Analysis of an inviscid zero-Mach number system in endpoint Besov spaces for finite-energy initial data

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Abstract

The present paper is the continuation of work [14], devoted to the study of an inviscid zero-Mach number system in the framework of *endpoint* Besov spaces of type $B^s_{\infty,r}(\mathbb{R}^d)$, $r \in [1,\infty]$, $d \geq 2$, which can be embedded in the Lipschitz class $C^{0,1}$. In particular, the largest case $B^1_{\infty,1}$ and the case of Hölder spaces $C^{1,\alpha}$ are permitted.

The local in time well-posedness result is proved, under an additional L^2 hypothesis on the initial inhomogeneity and velocity field. A new a priori estimate for parabolic equations in endpoint spaces $B^s_{\infty,r}$ is presented, which is the key to the proof.

In dimension two, we are able to give a lower bound for the lifespan, such that the solutions tend to be globally defined when the initial inhomogeneity is small. There we will show a refined a priori estimate in endpoint Besov spaces for transport equations with *non solenoidal* transport velocity field.

Keywords. Zero-Mach number system; endpoint Besov spaces; finite energy; well-posedness; parabolic regularity; lifespan.

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1 Introduction

In the present paper we will study the following *inviscid zero-Mach number system*:

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho v) &= 0, \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \nabla\Pi &= 0, \\ \operatorname{div}(v + \kappa\rho^{-1}\nabla\rho) &= 0, \end{cases} \tag{1}$$

where $\rho = \rho(t,x) \in \mathbb{R}^+$ stands for the mass density, $v = v(t,x) \in \mathbb{R}^d$ for the velocity field and $\Pi = \Pi(t,x)$ for the unknown pressure. The positive heat-conducting coefficient $\kappa = \kappa(\rho)$ depends smoothly on its variable. The time variable t and the space variable x belong to $\mathbb{R}^+$ (or to $[0,T]$) and $\mathbb{R}^d$, $d \geq 2$, respectively.

This model derives from the full compressible, heat-conducting and inviscid system as the Mach number tends to vanish (see e.g. [1, 15, 20, 21, 25]). In particular, this singular low-Mach number limit is rigorously justified in Alazard [1] for smooth enough solutions. We refer to the introduction of [14] or the previous literature for more details on the derivation of the system.

Interestingly, System (1) can also describe, for instance, a two-component incompressible inviscid mixture with diffusion effects between these two components. We refer to e.g. [16] for more physical backgrounds.

Notice that if we take simply $\kappa \equiv 0$ (i.e. we have no heat conduction), then System (1) reduces to the density-dependent Euler equations

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho v) = 0, \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \nabla\Pi = 0, \\ \operatorname{div} v = 0. \end{cases} \tag{2}$$

We refer to [2, 8, 9], among other works, for some well-posedness results for System (2).

Let us just mention here that, in [10], Danchin adopted mainly the functional framework of Besov spaces $B^s_{p,r}$, $1 < p < +\infty$, which can be embedded in the set of globally Lipschitz functions. There he considered e.g. the finite-energy initial velocity field case, the case with $p \in [2,4]$ or the case of small inhomogeneity. All the assumptions are, roughly speaking, due to the control of (the low frequencies of) the pressure term. In [11], Danchin and the first author treated the endpoint case $B^s_{\infty,r}$. They also proved a lower bound for the solutions in the case of space dimension $d = 2$: the *infinite* energy data were considered as well and one has to resort to the analysis the vorticity of the fluid.

When the fluid is supposed to be viscous, instead, System (1) becomes the viscous zero-Mach number system

$$\begin{cases} \partial_t\rho + \mathrm{div}\,(\rho v) & = & 0, \\ \partial_t(\rho v) + \mathrm{div}\,(\rho v \otimes v) - \mathrm{div}\,\sigma + \nabla\Pi & = & 0, \\ \mathrm{div}\,(v + \kappa\rho^{-1}\nabla\rho) & = & 0, \end{cases} \tag{3}$$

where we defined the viscous stress tensor

$$\sigma = 2\zeta S v + \eta \mathrm{div}\, v \mathrm{Id}\,, \quad Sv := \frac{1}{2}(\nabla v + (\nabla v)^T),$$

for two positive viscous coefficients ζ, η.

The viscous system (3) is the low-Mach number limit system of the full Navier-Stokes equations, and hence it describes for instance the motion of highly subsonic ideal gases. See [1, 7, 13, 18, 19, 22] and references therein for further results. Let us just mention that, Danchin and the second author [12] addressed the well-posedness issue in the general *critical* Besov spaces $B^{d/p_1}_{p_1,1} \times B^{d/p_2-1}_{p_2,1}$, with technical restrictions on the Lebesgue exponents p_1, p_2. Under a special relationship between the viscous coefficient and the heat-conduction (or diffusion) coefficient, [19] showed the global-in-time wellposedness result in dimension two.

To our knowledge, there are just few well-posedness results for the inviscid zero-Mach number system (1). We refer here that in [4], Beirão da Veiga, Serapioni and Valli proved existence of classical solutions on smooth bounded domains for our system (1).

In our previous work [14], instead, we investigate the well-posedness in the functional framework of Besov spaces. There, we reformulated System (1) by introducing new *divergence-free* velocity field. Similarly, let us immediately perform an *invertible* change of unknowns here, to introduce the set of equations (see (6) below) we will mainly work on: for the details we refer to [12, 14].

For notational simplicity, we introduce three coefficients, $a = a(\rho)$, $b = b(\rho)$ and $\lambda = \lambda(\rho)$, such that

$$\nabla a \,=\, \kappa\nabla\rho \,=\, -\rho\nabla b, \quad \lambda = \rho^{-1} > 0, \quad a(1) = b(1) = 0. \tag{4}$$

Then, we introduce the new divergence-free "velocity" u and the new "pressure" π as

$$u \,:=\, v + \kappa\rho^{-1}\nabla\rho \,=\, v - \nabla b, \qquad \pi \,=\, \Pi - \kappa\partial_t\rho \,=\, \Pi - \partial_t a\,. \tag{5}$$

Therefore, System (1) can be rewritten as the following system for the unknowns (ρ, u, π):

$$\begin{cases} \partial_t\rho + u\cdot\nabla\rho - \mathrm{div}\,(\kappa\nabla\rho) = 0, \\ \partial_t u + (u + \nabla b)\cdot\nabla u + \lambda\nabla\pi = h, \\ \mathrm{div}\, u = 0, \end{cases} \tag{6}$$

where the new nonlinear "source" term h reads as

$$h(\rho, u) \,=\, \rho^{-1}\mathrm{div}\,(v \otimes \nabla a) \,=\, -u\cdot\nabla^2 b - (u\cdot\nabla\lambda)\nabla a - (\nabla b\cdot\nabla\lambda)\nabla a - \mathrm{div}\,(\nabla b \otimes \nabla b)\,. \tag{7}$$

In the above mentioned work [14], we studied the well-posedness of the zero-Mach number system, in its reformulated version (6), in the setting of Besov spaces $B^s_{p,r}(\mathbb{R}^d)$ embedded in the class $C^{0,1}$ of globally Lispchitz functions, that is to say for

$$(C) \qquad\qquad s \,>\, 1 + \frac{d}{p}\,, \qquad\qquad \text{or} \qquad\qquad s \,=\, 1 + \frac{d}{p} \quad \text{and} \quad r \,=\, 1\,.$$

Such a restriction is in fact necessary, essentially due to the transport equation for the velocity field: preserving the initial regularity demands u to be at least locally Lipschitz with respect to the space variable. On the other hand, the non-linear source term h requires the control of this Besov norm on $\nabla^2\rho$: this is guaranteed by the smoothing effect of of the *parabolic equation* for the density. Due to technical reasons, we had to impose the additional condition in [14]

$$p \,\in\, [2, 4]\,. \tag{8}$$

This hypothesis (8) ensures that the "source" term h, composed of quadratic terms, belongs to $L^2(\mathbb{R}^d)$. Hence, regarding the pressure which satisfies an elliptic equation in divergence form

$$\mathrm{div}\,(\lambda\nabla\pi) = \mathrm{div}\,(h - (u + \nabla b)\cdot\nabla u),$$

the pressure term $\nabla\pi$ belongs to L^2 too. This gives control on the low frequencies of the pressure term.

Finally, under conditions (C) and (8), we proved local in time well-posedness of System (6) in spaces $B^s_{p,r}$, as well as a continuation criterion for its solutions and a bound from below for the lifespan in any space dimension $d \geq 2$.

In the present paper we propose a different study, rather in endpoint Besov spaces $B^s_{\infty,r}$ which still verifies condition (C) (with $p = +\infty$ of course), in the same spirit of work [11]. This functional framework includes, in particular, the case of Hölder spaces of type $C^{1,\alpha}$, and the case of $B^1_{\infty,1}$, which is the largest one embedded in the space of globally Lipschitz functions, and so the largest one in which one can expect to recover well-posedness for our system.

We will add a *finite-energy* hypothesis on the initial data, which is fundamental to control the pressure term, just as the above condition (8) assumed in [14].

Then we are able to prove the local in time well-posedness issue for System (6) in the adopted functional framework. The key point of the analysis is the proof of new a priori estimates for parabolic equations in spaces $B^s_{\infty,r}$ (see Proposition 4.1).

The global in time existence of solutions to the inviscid zero-Mach number system is still an open problem, even in the simpler case of space dimension $d = 2$. However, similarly as in [11], we are able to move a first step in this direction: by establishing an explicit lower bound for the lifespan of the solutions in dimension $d = 2$, we show that planar flows tend to be globally defined if the inital density is "close" (in an appropriate sense) to a constant state. Such a lower bound improves the one stated in [14], and it can be proved resorting to arguments similar as in Vishik [23] and Hmidi-Keraani [17]. More precisely, the *scalar vorticity* satisfies a transport equation, and then one aims at bounding it *linearly* with respect to the velocity field. Unluckily, in our case the transport velocity occurring in the vorticity equation is the original vector-field v of System (1), which is *not* divergence-free: hence, one can just bound the vorticity linearly in v and $\mathrm{div}\, v$ (see Proposition 4.4). Since the potential part of v just depends on the density term ρ, for which parabolic effect gives enough regularity to control $\mathrm{div}\, v$.

Let us conclude the introduction by pointing out that we decided to adopt the present functional framework, i.e. $B^s_{\infty,r}\cap L^2$, just for simplicity and clarity of exposition. Actually, combining the techniques of [14] with the ones in [10], it's easy to see that our results can be extended to any space $B^s_{p,r}$ which satisfies condition (C) for any $1 < p \leq +\infty$.

Before going on, we give an overview of the paper.

The next section is devoted to the statement of our main results.

In Section 3 we briefly present the tools we use in our analysis, namely Littlewood-Paley decomposition and paradifferential calculus, while in Section 4 we prove fundamental a priori estimates for parabolic and transport equations in endpoint Besov spaces.

Finally, Section 5 contains the proof of our results.

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2 Main Results

As explained in the introduction, in the sequel we will deal with system (6)-(7) in endpoint Besov spaces $B^s_{\infty,r}$ with the indices $s \in \mathbb{R}$ and $r \in [1,+\infty]$ satisfying (C) (for $p=+\infty$), i.e.

$$s > 1 \qquad\qquad \text{or} \qquad\qquad s = r = 1\,. \tag{9}$$

Recall that this is sufficient to ensure the embedding $B^s_{\infty,r} \hookrightarrow C^{0,1}$.

In order to ensure the velocity field u to belong to $B^s_{\infty,r}(\mathbb{R}^d)$, the source term h in the velocity equation, which involves two derivatives of the density $\nabla^2\rho$, should be in the same space. Nonetheless, new a priori estimates for parabolic equations in endpoint Besov spaces $B^s_{\infty,r}$ (see Proposition 4.1 below) will guarantee the gain of two orders of regularity for the density as time goes by. For this reason we take the initial inhomogeneity $\varrho_0 := \rho_0 - 1 \in B^s_{\infty,r}$. And hence we will get the density in the so-called Chemin-Lerner space $\widetilde{L}^\infty_T(B^s_{\infty,r})$ and $\widetilde{L}^1_T(B^s_{\infty,r})$. See Definition 3.6 for the definition of these time-dependent Besov spaces.

Moreover, in order to avoid vacuum regions, we will always suppose that the initial density satisfy

$$0 < \rho_* \le \rho_0 \le \rho^*\,.$$

By applying *maximum principle* on the parabolic equation $(6)_1$, one gets a priori that the density ρ (if it exists on the time interval $[0,T]$) keeps the same upper and lower bounds as the initial density ρ_0:

$$0 < \rho_* \le \rho(t,x) \le \rho^*, \quad \forall\, t \in [0,T],\, x \in \mathbb{R}^d.$$

Hence, applying the divergence operator to Equation $(6)_2$ gives an elliptic equation for π of the form

$$\mathrm{div}\,(\lambda\nabla\pi) = \mathrm{div}\,(h - v\cdot\nabla u)\,, \qquad \text{with} \quad \lambda = \lambda(\rho) \ge \lambda_* := (\rho^*)^{-1} > 0\,. \tag{10}$$

By a result in [10], we hence have a priori energy estimate for $\nabla\pi$ (independently on ρ):

$$\lambda_*\|\nabla\pi\|_{L^2} \le \|h - v\cdot\nabla u\|_{L^2}\,.$$

This gives *low frequency* informations for $\nabla\pi$.

