for the transfer of particles of size R towards larger values. It is worth to remark that for specific initial data such a transfer of particles could be nonexistent. For instance, if $g_0(\omega) = M\delta(\omega - R_0)$, the transfer of energy towards larger modes does not take place.

PROPOSITION 3.11. *Suppose that g is a weak solution of (1.7) in the sense of Definition 2.2. We define the test function $\varphi_R(\omega) = RQ\left(\frac{\omega}{R}\right)$, with $Q(s) = s$ for $0 \le s \le \frac{1}{2}$, $Q' \ge 0$, $Q(s) = 1$ for $s \ge \frac{3}{2}$, $Q \in C^2([0,\infty))$. There exists a constant $c_0 > 0$ independent of R such that, if $\int g_0 d\omega = 1$, $\int g_0 \omega d\omega = 1$, $\int g_0 \varphi_R d\omega \ge \frac{1}{2}$ and $\int g(T,\omega)\varphi_R d\omega \ge \frac{1}{4}$ we have $T \ge c_0 R^2$.*

REMARK 3.12. Given any initial configuration g_0 such that $\int g_0 d\omega = M$, $\int g_0 \omega d\omega = E$, a simple rescaling argument implies that, if $\int g_0 \varphi_R d\omega \ge \frac{E}{2}$ and $\int g(T,\omega)\varphi_R d\omega \ge \frac{E}{4}$ we have $T \ge c_0 \frac{R^2}{ME}$.

PROOF. We apply (2.3) with test function $\varphi = \varphi_R$, $t_* \in [0,T]$ and $\sigma = 0$. We then obtain:

(3.45)
$$\partial_t \left(\int_{[0,\infty)} g(t,\omega)\,\varphi(\omega)\,d\omega \right) = \iiint_{([0,\infty)^3} g_1 g_2 g_3\, \Delta_{\varphi_k,0}(\omega_1,\omega_2,\omega_3) d\omega_1 d\omega_2 d\omega_3$$

a.e. $t \in [0,T]$, with $\Delta_{\varphi_k,0}(\omega_1,\omega_2,\omega_3)$ as defined in (2.1), (2.4).

Let us write $I_R = [0,R]$ and $I_R^c = \mathbb{R} \setminus [0,R]$. We then split the domains of integration in the last integral of (3.45) as follows:

$$\begin{aligned}\iiint_{([0,\infty)^3} &= \int_{I_R}\int_{I_R}\int_{I_R} + \int_{I_R}\int_{I_R}\int_{I_R^c} + \int_{I_R}\int_{I_R^c}\int_{I_R} + \int_{I_R^c}\int_{I_R}\int_{I_R} + \\ &+ \int_{I_R}\int_{I_R^c}\int_{I_R^c} + \int_{I_R^c}\int_{I_R}\int_{I_R^c} + \int_{I_R^c}\int_{I_R^c}\int_{I_R} + \int_{I_R^c}\int_{I_R^c}\int_{I_R^c} \\ &\equiv \sum_{k=1}^{8} J_k\end{aligned}$$

We now claim that $J_k \geq -\frac{K}{R^2}\left(\int g\varphi_R d\omega\right)^2$ for $k = 1,...,8$. Indeed, in the case of J_1 we notice that the integrand vanishes unless $\max\{\omega_1,\omega_2\} \geq \frac{R}{2}$. A symmetrization argument allows to reduce the domain of integration to the set $\{\omega_1 \geq \omega_2\}$. Then $\omega_1 \geq \frac{R}{2}$. We then split the domain of integration in the variables (ω_2,ω_3) in the sets $I_{\frac{R}{8}} \times I_{\frac{R}{8}}$ and its complement. Then:

$$\begin{aligned}J_1 &= 2\int_{I_R\setminus I_{\frac{R}{2}}} d\omega_1 \iint_{I_{\frac{R}{8}}\times I_{\frac{R}{8}}} d\omega_2 d\omega_3 + 2\int_{I_R\setminus I_{\frac{R}{2}}} d\omega_1 \iint_{\{\omega_1\geq\omega_2\}\setminus\left(I_{\frac{R}{8}}\times I_{\frac{R}{8}}\right)^c} d\omega_2 d\omega_3 \\ &= J_{1,1} + J_{1,2}\end{aligned}$$

Using the definition of φ_R and using also that in the domain of integration of $J_{1,1}$ we have $(\omega_1 + \omega_2 - \omega_3) \geq \max\{\omega_2,\omega_3\}$ we obtain:

$$\begin{aligned}J_{1,1} \geq \int_{I_R\setminus I_{\frac{R}{2}}} d\omega_1 \iint_{I_{\frac{R}{8}}\times I_{\frac{R}{8}}} d\omega_2 d\omega_3 \frac{g_1 g_2 g_3 \min\left\{\sqrt{\omega_2},\sqrt{\omega_3}\right\}}{\sqrt{\omega_1\omega_2\omega_3}} \times \\ \times\left[\omega_3 + \varphi_R(\omega_1+\omega_2-\omega_3) - \omega_2 - \varphi_R(\omega_1)\right]\end{aligned}$$

Due to the symmetry of the integral:

$$\iint_{I_{\frac{R}{8}}\times I_{\frac{R}{8}}} d\omega_2 d\omega_3 \frac{g_2 g_3 \min\left\{\sqrt{\omega_2},\sqrt{\omega_3}\right\}}{\sqrt{\omega_2\omega_3}}\left[\omega_3 - \omega_2\right] = 0 \tag{3.46}$$

Therefore, using Taylor's expansion we obtain:

$$\begin{aligned}J_{1,1} \geq \int_{I_R\setminus I_{\frac{R}{2}}} d\omega_1 \iint_{I_{\frac{R}{8}}\times I_{\frac{R}{8}}} d\omega_2 d\omega_3 \frac{g_1 g_2 g_3 \min\left\{\sqrt{\omega_2},\sqrt{\omega_3}\right\}}{\sqrt{\omega_1\omega_2\omega_3}} \times \\ \times\left[\frac{\partial\varphi_R}{\partial\omega}(\omega_1)(\omega_2-\omega_3) - \frac{C}{R}(\omega_2-\omega_3)^2\right]\end{aligned} \tag{3.47}$$

for some $C > 0$ independent of R perhaps changing from line to line. Using (3.46) we obtain the cancellation of the integral of the first term between brackets in

(3.47). Then:

$$J_{1,1} \geq -\frac{C}{R^{\frac{3}{2}}} \int_{I_R \setminus I_{\frac{R}{2}}} d\omega_1 \iint_{\left(I_{\frac{R}{8}} \times I_{\frac{R}{8}}\right) \cap \{\omega_2 \geq \omega_3\}} d\omega_2 d\omega_3 g_1 g_2 g_3 \left(\omega_2\right)^{\frac{3}{2}}$$

whence, using the definition of φ_R :

$$J_{1,1} \geq -\frac{C}{R^2} \left(\int g\varphi_R d\omega\right)^2 \left(\int g d\omega\right) \geq -\frac{C}{R^2} \left(\int g\varphi_R d\omega\right)^2 \tag{3.48}$$

On the other hand, in the integrand of $J_{1,2}$, either ω_2 or ω_3 are larger than $\frac{R}{8}$. Then:

$$J_{1,2} \geq -\frac{C}{R^2} \left(\int g\varphi_R d\omega\right)^2 \tag{3.49}$$

Combining (3.48), (3.49) we obtain:

$$J_1 \geq -\frac{C}{R^2} \left(\int g\varphi_R d\omega\right)^2 \tag{3.50}$$

In order to estimate J_2 we use the fact that the for the integrand to be different from zero we need $\max\{\omega_1, \omega_2\} \geq \frac{R}{2}$. We can assume that, say $\omega_1 \geq \omega_2$. We then use that $\Phi \leq \sqrt{\omega_2}$ as well as the properties of φ_R to obtain:

$$J_2 \geq -\frac{C}{R^2} \left(\int g\varphi_R d\omega\right)^2 \tag{3.51}$$

The terms $J_3,\ J_4$ can be estimated in a similar manner. In order to estimate J_4 we split the integral as:

$$J_4 = \int_{I_R^c} d\omega_1 \iint_{I_{\frac{R}{8}} \times I_{\frac{R}{8}}} d\omega_2 d\omega_3 + \int_{I_R \setminus I_{\frac{R}{2}}} d\omega_1 \iint_{\left(I_{\frac{R}{8}} \times I_{\frac{R}{8}}\right)^c} d\omega_2 d\omega_3 = J_{4,1} + J_{4,2}$$

Notice that, using again the symmetry $\omega_2 \longleftrightarrow \omega_3$ as well as Taylor, we obtain:

$$\iint_{I_{\frac{R}{8}} \times I_{\frac{R}{8}}} d\omega_2 d\omega_3 \frac{g_2 g_3 \min\left\{\sqrt{\omega_2}, \sqrt{\omega_3}\right\}}{\sqrt{\omega_2\omega_3}} \left[\omega_3 + \varphi_R\left(\omega_1 + \omega_2 - \omega_3\right) - \omega_2 - \varphi_R\left(\omega_1\right)\right]$$

$$\geq \iint_{I_{\frac{R}{8}} \times I_{\frac{R}{8}}} d\omega_2 d\omega_3 \frac{g_2 g_3 \min\left\{\sqrt{\omega_2}, \sqrt{\omega_3}\right\}}{\sqrt{\omega_2\omega_3}} \left[\frac{\partial \varphi_R}{\partial \omega}\left(\omega_1\right)\left(\omega_2 - \omega_3\right) - \frac{C}{R}\left(\omega_2 - \omega_3\right)^2\right]$$

$$= -\frac{C}{R} \iint_{I_{\frac{R}{8}} \times I_{\frac{R}{8}}} d\omega_2 d\omega_3 \frac{g_2 g_3 \min\left\{\sqrt{\omega_2}, \sqrt{\omega_3}\right\}}{\sqrt{\omega_2\omega_3}} \left(\omega_2 - \omega_3\right)^2 \geq -\frac{C}{R^{\frac{1}{2}}} \int g\varphi_R d\omega$$

Using now that $\omega_1 \geq \frac{R}{2}$ in the integral $J_{4,1}$ we obtain $J_{4,1} \geq -\frac{C}{R^2}\left(\int g\varphi_R d\omega\right)^2$. On the other hand, $J_{4,2}$ can be estimated as $J_{1,2}$. Then:

$$J_3 + J_4 \geq -\frac{C}{R^2} \left(\int g\varphi_R d\omega\right)^2 \tag{3.52}$$

Finally, we notice that in the integrals $J_5,\ J_6,\ J_7,\ J_8$ there are at least two integration variables larger than R. Then:

$$J_5 + J_6 + J_7 + J_8 \geq -\frac{C}{R^2} \left(\int g\varphi_R d\omega\right)^2 \tag{3.53}$$

Combining (3.50), (3.51), (3.52), (3.53) and applying (3.45) we obtain:

$$\partial_t \left(\int_{[0,\infty)} \varphi(\omega)\, g(t, d\omega) \right) \geq -\frac{C}{R^2} \left(\int \varphi_R(\omega)\, g(s, d\omega) \right)^2 ds \quad , \quad a.e. \ \ t \in [0, T]$$

whence the result follows. □

Notice that Proposition 3.11 can be understood as an upper estimate for the rate of transfer of energy towards higher values of ω. Indeed, the assumption $\int g_0 \varphi_R d\omega \geq \frac{1}{2}$ in Proposition 3.11 means that a significant fraction of the energy of the system is in the region $\omega \leq R$. In order to reduce this amount of energy by a significant amount we need at least times of order R^2.

3.3. Detailed asymptotic behaviour of weak solutions.

The following result gives a more detailed information on the form in which $g(t, \cdot)$ approaches to $m\delta_0$ as $t \to \infty$ if $R_* = 0$ in Theorem 3.2.

THEOREM 3.13. *Let ρ, g, m, R^* as in Theorem 3.2. Suppose that $R^* = 0$. Then, one of the following alternatives hold:*

(i) There exists $t^* \geq 0$, $t^* < \infty$ such that $\int_{\{0\}} g(t, d\omega) > 0$ for a.e. $t \geq t^*$. Moreover, for a.e. t_1, t_2 such that $t_1 \leq t_2$ we have $\int_{\{0\}} g(t_1, d\omega) \leq \int_{\{0\}} g(t_2, d\omega)$.

(ii) There exist an unbounded set of times $A \subset [0, \infty)$ such that for any $t \in A$ there exist $\Omega(t)$, $\eta(t)$ such that:

$$\int_{(\Omega(t)(1-\eta(t)), \Omega(t)(1+\eta(t)))} g(t, d\omega) \geq m(1 - \eta(t)) \quad \textit{for} \ \ t \in A \tag{3.54}$$

$$\int_{\{0\}} g(t, d\omega) = 0 \quad \textit{for } t \geq 0 \tag{3.55}$$

with

$$\lim_{t\to\infty} {}_A \eta(t) = 0, \quad \lim_{t\to\infty} {}_A \Omega(t) = 0. \tag{3.56}$$

where, for a function $f : A \to \mathbb{R}$ we will say that $\lim_{t\to\infty} {}_A f(t) = 0$ iff for any $\delta > 0$ there exists L sufficiently large such that, for any $t \in A \cap \{t > L\}$ we have $|f(t)| \leq \delta$.

The density of the set A is one, in the sense that:

$$\lim_{T\to\infty} \frac{|A \cap [0, T]|}{T} = 1 \tag{3.57}$$

In the case (ii) the function $\Omega(t)$ defined in A has an additional property, namely that, for any $\varepsilon > 0$, there exists T_0 large such that, if $t \geq T_0$ we have $\Omega(s) \leq \Omega(t)(1 + \varepsilon)$ for any $s \geq t$, with $s, t \in A$.

REMARK 3.14. Theorem 3.13 states that, either a condensate appears in finite time, i.e. $\int_{\{0\}} g(t, dx)$ becomes positive at some finite t, or alternatively during most of the times $g(t, \cdot)$ can be approximated by means of a of Dirac mass at a distances $R(t)$ of the origin. Notice that we can reformulate (ii) using the rescaled measures:

$$G(t, \cdot) = \frac{1}{\Omega(t)} g\left(t, \frac{\cdot}{\Omega(t)}\right)$$

Then, the alternative (ii) in Theorem 3.13 implies that:

$$\sup_{t\in A\cap\{t>L\}} dist_*\left(G\left(t,\cdot\right),m\delta_1\right)\to 0 \ \text{ as } \ L\to\infty \ .$$

REMARK 3.15. The function $\Omega\left(t\right)$ is "almost-monotone", in the sense that for $\Omega\left(t_2\right)$ is smaller than $\Omega\left(t_1\right)$, plus some small error which can be made arbitrarily small, if $t_2\geq t_1$ with t_1 large. Nevertheless, it is important to take into account that alternative (ii) in Theorem 3.13 does not imply that $G\left(t,\cdot\right)$ can be approximated as $m\delta_1$ for any large t. In Chapter 4 we will construct a class of weak solutions for which the alternative (ii) of Theorem 3.13 holds. One of the properties of those particular solutions is the existence, for each of them of an unbounded set $B\subset[0,\infty)$, and a positive constant c_0 such that for any $t\in B$ and any $\Omega>0$ we have $dist_*\left(\frac{1}{\Omega}g\left(t,\frac{\cdot}{\Omega}\right),m\delta_1\right)\geq c_0>0$. Seemingly, such a set B exists for any weak solution of (1.7) in the sense of Definition 2.2 for which the alternative (ii) in Theorem 3.13 is satisfied. However, we will not prove this result in this paper with that degree of generality. The main difficulty proving the existence of these sets B for arbitrary solutions is to control the displacement of the mass of g for distributions that are close to a Dirac mass.

REMARK 3.16. We can choose initial values g_{in} such that alternative (i) in Theorem 3.13 holds (cf. Theorem 3.17), and also initial data g_{in} for which alternative (ii) is satisfied (cf. Theorem 4.1).

PROOF OF THEOREM `3.13` . Some of the methods required to prove Theorem 3.13 have been introduced in [**15**] in order to study singularity formation for the Nordheim equation. The auxiliary results needed to prove Theorem `3.13` have been included in Chapter 6.

Let $g\in C\left([0,\infty):\mathcal{M}_+\left([0,\infty):(1+\omega)^{\rho}\right)\right)$ be a weak solution of (1.7) in the sense of Definition 2.2. Suppose that for any $t\geq 0$:

$$\int_{\{0\}} g\left(t,d\omega\right)=0. \tag{3.58}$$

Given $a>0,\ \delta>0, R>0$ arbitrarily small, let us denote as $\mathcal{A}_{a,R,\delta}\subset[0,\infty)$ the set of times $t\geq 0$ with the property that there exists an interval of the form $\mathcal{I}_k\left(b,R\right)$ with $b=1+a$ such that:

$$\int_{\mathcal{I}_k(b,R)} g\left(t,d\omega\right)d\epsilon\geq(1-\delta)\int_{(0,R]} g\left(t,d\omega\right)d\epsilon \tag{3.59}$$

where $k=k\left(t\right)$. We now claim that the Lebesgue's measure of the complement of the set $\mathcal{A}_{a,R,\delta}$ satisfies:

$$\lim_{T\to\infty}\frac{\left|[0,T]\setminus\mathcal{A}_{a,R,\delta}\right|}{T}=0 \tag{3.60}$$

for some $C\left(a,\delta\right)$ depending on $a,\ \delta$ and the total mass of $g,\ n_0$, but not on R. Indeed, combining Lemmas 6.2 and 6.4 as well as the fact that Lemma 6.3 implies that only the alternatives (i) and (ii) can take place we would have:

$$\nu\int_{[0,T]\setminus\mathcal{A}_{a,R,\delta}(T)} dt\left(\int_{(0,R]} g\left(t,d\omega\right)\right)^3\leq\frac{2Bb^{\frac{7}{2}}MR}{\rho^2\left(\sqrt{b}-1\right)^2} \tag{3.61}$$

By assumption $R^* = 0$. Theorem 3.2, as well as (3.58) imply that, for any fixed $R > 0$ we have:

$$\int_{(0,R]} g(t, d\omega) \geq \frac{M}{2} > 0 \quad \text{with} \quad M = \int g_{in}(d\epsilon)$$

if $t \geq t^*$, t^* is sufficiently large. Combining (3.61) with this inequality we obtain:

$$\frac{1}{T}\int_{[t^*,T]\setminus\mathcal{A}_{a,R,\delta}(T)} dt \leq \frac{C(a,\delta,R,M)}{T}$$

whence:

$$\frac{1}{T}\int_{[0,T]\setminus\mathcal{A}_{a,R,\delta}(T)} dt \leq \frac{C(a,\delta,R,M)+t^*}{T}$$

and taking the limit $T \to \infty$, we obtain (3.60). We can then construct the set A as follows. We choose decreasing sequences $\{a_n\}$, $\{\delta_n\}$, $\{R_n\}$ converging to zero. The definition of the sets $\mathcal{A}_{a_n,R_n,\delta_n}$ then implies that $\mathcal{A}_{a_{n+1},R_{n+1},\delta_{n+1}} \subset \mathcal{A}_{a_n,R_n,\delta_n}$. We then choose T_n sufficiently large to have

$$\frac{|[0,T]\setminus\mathcal{A}_{a_n,R_n,\delta_n}|}{T} \leq \frac{1}{n} \quad \text{for } T \geq T_n$$

Actually, we will assume an stronger condition on the sequence $\{T_n\}$, namely:

$$\frac{|\mathcal{A}_{a_n,R_n,\delta_n}\cap[T_{n-1},T]|}{T} \geq 1-\frac{1}{n} \quad \text{for} \quad T \geq T_{n+1}$$

Notice that, due to (3.60) this is possible assuming that the sequence T_n increases fast enough in order to have $\frac{T_n - T_{n-1}}{T_{n+1}}$ sufficiently small, say smaller than 2^{-n}. We then define sets $\mathcal{B}_n$, A as:

$$\mathcal{B}_n = [0,T_n]\cup\mathcal{A}_{a_n,R_n,\delta_n} \quad , \quad A = \bigcap_n \mathcal{B}_n$$

Then, if $T_{\ell+1} \leq T$ we obtain $\mathcal{B}_{\ell+1}\cap[T_{\ell-1},T) \subset \mathcal{B}_{\ell-1}\cap[T_{\ell-1},T)$. On the other hand, $\mathcal{B}_n \cap [0,T) = [0,T)$ if $n < \ell - 1$. We then write:

$$A\cap[0,T] = [A\cap[0,T_{\ell-1}]]\cup[A\cap[T_{\ell-1},T]]$$

and:

$$\frac{|A\cap[0,T_{\ell-1}]|}{T} \leq \frac{T_{\ell-1}}{T} \leq 2\cdot 2^{-\ell+1} = 4\cdot 2^{-\ell} \tag{3.62}$$

$$A\cap[T_{\ell-1},T] = \mathcal{B}_{\ell-1}\cap[T_{\ell-1},T]$$

whence:

$$\frac{|A\cap[T_{\ell-1},T]|}{T} = \frac{|\mathcal{B}_{\ell-1}\cap[T_{\ell-1},T]|}{T} = \frac{|\mathcal{A}_{a_\ell,R_\ell,\delta_\ell}\cap[T_{\ell-1},T]|}{T} \geq 1-\frac{1}{n} \tag{3.63}$$

Combining (3.62), (3.63) we obtain (3.57).