One then considers the following *energy estimates*. First of all, the mass conservation law $(6)_1$ entails (provided $u \in L^\infty_T(L^\infty)$)

$$\frac{1}{2}\int_{\mathbb{R}^d} |\rho(t)-1|^2 + \int_0^t\int_{\mathbb{R}^d} \kappa|\nabla\rho|^2 = \frac{1}{2}\|\rho_0 - 1\|^2_{L^2}. \tag{11}$$

Now we rewrite the momentum conservation law $(6)_2$ into

$$\rho\partial_t u + \rho v\cdot\nabla u + \nabla\pi = \mathrm{div}\,(v\otimes\nabla a). \tag{12}$$

Then, using equation $(1)_1$ and $\mathrm{div}\, u = 0$, taking the L^2 scalar product of the previous relation by u entails

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d}\rho|u|^2 \equiv \int_{\mathbb{R}^d}(\rho\partial_t u + \rho v\cdot\nabla u)\cdot u = \langle \mathrm{div}\,(v\otimes\nabla a), u\rangle_{L^2(\mathbb{R}^d)}. \tag{13}$$

Recalling the definitions of a and b in (4), one bounds the above right-hand side by (up to a multiplicative constant depending on ρ_* and ρ^*)

$$\big(\Theta'(t)\,\|u\|^2_{L^2} + \|\nabla\rho\|^2_{L^2}\big)\,, \;\; \text{with} \;\; \Theta(t) := \int_0^t \big(\|\nabla\rho\|^2_{L^\infty} + \|\nabla\rho\|^4_{L^\infty} + \|\nabla^2\rho\|_{L^\infty} + \|\nabla^2\rho\|^2_{L^\infty}\big)\,d\tau\,.$$

Hence if $(\rho_0 - 1, u_0) \in L^2$ and $\Theta(T) < +\infty$ (this will be ensured by Besov regularity), then we gather

$$u, \rho - 1 \in L^\infty_T(L^2)\,, \quad \nabla\rho \in L^2_T(L^2) \quad \text{ and hence } \quad h \in L^1([0,T];L^2), \nabla\pi \in L^1([0,T];L^2).$$

To conclude, we have the following local-in-time wellposedness result for System (6).

Theorem 2.1. *Let $d \geq 2$ an integer and take $s \in \mathbb{R}$ and $r \in [1, +\infty]$ satisfying condition* (9).
Suppose that the initial data (ρ_0, u_0) fulfill

$$\rho_0 - 1\,,\, u_0 \;\in\; B^s_{\infty,r}(\mathbb{R}^d) \cap L^2\,, \qquad \rho_0 \;\in\; [\rho_*, \rho^*]\,, \qquad \operatorname{div} u_0 \;=\; 0\,. \tag{14}$$

Then there exist a positive time T and a unique solution $(\rho, u, \nabla\pi)$ to System (6) *such that $(\varrho, u, \nabla\pi) := (\rho - 1, u, \nabla\pi)$ belongs to the space $E^s_r(T)$, defined as the set of triplet $(\varrho, u, \nabla\pi)$ such that*

$$\left\{\begin{array}{c} \varrho \;\in\; \widetilde{C}\big([0,T]; B^s_{\infty,r}\big) \cap \widetilde{L}^1\big([0,T]; B^{s+2}_{\infty,r}\big) \cap C\big([0,T]; L^2\big)\,, \quad \rho_* \;\leq\; \varrho + 1 \;\leq\; \rho^*\,, \\ \nabla\varrho \;\in\; L^2\big([0,T]; L^2\big)\,, \\ u \;\in\; \widetilde{C}\big([0,T]; B^s_{\infty,r}\big)^d \cap C\big([0,T]; L^2\big)^d\,, \\ \nabla\pi \;\in\; \widetilde{L}^1\big([0,T]; B^s_{\infty,r}\big)^d \cap L^1\big([0,T]; L^2\big)^d\,, \end{array}\right. \tag{15}$$

with $\widetilde{C}_w([0,T]; B^s_{p,r})$ if $r = +\infty$.

Remark 2.2. Let us remark the well-posedness result for the original system (1). According to the change of variables (5), one knows

$$u = \mathcal{P}v, \quad \nabla b = \mathcal{Q}v\,, \qquad \text{where} \quad \widehat{\mathcal{Q}u}(\xi) = -(\xi/|\xi|^2)\,\xi \cdot \widehat{u}(\xi), \quad \mathcal{P} = I - \mathcal{Q}.$$

Hence, for the original system (1), if the initial datum (ρ_0, v_0) satisfies

$$0 < \rho_* \leq \rho_0 \leq \rho^*\,, \quad \nabla b(\rho_0) = \mathcal{Q}v_0, \qquad \rho_0 - 1, \mathcal{P}v_0 \in B^s_{\infty,r} \cap L^2,$$

then there exist a $T > 0$ and a unique solution $(\rho, v, \nabla\Pi)$ to System (1) such that $\rho_* \leq \rho \leq \rho^*$ and

$$\left\{\begin{array}{c} \varrho = \rho - 1 \in \widetilde{C}\big([0,T]; B^s_{\infty,r}\big) \cap \widetilde{L}^1\big([0,T]; B^{s+2}_{\infty,r}\big) \cap C\big([0,T]; L^2\big)\,, \quad \nabla\varrho \in L^2\big([0,T]; L^2\big)\,, \\ \mathcal{P}v \in \widetilde{C}\big([0,T]; B^s_{\infty,r}\big) \cap C\big([0,T]; L^2\big)\,, \quad v \in \widetilde{C}\big([0,T]; B^{s-1}_{\infty,r}\big)\,, \\ \nabla\Pi \in \widetilde{L}^1\big([0,T]; B^s_{\infty,r}\big) \cap L^1\big([0,T]; L^2\big)\,, \end{array}\right.$$

with $\widetilde{C}_w([0,T]; B^s_{p,r})$ if $r = +\infty$.

Remark 2.3. As said in the introduction, we can replace the Besov space $B^s_{\infty,r}$ in Theorem 2.1 by any general Besov space $B^s_{p,r}$, $p \in]1, +\infty]$ such that condition (C) is fulfilled.

The proof is quite standard, and it goes along the lines of the one in [14], with suitable modifications corresponding to the finite energy conditions. One can refer also to paper [10], where an analogous result is proved for the density-dependent Euler equations.

If $\rho \equiv 1$, System (6) becomes the classical Euler system. For this system, the global-in-time existence issue in dimension $d = 2$ has been well-known since 1933, due to the pioneering work Wolibner [24]. For non-homogeneous perfect fluids, see system (2), it's still open if its solutions exist globally in time. However, in [11] it was proved that, for initial densities close to a constant state, the lifespan of the corresponding solutions tends to *infinity*. In analogy, we have the following result for our system.

Theorem 2.4. *Let $d = 2$, and let us assume the hypotheses of Theorem 2.1.*
Then there exist $\ell > 5$ and $L > 0$ (depending only on ρ_*, ρ^*, s, r) such that the lifespan of the solution to System (6)*, given by Theorem 2.1, is bounded from below by the quantity*

$$\frac{L}{\Gamma_0}\log\left(\frac{L}{\Gamma_0^2}\log\left(1 + \frac{L}{\Big(1 + \|\varrho_0\|^{\ell}_{B^1_{\infty,1}}\Big)\,\|\varrho_0\|_{B^1_{\infty,1}}}\right)\right), \tag{16}$$

where we defined $\Gamma_0 = 1 + \|\varrho_0\|^2_{L^2} + \|u_0\|_{L^2 \cap B^1_{\infty,1}}$.

Remark 2.5. One easily see from (16) that if the initial density ρ_0 tends to 1, then this lower bound tends to infinity, which means the solutions tends to exist all the time.

Remark 2.6. We can just consider the limit Besov space norm $B^1_{\infty,1}$ in the statement of Theorem 2.4, and we will concentrate only on the $B^1_{\infty,1}$ case in the proof. In fact, similar as in the proof of the continuation criterion in [14], by classical commutator estimates and product estimates (see Proposition 3.4 and Proposition 3.7), one knows that if, on the time interval $[0,T^*]$, $T^*<+\infty$, one has

$$\|(\nabla\rho,u)\|_{L^\infty_{T^*}(L^\infty)}+\int_0^{T^*}\Big(1+\|\nabla u\|_{L^\infty}+\|\nabla\varrho\|^4_{L^\infty}+\|\nabla^2\varrho\|^2_{L^\infty}+\|\nabla\pi\|_{L^\infty}\Big)<+\infty,$$

then the solution (ϱ,u) with the initial data $(\varrho_0,u_0)\in B^s_{\infty,r}$ will be well defined in the solution space $E^s_r(T^*)$. On the other side, the above finiteness condition can be ensured if one already has the solution defined in the limit solution space $E^1_1(T^*)$.

Before going on, let us introduce some notations. We agree that in the sequel, C always denotes some "harmless" constant depending only on d,s,r,ρ_*,ρ^*, unless otherwise defined. Notation $A\lesssim B$ means $A\le CB$ and $A\sim B$ says A equals to B, up to a constant factor. For notational convenience, we denote

$$\varrho=\rho-1.$$

3 A brief review of Fourier analysis

In this section, we recall some definitions and results in Fourier analysis which will be used in this paper. Unless otherwise specified, all the presentation in this section have been proved in [3], Chapter 2.

Firstly, let's recall the Littlewood-Paley decomposition. Fix a smooth radial function χ supported in the ball $B(0,\frac{4}{3})$, such that it equals to 1 in a neighborhood of $B(0,\frac{3}{4})$ and is nonincreasing over $\mathbb{R}_+$. Define $\varphi(\xi)=\chi(\frac{\xi}{2})-\chi(\xi)$. The *non-homogeneous dyadic blocks* $(\Delta_j)_{j\in\mathbb{Z}}$ are defined by[1]

$$\Delta_j:=0\ \text{ if }\ j\le-2,\quad \Delta_{-1}:=\chi(D)\quad\text{and}\quad \Delta_j:=\varphi(2^{-j}D)\ \text{ if }\ j\ge0.$$

We also introduce the following low frequency cut-off operators:

$$S_ju:=\chi(2^{-j}D)=\sum_{j'\le j-1}\Delta_{j'}\quad\text{for}\quad j\ge0,\quad S_ju\equiv0\quad\text{for}\quad j\le0.$$

One hence defines *non-homogeneous Besov space* $B^s_{p,r}$ as follows:

Definition 3.1. Let $u\in\mathcal{S}'$, $s\in\mathbb{R}$, $(p,r)\in[1,\infty]^2$. We set

$$\|u\|_{B^s_{p,r}}:=\Big(\sum_j 2^{rjs}\|\Delta_ju\|^r_{L^p}\Big)^{\frac{1}{r}}\quad\text{if}\quad r<\infty\quad\text{and}\quad \|u\|_{B^s_{p,\infty}}:=\sup_j\left(2^{js}\|\Delta_ju\|_{L^p}\right).$$

The space $B^s_{p,r}$ is the subset of tempered distributions u such that $\|u\|_{B^s_{p,r}}$ is finite.

Recall that, for all $s\in\mathbb{R}$, we have the equivalence $H^s\equiv B^s_{2,2}$, while for all $s\in\mathbb{R}_+\backslash\mathbb{N}$, the space $B^s_{\infty,\infty}$ is actually the Hölder space C^s. If $s\in\mathbb{N}$, instead, we set $C^s_*:=B^s_{\infty,\infty}$, to distinguish it from the space C^s of the differentiable functions with continuous partial derivatives up to the order s. Moreover, the strict inclusion $C^s_b\hookrightarrow C^s_*$ holds, where C^s_b denotes the subset of C^s functions bounded with all their derivatives up to the order s. Finally, for $s<0$, the "negative Hölder space" C^s is defined as the Besov space $B^s_{\infty,\infty}$.

For spectrally localized functions, one has the following *Bernstein's inequalities*:

Lemma 3.2. *There exists a $C>0$ such that, for any $k\in\mathbb{Z}^+$, $\lambda\in\mathbb{R}^+$, $(p,q)\in[1,\infty]^2$ with $p\le q$, then*

$$\operatorname{Supp}\widehat{u}\subset B(0,\lambda)\Longrightarrow\|u\|_{L^q}\le C\lambda^{d(\frac{1}{p}-\frac{1}{q})}\|u\|_{L^p};$$
$$\operatorname{Supp}\widehat{u}\subset\{\xi\in\mathbb{R}^N\,/\,\lambda\le|\xi|\le2\lambda\}\Longrightarrow C^{-k-1}\lambda^k\|u\|_{L^p}\le\|\nabla^ku\|_{L^p}\le C^{k+1}\lambda^k\|u\|_{L^p}.$$

We remark explicitly that by previous lemma one has, for any $f\in L^2$,

$$\|\Delta_{-1}f\|_{L^\infty}\le C\|\Delta_{-1}f\|_{L^2}\le C\|f\|_{L^2}.$$

One also has the following embedding and interpolation results:

[1] In what follows we agree that $f(D)$ stands for the pseudo-differential operator $u\mapsto\mathcal{F}^{-1}(f(\xi)\mathcal{F}u(\xi))$.

Proposition 3.3. *Space $B^{s_1}_{p_1,r_1}$ is continuously embedded in Space $B^{s_2}_{p_2,r_2}$ whenever $1 \leq p_1 \leq p_2 \leq \infty$ and*

$$s_2 < s_1 - d/p_1 + d/p_2 \quad \text{or} \quad s_2 = s_1 - d/p_1 + d/p_2 \quad \text{and} \quad 1 \leq r_1 \leq r_2 \leq \infty.$$

Moreover, one has the following interpolation inequality:

$$\|\varrho\|_{B^{s+1}_{\infty,r}} \leq C\|\varrho\|^{1/2}_{B^s_{\infty,r}} \|\varrho\|^{1/2}_{B^{s+2}_{\infty,r}}\,, \qquad \|\nabla\pi\|_{B^{s-1}_{\infty,r}} \leq C\|\nabla\pi\|^{\gamma}_{L^2} \|\nabla\pi\|^{1-\gamma}_{B^s_{\infty,r}} \quad (0<\gamma<1)\,.$$

One also has the following classical commutator estimate:

Proposition 3.4. *If $s>0$, $r \in [1,\infty]$, then there exists a constant C depending only on d, s, r such that*

$$\int_0^t \left\|2^{js} \|[\varphi, \Delta_j]\nabla\psi\|_{L^\infty}\right\|_{\ell^r} d\tau \leq C \int_0^t \Big(\|\nabla\varphi\|_{L^\infty} \|\psi\|_{B^s_{\infty,r}} + \|\nabla\varphi\|_{B^{s-1}_{\infty,r}} \|\nabla\psi\|_{L^\infty}\Big)\, d\tau. \tag{17}$$

Let us recall the *Bony's paraproduct decomposition* (first introduced in [5]):

$$uv = T_u v + T_v u + R(u,v)\,, \tag{18}$$

where we defined the paraproduct operator T and the remainder R as

$$T_u v := \sum_j S_{j-1}u\Delta_j v \quad \text{and} \quad R(u,v) := \sum_j \sum_{|j'-j|\leq 1} \Delta_j u\, \Delta_{j'} v\,.$$

These operators enjoy the following continuity properties in the class of Besov spaces.