The definition of A implies the existence of $\Omega(t)$ with the properties stated in the Theorem. To conclude the description of the solutions given in the Theorem in the second case, it only remains to prove that for any $\varepsilon > 0$, there exists T_0 large such that $\Omega(s) \leq \Omega(t)(1+\varepsilon)$ if $s \geq t \geq T_0$. To this end we use the convex test function $\varphi(\omega) = \left(1-\frac{\omega}{\Omega(s)(1+\varepsilon)}\right)_+$. Lemma 2.24 implies that the function $t \to \int\left(1-\frac{\omega}{\Omega(s)(1+\varepsilon)}\right)_+ g(t,d\omega)$ is increasing. On the other hand, (3.54), (3.55), (3.56) imply that, assuming that $s \in A$ is large enough, we have

$\int\left(1-\frac{\omega}{\Omega(s)(1+\varepsilon)}\right)_{+}g\left(t,d\omega\right)\geq\frac{m\varepsilon}{1+\varepsilon}-\theta$, where $\theta>0$ can be made arbitrarily small if s is large enough. Indeed, this is due to the fact that $g\left(t,\cdot\right)$ can be approximated as $m\delta_{\Omega(s)}$ if $s\in A$ is sufficiently large and both the dispersion of the mass around $\omega=\Omega\left(s\right)$ and the amount of mass which is not close to this point can be made arbitrarily small for $t\geq T_0$ with T_0 depending on ε. On the other hand, if $\Omega\left(t\right)\geq\Omega\left(s\right)\left(1+\varepsilon\right)$ we would obtain $\int\left(1-\frac{\omega}{\Omega(s)(1+\varepsilon)}\right)_{+}g\left(t,d\omega\right)=0$. This would imply a contradiction.

In order to conclude the Proof of the Theorem it only remains to show that the function $t\to\int_{\{0\}}g\left(\bar{t},d\omega\right)$ is increasing. To this end we construct a family of convex test functions depending on two positive parameters M,a. The functions of the family which will be denotes as $\varphi_{M,a}\left(\omega\right)$ satisfy $\varphi_{M,a}\left(0\right)=M$, are increasing in M, decreasing in ω and such that the limit $\lim_{M\to\infty}\varphi_{M,a}\left(\omega\right)=\varphi_{\infty,a}\left(\omega\right)$ is finite for any $\omega>0$. We will assume also that the family $\{\varphi_{\infty,a}\left(\omega\right):a>0\}$ is increasing in a and it satisfies $\lim_{a\to0^+}\varphi_{M,a}\left(\omega\right)=0$ for any $\omega>0$. Using Lemma 2.24 we obtain $\int\varphi_{M,a}\left(\omega\right)g\left(t,d\omega\right)\geq\int\varphi_{M,a}\left(\omega\right)g\left(\bar{t},d\omega\right)$ *a.e.* $t\geq\bar{t}$ whence:

$$\int\varphi_{M,a}\left(\omega\right)g\left(t,d\omega\right)\geq\varphi_{M,a}\left(0\right)\int g\left(\bar{t},d\omega\right)=M\int g\left(\bar{t},d\omega\right)\quad,\quad a.e.\ t\geq\bar{t}\tag{3.64}$$

Taking the limit $a\to0^+$ we obtain, for any $M>0$ fixed:

$$\lim_{a\to0^+}\int_{(0,\infty)}\varphi_{M,a}\left(\omega\right)g\left(t,d\omega\right)=0$$

whence, using the fact that

$$\int\varphi_{M,a}\left(\omega\right)g\left(t,d\omega\right)=\int_{\{0\}}\varphi_{M,a}\left(0\right)g\left(t,d\omega\right)+\int_{(0,\infty)}\varphi_{M,a}\left(\omega\right)g\left(t,d\omega\right)$$

and taking the limit $a\to0^+$ it follows, using (3.64), as well as that $\varphi_{M,a}\left(0\right)=M$ that:

$$M\int_{\{0\}}g\left(t,d\omega\right)\geq M\int g\left(\bar{t},d\omega\right)\quad,\quad a.e.\ t\geq\bar{t}$$

whence the result follows. □

3.4. Finite time condensation.

THEOREM 3.17. *For $\rho<-1$, there exist $g_{in}\in\mathcal{M}_+\left([0,\infty):(1+\omega)^{\rho}\right)$ such that for any $g\in C\left([0,\infty):\mathcal{M}_+\left([0,\infty):(1+\omega)^{\rho}\right)\right)$, weak solution of (1.7) in the sense of Definition 2.2, with initial data g_{in}, the alternative (i) in Theorem 3.13 holds.*

PROOF. The existence of initial data $g_{in}\in\mathcal{M}_+\left([0,\infty):(1+\omega)^{\rho}\right)$ for which there exist $\bar{t}\geq0$ such that $\int_{\{0\}}g\left(\bar{t},d\omega\right)>0$ can be proved as in [**15**], Theorem 10.5. It would be possible to prove this result also using the arguments in [**34**]. The main difference among the results in [**15**] and [**34**] concerning the formation of condensate is that the initial data required in [**34**] must be assumed to behave like a suitable power law near the origin. On the contrary, in the case of the initial data considered in [**15**] it is possible to assume that the initial function f is bounded. Moreover the methods in [**34**]yield instantaneous condensation, in the sense that the solutions constructed there have $\int_{\{0\}}g\left(\bar{t},d\omega\right)>0$ for values of $\bar{t}$ arbitrarily small, due to the singular character of the initial data. On the contrary, the methods in [**15**] allow to obtain solutions satisfying $\int_{\{0\}} g(t, d\omega) = 0$ in some interval $0 \leq t < t^*$ and $\int_{\{0\}} g(\bar{t}, d\omega)$ for values $\bar{t} > t^*$ arbitrarily close to t^*. □

3.5. Finite time blow up of bounded mild solutions.

It is worth to remark that it is possible to have blow-up in finite time in the same manner starting from initially bounded solutions, as it happens for the Nordheim equation.

THEOREM 3.18. *Let $M > 0,\ E > 0,\ \nu > 0,\ \rho < -2$ be given. There exist $r = r(M, E, \nu) > 0, K^* = K^*(M, E, \nu) > 0, T_0 = T_0(M, E)$ and $\theta_* > 0$ independent on M, E, ν, such that for any $g_{in} \in L^\infty\left(\mathbb{R}^+; \sqrt{\omega}(1+\omega)^{\rho-\frac{1}{2}}\right)$ satisfying*

$$\int_{\mathbb{R}^+} g_{in}(d\omega) = M \ , \quad \int_{\mathbb{R}^+} \omega g_{in}(d\omega) = E, \tag{3.65}$$

$$\int_0^R g_{in}(d\omega) \geq \nu R^{\frac{3}{2}} \quad \text{for } 0 < R \leq r \quad , \quad \int_0^r g_{in}(d\omega) \geq K^*(r)^{\theta_*} \tag{3.66}$$

there exists a unique mild solution $g \in C\left([0, T_{\max}); L^\infty\left(\mathbb{R}^+; \sqrt{\omega}(1+\omega)^{\rho-\frac{1}{2}}\right)\right)$ of (1.7) defined for a maximal existence time $T_{\max} < T_0$ and such that $f(t, \omega) = \frac{g(t,\omega)}{\sqrt{\omega}}$ satisfies:

$$\lim_{t \to T_{\max}^-} \sup \|g(\cdot, t)\|_{L^\infty(\mathbb{R}^+)} = \infty.$$

THEOREM 3.19. *Let $M > 0, E > 0, \nu > 0, \rho < -2$ be given. There exist $r = r(M, E, \nu) > 0, K^* = K^*(M, E, \nu) > 0, T_0 = T_0(M, E)$ and $\theta_* > 0$ independent on $M,\ E, \nu$, such that for any $g_{in} \in L^\infty\left(\mathbb{R}^+; \sqrt{\omega}(1+\omega)^{\rho-\frac{1}{2}}\right)$ satisfying*

$$\int_{\mathbb{R}^+} g_{in}(d\omega) = M \ , \quad \int_{\mathbb{R}^+} \omega g_{in}(d\omega) = E \tag{3.67}$$

$$\int_0^R g_{in}(d\omega) \geq \nu R^{\frac{3}{2}} \quad \text{for } 0 < R \leq r \quad , \quad \int_0^r g_{in}(d\omega) \geq K^*(r)^{\theta_*} \tag{3.68}$$

there exists a weak solution $g \in C([0, \infty) : \mathcal{M}_+([0, \infty) : (1+\omega)^\rho))$ of (1.7) such that there exists $T_* > 0$ such that the following holds:

$$\sup_{0 \leq t \leq T_*} \|f(t, \cdot)\|_{L^\infty(\mathbb{R}^+)} < \infty \quad , \quad \sup_{T_* < t \leq T_0} \int_{\{0\}} g(t, d\omega) > 0 \tag{3.69}$$

where $g = \sqrt{\omega} f$.

The Proof of these Theorems is similar to the Proof of Theorems 2.3 and 10.5 in [**15**] respectively.

CHAPTER 4

Solutions without condensation: Pulsating behavior

4.1. Statement of the result

Our next goal is to prove the existence of a class of solutions of (1.7) for which the alternative (ii) in Theorem 3.13 holds. Moreover, the obtained solutions exhibit the behaviour described in Remark 3.15, namely the existence of an unbounded set of times B in which $g(t,\cdot)$ cannot be approximated by means of a Dirac mass in the form indicated in (3.54). This type of pulsating behavior cannot take place for the solutions of the Nordheim equation, as it follows from the results in [**15**], [**16**].

THEOREM 4.1. *There exist a class of weak solutions with interacting condensate of (1.7), in the sense of Definition 2.2, with $g \in C([0,\infty) : \mathcal{M}_+([0,\infty) : (1+\omega)^\rho))$, such that the alternative (ii) in Theorem 3.13 holds. Such solutions have the property $\int_{\{0\}} g(t,dx) = 0$ for any $t \geq 0$. Moreover, there exists a constant $c_1 > 0$ independent of g such that,*

$$\limsup_{t\to+\infty}\left(\inf_{a>0}\left(dist_*\left(\frac{1}{a}g\left(t,\frac{\cdot}{a}\right),m\delta_1\right)\right)\right) \geq c_1 \tag{4.1}$$

4.2. Proof of the result.

We now prove Theorem 4.1. To this end, we need to introduce some notation in order to describe the class of measures under consideration.

4.2.1. Notation. Our goal is to construct a weak solution of (1.7) with initial datum g_{in} which is a finite measure supported in a countable set of points. We will assume without loss of generality that $\int g_{in} = 1$. It is convenient to write the support as a disjoint union of countable sets. We define the following functions defined for $k \in \{1,2,3,...\}$:

$$\theta_\alpha(k) = k \quad \text{if } \alpha = 0 \quad , \quad \theta_\alpha(k) = 2k-1 \quad \text{if } \alpha \neq 0$$

We then set

$$x_\alpha(k) = 2^{-\alpha}\theta_\alpha(k) \quad , \quad k = 1,2,3,... \quad , \quad \alpha = 0,1,2,3,....$$

We introduce also the following disjoint sets of points:

$$\Omega_\alpha = \{x_\alpha(k) : k = 1,2,...\} \quad , \quad \alpha = 0,1,2,3,.... \tag{4.2}$$

We are interested in measures which have the mass $a_\alpha(k,t)$ at a given point $x_\alpha(k)$ at the time t, i.e., g has the form:

$$g(t,\cdot) = \sum_{\alpha=0}^{\infty}\sum_{k=1}^{\infty} a_\alpha(k)\,\delta_{x_\alpha(k)}(\cdot) \tag{4.3}$$

It is convenient to study separately the masses in each of the families. We will denote as $\mathcal{M}_+(\Omega_\alpha)$ the family of finite Radon measures supported in Ω_α. Then:

$$g = \sum_{\alpha=0}^{\infty} a_\alpha \tag{4.4}$$

where:

$$a_\alpha = \sum_{k=1}^{\infty} a_\alpha(k)\,\delta_{x_\alpha(k)} \in \mathcal{M}_+(\Omega_\alpha) \tag{4.5}$$

We will denote as m_α the total mass contained in the family Ω_α :

$$m_\alpha = \sum_{k=1}^{\infty} a_\alpha(k) = \langle a_\alpha, 1\rangle \tag{4.6}$$

We define the following auxiliary families of points:

$$\mathcal{Z}_\alpha = \bigcup_{\beta\le\alpha} \Omega_\beta \tag{4.7}$$

It is relevant to notice that the sets $\mathcal{Z}_\alpha$ can be obtained from Ω_0 by means of a rescaling. More precisely:

$$\mathcal{Z}_\alpha = 2^{-\alpha}\Omega_0\ ,\ \alpha \ge 0 \tag{4.8}$$

4.2.2. Heuristic description of the pulsating behaviour. The dynamics of the pulsating solutions will be characterized by the existence of two sequences of time intervals $\{t_n\}$, $\{s_n\}$ such that:

$$0 = t_0 < s_0 < t_1 < s_1 < t_2 < \ldots < t_n < s_n < t_{n+1} < s_{n+1} < \ldots \tag{4.9}$$

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \infty \tag{4.10}$$

During the time intervals $\{[t_n, s_n) : n = 0, 1, 2, 3, \ldots\}$ the solution $g(\cdot, t)$ will be in the so-called slow dynamics. The main characteristic of the dynamics of the is that, during these intervals most of the mass of g remains concentrated at the point $x_n(1)$, i.e., the smallest point of the family Ω_n. Therefore, we would have, approximately:

$$g(\cdot, t) \sim \delta(\cdot - x_n(1)) \quad , \quad t \in [t_n, s_n)$$

During the time intervals $\{[s_n, t_{n+1}) : n = 0, 1, 2, 3, \ldots\}$ the solution $g(\cdot, t)$ will pass through the so-called fast dynamics. These dynamics will be characterized by a fast transfer of most of the mass of g from $\omega = x_n(1)$ to $\omega = x_{n+1}(1)$. This transfer of mass will be characterized in a first stage by the spreading of the mass of g which at the beginning of this phase was mostly at $\omega = x_n(1)$ to the particles of the set $\mathcal{Z}_{n+1}$ defined in (4.7). During a second stage, this mass is transferred towards the particle $\omega = x_{n+1}(1)$ in such a way that at the time $t = t_{n+1}$ most of the mass of g is at such a point.

Something that will play a crucial role is the remainder of the mass which is not concentrated at the points $x_n(1)$ during the slow phases or at the points of the sets $\mathcal{Z}_{n+1}$ during the fast phases.

During the $n-$th fast phase, the main process taking place is the transfer of the mass from $\omega = x_n(1)$ to $\omega = x_{n+1}(1)$. More precisely, at the time $t = s_n$ most of the mass of g is concentrated at the point $\omega = x_n(1)$ and at the time $t = t_{n+1}$ most of the mass will be concentrated at the point $\omega = x_{n+1}(1)$.

During the forthcoming slow phase, which takes place in the time interval $[t_{n+1}, s_{n+1})$ the most relevant feature taking place is the interaction between the masses placed at the points $\omega = x_{n+1}(1)$, $\omega = x_{n+2}(1)$. The mass $a_{n+2}(1)$ whichis much smaller than the mass $a_{n+1}(1)$ can be described by means of a linear equation, which predicts a slow increase for $a_{n+2}(1)$. Due to the slow increment of $a_{n+2}(1)$ the mass $a_{n+2}(1)$ becomes comparable to the mass $a_{n+1}(1)$. This event marks the beginning of the next fast dynamics at time $t = s_{n+1}$.

The description of the masses during the fast stages cannot be approximated by a system of linear equations. On the contrary, it requires to study a system of nonlinear system of equations which describes the interactions between particles with many different values for the masses. However, using Proposition 2.21 it is possible to prove that after a sufficiently large time scale, most of the mass of g is transferred to $\omega = x_{n+2}(1)$. The end of this transfer will take place at the time $t = t_{n+2}$ and this will mark the starting time of a new slow phase.

The original distribution of masses at the points $x_n(k)$ at time $t = 0$ will be made in order to guarantee that most of the remainder of the mass, not contained in $\omega = x_{n+2}(1)$, will be at $\omega = x_{n+3}(1)$ at the time $t = t_{n+2}$. Therefore, the process described above would begin again and this iterative procedure would be repeated for arbitrarily long times.

In order to gain some intuition for the evolution of g we derive a set of approximated differential equations which describe the most relevant masses of g during each of the two phases. Suppose that we take an initial distribution of masses given by:

$$a_\alpha(k)(0) = \varepsilon_\alpha \delta_{k,1} \quad , \quad \text{with} \quad \sum_{\alpha=0}^{\infty} \varepsilon_\alpha = 1 \quad , \quad \varepsilon_\alpha > 0 \tag{4.11}$$

We will use in this Subsection the notation $<<$ to indicate that the term on the left is significantly smaller than the one on the right. We will use also the symbol $\approx$ to indicate that the quantities in both sides of this symbol are similar.

We first describe the slow phase which takes place in the interval $[t_n, s_n)$. During this phase most of the mass is concentrated in the families Ω_n, Ω_{n+1} although the mass in the second family will be much smaller, i. e. $m_{n+1}(t_n) << m_n(t_n)$. At the end of the previous fast dynamics phase we have $m_n(t_n) \approx a_n(1)(t_n) \approx 1$. We neglect the mass of g contained in the families Ω_α with $\alpha \geq (n+2)$ in the derivation of the approximate equations describing the evolution of g during this slow phase.

It is convenient to rescale the variables g, ω and t in order to bring the relevant points during this phase to the integers:

$$g = 2^{(n+1)}\bar{g} \quad , \quad \omega = 2^{-(n+1)}\bar{\omega} \quad , \quad t = t_n + 2^{-(n+1)}\bar{t} \tag{4.12}$$

This transformation brings the set $\Omega_n \cup \Omega_{n+1}$ to Ω_0 and it keeps invariant the equation (1.7). The evolution equation for measures $\bar{g}$ of the form

$$\bar{g}(\bar{t}, \cdot) = \sum_{k=1}^{\infty} \bar{a}_k(\bar{t})\, \delta_k(\cdot)$$

supported in Ω_0 is given by the countable set of equations:

$$\partial_{\bar{t}}\bar{a}_n(\bar{t}) = \sum_{k+m-\ell=n} \frac{\Phi_{k,m;\ell}}{\sqrt{km\ell}}\bar{a}_k\bar{a}_m\bar{a}_\ell - \left(\sum_{m,\ell=1}^{\infty} \frac{(2\Phi_{n,m;\ell}-\Phi_{\ell,m;n})}{\sqrt{nm\ell}}\bar{a}_m\bar{a}_\ell\right)\bar{a}_n, \tag{4.13}$$

for $n=1,2,\cdots$, where

$$\Phi_{k,m;\ell} = \min\left\{\sqrt{k},\sqrt{m},\sqrt{\ell},\sqrt{(k+m-\ell)_+}\right\}$$
$$\bar{a}_{2\ell-1}(\bar{t}) = a_{n+1}(\ell)(t) \quad , \quad \bar{a}_{2\ell}(\bar{t}) = a_n(\ell)(t) \quad , \quad \ell=1,2,...$$

During the slow phase under consideration we can approximate the evolution of the functions $\{\bar{a}_k\}$ using the equations.

$$\begin{aligned}
\partial_{\bar{t}}\bar{a}_1 &= \frac{1}{2}(\bar{a}_2)^2\bar{a}_1 + \frac{1}{2\sqrt{3}}(\bar{a}_2)^2\bar{a}_3 \\
\partial_{\bar{t}}\bar{a}_2 &= -\left(\bar{a}_1+\frac{1}{\sqrt{3}}\bar{a}_3\right)(\bar{a}_2)^2 \\
\partial_{\bar{t}}\bar{a}_3 &= \frac{1}{2}(\bar{a}_2)^2\bar{a}_1 + \frac{1}{2\sqrt{3}}(\bar{a}_2)^2\bar{a}_3
\end{aligned} \tag{4.14}$$

where we have neglected in (4.13) all the contributions due to the terms $\bar{a}_k,\ k\geq 4$. Notice that this approximation is consistent given the assumed relative size of the different masses.

Suppose that at the beginning of the slow phase (i.e. $\bar{t}=0$) we have $\bar{a}_k = \alpha_k,\ k=1,2,3$. By assumption $(\alpha_1+\alpha_3) << \alpha_2$. As long as $(\bar{a}_1+\bar{a}_3)$ remains small compared with $\bar{a}_2$ we can approximate the second equation in (4.14) as $\partial_{\bar{t}}\bar{a}_2=0$, i.e. $\bar{a}_2=\alpha_2$. Using this approximation we obtain from (4.14):

$$\begin{aligned}
\bar{a}_1(\bar{t}) &= \left(\frac{\sqrt{3}\alpha_1+\alpha_3}{\sqrt{3}+1}\right)e^{K\bar{t}} - \frac{(\alpha_3-\alpha_1)}{\sqrt{3}+1} \\
\bar{a}_3(\bar{t}) &= \left(\frac{\sqrt{3}\alpha_1+\alpha_3}{\sqrt{3}+1}\right)e^{K\bar{t}} - \frac{\sqrt{3}(\alpha_1-\alpha_3)}{\sqrt{3}+1}
\end{aligned}$$

These equations indicate that as $\bar{t}$ increases, the masses $\bar{a}_1(\bar{t})$ and $\bar{a}_3(\bar{t})$ become similar and simultaneously both of them increases exponentially. If we write $\varepsilon = \left(\sqrt{3}\alpha_1+\alpha_3\right)$ we obtain that $\bar{a}_1(\bar{t}),\ \bar{a}_3(\bar{t})$ become or order one, for times of order $\bar{t}\approx\frac{\log\left(\frac{1}{\varepsilon}\right)}{K}$. This marks the beginning of the next fast dynamics phase.

The description of the masses $\bar{a}_k$ cannot be made using a linear system of equations. During the fast phase the three masses $\bar{a}_1,\ \bar{a}_2,\ \bar{a}_3$ become comparable. It can then be readily seen from (4.13) that the mass of g is distributed by means of an involved nonlinear dynamics among many of the functions $\{\bar{a}_k\}$. It does not seem feasible to describe this dynamics by means of simple formulas. However, the result in Theorem 3.2 indicates that after a time $\bar{t}$ of order one most of the mass of g becomes concentrated at the value of $\bar{a}_1$.

The values of ε_α in (4.11) will be made in order to ensure that at the end of the fast phase which transfers the mass from $x_n(1)$ to $x_{n+1}(1)$ most of the remainder mass which is not contained in $x_{n+1}(1)$ is in the family Ω_{n+2}. Therefore an new slow phase begins which can be described as explained above. The process is then repeated infinitely often as $t\to\infty$.

It is important to take into account that the previous picture is an oversimplified description of the evolution of g. The main reason for this is that during the $n-$th fast dynamics phase, the mass of the families Ω_ℓ with ℓ larger than $(n+1)$ is transported to the points $x_\ell(k)$ with k large. Actually the values of k to which a meaningful fraction of the mass of the family Ω_ℓ is transported is much larger if $\ell >> n$. The consequence of this is that it could take very large times to arrive to a mass distribution which allows to approximate the dynamics of (4.13) by means of the system of three equations (4.14).

Another point to take into account is that in the previous heuristic description was assumed that only two families $\Omega_n,\ \Omega_{n+1}$ are relevant at each time. In a strict sense a careful analysis of the evolution of the mass in all the families $\Omega_\ell\ \ell > (n+1)$ will be needed. A key point in the whole construction is that arguments similar to the ones in the proof of Theorem 3.2 in the case $R_* > 0$, and more precisely, arguments like the ones in the proof of Lemma 3.6 will imply that the transfer of mass of the families Ω_n to Ω_ℓ with $\ell > n$ can be estimated by the amount of mass in the family Ω_ℓ. From this point of many of the arguments in the forthcoming pages can be thought as a some kind of continuous dependence result of Lemma 3.6. Indeed, Lemma 3.6 would imply, for the class of measures considered in this Section, that the transfer of mass from Ω_n to Ω_ℓ with $\ell > n$ vanishes if the mass at the family Ω_ℓ is zero. We will prove now that this transfer is small if the mass at the family Ω_ℓ is small.

4.2.3. Existence of global solutions in the class of measures supported in the sets $\bigcup_\beta \Omega_\beta$. Our next goal is to prove the existence of a class of global measured valued mild solutions of (2.2) such that $\mu\left([0,\infty) \setminus \bigcup_{\alpha=0}^{\infty} \Omega_\alpha\right) = 0$. The construction follows similar ideas to the proof of existence of mild solutions in Section 2.5. The main difference is that we will work with a particular class of measures. In order to use these measures, we need some results of functional analysis.

4.2.3.1. *Functional analysis preliminaries.* We are interested in proving some existence results for equation (1.7) in a class of measures $\mu \in \mathcal{M}_+([0,\infty))$ such that $\mu([0,\infty)) = \mu\left(\bigcup_{\alpha=0}^{\infty} \Omega_\alpha\right)$ where the sets Ω_α are as in (4.2). Notice that according to the definition of support of a measure given in (3.1), the support of a measure is a closed set, and since the closure of $\bigcup_{\alpha=0}^{\infty} \Omega_\alpha$ is $[0,\infty)$ the support of μ is different from $\bigcup_{\alpha=0}^{\infty} \Omega_\alpha$ in general. The following Theorem collects several classical results in [**42**] and [**6**] adapted to our particular setting.

THEOREM 4.2. *We will denote as the set of Radon measures $\mathcal{M}_+([0,\infty))$ the set of all the positive, continuous linear functionals $\Lambda : C_0([0,\infty)) \to \mathbb{R}$, where the topology of $C_0([0,\infty))$ is the topology of uniform convergence in compact sets. Let us denote the family of Borel sets of $[0,\infty)$ as $\mathcal{B}$. Then, there exists a unique Borel measure μ such that:*

a) $\Lambda f = \int_X f d\mu$ for every $f \in C_c(X)$.

b) $\mu(K) < \infty$ for every compact set $K \subset X$.

c) The relation $\mu(E) = \sup\{\mu(K) : K \subset E,\ K \text{ compact}\}$ holds for every $E \in \mathcal{B}$.

Moreover, the following additional properties hold:

i) For every $E \in \mathcal{B}$ we have: $\mu(E) = \inf\{\mu(V) : E \subset V,\ V \text{ open}\}$.

ii) If $E \in \mathcal{B}$, $A \subset E$, and $\mu(E) = 0$, then $\mu(A) = 0$.

iii) If $E\in\mathcal{B}$, and $\varepsilon>0$, there is a closed set F and an open set V such that $F\subset E\subset V$ and $\mu(V-F)<\varepsilon$.

iv) Suppose that we endow $\mathcal{M}_{+}([0,\infty))$ with the topology generated by the functionals $L_{\varphi}:\mathcal{M}_{+}([0,\infty))\to\mathbb{R}$, with $\varphi\in C_0([0,\infty))$. Then, for any $M>0$, the set $\left\{\mu\in\mathcal{M}_{+}([0,\infty)):\int\mu(d\omega)\le M\right\}$ is compact.

We will use extensively in the following Sections the fact that Radon measures can be characterized in two equivalent ways, namely as functions which assign values to the sets of the $\sigma-$algebra of Borel sets of $[0,\infty)$ or, alternatively as continuous functionals in $C_0([0,\infty))$. In order to avoid heavy notation we will use the same letter to denote the measure as set funtion and linear functional on $C_0([0,\infty))$.

We now define some functional spaces. From now on we will use the notation $\mathbb{R}_{+}$ to denote the set $[0,\infty)$. We remark that in the measures used until the rest of the paper we will have $\mu(\{0\})=0$.

Definition 4.3. Given $\theta>0,\ \rho^{*}>1$, we will denote as $\mathcal{X}_{\theta,\rho^{*}}$ the space of nonnegative Radon measures μ in $\mathcal{M}_{+}([0,\infty))$ such that $\mu\left([0,\infty)\setminus\bigcup_{\alpha=0}^{\infty}\Omega_{\alpha}\right)=0$, satisfying

$$\|\mu\|_{\theta,\rho^{*}}\equiv\sup_{\alpha\ge0}\left(\left(2^{\alpha}\right)^{\theta}\mu\left(\Omega_{\alpha}\right)\right)+\sup_{R\ge1}\left(\frac{\mu\left(\left[\frac{R}{2},R\right]\right)}{R^{1-\rho^{*}}}\right)<\infty \tag{4.15}$$

We will endow $\mathcal{X}_{\theta,\rho^{*}}$ with the weak topology of measures, i.e. the topology induced by the functionals $\mu\to\int\mu\varphi$ with $\varphi\in C_0(\mathbb{R}_{+})$.

Remark 4.4. Notice that, (4.15) implies that $\sum_{\alpha}\mu(\Omega_{\alpha})<\infty$. In particular, this implies the following representation formula for the measures $\mu\in\mathcal{X}_{\theta,\rho^{*}}$:

$$\mu=\sum_{\alpha=0}^{\infty}a_{\alpha}\quad,\quad a_{\alpha}(\ell)=\mu\left(\{x_{\alpha}(\ell)\}\right),\quad a_{\alpha}=\sum_{\ell=1}^{\infty}a_{\alpha}(\ell)\,\delta_{x_{\alpha}(\ell)} \tag{4.16}$$

where the convergences of the series are understood in the sense of the weak topology. Equivalently we can understand these measures as functions defined in Borel sets. We recall that given a Borel set B of $[0,\infty)$ we have $\delta_{x_0}(B)=1$ if $x_0\in B$ and $\delta_{x_0}(B)=0$ if $x_0\notin B$.

We will prove now the following result that will play a crucial role in the following.

Proposition 4.5. *Let $0<L<\infty,\ \theta>0,\ \rho^{*}>1$. The sets $\mathcal{K}_{\theta,\rho^{*}}(L)=\left\{\mu\in\mathcal{X}_{\theta,\rho^{*}}:\|\mu\|_{\theta,\rho^{*}}\le L<\infty\right\}\subset\mathcal{M}_{+}([0,\infty))$ are closed in the weak topology.*

Proof. Suppose that $\{\mu_m\}$ is a sequence of measures contained in $\mathcal{K}_{\theta,\rho^{*}}(R)$ such that $\mu_m\rightharpoonup\mu$ in the weak topology. We must prove that $\mu\left([0,\infty)\setminus\bigcup_{\alpha=0}^{\infty}\Omega_{\alpha}\right)=0$ as well as the inequality $\|\mu\|_{\theta,\rho^{*}}\le R$. We recall that for any Borel set A we have, using point (i) in Theorem (4.2),

$$\mu(A)=\inf\{\mu(U):A\subset U\ ,\ U\text{ open}\}.$$

Given any $\beta\ge0$ we define:

$$\mathcal{V}_{\beta}=\left[\mathbb{R}_{+}\setminus\bigcup_{\alpha=0}^{\beta}\Omega_{\alpha}\right]$$

Notice that (cf. (4.7)):

$$\mathcal{V}_{\beta} \cap \left[\bigcup_{\alpha=0}^{\infty} \Omega_{\alpha} \right] \subset \mathcal{Z}_{\beta+1}$$

Then, using that $\mu_m \in \mathcal{K}_{\theta,\rho^*}(L)$ we obtain

$$\begin{aligned} \mu_m(\mathcal{V}_\beta) &= \mu_m\left(\mathcal{V}_\beta \cap \left[\bigcup_{\alpha=0}^{\infty} \Omega_\alpha\right]\right) \leq \mu_m(\mathcal{Z}_{\beta+1}) \\ &\leq L \sum_{k=\beta+1}^{\infty} \left(2^{-k}\right)^{\theta} \leq CL\left(2^{-(\beta+1)}\right)^{\theta} \end{aligned} \tag{4.17}$$

for some $C > 0$ depending only in θ. Taking the limit $m \to \infty$ and using the weak convergence $\mu_m \rightharpoonup \mu$ we obtain:

$$\mu(\mathcal{V}_\beta) \leq CL\left(2^{-(\beta+1)}\right)^{\theta} \tag{4.18}$$

Then, using (4.18) as well as the fact that $[\mathbb{R}_+ \setminus \bigcup_{\alpha=0}^{\infty} \Omega_\alpha] = \bigcap_{\beta=0}^{\infty} \mathcal{V}_\beta$ we obtain:

$$\mu\left(\mathbb{R}_+ \setminus \bigcup_{\alpha=0}^{\infty} \Omega_\alpha\right) \leq \sum_{\beta \geq N} \mu(\mathcal{V}_{\beta,N}) \leq CL \sum_{\beta \geq N} \left(2^{-(1+N)}\right)^{\theta} \leq CL2^{-\theta(N+1)}$$

for $N \geq 1$ arbitrary, with $C > 0$ independent on N. Taking the limit $N \to \infty$ we obtain:

$$\mu\left(\mathbb{R}_+ \setminus \bigcup_{\alpha=0}^{\infty} \Omega_\alpha\right) = 0 \tag{4.19}$$

On the other hand, the measures μ_m satisfy the inequalities:

$$\sup_{\alpha \geq 0}\left((2^\alpha)^\theta \mu_m(\Omega_\alpha)\right) + \sup_{R \geq 1}\left(\frac{\mu_m\left(\left[\frac{R}{2}, R\right]\right)}{R^{1-\rho^*}}\right) \leq L$$

whence:

$$(2^\alpha)^\theta \mu_m(\Omega_\alpha) + \frac{\mu_m\left(\left[\frac{R}{2}, R\right]\right)}{R^{1-\rho^*}} \leq L$$

for any $\alpha \geq 0,\ R \geq 1$. Taking the limit $m \to \infty$, and then $\sup_{\alpha \geq 0}$ and $\sup_{R \geq 1}$ we obtain that:

$$\sup_{\alpha \geq 0}\left((2^\alpha)^\theta \mu(\Omega_\alpha)\right) + \sup_{R \geq 1}\left(\frac{\mu\left(\left[\frac{R}{2}, R\right]\right)}{R^{1-\rho^*}}\right) \leq L$$

This inequality, combined with (4.19) yields $\mu \in \mathcal{K}_{\theta,\rho^*}(L)$ and the Proposition follows. $\square$

Remark 4.6. The previous result encodes in a single functional analysis result one of the main ideas of the sought-for construction of measured valued solutions of (1.7). Notice that the class of measures satisfying (4.19) is dense in $\mathcal{M}_+([0,\infty))$ in the weak topology. The reason because the set of measures $\mathcal{K}_{\theta,\rho^*}(L)$ is closed in $\mathcal{M}_+([0,\infty))$, in spite of the fact that the closure of $\bigcup_{\alpha=0}^{\infty} \Omega_\alpha$ is $[0,\infty)$, is by the condition $\|\mu\|_{\theta,\rho^*} \leq L$ that yields a fast decay of the amount of mass concentrated in the sets Ω_α with large values of α.

4.2.3.2. *Study of some auxiliary operators.* We need to study the properties of the following auxiliary function $A_g(\omega_1)$. It is worth to compare this result with Lemma 2.7. Notice that, differently from Lemma 2.7, we assume that $\sigma = 0$. The function A_g can still be defined in spite of this due to the decay properties of the measures $g \in \mathcal{X}_{\theta,\rho^*}$ for small ω.

LEMMA 4.7. *Suppose that $g \in \mathcal{X}_{\theta,\rho^*}$ for some $\theta > \frac{1}{2}$, $\rho^* > 1$. Then $A_g(\omega_1)$ defined by means of*

$$A_g(\omega_1) = -\iint \Phi\left[\frac{2g_2g_3}{\sqrt{\omega_1\omega_2\omega_3}} - \frac{g_3g_4}{\sqrt{\omega_1\omega_3\omega_4}}\right] d\omega_3 d\omega_4 \tag{4.20}$$

where $\omega_2 = \omega_3 + \omega_4 - \omega_1$ defines a continuous function in $[0,\infty)$. Moreover, we have:

$$A_g(\omega_1) \geq 0 \quad , \quad \omega_1 \in [0,\infty) \tag{4.21}$$

PROOF. The function $\frac{\Phi}{\sqrt{\omega_1}}$ is continuous for $\omega_1 > 0$ and $(\omega_1,\omega_2) \in \mathbb{R}_+^2$. Therefore, each of the terms $\frac{2\Phi g_2g_3}{\sqrt{\omega_1\omega_2\omega_3}}$, $\frac{\Phi g_3g_4}{\sqrt{\omega_1\omega_3\omega_4}}$ are Radon measures in $\mathbb{R}_+^2$. In order to prove the convergence of each of the integrals we just notice that the condition (4.15) implies the estimate:

$$\int_{\left[\frac{R}{2},R\right]} g(d\omega) \leq C \|g\|_{\theta,\rho^*} \min\left\{R^{\theta}, R^{1-\rho^*}\right\} \quad \text{with } \theta > \frac{1}{2},\ \rho^* > 1 \tag{4.22}$$

Therefore, (4.20) defines a continuous function in $\{\omega_1 > 0\}$. We can define $A_g(0)$ as $\lim_{\omega_1} A_g(\omega_1)$. In order to prove the existence of this limit we consider separately the two additive terms. In the case of $J_2 = \iint \frac{\Phi g_3g_4}{\sqrt{\omega_1\omega_3\omega_4}}$ we decompose the integration region in the sets $Q_\delta = \{\omega_3 \geq \delta,\ \omega_4 \geq \delta\}$ with $\delta > 0$ small as well as its complementary $\mathbb{R}_+^2 \backslash Q_\delta$. The integrals $\iint_{Q_\delta}[\cdots]$ are independent of ω_1 if ω_1 is small, due to the definition of Φ. On the other hand, the term $\iint_{\mathbb{R}_+^2\backslash Q_\delta}[\cdots]$ converges to zero as $\delta \to 0$ due to (4.22). This implies the existence of the limit $\lim_{\omega_1\to 0} J_2$. In order to prove the existence of a similar limit for $J_1 = \iint \frac{2\Phi g_2g_3}{\sqrt{\omega_1\omega_2\omega_3}} d\omega_3 d\omega_4$ we first replace the variable of integration ω_4 by ω_2 by means of a change of variables. We now repeat a similar splitting argument of the integral in the sets Q_δ and $\mathbb{R}_+^2 \setminus Q_\delta$ and use the same argument to prove the existence of $\lim_{\omega_1\to 0} J_1$. This concludes the proof of the existence of the continuity of the function $A_g(\cdot)$ in $[0,\infty)$.

In order to prove (4.21) we rewrite $A_g(\omega_1)$. Notice that:

$$\iint \Phi\frac{2g_2g_3}{\sqrt{\omega_1\omega_2\omega_3}} d\omega_3 d\omega_4 = \iint \Phi\frac{g_2g_3}{\sqrt{\omega_1\omega_2\omega_3}} d\omega_3 d\omega_4 + \iint \Phi\frac{g_2g_4}{\sqrt{\omega_1\omega_2\omega_4}} d\omega_3 d\omega_4 \tag{4.23}$$

We now use the change of variables $\omega_2 = \omega_3 + \omega_4 - \omega_1$, $d\omega_2 = d\omega_4$ in the first integral and $\omega_2 = \omega_3 + \omega_4 - \omega_1$, $d\omega_2 = d\omega_3$ in the second one. Then, replacing the variable ω_2 by ω_4 in the first resulting integral and ω_2 by ω_3 in the second, we obtain that the integral in (4.23) becomes

$$\iint \frac{g_3g_4}{\sqrt{\omega_1\omega_3\omega_4}} \Psi d\omega_3 d\omega_4$$

where:

$$\Psi = \Psi_1 + \Psi_2$$

$$\Psi_1 = \chi_{\{\omega_3\geq\omega_4\}}\chi_{\{\omega_3\geq\omega_1\}}\sqrt{(\omega_1+\omega_4-\omega_3)_+} + \chi_{\{\omega_3\leq\omega_4\}}\chi_{\{\omega_3\geq\omega_1\}}\sqrt{\omega_1}+$$
$$+\chi_{\{\omega_3\geq\omega_4\}}\chi_{\{\omega_3\leq\omega_1\}}\sqrt{\omega_4} + \chi_{\{\omega_3\leq\omega_4\}}\chi_{\{\omega_3\leq\omega_1\}}\sqrt{\omega_3}$$

$$\Psi_2 = \chi_{\{\omega_3\leq\omega_4\}}\chi_{\{\omega_4\geq\omega_1\}}\sqrt{(\omega_1+\omega_3-\omega_4)_+} + \chi_{\{\omega_3\geq\omega_4\}}\chi_{\{\omega_4\geq\omega_1\}}\sqrt{\omega_1}+$$
$$+\chi_{\{\omega_3\leq\omega_4\}}\chi_{\{\omega_4\leq\omega_1\}}\sqrt{\omega_3} + \chi_{\{\omega_3\geq\omega_4\}}\chi_{\{\omega_4\leq\omega_1\}}\sqrt{\omega_4}$$

Notice that $\Psi \geq \Phi$ whence (4.21) follows. □

We now define a nonlinear operator in terms of any given measure $g \in \mathcal{X}_{\theta,\rho^*}$. Notice that it is possible to characterize Radon measures either by means of the measure of Borel sets or by means of the action of the measure as an element of the dual of the space of compactly supported continuous functions. We have decided to follow the second approach in the definition of $\mathcal{O}[g]$ in the following Lemma, even if it would be simpler to define the measure of the subsets of $\bigcup_{\alpha=0}^{\infty}\Omega_\alpha$ in order to obtain a definition consistent with the one given in Lemma 2.9.