Proposition 3.5. *For any $(s,p,r) \in \mathbb{R}\times[1,\infty]^2$ and $t>0$, the paraproduct operator T maps $L^\infty \times B^s_{p,r}$ in $B^s_{p,r}$, and $B^{-t}_{\infty,\infty} \times B^s_{p,r}$ in $B^{s-t}_{p,r}$. Moreover, the following estimates hold:*

$$\|T_u v\|_{B^s_{p,r}} \leq C\|u\|_{L^\infty}\|\nabla v\|_{B^{s-1}_{p,r}} \quad \text{and} \quad \|T_u v\|_{B^{s-t}_{p,r}} \leq C\|u\|_{B^{-t}_{\infty,\infty}}\|\nabla v\|_{B^{s-1}_{p,r}}.$$

For any (s_1,p_1,r_1) and (s_2,p_2,r_2) in $\mathbb{R}\times[1,\infty]^2$ such that $s_1+s_2>0$, $1/p := 1/p_1 + 1/p_2 \leq 1$ and $1/r := 1/r_1 + 1/r_2 \leq 1$ the remainder operator R maps $B^{s_1}_{p_1,r_1} \times B^{s_2}_{p_2,r_2}$ in $B^{s_1+s_2}_{p,r}$.

When solving evolutionary PDEs in Besov spaces, we have to localize the equations by Littlewood-Paley decomposition. So we will have estimates for the Lebesgue norm of each dyadic block *before* performing integration in time. This leads to the following definition, introduced for the first time in paper [6] by Chemin and Lerner.

Definition 3.6. For $s \in \mathbb{R}$, $(q,p,r) \in [1,+\infty]^3$ and $T \in [0,+\infty]$, we set

$$\|u\|_{\widetilde{L}^q_T(B^s_{p,r})} := \left\|\Big(2^{js}\|\Delta_j u(t)\|_{L^q_T(L^p)}\Big)_{j\geq -1}\right\|_{\ell^r}.$$

We also set $\widetilde{C}_T(B^s_{p,r}) = \widetilde{L}^\infty_T(B^s_{p,r}) \cap C([0,T];B^s_{p,r})$.

The relation between these classes and the classical $L^q_T(B^s_{p,r})$ can be easily recovered by Minkowski's inequality:

$$\begin{cases} \|u\|_{\widetilde{L}^q_T(B^s_{p,r})} \leq \|u\|_{L^q_T(B^s_{p,r})} & \text{if} \quad q \leq r \\ \|u\|_{\widetilde{L}^q_T(B^s_{p,r})} \geq \|u\|_{L^q_T(B^s_{p,r})} & \text{if} \quad q \geq r\,. \end{cases}$$

Combining the above proposition 3.5 with Bony's decomposition (18), we easily get the following product estimate in Chemin-Lerner space:

Corollary 3.7. *There exists a constant C depending only on d, s, p, r such that*

$$\|uv\|_{\widetilde{L}^q_T(B^s_{p,r})} \leq C\Big(\|u\|_{L^{q_1}_T(L^\infty)}\|v\|_{\widetilde{L}^{q_2}_T(B^s_{p,r})} + \|u\|_{\widetilde{L}^{q_3}_T(B^s_{p,r})}\|v\|_{L^{q_4}_T(L^\infty)}\Big)\,, \quad \frac{1}{q} := \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_3} + \frac{1}{q_4}.$$

One also has the estimates for the composition of functions in Besov spaces.

Proposition 3.8. *Let $F : \mathbb{R} \to \mathbb{R}$ be a smooth function. Then for any $s > 0$, $(q, p, r) \in [1, +\infty]^3$, we have*

$$\|\nabla(F(a))\|_{\widetilde{L}^q_T(B^{s-1}_{p,r})} \leq C\|\nabla a\|_{\widetilde{L}^q_T(B^{s-1}_{p,r})}.$$

If furthermore $F(0) = 0$, then $\|F(a)\|_{\widetilde{L}^q_T(B^s_{p,r})} \leq C\,\|a\|_{\widetilde{L}^q_T(B^s_{p,r})}$.

In the next section we will need also some notions about *homogeneous paradifferential calculus*: let us recall them.

The homogeneous dyadic blocks $(\dot{\Delta}_j)_{j\in\mathbb{Z}}$ are defined by

$$\dot{\Delta}_j := \varphi(2^{-j}D) \qquad \text{if} \quad j \in \mathbb{Z}.$$

The homogeneous low frequency cut-off are defined by:

$$\dot{S}_j u := \chi(2^{-j}D)u \qquad \text{for} \quad j \in \mathbb{Z}.$$

The homogeneous paraproduct operator and remainder operator are defined by:

$$\dot{T}_u v := \sum_{j\in\mathbb{Z}} \dot{S}_{j-1}u\dot{\Delta}_j v \qquad \text{and} \qquad \dot{R}(u, v) := \sum_{j\in\mathbb{Z}} \sum_{|j'-j|\leq 1} \dot{\Delta}_j u\, \dot{\Delta}_{j'} v.$$

Notice that, for all u and v in $\mathcal{S}'$, the sequence $(\dot{S}_{j-1}u\,\dot{\Delta}_j v)_{j\in\mathbb{Z}}$ is spectrally supported in dyadic annuli. The analogous of Proposition 3.5 holds true also in the homogeneous setting.

Finally, let us set $\dot{C}^s = \dot{B}^s_{\infty,\infty}$, for $s > 0$, to be the homogeneous Hölder space. Recall that, for any $u \in \dot{C}^s$, the equality $u = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u$ holds. For homogeneous Hölder spaces and time-dependent homogeneous Hölder spaces, we have the following characterization.

Proposition 3.9. *$\forall \epsilon \in (0, 1)$, there exists a constant C such that for all $u \in \mathcal{S}$,*

$$C^{-1}\|u\|_{\dot{C}^\epsilon} \leq \left\| \frac{\|u(x + y) - u(x)\|_{L^\infty_x}}{|y|^\epsilon} \right\|_{L^\infty_y} \leq C\|u\|_{\dot{C}^\epsilon}, \tag{19}$$

and

$$C^{-1}\|u\|_{\widetilde{L}^1_t(\dot{C}^\epsilon)} \leq \left\| \int_0^t \frac{\|u(\tau, x + y) - u(\tau, x)\|_{L^\infty_x}}{|y|^\epsilon}\, d\tau \right\|_{L^\infty_y} \leq C\|u\|_{\widetilde{L}^1_t(\dot{C}^\epsilon)}. \tag{20}$$

Proof. The proof of (19) can be found at page 75 of [3].

Let us just show the left-hand inequality of (20). Since

$$\dot{\Delta}_j u(t, x) = 2^{jd} \int_{\mathbb{R}^d} h(2^j y)(u(t, x - y) - u(t, x))\, dy,$$

then we easily find

$$\begin{aligned}\int_0^t 2^{j\epsilon}\|\dot{\Delta}_j u(\tau, \cdot)\|_{L^\infty_x} &\leq 2^{jd} \int_{\mathbb{R}^d} 2^{j\epsilon}|y|^\epsilon|h(2^j y)| \int_0^t \frac{\|u(\tau, x - y) - u(\tau, x)\|_{L^\infty_x}}{|y|^\epsilon}\, d\tau\, dy \\ &\leq C \left\| \int_0^t \frac{\|u(\tau, x + y) - u(\tau, x)\|_{L^\infty_x}}{|y|^\epsilon} \right\|_{L^\infty_y}.\end{aligned}$$

This relation gives us the left-hand side inequality of (20).

The inverse inequality follows immediately after similar changes with respect to time in the classical proof. □

Recall the classical a priori estimates for heat equations in homogeneous Besov spaces:

Proposition 3.10. *For any $s \in \mathbb{R}$, there exists a constant C_0 such that*

$$\|F\|_{\widetilde{L}^\infty_T(\dot{C}^s)\cap\widetilde{L}^1_T(\dot{C}^{2+s})} \leq C_0(\|f_0\|_{\dot{C}^s} + \|f\|_{\widetilde{L}^1_T(\dot{C}^s)}), \tag{21}$$

where $f_0, f, F \in \mathcal{S}(\mathbb{R}^d)$ and they are linked by the relation

$$F(t, x) = e^{t\Delta} f_0 + \int_0^t e^{(t-\tau)\Delta} f(\tau)\, d\tau.$$

For later use, it is also convenient to show an a priori estimate for the paraproduct $\dot{T}_v u$ in the space $\widetilde{L}^1_T(\dot{C}^s)$.

Lemma 3.11. *For any $s>0$, $\varepsilon>0$, $a,b>0$, there exists a constant $C_\varepsilon \sim \varepsilon^{-a/b}$ such that*

$$\|\dot{T}_v u\|_{\widetilde{L}^1_T(\dot{C}^s)} \le C_\varepsilon \int_0^T \|u(t)\|_{\dot{C}^s}^{\frac{a+b}{b}} \|v\|_{\dot{C}^{-a}} + \varepsilon \|v\|_{\widetilde{L}^1_T(\dot{C}^b)}. \tag{22}$$

Proof. First, let us notice that,

$$\|\dot{T}_v u\|_{\widetilde{L}^1_T(\dot{C}^s)} \le \int_0^T \sum_{j\in\mathbb{Z}} \|u\|_{\dot{C}^s} \|\dot{\Delta}_j v\|_{L^\infty}.$$

For any $\varepsilon>0$, $a,b>0$ and $t\in(0,T)$, we fix an integer

$$N_t = \left[\frac{1}{b}\log_2(\varepsilon^{-1}\|u(t)\|_{\dot{C}^s})\right] + 1,$$

then noticing that $\varepsilon^{-1}\|u(t)\|_{\dot{C}^s} \sim 2^{N_t b}$, we have

$$\begin{aligned}
\int_0^T \sum_{j\in\mathbb{Z}} \|u\|_{\dot{C}^s}\|\dot{\Delta}_j v\|_{L^\infty} &\le \int_0^T \sum_{j\le N_t} 2^{ja}\|u(t)\|_{\dot{C}^s} 2^{-ja}\|\dot{\Delta}_j v\|_{L^\infty} + \sum_{j\ge N_t+1} 2^{-jb}\|u(t)\|_{\dot{C}^s} 2^{jb}\|\dot{\Delta}_j v\|_{L^\infty} \\
&\lesssim \int_0^T 2^{N_t a}\|u(t)\|_{\dot{C}^s}\|v\|_{\dot{C}^{-a}} + \sum_{j\ge N_t+1} 2^{-(j-N_t)b}\,\varepsilon\, 2^{jb}\|\dot{\Delta}_j v\|_{L^\infty} \\
&\lesssim \int_0^T \varepsilon^{-a/b}\|u(t)\|_{\dot{C}^s}^{(a+b)/b}\|v\|_{\dot{C}^{-a}} + \varepsilon \sum_{j\ge 1} 2^{-jb}\, 2^{(j+N_t)b}\|\dot{\Delta}_{j+N_t} v\|_{L^\infty} \\
&\lesssim \varepsilon^{-a/b}\int_0^T \|u(t)\|_{\dot{C}^s}^{(a+b)/b}\|v\|_{\dot{C}^{-a}} + \varepsilon \sup_j \int_0^T 2^{jb}\|\dot{\Delta}_j v\|_{L^\infty}.
\end{aligned}$$

Thus the lemma follows. □

4 A priori estimates for parabolic and transport equations in endpoint Besov spaces

The present section is devoted to obtain new a priori estimates for parabolic and transport equations in the endpoint Besov spaces. These estimates will be the key point to get our results.

In the first subsection we will focus on the parabolic equations, and we will prove a priori estimates in Chemin-Lerner spaces based on endpoint Besov spaces $B^s_{\infty,r}$. This will be useful in the proof of Theorem 2.1.

In the second subsection we will prove refined a priori estimates in the enpoint space $B^0_{\infty,1}$ for linear transport equations. This is a generalization of such results in [17, 23], and it will be fundamental in getting lower bounds for the lifespan of the solution to the inviscid zero-Mach number system.

4.1 Parabolic estimates in $B^s_{\infty,r}$

The present subsection is devoted to state new a priori estimates for linear parabolic equations in endpoint Besov spaces $B^s_{\infty,r}$. For the reasons explained in the previous section, we will work in the time-dependent Besov spaces $\widetilde{L}^q_T(B^s_{\infty,r})$ defined above.

Proposition 4.1. *Let $\rho\in\mathcal{S}(\mathbb{R}^d)$ solve the following linear parabolic equation*

$$\begin{cases} \partial_t\rho - \operatorname{div}(\kappa\nabla\rho) = f, \\ \rho|_{t=0} = \rho_0, \end{cases} \tag{23}$$

with $\kappa, f, \rho_0 \in \mathcal{S}(\mathbb{R}^d)$ and

$$0<\rho_*\le\rho_0\le\rho^*, \quad 0<\kappa_*\le\kappa(t,x)\le\kappa^*.$$

If $s > 0$, then for any $\epsilon \in (0,1)$, there exists a constant C, depending on $d, s, r, \rho_*, \rho^*, \kappa_*, \kappa^*$, such that the following estimate holds true:

$$\|\rho\|_{\widetilde{L}^\infty_t(B^s_{\infty,r})\cap\widetilde{L}^1_t(B^{s+2}_{\infty,r})} \leq C \left[1 + \|\kappa\|^{2/\epsilon}_{L^\infty_t(B^\epsilon_{\infty,\infty})}\right] \times \Bigg(\|\rho_0\|_{B^s_{\infty,r}} + \|f\|_{\widetilde{L}^1_t(B^s_{\infty,r})} + \int_0^t \left(\left[1 + \|\kappa\|^{2/(1+\epsilon)}_{B^{1+\epsilon}_{\infty,\infty}}\right] \|\rho\|_{B^s_{\infty,r}} + \|\nabla\kappa\|_{L^\infty}\|\nabla\rho\|_{B^s_{\infty,r}} + \|\nabla\kappa\|_{B^s_{\infty,r}}\|\nabla\rho\|_{L^\infty}\right) d\tau \Bigg).$$

Remark 4.2. By Gronwall lemma, it is easy to get the following a priori estimate for ρ:

$$\|\rho\|_{\widetilde{L}^\infty_t(B^s_{\infty,r})\cap\widetilde{L}^1_t(B^{s+2}_{\infty,r})} \leq C_1 \exp\{C_1 K(t)\} \left(\|\rho_0\|_{B^s_{\infty,r}} + \|f\|_{\widetilde{L}^1_t(B^s_{\infty,r})}\right),$$

where $C_1 = C_1(t)$ depends on $d, s, r, \rho_*, \rho^*, \kappa_*, \kappa^*$ and $\|\kappa\|_{L^\infty_t(B^\epsilon_{\infty,\infty})}$, and

$$K(t) := \int_0^t \left(1 + \|\nabla\kappa\|^2_{L^\infty} + \|\nabla\kappa\|^{\max\{2/(1+s),1\}}_{B^s_{\infty,r}}\right) d\tau\,.$$

Proposition 4.1 will be proved in three steps. The strategy is the following.