Lemma 4.8. *Suppose that $g \in \mathcal{X}_{\theta,\rho^*}$ for some $\theta > 1$ and $\rho^* > 1$. Then, the following formula defines a mapping $\mathcal{O} : \mathcal{X}_{\theta,\rho^*} \to \mathcal{X}_{\theta,\rho^*}$:*

$$\mathcal{O}[g] = \iint \Phi \frac{g_2 g_3 g_4}{\sqrt{\omega_2\omega_3\omega_4}} d\omega_3 d\omega_4 \ , \ \omega_2 = \omega_3+\omega_4-\omega_1 \tag{4.24}$$

where the action of the measure $\mathcal{O}[g]$ acting over a test function $\varphi \in C_0(\mathbb{R}_+)$ is given by:

$$\langle \mathcal{O}[g], \varphi\rangle = \iiint \Phi \frac{g_2 g_3 g_4}{\sqrt{\omega_2\omega_3\omega_4}} \varphi(\omega_1)\, d\omega_3 d\omega_4 d\omega_1 \tag{4.25}$$

Moreover, we have the estimate:

$$\|(\mathcal{O}[g])\|_{\theta,\rho^*} \leq C\|g\|^3_{\theta,\rho^*} \tag{4.26}$$

Proof. Using the definition of the measure g_2 (i.e. the change of variables) we would have:

$$\begin{aligned}\langle \mathcal{O}[g], \varphi\rangle = \iiint \Phi(\omega_3+\omega_4-\omega_2, \omega_2, \omega_3, \omega_4) \frac{g_2 g_3 g_4}{\sqrt{\omega_2\omega_3\omega_4}} \times \\ \times\varphi(\omega_3+\omega_4-\omega_2)\, d\omega_2 d\omega_3 d\omega_4\end{aligned} \tag{4.27}$$

Using the definition of Φ as well as (4.22) we immediately obtain that (4.27) converges for any $\varphi \in C_0(\mathbb{R}_+)$. Moreover $|\langle \mathcal{O}[g], \varphi\rangle| \leq C\|g\|^3_{\theta,\rho^*}\|\varphi\|_\infty$ and therefore $\mathcal{O}[g] \in \mathcal{M}_+(\mathbb{R}_+)$. Notice that the constant C is independent of φ, due to the decay assumptions made for g for large and small values. Therefore, the operator $\langle \mathcal{O}[g], \varphi\rangle$ is well defined for any $\varphi \in C_b(\mathbb{R}_+)$.

In order to prove that $\mathcal{O}[g] \in \mathcal{X}_{\theta,\rho^*}$ let us show that $\mathcal{O}[g](\mathbb{R}_+ \setminus \bigcup_{\alpha=0}^{\infty}\Omega_\alpha) = 0$ as well as $\|\mathcal{O}[g]\|_{\theta,\rho^*} < \infty$. To this end we first approximate $\mathcal{O}[g]$ in the weak topology by a sequence $(\mathcal{O}[g])_N \in \mathcal{X}_{\theta,\rho^*}$ as follows. Suppose that $g = \sum_{\alpha=0}^{\infty} a_\alpha$, with $a_\alpha \in \mathcal{X}_{\theta,\rho^*}$ satisfying $a_\alpha(\mathbb{R}_+ \setminus \Omega_\alpha)$ (cf. (4.16)). We then define:

$$\begin{aligned}\langle (\mathcal{O}[g])_N , \varphi\rangle &= \sum_{\alpha\le N}\sum_{\beta\le N}\sum_{\gamma\le N}\iiint \Phi(\omega_3+\omega_4-\omega_2,\omega_2,\omega_3,\omega_4)\times \\ &\qquad \times \frac{a_{\alpha,2}a_{\beta,3}a_{\gamma,4}}{\sqrt{\omega_2\omega_3\omega_4}}\varphi(\omega_3+\omega_4-\omega_2)\,d\omega_2 d\omega_3 d\omega_4 \\ &= \iiint \Phi(\omega_3+\omega_4-\omega_2,\omega_2,\omega_3,\omega_4)\frac{G_{N,2}G_{N,3}G_{N,4}}{\sqrt{\omega_2\omega_3\omega_4}}\times \\ &\qquad \times \varphi(\omega_3+\omega_4-\omega_2)\,d\omega_2 d\omega_3 d\omega_4\end{aligned}$$

where:

$$G_N = \sum_{\alpha\le N} a_\alpha$$

We claim that $\lim_{N\to\infty}(\mathcal{O}[g])_N = \mathcal{O}[g]$ in the weak topology. To prove this, we use that $\Phi \le \min\{\sqrt{\omega_2},\sqrt{\omega_3},\sqrt{\omega_4}\}$, and obtain the estimate:

$$|\langle (\mathcal{O}[g])_N - \mathcal{O}[g], \varphi\rangle| \le 7\|\varphi\|_\infty \left(\int_{\bigcup_{\eta\ge N}\Omega_\eta} g\right)\left(\int \frac{g(\omega)}{\sqrt{\omega}}d\omega\right)^2, \tag{4.28}$$

where, in order to compute the difference $(\mathcal{O}[g])_N - \mathcal{O}[g]$ we have written $g = G_N + H_N$ with $H_N(\mathbb{R}_+ \setminus \bigcup_{\alpha=N}^\infty \Omega_\alpha) = 0$. The difference $g_2g_3g_4 - G_{N,2}G_{N,3}G_{N,4}$ can be written in terms of sums of products of functions G_N and H_N containing at least one measure H_N. Estimating Φ by one of the square roots, and using the fact that $H_N(\mathbb{R}_+ \setminus \bigcup_{\alpha=N}^\infty \Omega_\alpha) = 0$ and $H_N(\bigcup_{\alpha=N}^\infty \Omega_\alpha) = G_N(\bigcup_{\alpha=N}^\infty \Omega_\alpha)$ we obtain (4.28). Using the definition of $\mathcal{X}_{\theta,\rho^*}$ we then obtain:

$$|\langle (\mathcal{O}[g])_N - \mathcal{O}[g], \varphi\rangle| \le C\|\varphi\|_\infty \|g\|^3_{\theta,\rho^*}\sum_{\eta\ge N}\left(2^{-\eta}\right)^\theta \to 0 \quad \text{as} \quad N\to\infty$$

This gives the desired convergence $(\mathcal{O}[g])_N \rightharpoonup \mathcal{O}[g]$ as $N\to\infty$. We then obtain the representation formula:

$$\begin{aligned}\langle \mathcal{O}[g], \varphi\rangle &= \sum_{\alpha}\sum_{\beta}\sum_{\gamma}\iiint \Phi(\omega_3+\omega_4-\omega_2,\omega_2,\omega_3,\omega_4)\times \\ &\qquad \times \frac{a_{\alpha,2}a_{\beta,3}a_{\gamma,4}}{\sqrt{\omega_2\omega_3\omega_4}}\varphi(\omega_3+\omega_4-\omega_2)\,d\omega_2 d\omega_3 d\omega_4\end{aligned} \tag{4.29}$$

for any $\varphi \in C_0(\mathbb{R}_+)$. Our next goal is to prove:

$$\mathcal{O}[g]\left(\mathbb{R}_+ \setminus \bigcup_{\alpha=0}^\infty \Omega_\alpha\right) = 0 \tag{4.30}$$

Using again property (i) in Theorem (4.2) we have:

$$\begin{aligned}\mathcal{O}[g]\left(\mathbb{R}_+ \setminus \bigcup_{\alpha=0}^\infty \Omega_\alpha\right) &\le \mathcal{O}[g](\mathbb{R}_+ \setminus \mathcal{Z}_N) \\ &= \mathcal{O}[g]\left(\mathbb{R}_+ \setminus \bigcup_{x\in\mathcal{Z}_N}\left(x-\frac{\varepsilon}{2^N}, x+\frac{\varepsilon}{2^N}\right)\right) + \\ &\quad + \mathcal{O}[g]\left(\bigcup_{x\in\mathcal{Z}_N}\left[\left(x-\frac{\varepsilon}{2^N}, x+\frac{\varepsilon}{2^N}\right)\setminus\{x\}\right]\right)\end{aligned} \tag{4.31}$$

We can estimate the first term on the right hand side, using a nonnegative and continuous test function $\varphi_{\varepsilon,N}$ which takes the value 1 in the set $\mathbb{R}_+ \setminus \left[x - \frac{\varepsilon}{2^N}, x + \frac{\varepsilon}{2^N}\right]$ and vanishes in a neighbourhood of the points $\{x \in \mathcal{Z}_N\}$. Then:

$$\mathcal{O}[g]\left(\mathbb{R}_+ \setminus \bigcup_{x\in\mathcal{Z}_N}\left(x - \frac{\varepsilon}{2^N}, x + \frac{\varepsilon}{2^N}\right)\right) \leq \langle \mathcal{O}[g], \varphi_{\varepsilon,N}\rangle \tag{4.32}$$

We then use the representation formula (4.29) to compute the right-hand side of (4.32). We split the triple sum as follows:

$$\langle \mathcal{O}[g], \varphi_{\varepsilon,N}\rangle = \sum_{\substack{\alpha,\beta,\gamma \\ \max\{\alpha,\beta,\gamma\}\leq N}} \iiint [\cdots] + \sum_{\substack{\alpha,\beta,\gamma \\ \max\{\alpha,\beta,\gamma\}> N}} \iiint [\cdots] \tag{4.33}$$

The first term on the right vanishes, because, due to our choice of the function $\varphi_{\varepsilon,N}$, this term contains only contributions of points such that $\omega_3 + \omega_4 - \omega_2 \in \bigcup_{\sigma=N+1}^{\infty} \Omega_\sigma$, with $\omega_2 \in \Omega_\alpha$, $\omega_3 \in \Omega_\beta$, $\omega_4 \in \Omega_\gamma$. However, this set is empty because $\max\{\alpha,\beta,\gamma\} \leq N$. Indeed, if such a set of values of $(\omega_2, \omega_3, \omega_4)$ exists we would have:

$$2^{-\beta}\theta_\beta + 2^{-\gamma}\theta_\gamma - 2^{-\alpha}\theta_\alpha = 2^{-\sigma}\theta_\sigma \quad , \quad \sigma \geq N+1 \quad , \quad \max\{\alpha,\beta,\gamma\} \leq N \tag{4.34}$$

where $\theta_\alpha,\ \theta_\beta,\ \theta_\gamma,\ \theta_\sigma$ are positive integers and in addition θ_σ is an odd number. However, (4.34) implies:

$$\theta_\sigma = 2^{\sigma-\beta}\theta_\beta + 2^{\sigma-\gamma}\theta_\gamma - 2^{\sigma-\alpha}\theta_\alpha$$

and since $\sigma \geq \max\{\alpha,\beta,\gamma\} + 1$ this implies that θ_σ is an odd number, that would be a contradiction. Therefore:

$$\sum_{\substack{\alpha,\beta,\gamma \\ \max\{\alpha,\beta,\gamma\}\leq N}} \iiint [\cdots] = 0 \tag{4.35}$$

On the other hand, the last term in (4.33) can be estimated, using the fact that each of the integrals contains at least one index α, β, γ larger than N. Then:

$$\sum_{\substack{\alpha,\beta,\gamma \\ \max\{\alpha,\beta,\gamma\}> N}} \iiint [\cdots] \leq 3\,\|\varphi_{\eta,\varepsilon}\|_\infty \left(\int_{\bigcup_{\alpha\geq(N+1)}\Omega_\alpha} g\right)\left(\int \frac{g(\omega)}{\sqrt{\omega}}d\omega\right)^2 \tag{4.36}$$
$$\leq C\,\|g\|_{\theta,\rho^*}^3 \left(2^{-(N+1)}\right)^\theta$$

and this approaches to zero as $N \to \infty$.

We now estimate the last term in (4.31). To this end we use the regularity properties of the measure $\mathcal{O}[g]$. We first compute $\mathcal{O}[g](\{x\})$, $x \in \mathcal{Z}_N$ by means of:

$$\mathcal{O}[g](\{x\}) = \lim_{\delta\to 0} \mathcal{O}[g]((x-\delta, x+\delta)) \tag{4.37}$$

Notice that $\mathcal{O}[g]((x-\delta, x+\delta))$ can be estimated from below and above, using nonnegative continuous test functions $\varphi_1,\ \varphi_2$ satisfying $\varphi_1 \leq \varphi_2,\ \varphi_1 = 1$ in $\left[x - \frac{\delta}{2}, x + \frac{\delta}{2}\right]$, $\varphi_1 = 0$ in $\mathbb{R}_+ \setminus (x-\delta, x+\delta)$, $\varphi_2 = 1$ in $[x-\delta, x+\delta]$, $\varphi_2 = 0$ in

$\mathbb{R}_+ \setminus (x-2\delta, x+2\delta)$. We then compute $\langle \mathcal{O}[g], \varphi_1 \rangle$, $\langle \mathcal{O}[g], \varphi_2 \rangle$ using (4.29). Splitting the contribution of the terms of the sum yielding x and the rest in the interval of integration we obtain, for any of these functions:

$$|\langle \mathcal{O}[g], \varphi_k \rangle - M_x| \leq R(\delta) \quad , \quad k=1,2$$

where, using that $\varphi_k(x) = 1$, $k=1,2$

$$M_x = \sum_{\substack{x_\beta + x_\gamma = x_\alpha + x \\ x_\alpha \in \Omega_\alpha,\ x_\beta \in \Omega_\beta,\ x_\gamma \in \Omega_\gamma}} \sum \sum \Phi(x_\beta + x_\gamma - x_\alpha, x_\alpha, x_\beta, x_\gamma) \frac{a_\alpha(\{x_\alpha\}) a_\beta(\{x_\beta\}) a_\gamma(\{x_\gamma\})}{\sqrt{x_\alpha x_\beta x_\gamma}}$$

and, using $\|\varphi_k\|_\infty \leq 1$:

$$R(\delta) = \sum_{\substack{0<|x_\beta + x_\gamma - x_\alpha - x| < \delta \\ x_\alpha \in \Omega_\alpha,\ x_\beta \in \Omega_\beta,\ x_\gamma \in \Omega_\gamma}} \sum \sum \Phi(x_\beta + x_\gamma - x_\alpha, x_\alpha, x_\beta, x_\gamma) \frac{a_\alpha(\{x_\alpha\}) a_\beta(\{x_\beta\}) a_\gamma(\{x_\gamma\})}{\sqrt{x_\alpha x_\beta x_\gamma}}$$

Then, using arguments analogous to those yielding (4.35), (4.36) we obtain:

$$M_x \leq C \|g\|^3_{\theta,\rho^*} \left(2^{-\eta}\right)^\theta \quad \text{if} \quad x \in \Omega_\eta$$

We must estimate now the remainder $R(\delta)$. We claim that $\lim_{\delta \to 0} R(\delta) = 0$ for each x fixed. Indeed, let us denote as $\sigma = \max\{\alpha, \beta, \gamma, N\}$. We have $0 < |x_\beta + x_\gamma - x_\alpha - x| < \delta$. Then:

$$0 < |2^\sigma x_\beta + 2^\sigma x_\gamma - 2^\sigma x_\alpha - 2^\sigma x| < \delta 2^\sigma$$

Notice that $2^\sigma x_\beta + 2^\sigma x_\gamma - 2^\sigma x_\alpha - 2^\sigma x$ is an integer. Then its absolute value is larger than one, whence $1 \leq \delta\, 2^\sigma$. Since x and therefore N is fixed this implies:

$$\max\{\alpha, \beta, \gamma, N\} \geq \frac{\log\left(\frac{1}{\delta}\right)}{\log(2)} \to \infty \quad \text{as} \quad \delta \to 0$$

We can then estimate $R(\delta)$ as:

$$R(\delta) \leq C \|g\|^3_{\theta,\rho^*} \left(2^{-\sigma}\right)^\theta = C \|g\|^3_{\theta,\rho^*} (\delta)^\theta \to 0 \quad \text{as} \quad \delta \to 0$$

whence, using (4.37):

$$\mathcal{O}[g](\{x\}) = M_x \tag{4.38}$$

We can now estimate the last term in (4.31). A similar argument shows that, for ε small:

$$\mathcal{O}[g]\left(\left(x - \frac{\varepsilon}{2^N}, x + \frac{\varepsilon}{2^N}\right)\right) \leq M_x + C\|g\|^3_{\theta,\rho^*} \left(\frac{\varepsilon}{2^N}\right)^\theta \tag{4.39}$$

Using (4.38) as well as (4.39) we obtain:

$$\mathcal{O}[g]\left(\left(x - \frac{\varepsilon}{2^N}, x + \frac{\varepsilon}{2^N}\right) \setminus \{x\}\right) \leq C\|g\|^3_{\theta,\rho^*} \left(\frac{\varepsilon}{2^N}\right)^\theta$$

whence:

$$\mathcal{O}[g]\left(\bigcup_{x \in \mathcal{Z}_N} \left(x - \frac{\varepsilon}{2^N}, x + \frac{\varepsilon}{2^N}\right) \cap \{\omega \leq R_0\} \setminus \{x\}\right) \leq CR_0 \|g\|^3_{\theta,\rho^*} \left(\frac{\varepsilon}{2^N}\right)^\theta \cdot 2^N$$

where we use the fact that the number of points of $\mathcal{Z}_N \cap \{\omega \leq R_0\}$ can be estimated as $CR_0 2^N$. Since $\theta > 1$ it then follows that this measure converges to zero as $N \to \infty$.

On the other hand

$$\mathcal{O}[g]\left(\bigcup_{x\in\mathcal{Z}_N}\left(x-\frac{\varepsilon}{2^N},x+\frac{\varepsilon}{2^N}\right)\cap\{\omega>R_0\}\setminus\{x\}\right)$$

can be estimated as $C\|g\|_{\theta,\rho^*}^3 R_0^{1-\rho^*}$. This term can be made small choosing R_0 large. Therefore, all the terms on the right-hand side of (4.31) can be made arbitrarily small, whence (4.30) follows.