First of all, we localize the function ρ into countable functions ϱ_n, each of which is supported on some ball $B(x_n, \delta)$, with small radius $\delta \in (0,1)$ to be determined in the proof. Hence up to some small perturbation, ϱ_n verifies a heat equation with a *time-dependent* heat-conduction coefficient (see System (25) below). Consequently, changing the time variable and making use of estimates for the heat equation entail a control for ϱ_n in $\dot{E}^\epsilon$:

$$\dot{E}^\epsilon := \widetilde{L}^\infty_T(\dot{C}^\epsilon) \cap \widetilde{L}^1_T(\dot{C}^{2+\epsilon}), \quad \epsilon \in (0,1),$$

This will be done in Step 1.

In step 2, thanks to Proposition 3.9, we carry the result from $\{\varrho_n\}$ to ρ. Note that Maximum Principle applied to parabolic equations has already given us the control on low frequencies of the solution:

$$\|\rho\|_{L^\infty_t(L^\infty)} \leq \|\rho_0\|_{L^\infty} + \int_0^t \|f\|_{L^\infty}. \tag{24}$$

This already ensures the result in Hölder space C^ϵ.

Step 3 is devoted to handle general Besov spaces of form $B^s_{\infty,r}$: we again localize the system, but in *Fourier variables*; then we apply the result of Step 2 to $\rho_j := \Delta_j\rho$ and a careful calculation on commutator terms will yield the thesis.

We agree that in this subsection $\{\varrho_n(t,x)\}$ always denote localized functions of $\rho(t,x)$ in *x-space*, while ρ_j as usual, denotes $\Delta_j\rho$ (localization in the *phase space*).

Step 1: estimate for ϱ_n in $\dot{E}^\epsilon$

Let us take first a smooth partition of unity $\{\psi_n\}_{n\in\mathbb{N}}$ subordinated to a locally finite covering of $\mathbb{R}^d$. We suppose that the ψ_n's satisfy the following conditions:

(i) $\mathrm{Supp}\,\psi_n \subset B(x_n,\delta) \triangleq B_n$, with $\delta < 1$ to be determined later;

(ii) $\sum_n \psi_n \equiv 1$;

(iii) $0 \leq \psi_n \leq 1$, with $\psi_n \equiv 1$ on $B(x_n, \delta/2)$;

(iv) $\|\nabla^\eta\psi_n\|_{L^\infty} \leq C|\delta|^{-|\eta|}$, for $|\eta| \leq 3$;

(v) for each $x \in \mathbb{R}^d$, there are at most N_d (depending on the dimension d) elements in $\{\psi_n\}_{n\in\mathbb{N}}$ covering the ball $B(x, \delta/2)$.

Now by multiplying ψ_n to Equation (23), we get the equation for $\varrho_n \triangleq \rho\psi_n$, which is compactly supported on B_n:

$$\begin{cases} \partial_t\varrho_n - \bar{\kappa}_n\Delta\varrho_n = (\kappa - \bar{\kappa}_n)\Delta\varrho_n + \nabla\kappa\cdot\nabla\varrho_n + g_n, \\ \varrho_n|_{t=0} = \varrho_{0,n} = \psi_n\rho_0, \end{cases} \tag{25}$$

where

$$\bar{\kappa}_n(t) \triangleq \frac{1}{\mathrm{vol}(B_n)} \int_{B_n} \kappa(t,y)\,dy$$

is a function depending only on t, and

$$g_n = -2\kappa\nabla\psi_n \cdot \nabla\rho - (\kappa\Delta\psi_n + \nabla\kappa\cdot\nabla\psi_n)\rho + f\psi_n. \tag{26}$$

For convenience we suppose that there exists a positive constant C_κ such that

$$|\kappa(t,x) - \kappa(t,y)| \le C_0\|\kappa\|_{L^\infty_{t_0}(\dot{C}^\epsilon)}|x-y|^\epsilon, \quad \forall\, x,y\in\mathbb{R}^d,\ t\in[0,t_0]. \tag{27}$$

Notice that, by (27), we have $\bar{\kappa}_n \ge \kappa_* > 0$, which ensures that, for all $t\in[0,t_0]$,

$$\|\kappa/\bar{\kappa}_n - 1\|_{L^\infty(B_n)} \le \kappa_*^{-1}\left\|\frac{1}{\mathrm{vol}(B_n)}\int_{B_n}\kappa(t,x)-\kappa(t,y)\,dy\right\|_{L^\infty(B_n)} \le C_\kappa\|\kappa\|_{L^\infty_{t_0}(\dot{C}^\epsilon)}\kappa_*^{-1}\delta^\epsilon. \tag{28}$$

In order to get rid of the variable coefficient $\bar{\kappa}_n(t)$, let us make the one-to-one change in time variable

$$\tau \triangleq \tau(t) = \int_0^t \bar{\kappa}_n(t')\,dt'. \tag{29}$$

Therefore, the new unknown

$$\widetilde{\varrho}_n(\tau,x) \triangleq \rho_n(t,x),$$

satisfies (observe that $\frac{d\tau}{dt} = \bar{\kappa}_n(t)$)

$$\begin{cases} \partial_\tau\widetilde{\varrho}_n - \Delta\widetilde{\varrho}_n = \left(\frac{\widetilde{\kappa}(\tau)}{\widetilde{\kappa}_n(\tau)} - 1\right)\Delta\widetilde{\varrho}_n + \frac{\nabla\widetilde{\kappa}(\tau)}{\widetilde{\kappa}_n(\tau)}\cdot\nabla\widetilde{\varrho}_n + \frac{\widetilde{g}_n(\tau)}{\widetilde{\kappa}_n(\tau)}, \\ \widetilde{\varrho}_n|_{\tau=0} = \rho_{0,n}, \end{cases} \tag{30}$$

where $\widetilde{\kappa}(\tau,x) = \kappa(t,x)$, $\widetilde{\kappa}_n(\tau) = \bar{\kappa}_n(t)$, $\widetilde{\rho}(\tau,x) = \rho(t,x)$, $\widetilde{g}_n(\tau,x) = g_n(t,x)$.

This is a heat equation: in view of Proposition 3.10, we have to bound the "source" terms. Estimate (22) and

$$\|\dot{T}_u v + \dot{R}(u,v)\|_{\widetilde{L}^1_T(\dot{C}^\epsilon)} \le C\|u\|_{L^\infty_T(L^\infty)}\|v\|_{\widetilde{L}^1_T(\dot{C}^\epsilon)}$$

imply that the first source term of Equation (30) can be controlled by

$$\begin{aligned} \left\|\left(\frac{\widetilde{\kappa}(\tau,\cdot)}{\widetilde{\kappa}_n(\tau)} - 1\right)\Delta\widetilde{\varrho}_n(\tau,\cdot)\right\|_{\widetilde{L}^1_T(\dot{C}^\epsilon)} &\le C\,\|\widetilde{\kappa}/\widetilde{\kappa}_n - 1\|_{L^\infty_T(L^\infty(B_n))}\,\|\Delta\widetilde{\varrho}_n\|_{\widetilde{L}^1_T(\dot{C}^\epsilon)} \\ &\quad + C_{\eta_1}\int_0^T \|\widetilde{\kappa}/\widetilde{\kappa}_n - 1\|_{\dot{C}^\epsilon}^{\frac{2}{\epsilon}}\,\|\Delta\widetilde{\varrho}_n\|_{\dot{C}^{\epsilon-2}} + \eta_1\|\Delta\widetilde{\varrho}_n\|_{\widetilde{L}^1_T(\dot{C}^\epsilon)}, \end{aligned}$$

for any $\eta_1 \in (0,1)$ with $C_{\eta_1} \sim \eta_1^{\frac{\epsilon-2}{\epsilon}}$. Besides, Inequality (28) ensures that for all $\tau\in[0,\tau_0]$, with $\tau_0 = \tau(t_0)$,

$$\|\widetilde{\kappa}/\widetilde{\kappa}_n - 1\|_{L^\infty_{\tau_0}(B_n)} \le C_\kappa\|\kappa\|_{L^\infty_{t_0}(\dot{C}^\epsilon)}\kappa_*^{-1}\delta^\epsilon,$$

which implies, for $T\in[0,\tau_0]$,

$$\left\|\left(\frac{\widetilde{\kappa}(\tau,\cdot)}{\widetilde{\kappa}_n(\tau)} - 1\right)\Delta\widetilde{\varrho}_n(\tau,\cdot)\right\|_{\widetilde{L}^1_T(\dot{C}^\epsilon)} \le C_{\eta_1}\int_0^T \|\widetilde{\kappa}\|_{\dot{C}^\epsilon}^{\frac{2}{\epsilon}}\,\|\widetilde{\varrho}_n\|_{\dot{C}^\epsilon} + (CC_\kappa\|\kappa\|_{L^\infty_{t_0}(\dot{C}^\epsilon)}\kappa_*^{-1}\delta^\epsilon + \eta_1)\|\widetilde{\varrho}_n\|_{\widetilde{L}^1_T(\dot{C}^{2+\epsilon})}. \tag{31}$$

For any $\eta > 0$, there exists $C_\eta \sim \eta^{-1}$ such that

$$\|T_u v + R(u,v)\|_{\widetilde{L}^1_T(\dot{C}^\epsilon)} \le C_\eta\int_0^T \|u\|_{L^\infty}^2\|v\|_{\dot{C}^{\epsilon-1}} + \eta\|v\|_{\widetilde{L}^1_T(\dot{C}^{\epsilon+1})}.$$

Thus, also by use of Lemma 3.11 with $a = 1-\epsilon$ and $b = 1+\epsilon$, for any $\eta_2\in(0,1)$ we have the following (with $C_{\eta_2}\sim\eta_2^{-1}$):

$$\left\|\frac{\nabla\widetilde{\kappa}(\tau,\cdot)}{\widetilde{\kappa}_n(\tau)}\cdot\nabla\widetilde{\varrho}_n(\tau,\cdot)\right\|_{\widetilde{L}^1_T(\dot{C}^\epsilon)} \le C_{\eta_2}\int_0^T\left(\|\nabla\widetilde{\kappa}\|_{L^\infty}^2 + \|\nabla\widetilde{\kappa}\|_{\dot{C}^\epsilon}^{\frac{2}{1+\epsilon}}\right)\|\widetilde{\varrho}_n\|_{\dot{C}^\epsilon} + \eta_2\|\widetilde{\varrho}_n\|_{\widetilde{L}^1_T(\dot{C}^{2+\epsilon})}. \tag{32}$$

Now let us choose δ, η_1, η_2 such that

$$C_0 C C_\kappa \|\kappa\|_{L^\infty_{t_0}(\dot C^\epsilon)} \kappa_*^{-1} \delta^\epsilon \ , \quad C_0 \eta_1 \ , \quad C_0 \eta_2 \ \le \ 1/6, \tag{33}$$

with the same C_0 in (21). Then, from Proposition 3.10 and estimates (31), (32), for any $t \in [0, T_0]$ we get, for some "harmless" constant still denoted by C,

$$\|\widetilde{\varrho}_n\|_{L^\infty_T(\dot C^\epsilon)\cap \widetilde{L}^1_T(\dot C^{2+\epsilon})} \le C \left(\|\varrho_{0,n}\|_{\dot C^\epsilon} + \int_0^T \left(\|\widetilde{\kappa}\|^{\frac{2}{\epsilon}}_{\dot C^\epsilon} \ + \|\nabla \widetilde{\kappa}\|^2_{L^\infty} \ + \ \|\nabla \widetilde{\kappa}\|^{\frac{2}{1+\epsilon}}_{\dot C^\epsilon} \right) \|\widetilde{\varrho}_n\|_{\dot C^\epsilon} + \|\widetilde{g}_n\|_{\widetilde{L}^1_T(\dot C^\epsilon)} \right).$$

Since $\kappa_* \le \bar\kappa_n \le \kappa^*$, after transformation in time (29) we arrive at

$$\|\varrho_n\|_{L^\infty_T(\dot C^\epsilon)\cap \widetilde{L}^1_T(\dot C^{2+\epsilon})} \le C \left(\|\varrho_{0,n}\|_{\dot C^\epsilon} + \int_0^T K_1 \|\varrho_n\|_{\dot C^\epsilon} + \|g_n\|_{\widetilde{L}^1_T(\dot C^\epsilon)} \right) \tag{34}$$

for all $T \in [0, t_0]$, with

$$K_1 = \|\kappa\|^{\frac{2}{\epsilon}}_{\dot C^\epsilon} + \|\nabla\kappa\|^2_{L^\infty} + \|\nabla\kappa\|^{\frac{2}{1+\epsilon}}_{\dot C^\epsilon},$$

provided that we choose

$$\delta^{-\epsilon} \ = \ 1 \ + \ \widetilde{C}\, \|\kappa\|_{L^\infty_{t_0}(\dot C^\epsilon)} \qquad \text{for some constant } \widetilde{C} \text{ depending only on } d\,,\ \epsilon\,. \tag{35}$$

Step 2: Hölder estimates for ρ

Now we come back to consider $\rho = \sum_n \varrho_n$. By assumptions on the partition of unity $\{\psi_n\}$, for any x there exist N_d balls of our covering which cover the small ball $B(x, \delta/4)$. Therefore, from inequality (19) we have

$$\begin{aligned} \|\rho\|_{\widetilde{L}^1_t(\dot C^\epsilon)} &\le C \left\| \int_0^t \frac{\|\rho(\tau, x+y) - \rho(\tau, x)\|_{L^\infty_x}}{|y|^\epsilon} \right\|_{L^\infty_y} \\ &\le C \sup_{|y|>\delta/4} \int_0^t \frac{\|\rho(x+y) - \rho(x)\|_{L^\infty_x}}{|y|^\epsilon} + C \sup_{|y|\le\delta/4} \int_0^t \frac{\|\rho(x+y) - \rho(x)\|_{L^\infty_x}}{|y|^\epsilon}, \end{aligned}$$

whose second term can be controlled by

$$N_d C \sup_{|y-z|\le \delta/4} \int_0^t \frac{\sup_n \|\varrho_n(x+y) - \varrho_n(x+z)\|_{L^\infty_x}}{|y-z|^\epsilon}.$$

Thus we find

$$\|\rho\|_{\widetilde{L}^1_t(\dot C^\epsilon)} \le C\delta^{-\epsilon} \int_0^t \|\rho\|_{L^\infty} + N_d C \sup_n \|\varrho_n\|_{\widetilde{L}^1_t(\dot C^\epsilon)}.$$

Similarly, we have

$$\|\rho\|_{\widetilde{L}^\infty_t(\dot C^\epsilon)} \le C\delta^{-\epsilon} \|\rho\|_{L^\infty_t(L^\infty)} + C \sup_n \|\varrho_n\|_{\widetilde{L}^\infty_t(\dot C^\epsilon)}.$$

Since $\nabla^2\rho = \sum_n (\nabla^2 \varrho_n)$, from the same arguments as before we infer

$$\|\rho\|_{\widetilde{L}^1_t(\dot C^{2+\epsilon})} \le C \|\nabla^2\rho\|_{\widetilde{L}^1_t(\dot C^\epsilon)} \le C\delta^{-\epsilon} \int_0^t \|\nabla^2\rho\|_{L^\infty} + C \sup_n \|\varrho_n\|_{\widetilde{L}^1_t(\dot C^{2+\epsilon})}.$$

Therefore, to sum up, for all $t \in [0, t_0]$,

$$\begin{aligned} \|\rho\|_{\widetilde{L}^\infty_t(\dot C^\epsilon)\cap \widetilde{L}^1_t(\dot C^{2+\epsilon})} &\le C\delta^{-\epsilon} \left(\|\rho\|_{L^\infty_t(L^\infty)} + \int_0^t \|\nabla^2\rho\|_{L^\infty} \right) + C \sup_n \|\varrho_n\|_{\widetilde{L}^\infty_t(\dot C^\epsilon)\cap \widetilde{L}^1_t(\dot C^{2+\epsilon})} \\ &\le C\delta^{-\epsilon} \left(\|\rho\|_{L^\infty_t(L^\infty)} + \int_0^t \|\nabla^2\rho\|_{L^\infty} \right) + \\ &\quad + C \sup_n \left(\|\varrho_{0,n}\|_{\dot C^\epsilon} + \int_0^t K_1 \|\varrho_n\|_{\dot C^\epsilon} + \|g_n\|_{\widetilde{L}^1_t(\dot C^\epsilon)} \right), \end{aligned}$$

with the second inequality deriving from Estimate (34). Thanks to (24) and the fact that

$$\|\varrho_n\|_{C^\epsilon} = \|\rho\psi_n\|_{C^\epsilon} \le C\|\rho\|_{C^\epsilon}\|\psi_n\|_{C^\epsilon} \le C\delta^{-\epsilon}\|\rho\|_{C^\epsilon},$$

we thus have the following estimate for ρ in the nonhomogeneous Hölder space

$$\begin{aligned}\|\rho\|_{\widetilde{L}^\infty_t(C^\epsilon)\cap\widetilde{L}^1_t(C^{2+\epsilon})} &\le C\|\rho\|_{L^\infty_t(L^\infty)} + C\int_0^t \|\rho\|_{L^\infty} + \|\rho\|_{\widetilde{L}^\infty_t(\dot{C}^\epsilon)\cap\widetilde{L}^1_t(\dot{C}^{2+\epsilon})}\\ &\le C\delta^{-\epsilon}\left(\|\rho_0\|_{C^\epsilon} + \int_0^t (\|\nabla^2\rho\|_{L^\infty} + \|\rho\|_{L^\infty} + \|f\|_{L^\infty}) + \int_0^t K_1\|\rho\|_{C^\epsilon}\right)\\ &\quad + C\sup_n \|g_n\|_{\widetilde{L}^1_t(\dot{C}^\epsilon)},\end{aligned}$$

It rests us to bound g_n uniformly. In fact, starting from definition (26) of g_n, we follow the same method to get (31) and (32) and we arrive at

$$\begin{aligned}\|g_n\|_{\widetilde{L}^1_t(\dot{C}^\epsilon)} \le C\int_0^t &\left(\eta^{-1}(\|\kappa\nabla\psi_n\|^2_{L^\infty} + \|\kappa\nabla\psi_n\|^{\frac{2}{1+\epsilon}}_{\dot{C}^\epsilon}) + \|\kappa\Delta\psi_n + \nabla\kappa\cdot\nabla\psi_n\|_{L^\infty}\right)\|\rho\|_{\dot{C}^\epsilon}\\ &+ \eta\|\rho\|_{\widetilde{L}^1_t(\dot{C}^{2+\epsilon})} + C\int_0^t \|\kappa\Delta\psi_n + \nabla\kappa\cdot\nabla\psi_n\|_{\dot{C}^\epsilon}\|\rho\|_{L^\infty} + C\delta^{-\epsilon}\int_0^t \|f\|_{L^\infty} + C\|f\|_{\widetilde{L}^1_t(\dot{C}^\epsilon)}\\ &\le C_\eta\delta^{-2}\int_0^t \left(1 + \|\kappa\|^{\frac{2}{1+\epsilon}}_{C^\epsilon} + \|\kappa\|_{C^{1+\epsilon}}\right)\|\rho\|_{C^\epsilon} + \eta\|\rho\|_{\widetilde{L}^1_t(\dot{C}^{2+\epsilon})} + C\delta^{-\epsilon}\|f\|_{\widetilde{L}^1_t(C^\epsilon)},\end{aligned}$$

where we have used $\|f\|_{L^1_t(L^\infty)\cap\widetilde{L}^1_t(\dot{C}^\epsilon)} \le C\|f\|_{\widetilde{L}^1_t(C^\epsilon)}$.

We finally get a priori estimates for ρ:

$$\|\rho\|_{\widetilde{L}^\infty_t(C^\epsilon)\cap\widetilde{L}^1_t(C^{2+\epsilon})} \le C\delta^{-\epsilon}\left(\|\rho_0\|_{C^\epsilon} + \int_0^t \|\rho\|_{C^2} + \|f\|_{\widetilde{L}^1_t(C^\epsilon)}\right) + C\delta^{-2}\int_0^t K_2\|\rho\|_{C^\epsilon},$$

with

$$K_2 = 1 + \|\kappa\|^{\frac{2}{1+\epsilon}}_{C^{1+\epsilon}} \ge C\left(K_1 + 1 + \|\kappa\|^{\frac{2}{1+\epsilon}}_{C^\epsilon} + \|\kappa\|_{C^{1+\epsilon}}\right). \tag{36}$$

Thus, by a direct interpolation inequality, that is to say

$$\delta^{-\epsilon}\|\rho\|_{L^1_t(C^2)} \le C_\eta\delta^{-2}\int_0^t \|\rho\|_{C^\epsilon} + \eta\|\rho\|_{\widetilde{L}^1_t(C^{2+\epsilon})},$$

Gronwall's Inequality tells us

$$\|\rho\|_{\widetilde{L}^\infty_t(C^\epsilon)\cap\widetilde{L}^1_t(C^{2+\epsilon})} \le C\delta^{-\epsilon}\left(\|\rho_0\|_{C^\epsilon} + \|f\|_{\widetilde{L}^1_t(C^\epsilon)}\right) + C\delta^{-2}\int_0^t K_2\|\rho\|_{C^\epsilon}. \tag{37}$$

Step 3: the general case $B^s_{\infty,r}$

Now we want to deal with the general case $B^s_{\infty,r}$. Let us apply $\widetilde{\Delta_j} = \Delta_{j-1} + \Delta_j + \Delta_{j+1}$, $j \ge 0$, to System (23), yielding

$$\begin{cases} \partial_t\overline{\rho}_j - \operatorname{div}(\kappa\nabla\overline{\rho}_j) = \overline{f}_j - \overline{R}_j,\\ \overline{\rho}_j|_{t=0} = \overline{\rho}_{0,j}, \end{cases} \tag{38}$$

with

$$\overline{\rho}_j = \widetilde{\Delta_j}\rho, \quad \overline{f}_j = \widetilde{\Delta_j}f, \quad \overline{R}_j = \operatorname{div}([\kappa, \widetilde{\Delta_j}]\nabla\rho), \quad \overline{\rho}_{0,j} = \widetilde{\Delta_j}\rho_0.$$

We apply the a priori estimate (37) to the solution $\overline{\rho}_j$ of System (38), for some positive $\epsilon < \min\{s, 1\}$, entailing

$$\|\overline{\rho}_j\|_{\widetilde{L}^\infty_t(C^\epsilon)\cap\widetilde{L}^1_t(C^{2+\epsilon})} \le C\delta^{-\epsilon}\left(\|\overline{\rho}_{0,j}\|_{C^\epsilon} + \|\overline{f}_j - \overline{R}_j\|_{\widetilde{L}^1_t(C^\epsilon)}\right) + C\delta^{-2}\int_0^t K_2\|\overline{\rho}_j\|_{C^\epsilon}.$$

Let us notice that for $j \ge 0$, denoted by $\rho_j = \Delta_j\rho$ and $\rho_q = \Delta_q\rho$ as usual, then we have

$$\Delta_j\overline{\rho}_j = \rho_j \quad \text{and} \quad \Delta_q\overline{\rho}_j \equiv 0 \text{ if } |q - j| \ge 2.$$

Hence, due to the dyadic characterization of Hölder spaces, the above inequality gives

$$2^{j\epsilon}\|\rho_j\|_{L^\infty_t(L^\infty)}+2^{j(2+\epsilon)}\int_0^t\|\rho_j\|_{L^\infty}\ \le$$
$$\le C\delta^{-\epsilon}\left(2^{j\epsilon}\sum_{|j-q|\le 1}\left(\|\rho_{0,q}\|_{L^\infty}+\int_0^t\|f_q\|_{L^\infty}\right)+\|\overline{R}_j\|_{\widetilde{L}^1_t(C^\epsilon)}\right)+C\delta^{-2}2^{j\epsilon}\sum_{|j-q|\le 1}\int_0^t K_2\|\rho_q\|_{L^\infty}.$$

Finally, by use also of classical commutator estimates Proposition 3.4 to control the $\overline{R}_j$ term, one gets a priori estimate for ρ_j:

$$\|\rho_j\|_{L^\infty_t(L^\infty)}+2^{2j}\int_0^t\|\rho_j\|_{L^\infty}\le C\delta^{-2}\sum_{|j-q|\le 1}\int_0^t K_2\|\rho_q\|_{L^\infty}$$
$$+C\delta^{-\epsilon}\left(\sum_{|j-q|\le 2}\left(\|\rho_{0,q}\|_{L^\infty}+\int_0^t\|f_q\|_{L^\infty}\right)+2^{-js}c_j\int_0^t\|\nabla\kappa\|_{B^s_{\infty,r}}\|\nabla\rho\|_{L^\infty}+\int_0^t\|\nabla\kappa\|_{L^\infty}\|\nabla\rho\|_{B^s_{\infty,r}}\right),$$

for some suitable $(c_j)_j\in\ell^r$.

Therefore, we multiply both sides by 2^{js} (for $s>-1$) and then take ℓ^r norm, to arrive at

$$\|\rho\|_{\widetilde{L}^\infty_t(B^s_{\infty,r})\cap\widetilde{L}^1_t(B^{s+2}_{\infty,r})}\le$$
$$\le C\delta^{-2}\left(\|\rho_0\|_{B^s_{\infty,r}}+\|f\|_{\widetilde{L}^1_t(B^s_{\infty,r})}+\int_0^t K_2\|\rho\|_{B^s_{\infty,r}}+\int_0^t\|\nabla\kappa\|_{L^\infty}\|\nabla\rho\|_{B^s_{\infty,r}}+\|\nabla\kappa\|_{B^s_{\infty,r}}\|\nabla\rho\|_{L^\infty}\right).$$

The definitions (35) of δ and (36) of K_2 imply the result. This concludes the proof of Proposition 4.1.

4.2 Transport equations in $B^0_{\infty,1}$

We state and prove here new a priori estimates for transport equations

$$\begin{cases}\partial_t\omega+v\cdot\nabla\omega=g,\\ \omega|_{t=0}=\omega_0.\end{cases}\tag{39}$$

in the enpoint Besov space $B^0_{\infty,1}$.

First of all, let us recall the classical result in the setting of $B^s_{\infty,r}$ classes (see e.g. [3], Chapter 3).

Proposition 4.3. *Let $1\le r\le\infty$ and $\sigma>0$ ($\sigma>-1$ if $\operatorname{div} v=0$). Let $\omega_0\in B^\sigma_{\infty,r}$, $g\in L^1([0,T];B^\sigma_{\infty,r})$ and v be a time dependent vector field in $\mathcal{C}_b([0,T]\times\mathbb{R}^N)$ such that*

$$\begin{aligned}\nabla v&\in L^1([0,T];L^\infty)&&\text{if}\quad \sigma<1,\\ \nabla v&\in L^1([0,T];B^{\sigma-1}_{\infty,r})&&\text{if}\quad\sigma>1,\quad\text{or}\quad\sigma=r=1.\end{aligned}$$

Then equation (39) has a unique solution ω in

- *the space $\mathcal{C}([0,T];B^\sigma_{\infty,r})$ if $r<\infty$,*
- *the space $\left(\bigcap_{\sigma'<\sigma}\mathcal{C}([0,T];B^{\sigma'}_{\infty,\infty})\right)\bigcap\mathcal{C}_w([0,T];B^\sigma_{\infty,\infty})$ if $r=\infty$.*

Moreover, for all $t\in[0,T]$, we have

$$e^{-CV(t)}\|\omega(t)\|_{B^\sigma_{\infty,r}}\le\|\omega_0\|_{B^\sigma_{\infty,r}}+\int_0^t e^{-CV(t')}\|g(t')\|_{B^\sigma_{\infty,r}}\,dt'\tag{40}$$

$$\textit{with}\quad V'(t):=\begin{cases}\|\nabla v(t)\|_{L^\infty}&\text{if }\sigma<1,\\ \|\nabla v(t)\|_{B^{\sigma-1}_{\infty,r}}&\text{if }\sigma>1,\quad\text{or}\quad\sigma=r=1.\end{cases}$$

If $\omega=v$ then, for all $\sigma>0$ ($\sigma>-1$ if $\operatorname{div} v=0$), Estimate (40) holds with $V'(t):=\|\nabla\omega(t)\|_{L^\infty}$.