To conclude the proof of the Lemma it only remains to obtain (4.26). We first estimate in the formula for $\|(\mathcal{O}[g])\|_{\theta,\rho^*}$ the contributions from the regions where $\omega_1\geq\frac{1}{2}$. To this end, let $R\geq 1$, and define a continuous test function $\varphi=\varphi(\omega_1)$ such that $\varphi(\omega_1)=1$ if $\omega_1\geq\frac{R}{2}$, $\varphi(\omega_1)=0$ if $\omega_1\leq\frac{R}{4}$, $0\leq\varphi(\omega_1)\leq 1$ if $\omega_1\in\mathbb{R}_+$. Then:

$$\mathcal{O}[g]\left(\left[\frac{R}{2},R\right]\right)\leq\iiint\Phi(\omega_3+\omega_4-\omega_2,\omega_2,\omega_3,\omega_4)\times$$
$$\times\frac{g_2g_3g_4}{\sqrt{\omega_2\omega_3\omega_4}}\varphi(\omega_3+\omega_4-\omega_2)\,d\omega_2d\omega_3d\omega_4$$

Using the symmetry $\omega_3\leftrightarrow\omega_4$, as well as the fact that $\Phi\leq\sqrt{\omega_2}$ we obtain:

$$\mathcal{O}[g]\left(\left[\frac{R}{2},R\right]\right)\leq 2\iiint_{\{\omega_4\geq\omega_3\}}\frac{g_2g_3g_4}{\sqrt{\omega_3\omega_4}}\varphi(\omega_3+\omega_4-\omega_2)\,d\omega_2d\omega_3d\omega_4$$

Since the function $\varphi(\omega_3+\omega_4-\omega_2)$ vanishes for $\omega_3+\omega_4-\omega_2\leq\frac{R}{4}$, it follows that the set where the integrand does not vanishes is included in the set where $\omega_3+\omega_4\geq\frac{R}{4}$, and since $\omega_3\leq\omega_4$, we can then obtain an upper bound for the integral restricting the integration to the set $\left\{\omega_4\geq\frac{R}{8}\right\}$. Since $\varphi\leq 1$ it then follows that:

$$\mathcal{O}[g]\left(\left[\frac{R}{2},R\right]\right)\leq 2\iiint_{\left\{\omega_4\geq\frac{R}{8}\right\}}\frac{g_2g_3g_4}{\sqrt{\omega_3\omega_4}}d\omega_2d\omega_3d\omega_4\leq C\|g\|_{\theta,\rho^*}^2\int_{\left[\frac{R}{8},\infty\right)}\frac{g(\omega)}{\sqrt{\omega}}d\omega$$

Using then the definition of $\|g\|_{\theta,\rho^*}$ we obtain:

$$\mathcal{O}[g]\left(\left[\frac{R}{2},R\right]\right)\leq C\|g\|_{\theta,\rho^*}^3 R^{1-\rho^*}\quad,\quad R\geq 1 \tag{4.40}$$

We now derive estimates for the measures $\mathcal{O}[g](\Omega_\alpha)$. To this end we use the representation formula (4.29). We consider a family of functions $\psi_\varepsilon\in C_0(\mathbb{R})$ satisfying $\psi_\varepsilon(0)=1$, $\psi_\varepsilon(s)=0$ if $|s|\geq\varepsilon$, $0\leq\psi_\varepsilon\leq 1$. We then consider a sequence of test functions $\varphi_{\eta,\varepsilon}(\omega)=\sum_{\ell=1}^{\infty}\psi_\varepsilon(\omega-x_\eta(\ell))$. Notice that these test functions are not compactly supported, but they are $C_b(\mathbb{R}_+)$. Therefore, it is possible to define $\langle\mathcal{O}[g],\varphi_{\eta,\varepsilon}\rangle$. Our assumptions on ψ_ε as well as (4.29) imply:

$$\langle\mathcal{O}[g],\Omega_\eta\rangle\leq\langle\mathcal{O}[g],\varphi_{\eta,\varepsilon}\rangle \tag{4.41}$$

We compute $\langle\mathcal{O}[g],\varphi_{\eta,\varepsilon}\rangle$ using (4.29). We split the sum as:

$$\langle\mathcal{O}[g],\varphi_{\eta,\varepsilon}\rangle=\sum_{\substack{\alpha,\beta,\gamma\\ \max\{\alpha,\beta,\gamma\}<\eta}}\iiint[\cdots]+\sum_{\substack{\alpha,\beta,\gamma\\ \max\{\alpha,\beta,\gamma\}\geq\eta}}\iiint[\cdots] \tag{4.42}$$

We now claim that the first term on the right of (4.42) is identically zero if ε is sufficiently small. Indeed, the integrations in that term are restricted to those in:

$$\bigcup_{\ell=1}^{\infty}\left\{\left|\omega_3+\omega_4-\omega_2-x_\eta\left(\ell\right)\right|\leq\varepsilon:\left(\omega_2,\omega_3,\omega_4\right)\in\Omega_\alpha\times\Omega_\beta\times\Omega_\gamma\right\}$$

The elements of this set satisfy:

$$\left|2^{-\beta}\theta_\beta\left(j\right)+2^{-\gamma}\theta_\gamma\left(k\right)-2^{-\alpha}\theta_\alpha\left(m\right)-2^{-\eta}\theta_\eta\left(\ell\right)\right|\leq\varepsilon$$

or equivalently:

$$\left|2^{\eta-\beta}\theta_\beta\left(j\right)+2^{\eta-\gamma}\theta_\gamma\left(k\right)-2^{\eta-\alpha}\theta_\alpha\left(m\right)-\theta_\eta\left(\ell\right)\right|\leq\varepsilon 2^{\eta}$$

where $\theta_\eta\left(\ell\right)$ is a positive integer. However, since $\max\left\{\alpha,\beta,\gamma\right\}<\eta$ this set is empty, if ε is sufficiently small, whence:

$$\sum_{\substack{\alpha,\beta,\gamma\\ \max\{\alpha,\beta,\gamma\}<\eta}}\iiint\left[\cdots\right]=0 \tag{4.43}$$

In order to estimate the last term in (4.42) we use the fact that at least one of the indexes α,β,γ is larger than η. Suppose without loss of generality that such index is α. We then estimate Φ by $\sqrt{\omega_2}$ to arrive at the estimate:

$$\sum_{\substack{\alpha,\beta,\gamma\\ \max\{\alpha,\beta,\gamma\}\geq\eta}}\iiint\left[\cdots\right]\leq 3\left\Vert\varphi_{\eta,\varepsilon}\right\Vert_\infty\left(\int_{\bigcup_{\alpha\geq\eta}\Omega_\alpha}g\right)\left(\int\frac{g\left(\omega\right)}{\sqrt{\omega}}d\omega\right)^2$$

whence, since $\left\Vert\varphi_{\eta,\varepsilon}\right\Vert_\infty=1$:

$$\left\langle\mathcal{O}\left[g\right],\varphi_{\eta,\varepsilon}\right\rangle\leq C\left\Vert g\right\Vert_{\theta,\rho^*}^3\sum_{\alpha\geq\eta}\left(2^{-\alpha}\right)^\theta\leq C\left\Vert g\right\Vert_{\theta,\rho^*}^3\left(2^{-\eta}\right)^\theta$$

and using (4.41) we obtain:

$$\left\langle\mathcal{O}\left[g\right],\Omega_\eta\right\rangle\leq C\left\Vert g\right\Vert_{\theta,\rho^*}^3\left(2^{-\eta}\right)^\theta \tag{4.44}$$

Combining (4.40) and (4.44) we obtain (4.26). □

REMARK 4.9. Notice that Lemma 4.8 implies that the operator $\mathcal{O}\left[\cdot\right]$ transforms measures $g\in\mathcal{X}_{\theta,\rho^*}$ into measures in $\mathcal{X}_{\theta,\rho^*}$. This will allow to obtain mild solutions of (1.7) with values $g\left(t,\cdot\right)\in\mathcal{X}_{\theta,\rho^*}$ for $t\geq0$.

REMARK 4.10. It is interesting to remark that Lemma 4.8 implies also that, assuming that g is given by (4.3), it is possible to give the following representation formula for $\mathcal{O}\left[g\right]$:

$$\begin{aligned}\mathcal{O}\left[g\right]=\sum_{\gamma}\sum_{k=1}^{\infty}\Bigg[\sum_{\alpha,\beta,\eta}\sum_{\ell,j,m=1}^{\infty}&\frac{a_\eta\left(\ell\right)a_\alpha\left(j\right)a_\beta\left(m\right)}{\sqrt{x_\eta\left(\ell\right)x_\alpha\left(j\right)x_\beta\left(m\right)}}\times\\ &\times\Phi\left(x_\gamma\left(k\right),x_\eta\left(\ell\right),x_\alpha\left(j\right),x_\beta\left(m\right)\right)\delta_{\left(x_\gamma(k)+x_\eta(\ell)-x_\alpha(j)-x_\beta(m)\right),0}\Bigg]\delta_{x_\gamma(k)}\end{aligned} \tag{4.45}$$

The following definition is similar to Definition 2.10 with the only difference that we restrict the measures $g\left(t,\cdot\right)$ to be in $\mathcal{X}_{\theta,\rho^*}$. In addition to the discrete character of the measures, we assume also more stringent decay conditions at infinity because we are interested in solutions with finite mass.

DEFINITION 4.11. Given $\theta > 1,\ \rho^* > 1,\ T \in (0,\infty]$ and $g_{in} \in \mathcal{X}_{\theta,\rho^*}$ we will say that $g \in C([0,T] : \mathcal{X}_{\theta,\rho^*})$ is a mild solution of (1.7) with values in $\mathcal{X}_{\theta,\rho^*}$ and with initial value $g(\cdot,0) = g_{in}$ if the following identity holds in the sense of measures:

$$\begin{aligned} g(\omega_1,t) &= g_{in}(\omega_1)\exp\left(-\int_0^t A_g(\omega_1,s)\,ds\right) + \\ &\quad + \int_0^t \exp\left(-\int_s^t A_g(\omega_1,\xi)\,d\xi\right)\mathcal{O}[g](\cdot,s)\,ds \end{aligned} \tag{4.46}$$

for $0 \le t < T$, where $A_g(\cdot,s)$ is defined as in Lemma 4.7 for each $g(\cdot,s)$ and $\mathcal{O}[g](\cdot,s)$ is defined as in Lemma 4.8 for each $g(\cdot,s)$.

4.2.4. Proof of a local existence Theorem of measured mild solutions with values in the space $\mathcal{X}_{\theta,\rho^*}$.. As a first step we need to construct local measured valued solutions in $\mathcal{X}_{\theta,\rho^*}$ in the sense of Definition 4.11.

THEOREM 4.12. *Let $\theta > 1,\ \rho^* > 1$ and $g_0 \in \mathcal{X}_{\theta,\rho^*}$ there exists $T > 0$, and at least one mild solution of (1.7) in $C([0,T] : \mathcal{X}_{\theta,\rho^*})$ with initial value $g(\cdot,0) = g_{in}$ in the sense of the Definition 4.11. Moreover, the following identities hold:*

$$\int g(t,d\omega) = \int g_{in}(d\omega) \quad \textit{for any } t \in [0,T] \tag{4.47}$$

$$\int_{\{0\}} g(t,d\omega) = 0 \quad \textit{for any } t \in [0,T] \tag{4.48}$$

PROOF. We define a space of measures as:

$$Y(g_{in}) = \left\{ g \in C([0,T] : \mathcal{X}_{\theta,\rho^*}) : \sup_{0\le t\le T} \|g\|_{\theta,\rho^*} \le 2\|g_{in}\|_{\theta,\rho^*} \right\}$$

and define an operator $\mathcal{T} : Y(g_{in}) \to Y(g_{in})$ as the right-hand side of (4.46), or more precisely:

$$\begin{aligned} \mathcal{T}[g](t,\omega_1) &= g_{in}(\omega_1)\exp\left(-\int_0^t A_g(s,\omega_1)\,ds\right) + \\ &\quad + \int_0^t \exp\left(-\int_s^t A_g(\xi,\omega_1)\,d\xi\right)\left(\iint \Phi \frac{g_2 g_3 g_4}{\sqrt{\omega_2\omega_3\omega_4}}\,d\omega_3 d\omega_4\right) ds \\ &\equiv \mathcal{T}_1[g](t,\omega_1) + \mathcal{T}_2[g](t,\omega_1) \end{aligned}$$

Notice that the operator $\mathcal{T}[g]$ is well defined due to Lemmas 4.7, 4.8.

We now prove that the operator $\mathcal{T}$ brings $Y(g_{in})$ to itself if T is sufficiently small. To check this, we integrate $\mathcal{T}[g]$ in the interval $\left[\frac{R}{2},R\right]$. Then:

$$\int_{\left[\frac{R}{2},R\right]} \mathcal{T}[g](t,d\omega) = \int_{\left[\frac{R}{2},R\right]} \mathcal{T}_1[g](t,d\omega) + \int_{\left[\frac{R}{2},R\right]} \mathcal{T}_2[g](t,d\omega)$$

where, using (4.21) and the definition of $\|g\|_{\theta,\rho^*}$

$$\int_{\left[\frac{R}{2},R\right]} \mathcal{T}_1[g](t,d\omega) \le \int_{\left[\frac{R}{2},R\right]} g_{in}(d\omega) \le \|g_{in}\|_{\theta,\rho^*} R\min\left\{R^\theta, R^{-\rho^*}\right\} \tag{4.49}$$

We have also, using again (4.21), as well as the symmetry of the integral with respect to the symmetrization $\omega_3 \leftrightarrow \omega_4$:

$$\int_{\left[\frac{R}{2},R\right]} \mathcal{T}_2\left[g\right](t,d\omega) \leq 2\int_0^t \int_{\left[\frac{R}{2},R\right]} \left(\iint_{\{\omega_3\leq\omega_4\}} \Phi\frac{g_2g_3g_4}{\sqrt{\omega_2\omega_3\omega_4}}d\omega_3 d\omega_4\right) d\omega_1 ds$$

We estimate Φ by $\sqrt{\omega_1}$. Then:

(4.50)

$$\int_{\left[\frac{R}{2},R\right]} \mathcal{T}_2\left[g\right](t,d\omega) \leq 2\sqrt{R}\int_0^t \int_{\left[\frac{R}{2},R\right]} \left(\iint_{\{\omega_3\leq\omega_4\}} \frac{g_2g_3g_4}{\sqrt{\omega_2\omega_3\omega_4}}d\omega_3 d\omega_4\right) d\omega_1 ds$$

We now distinguish two cases. Suppose that $R\geq 1$. We then use that in the region of integration we have $\omega_4 \geq \frac{R}{4}$. Replacing the integration in ω_1 by the integration in ω_2 by means of a change of variables, we obtain the estimate:

$$\int_{\left[\frac{R}{2},R\right]} \mathcal{T}_2\left[g\right](t,d\omega) \leq 4\int_0^t \left(\int_{\mathbb{R}_+} \frac{g\left(s,d\omega\right)}{\sqrt{\omega}}\right)^2 \int_{\frac{R}{4}}^{\infty} g\left(s,d\omega\right) ds$$

Notice that, since $\rho^* > 1$ we have $\int_{\mathbb{R}_+} \frac{g(s,d\omega)}{\sqrt{\omega}} \leq C\left\|g\left(s,\cdot\right)\right\|_{\theta,\rho^*}$, as it can be seen decomposing the region of integration in dyadic intervals. On the other hand, a similar argument yields $\int_{\frac{R}{4}}^{\infty} g\left(s,d\omega\right) ds \leq CR^{1-\rho^*}\left\|g\left(s,\cdot\right)\right\|_{\theta,\rho^*}$ if $R\geq 1$. Then:

$$\int_{\left[\frac{R}{2},R\right]} \mathcal{T}_2\left[g\right](t,d\omega) \leq CR^{1-\rho^*}\int_0^t \left\|g\left(s,\cdot\right)\right\|_{\theta,\rho^*}^3 ds \ \text{ if } \ R\geq 1 \tag{4.51}$$

Suppose now that $R\leq 1$. Then (4.50) implies:

$$\begin{aligned} \int_{\left[\frac{R}{2},R\right]} \mathcal{T}_2\left[g\right](t,d\omega) &\leq 2\sqrt{R}\int_0^t \left(\int_{\mathbb{R}_+} \frac{g\left(s,d\omega\right)}{\sqrt{\omega}}\right)^3 ds \\ &\leq C\sqrt{R}\int_0^t \left\|g\left(s,\cdot\right)\right\|_{\theta,\rho^*}^3 ds \ \text{ if } \ R\leq 1 \end{aligned} \tag{4.52}$$

Combining (4.49), (4.51), (4.52) and using that $\theta = \frac{1}{2}$ we obtain:

$$\sup_{R>0}\frac{1}{\min\left\{R^{\theta},R^{-\rho^*}\right\}}\frac{1}{R}\int_{\left[\frac{R}{2},R\right]} \mathcal{T}\left[g\right](t,d\omega) \leq \left\|g_{in}\right\|_{\theta,\rho^*} + CT\sup_{0\leq t\leq T}\left\|g\left(t,\cdot\right)\right\|_{\theta,\rho^*}^3$$

Then the operator $\mathcal{T}\left[\cdot\right]$ maps $Y\left(g_{in}\right)$ into itself. Moreover, arguing as in the Proof of Lemma 2.20 we obtain that the operator $\mathcal{T}\left[\cdot\right]$ defines a continuous mapping from $Y\left(g_{in}\right)$ to $C\left(\left[0,T\right]:\mathcal{X}_{\theta,\rho^*}\right)$ in the weak topology. Notice that in this case $\sigma = 0$, and therefore some of the functions appearing in the integrals defining A_g and $\mathcal{O}\left[g\right]$ are singular near $\omega = 0$. However, the contribution to those integrals of the regions close to the origin can be made estimated if $g\in C\left(\left[0,T\right]:\mathcal{X}_{\theta,\rho^*}\right)$ using the fact that $\left\|g\left(t,\cdot\right)\right\|_{\theta,\rho^*}$ is bounded. Therefore, it is possible to adapt the argument in the Proof of Lemma 2.20 to prove the desired continuity of the operator $\mathcal{T}\left[\cdot\right]$. Moreover, since the set $\mathcal{X}_{\theta,\rho^*}$ is closed in $\mathcal{M}_+\left(\left[0,\infty\right)\right)$ it follows that it is compact in the weak topology. Therefore, applying also Arzela-Ascoli as in the Proof of Lemma 2.20. The existence of solutions then follows using Schauder's Theorem. To prove the identity (4.47), we can argue as in the Proof of Proposition 2.15, in order to show that g is also a weak solution of (1.7) in the sense of Definition 2.2. This follows from Proposition 2.15 due to the fact that mild measured values solutions with values in $\mathcal{X}_{\theta,\rho^*}$ in the sense of Definition 4.11 are also mild measured valued solutions in the sense of Definition 2.10. Taking a sequence of test functions converging to 1 in $\omega \geq 0$ we obtain (4.47). Finally, we notice that (4.48) follows by construction, since $0 \in \left[\mathbb{R}_+ \setminus \bigcup_{\alpha=0}^{\infty} \Omega_\alpha\right]$, whence the result follows. $\square$

Actually, it turns out that the solutions can be extended as long as $\|g\|_{\theta,\rho^*}$ remains bounded.

THEOREM 4.13. *Let $g \in C\left([0,T] : \mathcal{X}_{\theta,\rho^*}\right)$ the mild solution of (1.7) obtained in Theorem 4.12. Suppose that $\sup_{0\leq t\leq T} \|g(t,\cdot)\|_{\theta,\rho^*} < \infty$. Then, there exists $\delta > 0$ and $\tilde{g} \in C\left([0,T+\delta] : \mathcal{X}_{\theta,\rho^*}\right)$ such that $g(t,\cdot) = \tilde{g}(t,\cdot)$ for $t \in [0,T]$ and $\tilde{g}$ is a mild solution of (1.7) in the interval $t \in [0,T+\delta]$.*

PROOF. We just construct a mild solution in the time interval $[T, T+\delta]$ with initial datum $\tilde{g}(T,\cdot) \in \mathcal{X}_{\theta,\rho^*}$. Such solution is well defined for $\delta > 0$ as it can be seen using the argument in the Proof of Theorem 4.12. The function $\tilde{g}$ obtained combining the values of g in $t \in [0,T]$ and $\tilde{g}$ for $t \in [T, T+\delta]$ gives the provides mild solution of (1.7) as it can be seen using Definition 4.11. $\square$

4.2.5. Global existence of measure mild solutions with values in $\mathcal{X}_{\theta,\rho^*}$.. We will now prove that, if the masses $m_\alpha = \sum_{k=1}^{\infty} a_\alpha(k)$ contained in each of the families Ω_α decrease fast enough as $\alpha \to \infty$, the mild solutions obtained in Theorem 4.12 are globally defined in time. To this end, we first need to prove the following result which will has a consequence that the mass cannot propagate from the families $\{\Omega_\beta\}_{\beta>\alpha}$ to the family Ω_α unless some meaningful amount of mass is already present in this last family.

In order to prove a global well posedness results in the space $C\left([0,\infty) : \mathcal{X}_{\theta,\rho^*}\right)$ we need the following auxiliary Lemmas.

LEMMA 4.14. *Let $g \in C\left([0,T] : \mathcal{X}_{\theta,\rho^*}\right)$ be the mild solution of (1.7) obtained in Theorem 4.12. Let us write:*

$$m_\gamma = \sum_{k=1}^{\infty} a_\gamma(k) \quad , \quad M_{\gamma+1} = \sum_{\eta\geq\gamma+1} m_\eta \quad , \quad S_{\gamma+1} = \sum_{\alpha>\gamma} \frac{m_\alpha}{\sqrt{x_\alpha(1)}} \tag{4.53}$$

Then, the following inequality holds a.e. $t \in [0,T]$:

$$\partial_t m_\gamma \leq \frac{6 m_\gamma}{x_\gamma(1)} + 6 M_{\gamma+1}\left(S_{\gamma+1} + \frac{1}{\sqrt{x_\gamma(1)}}\right)^2 \tag{4.54}$$

PROOF. Notice that the Definition of mild solution (cf. Definition 4.11) implies the following identity, in the sense of measures:

$$\partial_t g(t,\cdot) = \mathcal{O}[g](\cdot,t) - A_g(t,\cdot)\, g(t,\cdot) \quad , \quad a.e.\ t \in [0,T]$$

Due to Lemma 4.7 we have $A_g(\omega_1,\cdot) \geq 0$. Then:

$$\partial_t g(t,\cdot) \leq \mathcal{O}[g](t,\cdot) \tag{4.55}$$

Due to the definition of $\mathcal{X}_{\theta,\rho^*}$ implies that for each $t \geq 0$ the measure g has the form (4.3), (4.6). Then $\mathcal{O}[g](\cdot,t)$ is given by (4.45). We then use that:

$$\begin{aligned} \Phi\left(x_\gamma(k), x_\eta(\ell), x_\alpha(j), x_\beta(m)\right) &\leq \min\left\{x_\eta(\ell), x_\alpha(j), x_\beta(m)\right\} \\ &= \Phi\left(x_\eta(\ell), x_\alpha(j), x_\beta(m)\right) \end{aligned}$$

Combining then (4.55), (4.45) we obtain:

$$\partial_t g(\cdot,t) \leq \sum_{\gamma}\sum_{k=1}^{\infty}\left[\sum_{\alpha,\beta,\eta}\sum_{\ell,j,m=1}^{\infty}\frac{a_\eta(\ell)\,a_\alpha(j)\,a_\beta(m)}{\sqrt{x_\eta(\ell)\,x_\alpha(j)\,x_\beta(m)}}\times\right.$$
$$\left.\times\Phi\left(x_\eta(\ell),x_\alpha(j),x_\beta(m)\right)\delta_{(x_\gamma(k)+x_\eta(\ell)-x_\alpha(j)-x_\beta(m)),0}\right]\delta_{x_\gamma(k)}$$

Adding the contributions associated to the family γ and using the definition of m_γ in (4.53) we obtain:

$$\partial_t m_\gamma \leq \sum_{\alpha,\beta,\eta}\sum_{\ell,j,s=1}^{\infty}\frac{a_\eta(\ell)\,a_\alpha(j)\,a_\beta(s)}{\sqrt{x_\eta(\ell)\,x_\alpha(j)\,x_\beta(s)}}\Phi\left(x_\eta(\ell),x_\alpha(j),x_\beta(s)\right)\times$$
$$\times\sum_{k=1}^{\infty}\delta_{(x_\gamma(k)+x_\eta(\ell)-x_\alpha(j)-x_\beta(s)),0}$$

We now claim that:

$$\sum_{k=1}^{\infty}\delta_{(x_\gamma(k)+x_\eta(\ell)-x_\alpha(j)-x_\beta(s)),0} \leq F(\alpha,\beta,\eta;\gamma)$$

where $F(\alpha,\beta,\eta;\gamma)=1$ if $\gamma\leq\max\{\alpha,\beta,\eta\}$ and $F(\alpha,\beta,\eta;\gamma)=0$ otherwise. This can be proved with the same argument yielding (4.35) in the Proof of Lemma 4.8. Then:

$$\partial_t m_\gamma \leq \sum_{\alpha,\beta,\eta}\sum_{\ell,j,m=1}^{\infty}\frac{a_\eta(\ell)\,a_\alpha(j)\,a_\beta(m)}{\sqrt{x_\eta(\ell)\,x_\alpha(j)\,x_\beta(m)}}\Phi\left(x_\eta(\ell),x_\alpha(j),x_\beta(m)\right)F(\alpha,\beta,\eta;\gamma)$$

Using the symmetry in the indexes we obtain:

$$\partial_t m_\gamma \leq 6\sum_{\alpha\leq\beta\leq\eta}\sum_{\ell,j,m=1}^{\infty}\frac{a_\eta(\ell)\,a_\alpha(j)\,a_\beta(m)}{\sqrt{x_\eta(\ell)\,x_\alpha(j)\,x_\beta(m)}}\Phi\left(x_\eta(\ell),x_\alpha(j),x_\beta(m)\right)F(\alpha,\beta,\eta;\gamma)$$

Then, using $\Phi\left(x_\eta(\ell),x_\alpha(j),x_\beta(m)\right)\leq\sqrt{x_\eta(\ell)}$:

$$\partial_t m_\gamma \leq 6\sum_{\alpha\leq\beta\leq\eta}\sum_{\ell,j,m=1}^{\infty}\frac{a_\eta(\ell)\,a_\alpha(j)\,a_\beta(m)\,F(\alpha,\beta,\eta;\gamma)}{\sqrt{x_\eta(\ell)\,x_\alpha(j)\,x_\beta(m)}}\sqrt{x_\eta(\ell)}\leq$$
$$\leq 6\sum_{\alpha\leq\beta\leq\eta}\sum_{\ell,j,m=1}^{\infty}\frac{a_\eta(\ell)\,a_\alpha(j)\,a_\beta(m)\,F(\alpha,\beta,\eta;\gamma)}{\sqrt{x_\alpha(j)\,x_\beta(m)}}$$

whence:

$$\partial_t m_\gamma \leq 6\sum_{\alpha\leq\beta\leq\eta}m_\eta\sum_{j,m=1}^{\infty}\frac{a_\alpha(j)\,a_\beta(m)\,F(\alpha,\beta,\eta;\gamma)}{\sqrt{x_\alpha(j)\,x_\beta(m)}}$$

Then, using the estimates $x_\alpha(j)\geq x_\alpha(1)$, $x_\beta(m)\geq x_\beta(1)$ and adding in j,m :

$$\partial_t m_\gamma \leq 6\sum_{\alpha\leq\beta\leq\eta\ ,\ \eta\geq\gamma}\frac{m_\eta m_\alpha m_\beta}{\sqrt{x_\alpha(1)\,x_\beta(1)}}$$

We now split the sum in two cases:

$$\partial_t m_\gamma \le 6 \sum_{\alpha\le\beta\le\eta=\gamma} \frac{m_\eta m_\alpha m_\beta}{\sqrt{x_\alpha(1)\, x_\beta(1)}} + 6 \sum_{\alpha\le\beta\le\eta\ ,\ \eta>\gamma} \frac{m_\eta m_\alpha m_\beta}{\sqrt{x_\alpha(1)\, x_\beta(1)}} \tag{4.56}$$
$$\le 6 m_\gamma \sum_{\alpha\le\beta\le\gamma} \frac{m_\alpha m_\beta}{\sqrt{x_\alpha(1)\, x_\beta(1)}} + 6 M_{\gamma+1} \sum_{\alpha\le\beta} \frac{m_\alpha m_\beta}{\sqrt{x_\alpha(1)\, x_\beta(1)}}$$

where $M_{\gamma+1}$ is as in (4.53). Then:

$$6 m_\gamma \sum_{\alpha\le\beta\le\gamma} \frac{m_\alpha m_\beta}{\sqrt{x_\alpha(1)\, x_\beta(1)}} \le \frac{6 m_\gamma}{x_\gamma(1)} \sum_{\alpha\le\beta\le\gamma} m_\alpha m_\beta \le \frac{6 m_\gamma}{x_\gamma(1)} \tag{4.57}$$

where we use the fact that the total mass of g is bounded. On the other hand:
(4.58)

$$\sum_{\alpha\le\beta} \frac{m_\alpha m_\beta}{\sqrt{x_\alpha(1)\, x_\beta(1)}} \le \left(\sum_\alpha \frac{m_\alpha}{\sqrt{x_\alpha(1)}} \right)^2 = \left(\sum_{\alpha>\gamma} \frac{m_\alpha}{\sqrt{x_\alpha(1)}} + \sum_{\alpha\le\gamma} \frac{m_\alpha}{\sqrt{x_\alpha(1)}} \right)^2$$
$$\le \left(S_{\gamma+1} + \frac{1}{\sqrt{x_\gamma(1)}} \right)^2$$

where we define $S_{\gamma+1}$ as in (4.53). Plugging (4.57), (4.58) into (4.56) we obtain (4.54) and the Lemma follows. □

Lemma 4.15. *Suppose that $g \in C([0,T] : \mathcal{X}_{\theta,\rho^*})$ solves (1.7) in the sense of Definition 4.11. Suppose that $g_0 = g(0,\cdot)$ satisfies*

$$\int_{[0,R2^{-\gamma}]} g_0 d\omega \ge 1-\eta \tag{4.59}$$

for suitable R and η. Let us assume that g has the form (4.3)-(4.6). Then, the following inequality holds:

$$\partial_t \left[a_\gamma(1)(t) + R_\gamma(t) \right] \ge \frac{a_\gamma(1)(t) \left[(1-2\eta) - a_\gamma(1)(t) \right]^2}{L^2 2^\gamma} \tag{4.60}$$

where:

$$R_\gamma(t) \le 3 M_{\gamma+1}(t) \ .$$

Proof. We use (2.27). The function $\mathcal{G}_\varphi(\omega_1\, \omega_2\, \omega_3)$ can be written as (cf. Lemma 2.23):

$$\mathcal{G}_\varphi(\omega_1,\omega_2,\omega_3) = \frac{1}{3}\left[\sqrt{\omega_-} H^1_\varphi(\omega_1,\omega_2,\omega_3) + \sqrt{(\omega_0+\omega_- - \omega_+)_+} H^2_\varphi(\omega_1,\omega_2,\omega_3) \right]$$

Using the convex test function $\varphi(\omega) = 2\left(\frac{3}{2} - \frac{\omega}{x_\gamma(1)}\right)_+$ we deduce by Lemma 2.23:

(4.61)
$$\mathcal{G}_\varphi(\omega_1,\omega_2,\omega_3) \ge \frac{\sqrt{\omega_-}}{3} \left[\varphi(\omega_+ + \omega_- - \omega_0) + \varphi(\omega_+ + \omega_0 - \omega_-) - 2\varphi(\omega_+) \right] \ge 0$$

We split the measure g in two pieces:

$$g = g_\gamma + \tilde{g}_\gamma$$

where:

$$g_\gamma = \sum_{\alpha\le\gamma} a_\alpha \quad , \quad \tilde{g}_\gamma = \sum_{\alpha>\gamma} a_\alpha$$

We now use the monotonicity formula (2.29). Then, using also (4.61) and ignoring all the terms containing $\tilde{g}_\gamma$ (since they are nonnegative):

$$\begin{aligned}\frac{d}{dt}\left(\int_0^\infty g(\omega)\varphi(\omega)\,d\omega\right) &= 6\iiint_{\{\omega_-\leq\omega_0\leq\omega_+\}}\frac{g(\omega_-)\,g(\omega_0)\,g(\omega_+)}{\sqrt{\omega_-\omega_0\omega_+}}\times\\ &\qquad\times\mathcal{G}_\varphi(\omega_-,\omega_0,\omega_+)\,d\omega_-d\omega_0d\omega_+\\ &\geq 6\iiint_{\{\omega_-\leq\omega_0\leq\omega_+\leq L\}}\frac{g_\gamma(\omega_-)\,g_\gamma(\omega_0)\,g_\gamma(\omega_+)}{\sqrt{\omega_-\omega_0\omega_+}}\times\\ &\qquad\times\mathcal{G}_\varphi(\omega_-,\omega_0,\omega_+)\,d\omega_-d\omega_0d\omega_+\end{aligned}$$

Notice that the smallest particle with mass in g_γ is $x_\gamma(1)$. Then we have the following inequalities in the sense of measures for $\omega_-\leq\omega_0\leq\omega_+\leq L$:

$$\begin{aligned}&\frac{g_\gamma(\omega_-)\,g_\gamma(\omega_0)\,g_\gamma(\omega_+)}{\sqrt{\omega_-\omega_0\omega_+}}\mathcal{G}_\varphi(\omega_-,\omega_0,\omega_+)\\ &\geq\frac{g_\gamma(\omega_-)\,g_\gamma(\omega_0)\,g_\gamma(\omega_+)}{3\sqrt{\omega_0\omega_+}}\left[\varphi(\omega_++\omega_--\omega_0)+\varphi(\omega_++\omega_0-\omega_-)-2\varphi(\omega_+)\right]\\ &\geq\frac{g_\gamma(\omega_-)\,g_\gamma(\omega_0)\,g_\gamma(\omega_+)}{3L}\left[\varphi(\omega_++\omega_--\omega_0)+\varphi(\omega_++\omega_0-\omega_-)-2\varphi(\omega_+)\right]\end{aligned}$$

This expression is nonnegative at every point. Moreover, if $\omega_+=\omega_0$ and $\omega_+>\omega_-$, and using the fact that φ vanishes for the points $x_\gamma(k)\,,\ k\geq 2, x_\alpha(j)\,,\ j\geq 1,\ \alpha<\gamma$ it then follows that:

$$\begin{aligned}&6\iiint_{\{\omega_-\leq\omega_0\leq\omega_+\leq L\}}\frac{g_\gamma(\omega_-)\,g_\gamma(\omega_0)\,g_\gamma(\omega_+)}{\sqrt{\omega_-\omega_0\omega_+}}\mathcal{G}_\varphi(\omega_-,\omega_0,\omega_+)\,d\omega_-d\omega_0d\omega_+\\ &\geq\frac{2a_\gamma(1)}{L}\sum_{\alpha\leq\gamma}\sum_{j=1}^\infty\left(a_\alpha(j)\left(1-\delta_{j,1}\delta_{\alpha,\gamma}\right)\right)\varphi(x_\gamma(1))\end{aligned}$$

Since $\varphi(x_\gamma(1))=1$, we deduce:

$$\begin{aligned}&6\iiint_{\{\omega_-\leq\omega_0\leq\omega_+\leq L\}}\frac{g_\gamma(\omega_-)\,g_\gamma(\omega_0)\,g_\gamma(\omega_+)}{\sqrt{\omega_-\omega_0\omega_+}}\mathcal{G}_\varphi(\omega_-,\omega_0,\omega_+)\,d\omega_-d\omega_0d\omega_+\\ &\geq\frac{2a_\gamma(1)}{L}\sum_{\alpha\leq\gamma}\sum_{j=1,\{x_\alpha(j)\leq L\}}^\infty\left(a_\alpha(j)\left(1-\delta_{j,1}\delta_{\alpha,\gamma}\right)\right)^2\\ &\geq\frac{2a_\gamma(1)}{L}\sum_{\alpha\leq\gamma}\left[\int_{\{x\leq L\}\setminus\{x_\gamma(1)\}}g_\gamma\right]^2\geq\frac{2a_\gamma(1)}{LN_\gamma}\left(\sum_{\alpha\leq\gamma}\int_{\{x\leq L\}\setminus\{x_\gamma(1)\}}g_\gamma\right)^2\\ &\geq\frac{2a_\gamma(1)}{LN_\gamma}\left(\sum_{\alpha\leq\gamma}\left[(1-2\eta)-a_\gamma(1)\right]\right)^2\end{aligned}$$

where N_γ is the number of elements of $\mathcal{Z}_\gamma$ which are smaller than L. We have used Jensen's inequality in the sum. Notice that $N_\gamma \leq L2^\gamma$. Therefore:

$$6\iiint_{\{\omega_-\leq\omega_0\leq\omega_+\leq L\}} \frac{g_\gamma(\omega_-)\, g_\gamma(\omega_0)\, g_\gamma(\omega_+)}{\sqrt{\omega_-\omega_0\omega_+}} \mathcal{G}_\varphi(\omega_-,\omega_0,\omega_+)\, d\omega_- d\omega_0 d\omega_+$$
$$\geq \frac{2a_\gamma(1)}{L^2 2^\gamma}\left(\sum_{\alpha\leq\gamma}\left[(1-2\eta)-a_\gamma(1)\right]\right)^2$$

We then have:

$$\frac{d}{dt}\left(\int_0^\infty g(\omega)\,\varphi(\omega)\,d\omega\right) \geq \frac{2a_\gamma(1)}{L^2 2^\gamma}\left(\sum_{\alpha\leq\gamma}\left[(1-2\eta)-a_\gamma(1)\right]\right)^2$$

We then write $\int_0^\infty g\varphi d\omega = \int_0^\infty g_\gamma\varphi d\omega + \int_0^\infty \tilde{g}_\gamma\varphi d\omega$. Let us denote as $R_\gamma(t)$ the quantity $\int_0^\infty \tilde{g}_\gamma\varphi d\omega$. Then, since $\varphi \leq 3$ we obtain:

$$R_\gamma(t) = \int_0^\infty \tilde{g}_\gamma\varphi d\omega \leq 3\int_0^\infty \tilde{g}_\gamma d\omega = 3M_{\gamma+1}$$

This concludes the Proof of Lemma 4.15. □

Lemma 4.16. *Suppose that the set of functions* $\{a_\gamma(1)\}$, $\{m_\gamma\}$ *defined above for* $\gamma = 0,1,2,...$ *satisfy the following set of inequalities:*

$$\frac{db_\gamma}{dt} \geq C_1(\gamma)\left[b_\gamma\left[(1-2\eta_\gamma)-b_\gamma\right]^2 - BM_{\gamma+1}\right]$$

where $b_\gamma = a_\gamma(1) + R_\gamma$, $R_\gamma \leq 3M_{\gamma+1}$, $\lim_{\gamma\to\infty}\eta_\gamma = 0$, *as well as:*

$$\frac{dm_\gamma}{dt} \leq C_2(\gamma)\, m_\gamma + 6M_{\gamma+1}\left(S_{\gamma+1} + \frac{1}{\sqrt{x_\gamma(1)}}\right)^2$$

where $S_{\gamma+1}$ *is as in (4.53). Then, there exists a sequence* $\{\varepsilon_\gamma\}$ *of positive numbers satisfying* $\sum_{\gamma=0}^\infty \varepsilon_\gamma = 1$, *such that, if we assume that:*

$$a_\gamma(\cdot)(0) = \varepsilon_\gamma\delta_{x_\gamma(1)}$$

there exists a solution g *of (1.7) in the sense of Definition 4.11 globally defined in time and there exists an increasing sequence of times* $\{t_n\}$ *such that* $\lim_{n\to\infty} t_n = \infty$ *and:*

$$a_n(1)(t) \geq 1 - 4\eta_n \quad \textit{for} \quad t_n \leq t \leq 2t_n \tag{4.62}$$

Remark 4.17. Notice that the Theorem implies

$$\sup_{t_n\leq t\leq 2t_n}\left(\sum_{\alpha\neq\gamma} a_\alpha(\cdot)(t) + \sum_{\ell=2}^\infty a_\gamma(\ell)(t)\right) \to 0 \text{ as } n\to\infty$$

Remark 4.18. The result can be reformulated also in terms of the weak topology of measures for g. We recall $dist_*$ denotes the distance associated to the weak topology of measures (cf. Notation 2.1). Then:

$$\sup_{t_n\leq t\leq 2t_n}\left[dist_*\left(2^n g\left(2^n(\cdot), t\right), \delta_1\right)\right] \to 0 \text{ as } n\to\infty$$

PROOF. We will assume, without loss of generality that $C_1(\gamma) \geq 0,\ C_2(\gamma) \geq 0$. We need to guarantee the existence of several inequalities for a suitable range of times. These inequalities are:

$$BM_{\gamma+1} \leq \frac{1}{8}\min\left\{b_\gamma, \eta_\gamma, \frac{Bb_\gamma}{3}\right\}$$

$$S_{\gamma+1} \leq \frac{1}{\sqrt{x_\gamma(1)}}$$

$$6M_{\gamma+1}\left(\frac{2}{\sqrt{x_\gamma(1)}}\right)^2 \leq C_2(\gamma)\, m_\gamma \tag{4.63}$$

As long as these inequalities are satisfied we have:

$$\frac{db_\gamma}{dt} \geq \frac{C_1(\gamma)\, b_\gamma}{2}\left[(1-2\eta_\gamma)-b_\gamma\right]^2 \tag{4.64}$$

$$\frac{dm_\gamma}{dt} \leq 2C_2(\gamma)\, m_\gamma \tag{4.65}$$

We need to precise sufficient conditions to have (4.63). Notice that if (4.63) holds we have $\frac{db_\gamma}{dt} \geq 0$. Moreover, the first inequality in (4.63) guarantees also that $a_\gamma(1)(t)$ is comparable to $b_\gamma(t)$. Then $b_\gamma(0) \geq \frac{\varepsilon_\gamma}{2}$. We can assume that $\eta_\gamma \geq \frac{\varepsilon_\gamma}{2}$ and also that $B \geq 3$. Then, since $m_\gamma \geq a_\gamma(1)$ and $a_\gamma(1)(t)$ is comparable to $b_\gamma(t)$, we would have the first and the third inequalities in (4.63) if we have:

$$BM_{\gamma+1} \leq \varepsilon_\gamma \quad \text{and} \quad 6M_{\gamma+1}\left(\frac{2}{\sqrt{x_\gamma(1)}}\right)^2 \leq C_2(\gamma)\,\varepsilon_\gamma \tag{4.66}$$

Notice that (4.65) implies:

$$m_\gamma(t) \leq m_\gamma(0)\exp\left(2C_2(\gamma)\, t\right)$$

and since $m_\gamma(0) = \varepsilon_\gamma$ we obtain:

$$m_\gamma(t) \leq \varepsilon_\gamma \exp\left(2C_2(\gamma)\, t\right)$$

Therefore:

$$M_{\gamma+1} \leq \sum_{\alpha \geq \gamma+1} \varepsilon_\alpha \exp\left(2C_2(\alpha)\, t\right) \tag{4.67}$$

and:

(4.68)

$$S_{\gamma+1} \leq \sum_{\alpha\geq\gamma+1}\frac{m_\alpha}{\sqrt{x_\alpha(1)}} \leq \sum_{\alpha\geq\gamma+1}\frac{\varepsilon_\alpha\exp\left(2C_2(\alpha)\, t\right)}{\sqrt{x_\alpha(1)}} = \sum_{\alpha\geq\gamma+1}\varepsilon_\alpha 2^{\frac{\alpha}{2}}\exp\left(2C_2(\alpha)\, t\right)$$

Using (4.64) as well as the fact that $b_\gamma(0) \geq \frac{\varepsilon_\gamma}{2}$ we obtain that, if (4.64) holds during the time interval $0 \leq t \leq 2t_\gamma$ for we would have $b_\gamma(t) \geq 1 - 3\eta_\gamma$ for $t_\gamma \leq t \leq 2t_\gamma$, where:

$$t_\gamma(\varepsilon_\gamma, \eta_\gamma) = \frac{C_1(\gamma)}{2(1-2\eta_\gamma)^2}\left[\frac{(1-2\eta_\gamma)}{\eta_\gamma} - \frac{2(1-2\eta_\gamma)}{2+4\eta_\gamma-\varepsilon_\gamma} + \right.$$
$$\left. + \log\left(\frac{(1-3\eta_\gamma)(2+4\eta_\gamma-\varepsilon_\gamma)}{\eta_\gamma\varepsilon_\gamma}\right)\right]$$

Notice that t_γ tends to infinity if η_γ or ε_γ approach zero.