Then, the Besov norm of the solution grows in an exponential way with respect to the norm of the transport field v. Nevertheless, if v is *divernge-free* then the $B^0_{\infty,r}$ norm of ω grows linearly in v. This was proved first by Vishik in [23], and then by Hmidi and Keraani in [17]. Here we generalize their result to the case when v is not divergence free. Of course, we will get a growth also on $\operatorname{div} v$, which is still suitable for our scopes (see subsection 5.2).

Proposition 4.4. *Let us consider the linear transport equation* (39).

For any $\beta > 0$, there exists a constant C, depending only on d and β, such that the following a priori estimate holds true:

$$\|\omega(t)\|_{B^0_{\infty,1}} \leq C\Big(\|\omega_0\|_{B^0_{\infty,1}} + \|g\|_{L^1_t(B^0_{\infty,1})}\Big)\Big(1+\mathcal{V}(t)\Big),$$

with $\mathcal{V}(t)$ defined by

$$\mathcal{V}(t) := \int_0^t \|\nabla v\|_{L^\infty} + \|\operatorname{div} v\|_{B^\beta_{\infty,\infty}}\, dt'\,.$$

Proof. We will follow the proof of [17]. Firstly we can write the solution ω of the transport equation (39) as a sum: $\omega = \sum_{k\geq -1}\omega_k$, with ω_k satisfying

$$\begin{cases} \partial_t\omega_k + v\cdot\nabla\omega_k = \Delta_k g, \\ \omega_k|_{t=0} = \Delta_k\omega_0. \end{cases} \tag{41}$$

We obviously have from above that

$$\|\omega_k(t)\|_{L^\infty} \leq \|\Delta_k\omega_0\|_{L^\infty} + \int_0^t \|\Delta_k g\|_{L^\infty}\, dt'. \tag{42}$$

By classical transport estimates in Proposition 4.4, for any $\epsilon \in (0,1)$, we have

$$\|\omega_k(t)\|_{B^\epsilon_{\infty,1}} \leq \Big(\|\Delta_k\omega_0\|_{B^\epsilon_{\infty,1}} + \|\Delta_k g\|_{L^1_t(B^\epsilon_{\infty,1})}\Big)\exp\Big(C\|\nabla v\|_{L^1_t(L^\infty)}\Big). \tag{43}$$

In order to get a priori estimates in Besov space $B^{-\epsilon}_{\infty,1}$, after applying the operator Δ_j to Equation (41), we write the commutator $[v,\Delta_j]\cdot\nabla\omega_k$ as follows (recalling Bony's decomposition (18) and denoting $\widetilde{v} := v - \Delta_{-1}v$)

$$\begin{aligned}&[T_{\widetilde{v}},\Delta_j]\cdot\nabla\omega_k + T_{\Delta_j\nabla\omega_k}\widetilde{v} + R(\Delta_j\nabla\omega_k,\widetilde{v}) - \Delta_j(T_{\nabla\omega_k}\widetilde{v})\\ &\qquad -\Delta_j\operatorname{div}\left(R(\omega_k,\widetilde{v})\right) + \Delta_j R(\omega_k,\operatorname{div}\widetilde{v}) + [\Delta_{-1}v,\Delta_j]\cdot\nabla\omega_k.\end{aligned}$$

Then, for all $\beta > \epsilon$, the L^∞-norm of all the above terms can be bounded by (for some nonnegative sequence $\|(c_j)\|_{\ell^1} = 1$):

$$C(d,\beta)\, 2^{-j\epsilon}\, c_j\, \mathcal{V}'(t)\|\omega_k\|_{B^{-\epsilon}_{\infty,1}}.$$

Thus, we have the following a priori estimate in the space $B^{-\epsilon}_{\infty,1}$:

$$\|\omega_k(t)\|_{B^{-\epsilon}_{\infty,1}} \leq \Big(\|\Delta_k\omega_0\|_{B^{-\epsilon}_{\infty,1}} + \|\Delta_k g\|_{L^1_t(B^{-\epsilon}_{\infty,1})}\Big)\exp\Big(C\mathcal{V}(t)\Big). \tag{44}$$

On the other side, one has the following, for some positive integer N to be determined hereafter:

$$\|\omega\|_{B^0_{\infty,1}} \leq \sum_{j,k\geq -1}\|\Delta_j\omega_k\|_{L^\infty} = \sum_{|j-k|<N}\|\Delta_j\omega_k\|_{L^\infty} + \sum_{|j-k|\geq N}\|\Delta_j\omega_k\|_{L^\infty}.$$

Estimate (42) implies

$$\sum_{|j-k|<N}\|\Delta_j\omega_k\|_{L^\infty} \leq N\sum_k\Big(\|\Delta_k\omega_0\|_{L^\infty} + \|\Delta_k g\|_{L^1_t(L^\infty)}\Big) \leq N\Big(\|\omega_0\|_{B^0_{\infty,1}} + \|g\|_{L^1_t(B^0_{\infty,1})}\Big),$$

while Estimates (43) and (44) entail the following (for some nonnegative sequence $(c_j)\in\ell^1$):

$$\|\Delta_j\omega_k\|_{L^\infty} \leq 2^{-\epsilon|k-j|}c_j\Big(\|\Delta_k\omega_0\|_{L^\infty} + \|\Delta_k g\|_{L^1_t(L^\infty)}\Big)\exp\Big(C\mathcal{V}(t)\Big),$$

which issues immediately

$$\sum_{|j-k|\geq N}\|\Delta_j\omega_k\|_{L^\infty} \leq 2^{-N\epsilon}\Big(\|\omega_0\|_{B^0_{\infty,1}} + \|g\|_{L^1_t(B^0_{\infty,1})}\Big)\exp\Big(C\mathcal{V}(t)\Big).$$

Therefore, for any $\beta > 0$, we can choose $\epsilon \in (0,1)$ and $N \in \mathbb{N}$ such that $\epsilon < \beta$ and $N\epsilon\log 2 \sim 1 + C\mathcal{V}(t)$. Thus the lemma follows from the above estimates. $\square$

5 Proof of the main results

We are now ready to tackle the proof our main results, which this section is devoted to.

First of all we will focus on the proof of Theorem 2.1; in the second part, instead, we will deal with Theorem 2.4.

5.1 Proof of the local in time well-posedness result

In this subsection we will prove Theorem 2.1. We will follow the standard procedure: in Step 1 we construct a sequence of approximate solutions having uniform bounds, and in Step 2 we prove the convergence of this sequence.

For the sake of conciseness, we will present the proof just for $r = 1$, for which we can use classical time-dependent spaces $L^q_T(B^s_{\infty,1})$. The general case is just more technical, but it doesn't involve any novelty: it can be treated as in [14], by use of refined commutator and product estimates in Chemin-Lerner spaces.

Let us make some simplifications in the coming proof. We always suppose the existence time $T^* \leq 1$ and that all the constants appearing in the sequel, such as C, C_M, C_E, are bigger than 1. We always denote $f^n = f(\rho^n)$ and $\delta f^n = f(\rho^n) - f(\rho^{n-1})$ for $f = f(\rho)$.

Step 1: construction of a sequence of approximate solutions

As usual, after fixing $(\varrho^0, u^0, \nabla\pi^0) = (\varrho_0, u_0, 0)$, we consider inductively the n-th approximate solution (ϱ^n, u^n, π^n) to be the unique global solution of the following linear system:

$$\begin{cases} \partial_t \varrho^n + u^{n-1} \cdot \nabla \varrho^n - \operatorname{div}(\kappa^{n-1} \nabla \varrho^n) = 0, \\ \partial_t u^n + (u^{n-1} - \kappa^{n-1}(\rho^n)^{-1} \nabla \rho^n) \cdot \nabla u^n + \lambda^n \nabla \pi^n = h^{n-1}, \\ \operatorname{div} u^n = 0, \\ (\varrho^n, u^n)|_{t=0} = (\varrho_0, u_0), \end{cases} \tag{45}$$

where $b^{n-1} = b(\rho^{n-1})$, $\kappa^{n-1} = \kappa(\rho^{n-1})$, $\lambda^{n-1} = \lambda(\rho^{n-1})$ and

$$h^{n-1} = (\rho^{n-1})^{-1}\Big(\Delta b^{n-1} \nabla a^{n-1} + u^{n-1} \cdot \nabla^2 a^{n-1} + \nabla b^{n-1} \cdot \nabla^2 a^{n-1}\Big), \quad a^{n-1} = a(\rho^{n-1}). \tag{46}$$

It is easy to see that by testing $(45)_2$ by $\rho^n u^n$, one should has the energy identity for u^n

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} \rho^n |u^n|^2 = \int_{\mathbb{R}^d} \rho^n h^{n-1} \cdot u^n. \tag{47}$$

In this paragraph, one denotes

$$M := \|(\varrho_0, u_0)\|_{B^s_{\infty,1}}, \quad E_0 := \|\varrho_0\|_{L^2} + \|u_0\|_{L^2}.$$

We aim at proving that there exist a sufficiently small parameter τ (to be determined later), a positive time T^* (which may depend on τ), a positive constant C_M (which may depend on M) and a positive constant C_E such that the uniform estimates for the solution sequence $(\rho^n, u^n, \nabla\pi^n)$ hold:

$$\rho_* \leq \rho^{n-1} := 1 + \varrho^{n-1}, \|\varrho^{n-1}\|_{L^\infty_{T^*}(B^s_{\infty,1})} \leq C_M, \quad \|\varrho^{n-1}\|_{L^2_{T^*}(B^{s+1}_{\infty,1}))\cap L^1_{T^*}(B^{s+2}_{\infty,1})} \leq \tau, \tag{48}$$

$$\|u^{n-1}\|_{L^\infty_{T^*}(B^s_{\infty,1})} \leq C_M, \ \|u^{n-1}\|_{L^2_{T^*}(B^s_{\infty,1})\cap L^1_{T^*}(B^s_{\infty,1})} \leq \tau, \ \|\nabla\pi^{n-1}\|_{L^1_{T^*}(B^s_{\infty,1}\cap L^2)} \leq \tau^{1/2(d+2)}, \tag{49}$$

$$\|\varrho^n\|_{L^\infty_{T^*}(L^2)} + \|\nabla\varrho^n\|_{L^2_{T^*}(L^2)} + \|u^n\|_{L^\infty_{T^*}(L^2)} \leq C_E E_0. \tag{50}$$

Firstly, by choosing small T^*, Estimates (48), (49) and (50) all hold true for $n = 0$. Next we suppose the $(n-1)$-th element $(\varrho^{n-1}, u^{n-1}, \nabla\pi^{n-1})$ to belong to Space E, defined as the set of the triplet $(\varrho, u, \nabla\pi)$ belonging to

$$\Big(C(\mathbb{R}^+; B^s_{\infty,1} \cap L^2) \cap L^2_{\mathrm{loc}}(H^1) \cap \widetilde{L}^1_{\mathrm{loc}}(B^{s+2}_{\infty,1})\Big) \times \Big(C(\mathbb{R}^+; B^s_{\infty,1} \cap L^2)\Big) \times \Big(L^1_{\mathrm{loc}}(B^s_{\infty,1}) \cap L^1_{\mathrm{loc}}(L^2)\Big), \tag{51}$$

such that the inductive assumptions are satisfied. We then just have to show that the n-th unknown $(\varrho^n, u^n, \nabla\pi^n)$ defined by System (45) belongs to the same space E.

According to Proposition 4.1 (or Remark 4.2), ϱ^n belongs to $C(\mathbb{R}^+; B^s_{\infty,1}) \cap L^1_t(B^{s+2}_{\infty,1})$ for any finite $t > 0$. On the other hand, since $u^{n-1} \in L^\infty_{\mathrm{loc}}(L^\infty)$, energy inequality (11) for ϱ^n follows and $\varrho^n \in C(\mathbb{R}^+; L^2) \cap L^2_{\mathrm{loc}}(H^1)$.

As in [14], we introduce ϱ_L to be the solution of the free heat equation with initial datum ϱ_0, which satisfies

$$\|\varrho_L\|_{L^\infty_T(B^s_{\infty,1})} + \|\varrho_L\|_{L^1_T(B^{s+2}_{\infty,1})} \leq C_T \|\varrho_0\|_{B^s_{\infty,1}}, \quad \forall T > 0, \quad C_T \text{ depends on } T, \tag{52}$$

$$\|\varrho_L\|_{L^2_{T^*}(B^{s+1}_{\infty,1}) \cap L^1_{T^*}(B^{s+2}_{\infty,1})} \leq \tau^2 \quad \text{for small enough } T^*. \tag{53}$$

Correspondingly, the remainder $\bar\varrho^n := \varrho^n - \varrho_L$ verifies the following system:

$$\begin{cases} \partial_t \bar\varrho^n + u^{n-1} \cdot \nabla \bar\varrho^n - \mathrm{div}\,(\kappa^{n-1} \nabla \bar\varrho^n) = -u^{n-1} \cdot \nabla \varrho_L + \mathrm{div}\,((\kappa^{n-1} - 1)\nabla \varrho_L), \\ \bar\varrho^n|_{t=0} = 0. \end{cases} \tag{54}$$

Proposition 4.1 (or Remark 4.2) thus implies that

$$\|\bar\varrho^n\|_{L^\infty_t(B^s_{\infty,1}) \cap L^1_t(B^{s+2}_{\infty,1})} \leq \Big(C^{n-1}(t)\, e^{C^{n-1}(t)\mathcal{K}^{n-1}(t)}\Big) \|f^n\|_{L^1_t(B^s_{\infty,1})},$$

where $C^{n-1}(t)$ depends on $\|\varrho^{n-1}\|_{L^\infty_t(B^s_{\infty,1})}$, and

$$\mathcal{K}^{n-1}(t) := t + \|\nabla \kappa^{n-1}\|^2_{L^2_t(B^s_{\infty,1})},$$
$$f^n := -u^{n-1} \cdot \nabla \bar\varrho^n - u^{n-1} \cdot \nabla \varrho_L + \mathrm{div}\,((\kappa^{n-1} - 1)\nabla \varrho_L).$$

Inductive assumptions and product estimates in Proposition 3.5 entail hence

$$C^{n-1}(t)\, e^{C^{n-1}(t)\mathcal{K}^{n-1}(t)} \leq C_{\mathcal{K}}, \quad \forall t \in [0, T^*],\ C_{\mathcal{K}} \text{ depending only on } M,$$
$$\|f^n\|_{L^1_{T^*}(B^s_{\infty,1})} \leq \|u^{n-1}\|_{L^2_{T^*}(B^s_{\infty,1})} \|\nabla \bar\varrho^n\|_{L^2_{T^*}(B^s_{\infty,1})} + CC_M \tau^2.$$

Therefore by the interpolation inequality

$$\|\nabla \bar\varrho^n\|_{L^2_{T^*}(B^s_{\infty,1})} \leq C \|\bar\varrho^n\|^{1/2}_{L^\infty_{T^*}(B^s_{\infty,1})} \|\bar\varrho^n\|^{1/2}_{L^1_{T^*}(B^{s+2}_{\infty,1})},$$

the following smallness statement pertaining to $\bar\varrho^n$

$$\|\bar\varrho^n\|_{L^2_{T^*}(B^{s+1}_{\infty,1})} \leq \|\bar\varrho^n\|_{L^\infty_{T^*}(B^s_{\infty,1})} + \|\bar\varrho^n\|_{L^1_{T^*}(B^{s+2}_{\infty,1})} \leq \tau^{3/2}\,. \tag{55}$$

is verified. Hence inductive assumption (48) holds for ϱ^n.