We would have (4.64), (4.65) for $0 \leq t \leq 2t_\gamma$ if the inequalities (4.63) hold in the same time interval. Sufficient condition for this are the inequalities (4.66) as well as the second inequality in (4.63). Using (4.67), (4.68) we would have those inequalities for $0 \leq t \leq 2t_\gamma$ if:

$$B \sum_{\alpha \geq \gamma+1} \varepsilon_\alpha \exp\left(4C_2(\alpha)\, t_\gamma(\varepsilon_\gamma, \eta_\gamma)\right) \leq \varepsilon_\gamma$$

$$6\left(\frac{2}{\sqrt{x_\gamma(1)}}\right)^2 \sum_{\alpha \geq \gamma+1} \varepsilon_\alpha \exp\left(4C_2(\alpha)\, t_\gamma(\varepsilon_\gamma, \eta_\gamma)\right) \leq C_2(\gamma)\, \varepsilon_\gamma$$

$$\sum_{\alpha \geq \gamma+1} \varepsilon_\alpha 2^{\frac{\alpha}{2}} \exp\left(4C_2(\alpha)\, t_\gamma(\varepsilon_\gamma, \eta_\gamma)\right) \leq \frac{1}{\sqrt{x_\gamma(1)}}$$

The three inequalities hold if we have:

(4.69)
$$\sum_{\alpha \geq \gamma+1} \varepsilon_\alpha 2^{\frac{\alpha}{2}} \exp\left(4C_2(\alpha)\, t_\gamma(\varepsilon_\gamma, \eta_\gamma)\right) \leq \min\left\{\frac{\varepsilon_\gamma}{B}, \frac{C_2(\gamma)\, 2^{-\gamma}\varepsilon_\gamma}{24}, 2^{\frac{\gamma}{2}}\right\} \equiv Q(\gamma; \varepsilon_\gamma)$$

We can now construct the sequence ε_γ inductively. We will assume that $\varepsilon_0 \geq \frac{1}{2}$. We will then select ε_α inductively for $\alpha \geq 1$, as any positive number satisfying the inequalities:

$$\varepsilon_\alpha \leq \min\left\{\frac{1}{2^\alpha} \frac{\min\left\{Q(\gamma; \varepsilon_\gamma), \frac{1}{2}\right\}}{2^{\frac{\alpha}{2}} \exp\left(4C_2(\alpha)\, t_\gamma(\varepsilon_\gamma, \eta_\gamma)\right)} : 0 \leq \gamma \leq \alpha - 1\right\} \tag{4.70}$$

where we replace in these inequalities that $\varepsilon_0 = \frac{1}{2}$. Since $\exp\left(4C_2(\alpha)\, t_\gamma(\varepsilon_\gamma, \eta_\gamma)\right) \geq 1$, these inequalities imply that:

$$\sum_{\alpha \geq 1} \varepsilon_\alpha \leq \frac{1}{2} \sum_{\alpha \geq 1} \frac{1}{2^\alpha} = \frac{1}{4} < \frac{1}{2}$$

We then choose ε_0 as $\left(1 - \sum_{\alpha \geq 1} \varepsilon_\alpha\right)$. Then $\sum_{\alpha \geq 0} \varepsilon_\alpha = 1$. Notice that $\varepsilon_0 \geq \frac{1}{2}$ and since the right-hand side of (4.70) is increasing in ε_0, it follows that this inequalities hold with this new choice of ε_0, since they were valid with $\varepsilon_0 = \frac{1}{2}$. Moreover, since $a_\gamma(1) \geq b_\gamma - 3M_{\gamma+1}$ it then follows that $a_\gamma(1)(t) \geq 1 - 4\eta_\gamma$ if $t_\gamma \leq t \leq 2t_\gamma$. □

4.2.6. Proof of Theorem 4.1.

Proof of Theorem 4.1. Lemma 4.16 combined with Lemmas 4.14 and 4.15 imply the existence of $g \in C([0,\infty) : \mathcal{X}_{\theta,\rho^*})$ which solves (1.7) in the sense of Definition 4.11. Notice that, by construction, this solution satisfies $\int_{\{0\}} g(t, d\omega) = 0$ for all $t > 0$. Using then Theorem 3.13 we obtain that the alternative (ii) holds.

It only remains to prove (4.1). The construction of the family implies that for t sufficiently large, most of the mass of the measure g is contained in $\Omega_J \cup \Omega_{J+1}$, with J depending of t and the mass contained out of this set tends to zero as $t \to \infty$. Rescaling the unit of length we may assume that $x_J(1) = 1$. We denote as $\tilde{g}$ the measure g using this new length scale. The construction of g implies, the existence for any $\delta > 0$, of $t_1 < t_2$ both sufficiently large, such that: $\int_{(1-\delta,1+\delta)} \tilde{g}(t_1) \geq m(1-\delta)$ and $\int_{(1/2-\delta,1/2+\delta)} \tilde{g}(t_2) \geq m(1-\delta)$. Then, by continuity, there exists $t^* \in (t_1, t_2)$

such that $\int_{(1/2-\delta,1/2+\delta)} \tilde{g} = 1/2$. Since the integral of $\tilde{g}$ over $\cup_{k\neq J,J+1}\Omega_k$ tends to zero as $t \to +\infty$, we obtain, using the definition of $\tilde{g}$:

$$\inf_{a>0} \left(dist_* \left(\frac{1}{a} g \left(t^*, \frac{\cdot}{a} \right), m\delta_1 \right) \right) \geq c_1 > 0. \tag{4.71}$$

whence the result follows. $\square$

CHAPTER 5

Heuristic arguments and open problems.

We present in this Chapter several heuristic arguments and formal calculations concerning long time asymptotic properties of the solutions of equation (1.3), (1.4). To deal with this problem scaling arguments have been repeatedly used in the physical literature, cf. in particular [**41**], [**12**], [**53**]. From this point of view, the main goal of this Chapter is to formulate some precise PDE's problems covering several different cases.

5.1. Transport of the energy towards large values of ω.

5.1.1. Weak solutions with interacting condensates. The case of finite particle mass. Notice that Corollary 3.9 implies that for general initial data, the energy of the solution is transported towards large values of ω. We remark that it is possible to derive heuristically one equation that describes the transfer of energy towards larger scales. Let us assume by definiteness that g has the following two scale form:

$$g(t,\cdot) = g_{comp}(t,\cdot) + \frac{1}{R^2} G\left(t, \frac{\cdot}{R}\right) \quad , \quad g_{comp}(t,\cdot) = M\delta_0(\cdot) \tag{5.1}$$

where $R >> 1$. Notice that the form (5.1) implicitly assumes that most of the mass of g concentrates in $\omega = 0$. On the other hand, the energy of the solution $\int \omega g d\omega$ is at distances of order R from the origin. Notice that Theorem 3.2 indicates that after a transient state most of the mass of the solutions should concentrate at $\omega = 0$. We are assuming in (5.1) that the value of R_* in Theorem 3.2 is $R_* = 0$. If $R_* > 0$ and $R >> R_*$ it would be possible to argue in a similar manner, although in such a case a fraction of the energy would remain trapped at distances of order R_* from the origin.

Assuming that g has the form (5.1) we can derive an evolution equation for G as follows. We use in (3.45) a test function φ with the form $\varphi(\omega) = \psi\left(\frac{\omega}{R}\right)$, with ψ compactly supported in $(0,\infty)$. Then:

$$\partial_t \left(\int_{[0,\infty)} g(t,\omega)\varphi(\omega)\, d\omega \right) = \frac{1}{R}\partial_t \left(\int_{[0,\infty)} G(t,\tilde{\omega})\psi(\tilde{\omega})\, d\tilde{\omega} \right) \tag{5.2}$$

On the other hand we have:

$$
\begin{aligned}
g_1 g_2 g_3 = {} & M^3 \delta_0(\omega_1)\delta_0(\omega_2)\delta_0(\omega_2) + \frac{M^2}{R^2}\delta_0(\omega_1)\delta_0(\omega_2) G\left(t, \frac{\omega_3}{R}\right) + \\
& + \frac{M^2}{R^2}\delta_0(\omega_1)\delta_0(\omega_3) G\left(t, \frac{\omega_2}{R}\right) + \frac{M^2}{R^2}\delta_0(\omega_2)\delta_0(\omega_3) G\left(t, \frac{\omega_1}{R}\right) + \\
& + \frac{M}{R^4}\delta_0(\omega_1) G\left(t, \frac{\omega_2}{R}\right) G\left(t, \frac{\omega_3}{R}\right) + \frac{M}{R^4}\delta_0(\omega_2) G\left(t, \frac{\omega_1}{R}\right) G\left(t, \frac{\omega_3}{R}\right) + \\
& + \frac{M}{R^4}\delta_0(\omega_3) G\left(t, \frac{\omega_1}{R}\right) G\left(t, \frac{\omega_2}{R}\right) + \\
& + \frac{1}{R^6} G\left(t, \frac{\omega_1}{R}\right) G\left(t, \frac{\omega_2}{R}\right) G\left(t, \frac{\omega_3}{R}\right) = \sum_{k=1}^{8} I_k
\end{aligned}
$$

The contribution of the term I_1 in the integral of the right-hand side of (3.45) vanishes. Notice that for the test functions under consideration this would hold also if $M\delta_0$ is replaced by a distribution supported in regions ω of order one. The contribution of the term I_2 is $o\left(\frac{1}{R^2}\right)$ if $G(t, \bar{\omega})$ contains a small amount of mass in regions with $\bar{\omega}$ small. The contribution of the terms $I_3,\ I_4$ also vanishes. Actually, if g_{comp} is replaced by a distribution supported in values with ω of order one we would obtain terms containing the derivatives of the test function ψ. However, the contribution of those terms would be $o\left(\frac{1}{R^2}\right)$ due to the factor $\frac{1}{R^2}$, as well as the form of the test function φ which gives an additional term $\frac{1}{R}$ upon differentiation. More precisely, if we assume that g_{comp} is supported in a bounded range of values of ω we would obtain the following contributions due to I_3 in the integral on the right-hand side of (3.45):

$$
\begin{aligned}
& \frac{1}{R^3} \iiint_{([0,\infty))^3} \frac{g_{comp,1} g_{comp,3} G\left(t, \frac{\omega_2}{R}\right) \Phi}{\sqrt{\omega_1 \omega_2 \omega_3}} \times \\
& \qquad \times \left[\psi'\left(\frac{\omega_1}{R}\right)(\omega_1 - \omega_3) + O\left(\frac{(\omega_1 - \omega_3)^2}{R}\right)\right] d\omega_1 d\omega_2 d\omega_3
\end{aligned}
$$

The integral of the term containing $(\omega_1 - \omega_3)$ vanishes by symmetry and the last term gives then a contribution of order $O\left(R^{-4}\right)$. The one of I_4 is similar.

The term I_8 in the right-hand side of (3.45) can be computed by means of rescaling arguments. It turns out to be of order $\frac{1}{R^4} = o\left(\frac{1}{R^3}\right)$.

The main contribution to the integral on the right of (3.45) is due to the terms $I_5,\ I_6,\ I_7$. The terms $I_5 + I_6$ yield:

$$
\frac{2M}{R^3} \iint_{([0,\infty))^2} \frac{G(t, \tilde{\omega}_2) G(t, \tilde{\omega}_3)}{\sqrt{\tilde{\omega}_2 \tilde{\omega}_3}} \left[\psi(\tilde{\omega}_2 - \tilde{\omega}_3) + \psi(\tilde{\omega}_3) - \psi(\tilde{\omega}_2)\right] d\tilde{\omega}_2 d\tilde{\omega}_3 \tag{5.3}
$$

On the other hand, the contribution of the term I_7 is:

$$
\frac{M}{R^3} \iint_{([0,\infty))^2} \frac{G(t, \tilde{\omega}_1) G(t, \tilde{\omega}_2)}{\sqrt{\tilde{\omega}_1 \tilde{\omega}_2}} \left[\psi(\tilde{\omega}_1 + \tilde{\omega}_2) - \psi(\tilde{\omega}_1) - \psi(\tilde{\omega}_2)\right] d\tilde{\omega}_1 d\tilde{\omega}_2 \tag{5.4}
$$

It then follows, combining (5.2)-(5.4) that to the leading order, the evolution equation for G is given by:

$$
\begin{aligned}
(5.5)\quad \partial_t\left(\int_{[0,\infty)} G(t,\tilde{\omega})\,\psi(\tilde{\omega})\,d\tilde{\omega}\right) &= \frac{2M}{R^2}\iint_{([0,\infty))^2}\frac{G(t,\tilde{\omega}_2)\,G(t,\tilde{\omega}_3)}{\sqrt{\tilde{\omega}_2\tilde{\omega}_3}}\times \\
&\quad\times\left[\psi(\tilde{\omega}_2-\tilde{\omega}_3)+\psi(\tilde{\omega}_3)-\psi(\tilde{\omega}_2)\right]d\tilde{\omega}_2 d\tilde{\omega}_3+ \\
&\quad+\frac{M}{R^2}\iint_{([0,\infty))^2}\frac{G(t,\tilde{\omega}_1)\,G(t,\tilde{\omega}_2)}{\sqrt{\tilde{\omega}_1\tilde{\omega}_2}}\times \\
&\quad\times\left[\psi(\tilde{\omega}_1+\tilde{\omega}_2)-\psi(\tilde{\omega}_1)-\psi(\tilde{\omega}_2)\right]d\tilde{\omega}_1 d\tilde{\omega}_2
\end{aligned}
$$

Equation (5.5) is the weak formulation of the following coagulation - fragmentation equation:

$$
\begin{aligned}
(5.6)\quad \frac{R^2}{2M}\partial_t G(t,\tilde{\omega}) &= \frac{G(t,\tilde{\omega})}{\sqrt{\tilde{\omega}}}\int_{\tilde{\omega}}^{\infty}\frac{G(t,\xi)\,d\xi}{\sqrt{\xi}}-\frac{G(t,\tilde{\omega})}{\sqrt{\tilde{\omega}}}\int_0^{\tilde{\omega}}\frac{G(t,\xi)\,d\xi}{\sqrt{\xi}}+ \\
&\quad+\int_0^{\infty}\frac{G(t,\tilde{\omega}+\xi)\,G(t,\xi)\,d\xi}{\sqrt{(\tilde{\omega}+\xi)\,\xi}}+\frac{1}{2}\int_0^{\tilde{\omega}}\frac{G(t,\tilde{\omega}-\xi)\,G(t,\xi)\,d\xi}{\sqrt{(\tilde{\omega}-\xi)\,\xi}}- \\
&\quad-\frac{G(t,\tilde{\omega})}{\sqrt{\tilde{\omega}}}\int_0^{\infty}\frac{G(t,\xi)\,d\xi}{\sqrt{\xi}}
\end{aligned}
$$

Equation (5.6) can be expected to describe the flux of energy of the solution towards $\tilde{\omega}\to\infty$. More precisely, (5.6) describes the distribution G which describes the part of the distribution g in which the energy of the initial distribution is concentrated. Notice that the characteristic time scale for the equation (5.6) is of order $\frac{R^2}{2M}$. This agrees with the result obtained in Proposition 3.11. Moreover, due to the fact that the energy associated to the solution of (1.7), (1.8) escapes to large values of ω as $t\to\infty$, we can expect the asymptotics of (5.6) to be given by a self-similar behaviour with the form:

$$
(5.7)\qquad G(t,\tilde{\omega}) = \frac{R^2}{2Mt+R^2}\Phi(y)\quad,\quad y=\frac{R\tilde{\omega}}{\sqrt{2Mt+R^2}}
$$

Notice that such a rescaling indicates that the energy of the initial distribution g, which is concentrated at values $\omega\approx R$ for $t=0$, would be concentrated at distances $\omega\approx\sqrt{2Mt+R^2}$ for arbitrary values of $t\geq 0$.

Equation (5.6) is reminiscent of the equation which has been obtained in several papers considering the linearization of the isotropic Nordheim equation near Bose-Einstein condensates (cf. [**12**], [**26**], [**44, 45**], [**46**]). The equations derived in those papers contain additional terms which are due to the fact that the Nordheim equation contains, besides the cubic terms in (1.3) additional quadratic terms. Moreover, in some of these papers, it is assumed that M is a function of t, which is due to the fact that the mass of the condensate it is assumed to change in time. In the case of the Nordheim equation the energy of the distribution is expected to remain in bounded regions of ω, and therefore the previous analysis would be meaningless. A nonisotropic version of (5.6) has been obtained also in [**12**], linearizing also near a condensate at $\omega=0$. However, some of the analysis in [**12**] and [**41**] can be used in order to describe the behaviour which can be expected for the solutions of (5.6). In particular the paper [**41**] has obtained the rescaling laws for the transfer of energy towards infinity by means of dimensional arguments. In order to describe asymptotically how this transfer takes place, we notice that there exists a set of "thermal equilibria" for (5.6) having the form:

$$G\left(\tilde{\omega}\right)=\frac{a}{\sqrt{\tilde{\omega}}}\quad,\quad a\geq 0 \tag{5.8}$$

The first and last integrals in (5.6) are divergent for if G is as in (5.8) but the integrals can be made meaningful combining the first and last integrals on the right-hand side of (5.6). Notice that the stationary solutions correspond to a balance between the aggregation and the fragmentation terms in (5.6).

The asymptotics of the function $\Phi\left(y\right)$ as $y\rightarrow 0$ can be expected to be given by the equilibria (5.8). We would then have $\Phi\left(y\right)\sim\frac{a}{\sqrt{y}}$ as $y\rightarrow 0$ for some $a>0$. Notice that this implies the following behaviour for the function g:

$$g\left(t,\omega\right)\sim\frac{a}{\left(2Mt+R^{2}\right)^{\frac{3}{4}}}\frac{1}{\sqrt{\omega}}\quad,\quad\left(2Mt+R^{2}\right)\rightarrow\infty\quad,\quad 1<<\omega<<\sqrt{2Mt+R^{2}}$$

5.1.2. Weak solutions with interacting condensates. The case of infinite particle mass. We remark that the well posedness Theorem 2.16 allows to obtain solutions of (2.2) for initial data $g_{in}\left(\omega\right)$ bounded as $\omega^{-\rho}$ as $\omega\rightarrow\infty$, with $\rho<-\frac{1}{2}$. In particular this suggests that it is possible to obtain global measured valued solutions of (2.2) for a large class of nonintegrable initial data, although we have proved existence of global solutions only for $\rho<-1$ (cf. Remark 2.17) . Although many of the results of this paper apply only to solutions satisfying $\int g\left(d\omega\right)<\infty$, and the results in Subsection 5.1.1 require finite energy (i.e. $\rho>2$), it is interesting to remark that the arguments leading to the equation (5.6) can be adapted to cover also the case in which $g_{in}\left(\omega\right)\sim\frac{K}{\omega^{\rho}}$ as $\omega\rightarrow\infty$ with $K>0,\ \frac{1}{2}<\rho<2$. The main difference between this case and the one studied above, is the fact that the number of particles with small values of ω increases without limit as $t\rightarrow\infty$. We need to modify the ansatz (5.1) in order to take into account that $M=M\left(t\right)$ changes in time. Suppose that the characteristic length scale for the particles aggregating (or fragmenting) their energy is $R=R\left(t\right)$. If we assume that the function $g\left(\omega\right)$ behaves like the power law $\omega^{-\rho}$ as $\omega\rightarrow\infty$ we should have the following rescaling for "large" energies:

$$g\left(t,\cdot\right)\simeq\frac{1}{R^{\rho}}G\left(t,\frac{\cdot}{R}\right)\quad,\quad R=R\left(t\right) \tag{5.9}$$

On the other hand, the number of small particles will be denoted as $M=M\left(t\right)$. Notice that this amounts to approximate $g\left(t,\cdot\right)$ in all the regions as:

$$g\left(t,\cdot\right)=M\left(t\right)\delta_{0}+\frac{1}{R^{\rho}}G\left(t,\frac{\cdot}{R}\right) \tag{5.10}$$

We have to distinguish two different cases. If $1<\rho$ the number of particles of the system is finite and therefore we would have $M\left(t\right)\rightarrow M\left(\infty\right)=\int g_{in}\left(d\omega\right)$. In this case the dynamics of the particles with large energies would be given by a solution of (1.7) with the form (5.9). The rescaling properties of (5.9) give the rescaling $R=t^{\frac{1}{\rho}}$. The distribution of particles containing most of the energy would be given then by a selfsimilar solution of the equation (1.7) with one of the functions g replaced by $M\left(\infty\right)\delta_{0}$ and the two remaining functions g replaced by the ansatz (5.9). This corresponds to one self-similar solution of the coagulation-fragmentation model (5.6) with "fat tails".

Suppose now that $\frac{1}{2} < \rho < 1$. We will ignore critical cases in which logarithmic corrections can be expected. Using the approximation (5.10) into (1.7) we can derive formally one equation for the change of $M(t)$. To this end, we need to integrate (1.7) in regions where ω is bounded. Using the fact that the increment of $M(t)$ is due mainly to interactions of two particles described by the distribution G with one particle with ω of order one we obtain the rescaling:

$$\frac{[M]}{[t]} \simeq \frac{[M]}{[R]^{2\rho}}[R] \tag{5.11}$$

On the other hand, the equation (1.7) yields, assuming that the change of G is also due mostly to the interaction between two particles described by G with one particle described by the distribution $M(t)\,\delta_0$:

$$\frac{1}{[t]\,[R]^{\rho}} \simeq \frac{[M]}{[R]^{2\rho}} \tag{5.12}$$

Combining (5.11), (5.12) we obtain the following scaling laws for ω and the particles of order one:

$$R = t^{\frac{1}{2\rho-1}} \quad , \quad M = t^{\frac{1-\rho}{2\rho-1}}$$

This gives the characteristic rescaling for the self-similar solutions describing the distribution of frequencies for large particles.

5.1.3. Weak solutions with non interacting condensate. The results of this paper refer mainly to the weak solutions of (1.7) in the sense of Definition 2.2. However, as it has been explained in Sections 2.2 and 2.7, other types of solutions are possible.

In this Subsection we speculate about the possible long time asymptotics of the solutions of (1.7) obtained in Theorem 2.30 assuming that they could be extended for arbitrarily long times. As we have indicated in the previous Subsections, in the case of the weak solutions which satisfy Definition 2.2 the long time asymptotics can be described using the coagulation-aggregation model (5.6). This model is a simplification of the original model (1.7), which is possible due to the fact that the most relevant process in order to determine the behaviour of particles with large values of ω is the interaction of two particles with one particle with $\omega = 0$, or at least ω small.

For the solutions obtained in Theorem 2.30 the particles escaping towards $\omega = 0$ do not interact any longer with the remaining particles of the system. We examine the long time asymptotics of solutions such that $g_0(\omega) \sim \frac{1}{\omega^{\rho}},\ \rho > \frac{1}{2}$. We can then look for self-similar solutions of (1.7) with the form:

$$g(t,\omega) = \frac{1}{t^{\alpha}} G\left(\frac{\omega}{t^{\beta}}\right) \quad , \quad \xi = \frac{\omega}{t^{\beta}}$$

Due to the differential equation (1.7) we must assume that $2\alpha - \beta = 1$. If we denote the left-hand side of (1.7) as $\mathcal{Q}[g]$ we obtain the following equation for G :

$$-\alpha G - \beta\xi\frac{\partial G}{\partial \xi} = \mathcal{Q}[G] \tag{5.13}$$

We are interested in solutions of (5.13) with the behaviour $G(\xi) \sim \frac{1}{\xi^{\rho}}$. This requires $\alpha = \beta\rho$. This identity implies also the asymptotics $g(t,\omega) \sim \frac{1}{\omega^{\rho}}$ as $\omega \to \infty$ for each fixed t. Notice that the integral term $\mathcal{Q}[G]$ can be expected to behave then as $\frac{1}{\xi^{3\rho-1}}$, and since $\rho > \frac{1}{2}$ this implies that the contribution of the integral term is negligible compared with the left-hand side of (5.13). Combining the constraints for α, β we obtain $\alpha = \frac{\rho}{2\rho-1}$, $\beta = \frac{1}{2\rho-1}$. These exponents determine the long time asymptotics of the solutions with this initial data. It is interesting to remark that the type of solution derived does not depend in the finiteness of the mass. If $\rho > 1$ the mass of the solutions is finite. We can readily see that in that case:

$$M(t) = \int g(t, d\omega) = \frac{1}{t^{\alpha-\beta}} \int G(d\omega) = \frac{M_0}{t^{\frac{\rho-1}{2\rho-1}}} \to 0 \quad \text{as} \quad t \to \infty$$

A rigorous construction of the self-similar solutions described in this and previous Sections is not currently available. Their construction would provide further insight in the long time asymptotics of the solutions of this problem.

5.2. Open problems.

In this paper we have obtained several mathematical results for the Weak Turbulence Equation associated to the cubic nonlinear Schrödinger equation in three space dimensions. Nevertheless, there are many questions which still remain unsolved as well as some questions which arise naturally from the results in this paper. We list here some open problems suggested by the results of this paper:

- Uniqueness of weak solutions defined in Definitions 2.2 , 2.5 or more generally in Definition 2.42 (cf. Sections 2.1, 2.2 and 2.7).
- Existence of weak solutions in the sense of Definitions 2.5 and 2.42 (cf. Sections 2.2 and 2.7).
- Our construction of weak solutions in the sense of Definition 2.2 takes as a starting point the solution of the weak turbulence equation with a regularized kernel Φ_σ which has the property that cuts the singular terms of Φ for small values of ω. For this regularized kernels the solutions are global and it is possible to consider their limit as $\sigma \to 0$. It is a natural question to determine if using different approximating kernels Φ_σ in order to approximate the kernel Φ it is possible to derive limit solutions which are weak solutions of (1.7), (1.8) in a sense different from Definition 2.2. In particular this includes to obtain (if they exist) approximating kernels Φ_σ for which the corresponding solutions could yield as limit weak solutions in the sense of Definitions 2.5 and 2.42 (cf. Sections 2.2 and 2.7).
- Smoothing effects. The results in the physical literature suggest that weak solutions in the sense of Definition 2.2 yield an asymptotics $g(t, \omega) \sim a(t)\, \omega^{-1/2}$ as $\omega \to 0^+$ if there is a condensate. On the other hand, we can expect similarly the behaviour $g(t, \omega) \sim a(t)\, \omega^{-\frac{2}{3}}$ as $\omega \to 0^+$ for the weak solutions in the sense of Definition 2.5. It is unlikely that this asymptotics holds pointwise. Most likely the asymptotics takes place in some kind of averaged sense or in some suitable weak topology. These issues are closely related to the stability analysis of the Kolmogorov-Zakharov and Rayleigh-Jeans solutions (cf. Sections 2.7).
- Prove of disprove global existence of weak solutions of the Wave Turbulence Equation without finite mass (cf. Section 2.5).
- Prove or disprove rigorously the asymptotic results conjectured for the transfer of mass and energy in Section 5.1. More generally, to derive information about the asymptotics of the solutions of weak turbulence theory.

- We have obtained a family of solutions yielding "pulsating" behaviour. We have proved also that the solutions of weak turbulence with finite mass and that do not develop a condensate in finite time have the property that g can be approximated by a Dirac mass at a positive distance of the origin during most of the times. Our result does not rule out the possibility of the Dirac mass moving continuously towards $\omega = 0$, although in the case of the pulsating solutions constructed in Chapter 4 this is not the case. Is it possible to prove that the pulsating behavior occurs for any solution of weak turbulence that does not develop a condensate in finite time? (Cf. Remark 3.15).

There exists some long standing open questions, although not completely precisely formulated in some cases.

- Derive a precise asymptotics of the solutions near blow-up (and condensate formation). Numerical simulations in the Physical literature suggest self-similar behaviour (cf. ([**21**], [**26**], [**44, 45**]).
- Establish the precise mathematical connection between the cubic nonlinear Schrödinger equation and the equation for weak turbulence. This could yield a very large class of problems ranging from the approximation of particular solutions of NLS by means of solutions of weak turbulence equations, to the precise statistical conditions which ensure the validity of weak turbulence theory. Many issues related to this problem could be complicated due to the presence of the condensate.
- The mathematical theory of the Cauchy problem for non isotropic weak turbulence is a widely open area. One of the questions considered in the physical literature is the dynamics of non isotropic perturbations of the isotropic KZ solutions (cf. in particular [**4**], [**53**]).

CHAPTER 6

Auxiliary results.

In this Chapter we give several auxiliary results which can be proved by means of minor adaptions of some of the arguments used in [**15**] in the proof of blow up for the Nordheim equation. Therefore, we will just state here the results used emphasizing the points where differences with [**15**] arise. The results of this Chapter are used in the proof of Theorems 2.27 and 3.13.

PROPOSITION 6.1. *Suppose that* $g \in C([0,\infty) : \mathcal{M}_+([0,\infty) : (1+\omega)^\rho))$ *is a weak solution of (1.7) in the sense of Definition 2.2. There exists a positive constant* $B < \infty$, *independent on* g *such that, for any* $T > 0$ *and* $R \in (0,1)$ *we have:*

$$\int_0^T dt \int_{\left[0,\frac{R}{2}\right]^3} \left[\prod_{m=1}^3 g_m d\omega_m\right] \frac{(\omega_0)^{\frac{3}{2}}}{(\omega_+)^{\frac{3}{2}}} \left(\frac{\omega_0 - \omega_-}{\omega_0}\right)^2 \leq BR \int g d\epsilon \tag{6.1}$$

where the functions $\omega_-,\ \omega_0,\ \omega_+$ *are as in Definition 2.22.*

PROOF. It is similar to the Proof of Proposition 5.1 of [**15**]. The main idea is to use in (2.3) the test function $\varphi(\omega) = \psi\left(\frac{\omega}{R}\right)$, $R > 0$, $\omega > 0$ with $\psi(s) = s^\theta$ for $0 < s < 1$, $\psi(s) = 1$, $s \geq 1$, $0 < \theta < 1$. The monotonicity property described in Subsection 2.5.2 then yields several inequalities, which can be transformed in (6.1) after some computations. The only difference with the argument in [**15**] is that in that paper, an estimate for some additional quadratic terms, analogous to the ones appearing in the classical Boltzmann equation, must be obtained, and this results in an additional term on the right-hand side of (6.1). These terms are not present in (1.7) and this results in the simpler estimate (6.1), which contains only one term on the right-hand side, due to the contribution of the initial value g_{in}. □

It is now possible to reformulate the estimate (6.1) in a form that makes clearer the fact that this estimate basically allows to control the mass associated to the product measure $\prod_{m=1}^3 g_m d\omega_m$ contained outside the diagonal set $\{(\omega_1,\omega_2,\omega_3) : \omega_1 = \omega_2 = \omega_3\}$. We define the following family of sets:

$$\mathcal{S}_{R,\rho} = \left\{(\omega_1,\omega_2,\omega_3) \in [0,R]^3 : |\omega_0 - \omega_-| > \rho\omega_0\right\},\ 0 < R \leq 1,\ 0 < \rho < 1\ . \tag{6.2}$$

We then have the following result:

LEMMA 6.2. *Suppose that* $g \in L_+^\infty([0,T];\mathcal{M}_+([0,1]))$, *satisfies (6.1) for any* $0 \leq R \leq 1$ *and* $T > 0$. *Suppose also that* $\int_{\{0\}} g(\omega,t)\, d\epsilon = 0$ *for any* $t \in [0,T]$. *Let* $0 < \rho < 1$ *and* $\mathcal{S}_{R,\rho}$ *as in (6.2). Then, for any* $T > 0$ *we have:*

$$\int_0^T dt \int_{\mathcal{S}_{R,\rho}} \left[\prod_{m=1}^3 g_m d\omega_m\right] \leq \frac{2Bb^{\frac{7}{2}}MR}{\rho^2\left(\sqrt{b}-1\right)^2},\quad R \in \left[0,\frac{1}{2}\right] \tag{6.3}$$

with $b=\frac{1}{1-\rho}$ *and* B *as in (6.1).*

Proof. It is an adaptation of the proof of Lemma 5.4 of [**15**]. $\square$

We recall now a Key Measure Theory result that has been used in [**15**]. In order to formulate it we need some additional notation. Given $b>1$, we define a sequence of intervals $\{\mathcal{I}_k\}_{k=0}^{\infty}$ contained in the interval $[0,1]$ by means of:

$$\mathcal{I}_k(b)=b^{-k}\left(\frac{1}{b},1\right] \quad , \quad k=0,1,2,... \quad , \quad b=1+a>1 \tag{6.4}$$

Notice that $\bigcup_{k=0}^{\infty}\mathcal{I}_k(b)=(0,1]$, $\mathcal{I}_k(b)\cap\mathcal{I}_j(b)=\emptyset$ if $k\neq j$.

We need to define also some "extended" intervals:

$$\mathcal{I}_k^{(E)}(b)=\mathcal{I}_{k-1}(b)\cup\mathcal{I}_k(b)\cup\mathcal{I}_{k+1}(b) \quad , \quad k=0,1,2,... \tag{6.5}$$

where, by convenience, we assume that $\mathcal{I}_{-1}(b)=\emptyset$.

We will write $\mathcal{I}_k=\mathcal{I}_k(b)$, $\mathcal{I}_k^{(E)}=\mathcal{I}_k^{(E)}(b)$ if the dependence of the intervals in b is clear in the argument.

We also define for further reference a family $\mathcal{P}_b$ of unions of elements of the family $\{\mathcal{I}_k(b)\}$. We define:

$$\mathcal{P}_b=\left\{A\subset[0,1]: A=\bigcup_{j=1}^{\infty}\mathcal{I}_{k_j}(b) \text{ for some sequence } \{k_j\}\subset\{1,2,...\}\right\} \tag{6.6}$$

Given $A\in\mathcal{P}_b$ we can define an extended set $A^{(E)}$ as follows. Suppose that $A=\bigcup_{j=1}^{\infty}\mathcal{I}_{k_j}(b)$. We then define:

$$A^{(E)}=\bigcup_{j=1}^{\infty}\mathcal{I}_{k_j}^{(E)}(b) \tag{6.7}$$

Notice that given a measure $g\in\mathcal{M}_+([0,1])$, such that $\int_{\{0\}}g(\omega)\,d\epsilon=0$, we have:

$$\int_{[0,1]}g(\omega)\,d\epsilon=\sum_{k=0}^{\infty}\int_{\mathcal{I}_k(b)}g(\omega)\,d\epsilon \tag{6.8}$$

We need also a rescaled version of the sets $\{\mathcal{I}_k(b)\}$, $\left\{\mathcal{I}_k^{(E)}(b)\right\}$, $\mathcal{P}_b$. Given $R\in(0,1]$ and $b>1$ we define two families of intervals $\{\mathcal{I}_k(b,R)\}$, $\left\{\mathcal{I}_k^{(E)}(b,R)\right\}$ by means of:

$$\mathcal{I}_k(b,R)=R\mathcal{I}_k(b) \quad , \quad \mathcal{I}_k^{(E)}(b,R)=R\mathcal{I}_k^{(E)}(b) \quad , \quad k=0,1,2,... \tag{6.9}$$

with $\{\mathcal{I}_k(b)\}$, $\left\{\mathcal{I}_k^{(E)}(b)\right\}$ as in (6.4), (6.5). We define also a class of sets $\mathcal{P}_b(R)$ as follows:

$$\mathcal{P}_b(R)=\{A\subset[0,R]: A=RB, \quad B\in\mathcal{P}_b\} \tag{6.10}$$

where $\mathcal{P}_b$ is as in (6.7). We can also define the concept of extended sets. Given $A\in\mathcal{P}_b(R)$, with the form $A=RB$, $B\in\mathcal{P}_b$ we define:

$$A^{(E)}=RB^{(E)} \tag{6.11}$$

The following result has been proved in [**15**].

Lemma 6.3. *(Lemma 6.3 of* [**15**]*). Suppose that* $b>1,\ 0<R\leq 1$*. We define intervals* $\{\mathcal{I}_k(b,R)\}$, $\left\{\mathcal{I}_k^{(E)}(b,R)\right\}$ *as in (6.9). Let* $\mathcal{P}_b(R)$ *as in (6.10) and* $A^{(E)}$ *as in (6.11) for* $A\in\mathcal{P}_b(R)$*. Given* $0<\delta<\frac{2}{3}$*, we define* $\eta=\min\left\{\left(\frac{1}{3}-\frac{\delta}{2}\right),\frac{\delta}{6}\right\}>0$*. Then, for any* $g\in\mathcal{M}^+[0,R]$ *satisfying* $\int_{\{0\}}g(d\epsilon)=0$*, at least one of the following statements is satisfied:*

(i) Either there exist an interval $\mathcal{I}_k(b,R)$ *such that:*

$$\int_{\mathcal{I}_k^{(E)}(b,R)} g(d\omega)\geq(1-\delta)\int_{[0,R]}g(d\omega)\quad , \tag{6.12}$$

(ii) or, either there exist two sets $\mathcal{U}_1,\mathcal{U}_2\in\mathcal{P}_b(R)$ *such that* $\mathcal{U}_2\cap\mathcal{U}_1^{(E)}=\emptyset$ *and:*

$$\min\left\{\int_{\mathcal{U}_1}g(d\omega),\int_{\mathcal{U}_2}g(d\omega)\right\}\geq\eta\int_{[0,R]}g(d\omega). \tag{6.13}$$

Moreover, in the case (ii) the set $\mathcal{U}_1$ *can be written in the form:*

$$\mathcal{U}_1=\bigcup_{j=1}^{L}\mathcal{I}_{k_j}(b,R) \tag{6.14}$$

for some sequence $\{k_j\}$ *and some finite* L*. We have:*

$$\mathcal{I}_{k_m}(b,R)\cap\left(\bigcup_{j=1}^{m-1}\mathcal{I}_{k_j}^{(E)}(b,R)\right)=\emptyset\quad,\quad m=2,3,...L, \tag{6.15}$$

and also:

$$\sum_{j=1}^{L}\left(\int_{\mathcal{I}_{k_j}(b,R)}g(d\omega)\right)^2\leq\left(\int_{\mathcal{I}_{k_1}(b,R)}g(d\omega)\right)^2+\sum_{j=2}^{L}\int_{\mathcal{I}_{k_1}(b,R)}g(d\omega)\int_{\mathcal{I}_{k_j}(b,R)}g(d\omega), \tag{6.16}$$

$$\int_{\mathcal{I}_{k_1}(b,R)}g(d\omega)<(1-\delta)\int_{[0,R]}g(d\omega). \tag{6.17}$$

This Lemma basically states that either the measure g is concentrated in one of the intervals $\mathcal{I}_k^{(E)}(b,R)$, or alternatively its mass is spread among some sets "sufficiently separated". Using this Lemma we can then obtain the following result, which has been proved also in [**15**].

Lemma 6.4. *Let* $0<\delta<\frac{2}{3}$*,* $0<\rho<1$*. For any* $R\in(0,1)$ *we define* $\mathcal{S}_{R,\rho}$ *as in (6.2). Let us assume also that* $b=\frac{1}{(1-\rho)}$*. There exists* $\nu=\nu(\delta)>0$ *independent on* R *and* ρ *such that, for any* $g\in\mathcal{M}^+[0,R]$ *satisfying* $\int_{\{0\}}g(d\omega)=0$ *if the alternative (ii) in Lemma 6.3 takes place we have:*

$$\int_{\mathcal{S}_{R,\rho}}\left[\prod_{m=1}^{3}g_m(d\omega_m)\right]\geq\nu\left(\int_{(0,R]}g(d\omega)\right)^3>0. \tag{6.18}$$

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Index