We will bound u^n and $\nabla \pi^n$ in the following steps:

(i) Energy Identity (47) holds. For bounding $\|h^{n-1}\|_{L^2}$ we use the following inequalities

$$\|\Delta b \nabla a\|_{L^2} \lesssim \|b\|_{B^{s+1}_{\infty,1}} \|\nabla a\|_{L^2} \lesssim \|\varrho\|_{B^{s+1}_{\infty,1}} \|\nabla \rho\|_{L^2},$$
$$\|u \cdot \nabla^2 a\|_{L^2} \lesssim \|u\|_{L^2} \|\varrho\|_{B^{s+1}_{\infty,1}}.$$

Thus the induction assumptions imply (50).

(ii) Standard estimates for transport equation in Besov spaces and inductive assumptions ensure that

$$\|u^n\|_{L^\infty_t(B^s_{\infty,1})} \leq CC_M e^{CC_M\tau} \Big(\|u_0\|_{B^s_{p,r}} + \tau + \|\nabla \pi^n\|_{L^1_t(B^s_{\infty,1})}\Big) \leq CC_M(1 + \Pi^n), \tag{56}$$

where we have defined

$$\Pi^n := \|\nabla \pi^n\|_{L^1_{T^*}(B^s_{\infty,1})}.$$

(iii) Consider the elliptic equation satisfied by π^n:

$$\mathrm{div}\,(\lambda^n \nabla \pi^n) = \mathrm{div}\,\Big(h^{n-1} - (u^{n-1} - \kappa^{n-1}(\rho^n)^{-1} \nabla \rho^n) \cdot \nabla u^n\Big).$$

By view of $\|\nabla u^n\|_{L^2_{T^*}(L^\infty)} \le (T^*)^{1/2}\|u^n\|_{L^\infty_{T^*}(B^s_{\infty,1})}$, inductive assumptions (49) and (50) imply

$$\begin{aligned}\|\nabla\pi^n\|_{L^1_{T^*}(L^2)} &\le C\Big\|h^{n-1} - (u^{n-1} - \kappa^{n-1}(\rho^n)^{-1}\nabla\rho^n)\cdot\nabla u^n\Big\|_{L^1_{T^*}(L^2)} \\ &\le CC_E E_0(\tau + \tau\|u^n\|_{L^\infty_{T^*}(B^s_{\infty,1})}) \\ &\le CC_E C_M E_0\tau(1+\Pi^n), \quad \text{if } (T^*)^{1/2}\le\tau. \end{aligned} \tag{57}$$

(iv) Now, interpolation between $B^s_{\infty,1}$ and L^2 entails hence (with some appropriated C_Π and $0<\gamma<1$)

$$\|\nabla\pi^n\|_{L^1_{T^*}(B^{s-1/2}_{\infty,1})} \le C\|\nabla\pi^n\|^\gamma_{L^1_{T^*}(L^2)}\|\nabla\pi^n\|^{1-\gamma}_{L^1_{T^*}(B^s_{\infty,1})} \le C_\Pi(1+\Pi^n)\tau^\gamma.$$

(v) Let's consider the following equation

$$\Delta\pi^n = \nabla\log\rho^n\cdot\nabla\pi^n + \rho^n \mathrm{div}\Big(h^{n-1} - (u^{n-1} - \kappa^{n-1}(\rho^n)^{-1}\nabla\rho^n)\cdot\nabla u^n\Big).$$

According to product estimates in Proposition 3.5, for some appropriated C_Π,

$$\|\Delta\pi^n\|_{L^1_{T^*}(B^{s-1}_{\infty,1})} \le CC_M\|\nabla\pi^n\|_{L^1_{T^*}(B^{s-1/2}_{\infty,1})} + CC_M\tau + CC_M\tau^2 \le C_\Pi(1+\Pi^n)\tau^\gamma. \tag{58}$$

Notice that such an inequality is true also in the endpoint case $s=1$, for which we have to estimate the $B^0_{\infty,1}$ norm of $\Delta\pi^n$. This space is no more an algebra, but we can overcome the problem using the $B^{1/2}_{\infty,1}$ regularity.

(vi) By decomposing $\nabla\pi^n$ into low frequency part and high frequency part (and using Bernstein's inequality Lemma 3.2), one has

$$\Pi^n \lesssim \|\nabla\Delta_{-1}\pi^n\|_{L^1_{T^*}(L^2)} + \|\Delta\pi^n\|_{L^1_{T^*}(B^{s-1}_{\infty,1})} \lesssim \|\nabla\pi^n\|_{L^1_{T^*}(L^2)} + \|\Delta\pi^n\|_{L^1_{T^*}(B^{s-1}_{\infty,1})}.$$

Thus, the above two estimates (57) and (58) imply, for τ and T^* small enough, the inductive assumption (49) for π^n. Furthermore, (56) entails the inductive assumption (49) for u^n.

Step 2: convergence of the approximate solution sequence

Let us turn to establish that the above sequence converges to the solution. Let's introduce the difference sequence

$$(\delta\varrho^n, \delta u^n, \nabla\delta\pi^n) = (\varrho^n - \varrho^{n-1}, u^n - u^{n-1}, \nabla\pi^n - \nabla\pi^{n-1}), \quad n\ge 1.$$

When $n\ge 2$, it verifies the following system:

$$\left\{\begin{array}{c} \partial_t\delta\varrho^n + u^{n-1}\cdot\nabla\delta\varrho^n - \mathrm{div}\,(\kappa^{n-1}\nabla\delta\varrho^n) = F^{n-1}, \\ \partial_t\delta u^n + (u^{n-1} - \kappa^{n-1}\nabla\log\rho^n)\cdot\nabla\delta u^n + \lambda^n\nabla\delta\pi^n = H_e^{n-1}, \\ \mathrm{div}\,\delta u^n = 0, \\ (\delta\varrho^n, \delta u^n)|_{t=0} = (0,0), \end{array}\right. \tag{59}$$

where we have set

$$\begin{aligned} &F^{n-1} = -\delta u^{n-1}\cdot\nabla\varrho^{n-1} + \mathrm{div}\,(\delta\kappa^{n-1}\nabla\varrho^{n-1}), \\ &H_e^{n-1} = \delta h^{n-1} - (\delta u^{n-1} - \delta\kappa^{n-1}\nabla\log\rho^n - \kappa^{n-2}\nabla\delta(\log\rho)^n)\cdot\nabla u^{n-1} - \delta\lambda^n\nabla\pi^{n-1}, \end{aligned}$$

with $\delta h^{n-1} = h^{n-1} - h^{n-2}$.

We will consider the difference sequence in the energy space. Let's do some analysis first: one needs H_e^{n-1} in $L^1_{T^*}(L^2)$ and hence

$$(\rho^{n-1})^{-1}\Delta\delta b^{n-1}\nabla a^{n-1} \quad \text{and} \quad (\rho^{n-1})^{-1}\nabla b^{n-2}\cdot\nabla^2\delta a^{n-1} \text{ in } L^1_{T^*}(L^2).$$

We only have $\nabla\varrho^n$ in $L^\infty_{T^*}(L^\infty)$, and thus we need $\nabla^2\delta\varrho^n$ in $L^1_{T^*}(L^2)$: this property follows from the energy inequality of the equation of $\nabla\delta\varrho^n$

$$\partial_t\nabla\delta\varrho^n + u^{n-1}\cdot\nabla^2\delta\varrho^n - \mathrm{div}\,(\kappa^{n-1}\nabla^2\delta\varrho^n) = -\nabla\delta\varrho^n\cdot\nabla u^{n-1} + \mathrm{div}\,(\nabla\delta\varrho^n\otimes\nabla\kappa^{n-1}) + \nabla F^{n-1}. \tag{60}$$

In the above, the first two terms of the right-hand side are of lower order, while the third one is in $L^2_{\mathrm{loc}}(H^{-1})$, thus taking L^2 inner product works.

Now we begin to make the above analysis in detail.
Since $\delta\varrho^n \in E$, the energy equality for Equation $(59)_1$ holds for $n \geq 2$:

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} |\delta\varrho^n|^2 + \int_{\mathbb{R}^d} \kappa^{n-1}|\nabla\delta\varrho^n|^2 = -\int_{\mathbb{R}^d} \delta u^{n-1}\cdot\nabla\rho^{n-1}\,\delta\varrho^n - \int_{\mathbb{R}^d} \delta\kappa^{n-1}\nabla\rho^{n-1}\cdot\nabla\delta\varrho^n, .$$

Thus integration in time and the uniform estimates for solution sequence give

$$\|\delta\varrho^n\|_{L^\infty_{T^*}(L^2)} + \|\nabla\delta\varrho^n\|_{L^2_{T^*}(L^2)} \leq C(\|\delta\varrho^{n-1}\|_{L^\infty_{T^*}(L^2)} + \|\delta u^{n-1}\|_{L^2_{T^*}(L^2)})\,\tau. \tag{61}$$

Similarly, energy equality holds for $\nabla\delta\varrho^n$, $n \geq 2$ (in fact, it's not clear that $\delta\varrho^1 \in L^2_{\mathrm{loc}}(H^2)$):

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} |\nabla\delta\varrho^n|^2 + \int_{\mathbb{R}^d} \kappa^{n-1}|\nabla^2\delta\varrho^n|^2 = -\int \nabla\delta\varrho^n\cdot\nabla u^{n-1}\cdot\nabla\delta\varrho^n + \nabla\delta\varrho^n\cdot\nabla^2\delta\varrho^n\cdot\nabla\kappa^{n-1} + F^{n-1}\Delta\delta\varrho^n.$$

Integrating in time and the inductive assumptions also imply

$$\|\nabla\delta\varrho^n\|_{L^\infty_{T^*}(L^2)} + \|\nabla^2\delta\varrho^n\|_{L^2_{T^*}(L^2)} \leq C\tau\|(\delta\varrho^{n-1}, \delta u^{n-1})\|_{L^\infty_{T^*}(L^2)} + CC_M\|\nabla\delta\varrho^{n-1}\|_{L^2_{T^*}(L^2)}.$$

By controlling $\|\nabla\delta\varrho^{n-1}\|_{L^2_{T^*}(L^2)}$ above by (61), one sums up these two inequalities, entailing

$$\|\delta\varrho^n\|_{L^\infty_{T^*}(H^1)} + \|\nabla\delta\varrho^n\|_{L^2_{T^*}(H^1)} \leq CC_M\tau\|(\delta\varrho^{n-1}, \delta\varrho^{n-2}, \delta u^{n-1}, \delta u^{n-2})\|_{L^\infty_{T^*}(L^2)}. \tag{62}$$

Now we turn to δu^n. We rewrite δh^{n-1} as

$$\frac{1}{\rho^{n-1}}\Big(\Delta\delta b^{n-1}\nabla a^{n-1} + \Delta b^{n-2}\nabla\delta a^{n-1} + \delta u^{n-1}\cdot\nabla^2 a^{n-1} + u^{n-2}\cdot\nabla^2\delta a^{n-1} + \nabla\delta b^{n-1}\cdot\nabla^2 a^{n-1} + \nabla b^{n-2}\cdot\nabla^2\delta a^{n-1}\Big) + \Big((\rho^{n-1})^{-1} - (\rho^{n-2})^{-1}\Big)(\Delta b^{n-2}\cdot\nabla a^{n-2} + u^{n-2}\cdot\nabla^2 a^{n-2} + \nabla b^{n-2}\cdot\nabla^2 a^{n-2}).$$

From the inductive estimates we also have that

$$\|\delta h^{n-1}\|_{L^1_{T^*}(L^2)} \leq CC_M\tau(\|\delta\varrho^{n-1}\|_{L^2_{T^*}(H^2)} + \|\delta u^{n-1}\|_{L^\infty_{T^*}(L^2)}) + CC_E E_0\tau\|\delta\varrho^{n-1}\|_{L^\infty_{T^*}(L^2)},$$

and

$$\|H_e^{n-1}\|_{L^1_{T^*}(L^2)} \leq C(C_M + C_E E_0)\tau(\|\delta\varrho^{n-1}\|_{L^2_{T^*}(H^2)} + \|\delta u^{n-1}\|_{L^\infty_{T^*}(L^2)}) + C\tau^{1/2(d+2)}\|\delta\varrho^n\|_{L^\infty_{T^*}(L^2)}.$$

By view of the density equation for ρ^n and $\operatorname{div}\delta u^n = 0$, one has

$$\|\delta u^n\|_{L^\infty_{T^*}(L^2)} \leq C\|H_e^{n-1}\|_{L^1_{T^*}(L^2)}. \tag{63}$$

Combining Estimate (62) and (63) entails, for sufficiently small τ (depending only on d, C_M, C_E, E_0),

$$\|\delta\varrho^n\|_{L^\infty_{T^*}(H^1)\cap L^2_{T^*}(H^2)} + \|\delta u^n\|_{L^\infty_{T^*}(L^2)} \leq \frac{1}{6}\|(\delta\varrho^{n-1}, \delta\varrho^{n-2}, \delta\varrho^{n-3}, \delta u^{n-1}, \delta u^{n-2}, \delta u^{n-3})\|_{L^\infty_{T^*}(L^2)}.$$

Thus $\sum \|(\delta\varrho^n, \delta u^n)\|_{L^\infty_{T^*}(L^2)}$ converges. Since $\delta\varrho^n \in C(\mathbb{R}^+; L^2)$, the Cauchy sequences $\{\varrho^n\}$ and $\{u^n\}$ converge respectively to ϱ and u in $C([0,T^*]; L^2)$. It is also easy to see that

$$\sum_{n\geq 2} \|\delta\varrho^n\|_{L^\infty_{T^*}(H^1)\cap L^2_{T^*}(H^2)}, \quad \sum_{n\geq 2} \|\delta h^n\|_{L^1_{T^*}(L^2)}, \quad \sum_{n\geq 2} \|H_e^{n-1}\|_{L^1_{T^*}(L^2)} < +\infty.$$

Rewrite the elliptic equation for π^n

$$\begin{aligned} \operatorname{div}(\lambda^n \nabla\delta\pi^n) &= \operatorname{div} H_e^{n-1} - \operatorname{div}\left((u^{n-1} - \kappa^{n-1}\nabla\log\rho^n)\cdot\nabla\delta u^n\right), \\ &= \operatorname{div} H_e^{n-1} - \operatorname{div}\Big(\delta u^n\cdot\nabla(u^{n-1} - \kappa^{n-1}\nabla\log\rho^n) + \delta u^n \operatorname{div}(\kappa^{n-1}\nabla\log\rho^n)\Big). \end{aligned}$$

We hence get

$$\|\nabla\delta\pi^n\|_{L^1_{T^*}(L^2)} \leq C(\|H_e^{n-1}\|_{L^1_{T^*}(L^2)} + C_M\|\delta u^n\|_{L^\infty_{T^*}(L^2)}).$$

Thus $\sum_2^\infty \|\nabla\delta\pi^n\|_{L^1_{T^*}(L^2)}$ also converges and hence $\nabla\pi^n$ converges to the unique limit $\nabla\pi$ in $L^1_{T^*}(L^2)$.

Finally, one easily checks that the limit $(\rho, u, \nabla\pi)$ solves System (6) and is in $E(T^*)$ by Fatou property. The proof of the uniqueness is quite similar and we omit it.

5.2 Lower bounds for the lifespan in dimension $d = 2$

In this section, we aim to get a lower bound for the lifespan of the solution in the case of dimension $d = 2$. The idea is to resort to the vorticity in order to control the *high frequencies* of the velocity field, as done in [11] in the context of incompressible Euler equations with variable density.

We define the (scalar) vorticity ω of the fluid as in the classical case:

$$\omega := \partial_1 u^2 - \partial_2 u^1 \equiv \partial_1 v^2 - \partial_2 v^1 . \tag{64}$$

According to $(1)_2$, it satisfies the following transport equation:

$$\partial_t\omega + v\cdot\nabla\omega + \omega\,\Delta b + \nabla\lambda\wedge\nabla\Pi = 0\,, \tag{65}$$

where (recalling the change of variables (4) and (5))

$$\begin{gathered} v = u + \nabla b, \quad a = a(\rho),\, b = b(\rho),\, \lambda = \lambda(\rho), \\ \nabla\Pi = \nabla\pi + \nabla\partial_t a, \quad \nabla\lambda\wedge\nabla\Pi = \partial_1\lambda\,\partial_2\Pi - \partial_2\lambda\,\partial_1\Pi. \end{gathered}$$

The key to the proof of Theorem 2.4 will be bounding, by use of Proposition 4.4, the $B^0_{\infty,1}$ norm of the vorticity *linearly* (but not exponentially) with respect to the velocity field.

Similarly as in [14], let us introduce the following notations:

$$\begin{aligned} &R_0 = \|\varrho_0\|_{B^1_{\infty,1}}, \quad U_0 = \|u_0\|_{B^1_{\infty,1}}, \\ &R(t) = \|\varrho\|_{L^\infty_t(B^1_{\infty,1})}, \quad S(t) = \|\varrho\|_{L^1_t(B^3_{\infty,1})}, \quad U(t) = \|u\|_{L^\infty_t(B^1_{\infty,1})}. \end{aligned}$$

First of all, we apply Proposition 4.1 to the density equation $(6)_1$: it's easy to see that we get (noticing $S'(t) = \|\varrho(t)\|_{B^3_{\infty,1}}$)

$$R + S \leq C(1+R^3)\Big(R_0 + \int_0^t UR + UR^{1/2}(S')^{1/2} + (1 + R^{(2-\epsilon)/(1+\epsilon)}(S')^{\epsilon/(1+\epsilon)})R + R^{3/2}(S')^{1/2}\Big).$$

Hence by use of Young's inequality, one arrives at

$$R + S \leq C(1+R^3)R_0 + C(1+R^6)\int_0^t \left(UR + U^2R + R + R^3\right)d\tau.$$

If we define now

$$T_R := \sup\Big\{t > 0 \;\Big|\; R^6 \leq 1,\, \int_0^t R^3(\tau)\,d\tau \leq 2\,R_0\Big\}, \tag{66}$$

then, for all $t \in [0, T_R]$ we find

$$R + S \leq CR_0 \exp\Big(C\int_0^t (1+U^2)\Big)\,. \tag{67}$$

We now estimate the velocity field. Let us summarise the following inequalities for the non-linear terms in the momentum equation, which will be frequently used in the sequel:

$$\|\nabla^2 b(\rho)\|_{B^1_{\infty,1}} \lesssim \|b\|_{B^3_{\infty,1}} \lesssim \|\varrho\|_{B^3_{\infty,1}} = S'; \tag{68}$$

$$\|\Delta b \nabla a\|_{L^2} \lesssim \|b\|_{B^2_{\infty,1}}\|\nabla a\|_{L^2} \lesssim \|\varrho\|_{B^2_{\infty,1}}\|\nabla\rho\|_{L^2} \lesssim R^{1/2}(S')^{1/2}\|\nabla\rho\|_{L^2} \leq R\|\nabla\rho\|^2_{L^2} + S'; \tag{69}$$

$$\|(u+\nabla b)\cdot\nabla u\|_{L^2} \lesssim \|\nabla u\|_{L^\infty}(\|\nabla\rho\|_{L^2} + \|u\|_{L^2}) \lesssim U(\|\nabla\rho\|_{L^2} + \|u\|_{L^2}). \tag{70}$$

Similarly as the above inequality (69), one has also

$$\|\nabla b\cdot\nabla^2 a\|_{L^2} \lesssim R\|\nabla\rho\|^2_{L^2} + S', \quad \|u\cdot\nabla^2 a\|_{L^2} \lesssim R\|u\|^2_{L^2} + S'. \tag{71}$$

Now, by separating low and high frequencies, we find the following bound for the velocity:

$$U(t) \;\leq\; C\left(\|u\|_{L^2} \;+\; \|\omega\|_{B^0_{\infty,1}}\right). \tag{72}$$

From the energy inequality for equation (12) of u, i.e.

$$\|u(t)\|_{L^2} \;\leq\; C\left(\|u_0\|_{L^2} \;+\; \int_0^t \|\text{div}\;(v\otimes\nabla a)\,\|_{L^2}\,d\tau\right)\,,$$

due to Inequalities (69) and (71), it follows that

$$\|u(t)\|_{L^2} \;\leq\; C\left(\|u_0\|_{L^2} \;+\; \int_0^t\Big(R(\|\nabla\rho\|^2_{L^2} + \|u\|^2_{L^2}) \;+\; S'\Big)\,d\tau\right). \tag{73}$$

Now, applying Proposition 4.4 with $\beta = 1$ to Equation (65), we find

$$\|\omega(t)\|_{B^0_{\infty,1}} \lesssim \left(\|\omega_0\|_{B^0_{\infty,1}} \;+\; \int_0^t\Big\|\nabla\lambda\wedge\nabla\Pi + \omega\,\Delta b\Big\|_{B^0_{\infty,1}}\,d\tau\right)\left(1+\int_0^t\Big(\|\nabla u\|_{L^\infty} + \|\nabla^2 b\|_{B^1_{\infty,1}}\Big)\,d\tau\right).$$

By use of Bony's paraproduct decomposition (see also [11]), one has

$$\begin{aligned}\|\nabla\lambda\wedge\nabla\pi\|_{B^0_{\infty,1}} &\lesssim \|\nabla\rho\|_{B^0_{\infty,1}}\,\|\nabla\Pi\|_{B^0_{\infty,1}}\\ \|\omega\,\Delta b\|_{B^0_{\infty,1}} &\lesssim \|\omega\|_{B^0_{\infty,1}}\,\|\Delta b\|_{B^1_{\infty,1}}\,.\end{aligned}$$

Hence, by virtue of the relation $\|\omega\|_{B^0_{\infty,1}} \lesssim U$, we get

$$\|\omega(t)\|_{B^0_{\infty,1}} \leq C\left(U_0 + \int_0^t\Big(R\|\nabla\Pi\|_{B^0_{\infty,1}} + US'\Big)\,d\tau\right)\left(1+\int_0^t\|\nabla u\|_{L^\infty}d\tau + S\right). \tag{74}$$

It remains us to deal with the pressure term. First of all, from the density equation we have

$$\|\nabla\Pi\|_{B^0_{\infty,1}} \;\leq\; \|\nabla\pi\|_{B^0_{\infty,1}} \;+\; \|\partial_t\nabla a\|_{B^0_{\infty,1}} \;\lesssim\; \|\nabla\pi\|_{B^0_{\infty,1}} \;+\; U\,S' \;+\; S'\,.$$

We next bound π which satisfies the following elliptic equation:

$$\text{div}\,(\lambda\nabla\pi) = \text{div}\,(h - v\cdot\nabla u) = \text{div}\,(h - u\cdot\nabla v + u\,\text{div}\,v).$$

Similarly as in Step 1, Subsection 5.1, by decomposing $\nabla\pi$ into the high and low frequency part separately, one arrives at

$$\begin{aligned}\|\nabla\pi\|_{B^1_{\infty,1}} &\lesssim \|\nabla\Delta_{-1}\pi\|_{L^\infty} + \|\Delta\pi\|_{B^0_{\infty,1}}\\ &\lesssim \|\nabla\pi\|_{L^2} + \|\lambda^{-1}[-\nabla\lambda\cdot\nabla\pi + \text{div}\,(h - v\cdot\nabla u)]\|_{B^0_{\infty,1}}\end{aligned}$$

$$\lesssim \|h - v\cdot\nabla u\|_{L^2} + \|\nabla\rho\|_{B^0_{\infty,1}}\|\nabla\pi\|_{B^{\frac12}_{\infty,1}}$$
$$+ (1+\|\nabla\rho\|_{B^0_{\infty,1}})(\|h\|_{B^1_{\infty,1}} + \|\mathrm{div}\,(v\cdot\nabla u)\|_{B^0_{\infty,1}}).$$

By the following interpolation inequality (recall also Lemma 3.3),

$$\|\nabla\pi\|_{B^{1/2}_{\infty,1}} \lesssim \|\nabla\pi\|_{L^2}^{1/(d+2)}\|\nabla\pi\|_{B^1_{\infty,1}}^{(d+1)/(d+2)},$$

one derives that, for some $\delta > 1$,

$$\|\nabla\pi\|_{B^1_{\infty,1}} \;\le\; C\left((1+R^\delta)\|h-(u+\nabla b)\cdot\nabla u\|_{L^2} \;+\; (1+R)\left(\|h\|_{B^1_{\infty,1}} + \|\mathrm{div}\,(v\cdot\nabla u)\|_{B^0_{\infty,1}}\right)\right).$$

Then, by the product estimates in Proposition 3.5, one finally bounds $\nabla\pi$ as follows:

$$\|\nabla\pi\|_{B^1_{\infty,1}} \le C(1+R^\delta)\big(R(\|\nabla\rho\|^2_{L^2}+\|u\|^2_{L^2}) + U(\|\nabla\rho\|_{L^2}+\|u\|_{L^2}) + (1+R^2)(US'+S'+U^2)\big).$$

Let us define

$$X(t) \;:=\; U(t) + \|u(t)\|_{L^2} \;=\; \|u(t)\|_{L^2\cap B^1_{\infty,1}}.$$

So we get

$$\|\nabla\Pi\|_{B^0_{\infty,1}},\ \|\nabla\pi\|_{B^1_{\infty,1}} \;\le\; C\,\left(1+R^{\delta+2}\right)\Big(\|\nabla\rho\|^2_{L^2} \;+\; S' \;+\; X^2 \;+\; XS'\Big).$$

Therefore, Estimate (74) for the vorticity becomes (denoting $X(0) = X_0$)

$$\|\omega(t)\|_{B^0_{\infty,1}} \;\lesssim\; \left(1+S+\int_0^t X\,d\tau\right) \times\left(X_0 + \int_0^t (1+R^{\delta+3})\Big(R\,\|\nabla\rho\|^2_{L^2} \;+\; R\,S' \;+\; R\,X^2 \;+\; X\,S'\Big)d\tau\right).$$

Keeping in mind (67) and introducing $\ell := \delta + 4 > 5$, from relation (72) we finally find, for $t\in[0,T_R]$,

$$X(t) \le C\Bigg(X_0 + \underbrace{R_0(1+R_0^\ell)e^{C\int_0^t(1+X^2)}}_{\Gamma_1}\Big(\int_0^t \|\nabla\rho\|^2_{L^2}+1\Big) \;+\; \underbrace{(1+R_0^\ell)e^{C\int_0^t(1+X^2)}\int_0^t XS'\,d\tau}_{\Gamma_2}\Bigg) \times\left(1+S+\int_0^t X\,d\tau\right).$$

We define T_X as the quantity

$$T_X := \sup\Big\{t \,|\, \Gamma_1(t)\le 1, \quad \Gamma_2(t) \le 1 + \|\varrho\|^2_{L^2} + X_0\Big\}.$$

Then, noticing $S \le \Gamma_1$, one easily arrives at the following bound for $X(t)$, with $t\in[0,T_R]\cap[0,T_X]$:

$$X(t) \;\le\; C(1+\|\varrho_0\|^2_{L^2}+X_0)\left(1+\int_0^t X(\tau)d\tau\right).$$

Hence, since $\Gamma_0 := (1+\|\varrho_0\|^2_{L^2}+X_0)$, then by Gronwall's lemma we get $X(t) \;\le\; C\Gamma_0\, e^{C\Gamma_0 t}$. After a long but straightforward calculation (omitted here), one can check that T, defined by relation (16) where we take a small enough constant L, satisfies

$$R^6 \le 1, \qquad \int_0^T R^3 \le 2R_0, \quad \Gamma_1(T)\le 1, \quad \Gamma_2(T) \le 1 + \|\varrho_0\|^2_{L^2} + X_0.$$

This completes the proof of Theorem 2.4.

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