The well-posedness issue for an inviscid zero-Mach number system in general Besov spaces

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Abstract

The present paper is devoted to the study of a zero-Mach number system with heat conduction but no viscosity. We work in the framework of general non-homogeneous Besov spaces $B^s_{p,r}(\mathbb{R}^d)$, with $p \in [2,4]$ and for any $d \geq 2$, which can be embedded into the class of globally Lipschitz functions.

We prove a local in time well-posedness result in these classes for general initial densities and velocity fields. Moreover, we are able to show a continuation criterion and a lower bound for the lifespan of the solutions.

The proof of the results relies on Littlewood-Paley decomposition and paradifferential calculus, and on refined commutator estimates in Chemin-Lerner spaces.

Keywords. Zero-Mach number system; well-posedness; Besov spaces; Chemin-Lerner spaces; continuation criterion; lifespan.

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1 Introduction

The free evolution of a compressible, effectively heat-conducting but inviscid fluid obeys the following equations:

$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\rho v) &= 0, \\ \partial_t(\rho v) + \mathrm{div}\,(\rho v \otimes v) + \nabla p &= 0, \\ \partial_t(\rho e) + \mathrm{div}\,(\rho v e) - \mathrm{div}\,(k \nabla \vartheta) + p\,\mathrm{div}\, v &= 0, \end{cases} \tag{1}$$

where $\rho = \rho(t,x) \in \mathbb{R}^+$ stands for the mass density, $v = v(t,x) \in \mathbb{R}^d$ for the velocity field and $e = e(t,x) \in \mathbb{R}^+$ for the internal energy per unit mass. The time variable t belongs to $\mathbb{R}^+$ or to $[0,T]$ and the space variable x is in $\mathbb{R}^d$ with $d \geq 2$. The scalar functions $p = p(t,x)$ and $\vartheta = \vartheta(t,x)$ denote the pressure and temperature respectively. The heat-conducting coefficient $k = k(\rho, \vartheta)$ is supposed to be smooth in both its variables.

We supplement System (1) with the following two state equations:

$$p = R\rho\vartheta, \quad e = C_v \vartheta,$$

where R, C_v denote the ideal gas constant and the specific heat capacity at constant volume, respectively. That is, we restrict ourselves to (so-called) ideal gases.

In this paper, we will consider highly subsonic ideal gases strictly away from vacuum, and correspondingly, we will work with the *inviscid zero-Mach number system* (see (3) or (7) below) which derives from System (1) by letting the Mach number go to zero. For completeness, in the following we derive the zero-Mach number system *formally.*

Just as in [13], suppose (ρ, v, p) to be a solution of System (1) and define the dimensionless Mach number ε to be the ratio of the velocity v by the reference sound speed. Then the rescaled triplet

$$\left(\rho_\varepsilon(t,x) = \rho\Big(\frac{t}{\varepsilon}, x\Big), \quad v_\varepsilon(t,x) = \frac{1}{\varepsilon} v\Big(\frac{t}{\varepsilon}, x\Big), \quad p_\varepsilon(t,x) = p\Big(\frac{t}{\varepsilon}, x\Big)\right)$$

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satisfies the following non-dimensional system

$$\begin{cases} \partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon v_\varepsilon) & = & 0, \\ \partial_t(\rho_\varepsilon v_\varepsilon) + \operatorname{div}(\rho_\varepsilon v_\varepsilon \otimes v_\varepsilon) + \frac{\nabla p_\varepsilon}{\varepsilon^2} & = & 0, \\ \frac{1}{\gamma-1}(\partial_t p_\varepsilon + \operatorname{div}(p_\varepsilon v_\varepsilon)) - \operatorname{div}(k_\varepsilon \nabla \vartheta_\varepsilon) + p_\varepsilon \operatorname{div} v_\varepsilon & = & 0. \end{cases} \tag{2}$$

Here $\gamma := C_p/C_v = 1 + R/C_v$ represents the adiabatic index and the constant C_p denotes the specific heat capacity at constant pressure. The rescaled temperature and heat-conducting coefficient are given by

$$\vartheta_\varepsilon(t,x) = \vartheta\Big(\frac{t}{\varepsilon}, x\Big), \quad k_\varepsilon(t,x) = \frac{1}{\varepsilon} k\Big(\frac{t}{\varepsilon}, x\Big).$$

Now let ε go to 0, that is, the pressure p_ε equals to a positive constant P_0 by Equations $(2)_2$ and $(2)_3$: thus System (2) becomes formally the following zero Mach number system immediately (see [2], [20] and [24] for detailed computations):

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) & = & 0, \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \nabla \Pi & = & 0, \\ \operatorname{div} v - \frac{\gamma-1}{\gamma P_0} \operatorname{div}(k \nabla \vartheta) & = & 0, \end{cases} \tag{3}$$

where $\Pi = \Pi(t,x)$ is a new unknown function.

Let us make reference to some results on the *incompressible limit* of Euler equations: namely, incompressible Euler equations can be viewed as compressible Euler equations when the Mach number tends to vanish. For the *isentropic* Euler system, there are many early works such as [14, 17, 18, 22, 23], etc. At the beginning of this century, [1, 21] treated the *non-isentropic* case. Later Alazard [2] generalized the study to various models, which include the case of the low Mach number limit from System (1) to System (3).

Next, we will reformulate System (3) into a new system (see System (7) below) with a new *divergence-free* velocity field. We will mainly deal with this new system in this paper. Before going into details, let us suppose here that the density ρ always has positive lower bound and converges to some constant (say "1") at infinity, in the sense detailed below (see (15)). We also mention that in the following, the fact that $\rho\vartheta \equiv P_0/R$ is a positive constant will be used thoroughly.

Similarly as in [13], for notational simplicity, we set α to be the positive constant defined by

$$\alpha = \frac{\gamma-1}{\gamma P_0} = \frac{R}{C_p P_0} = \frac{1}{C_p \rho\vartheta}.$$

Then we define the following two coefficients, always viewed as regular functions of ρ only:

$$\kappa = \kappa(\rho) = \alpha k \vartheta \quad \text{and} \quad \lambda = \lambda(\rho) = \rho^{-1}. \tag{4}$$

One furthermore introduces two scalar functions $a = a(\rho)$ and $b = b(\rho)$, such that

$$\nabla a = \kappa \nabla \rho = -\rho \nabla b, \quad a(1) = b(1) = 0. \tag{5}$$

We then define the new "velocity" u and the new "pressure" π respectively as

$$u = v - \alpha k \nabla \vartheta \equiv v - \nabla b \equiv v + \kappa \rho^{-1} \nabla \rho, \qquad \pi = \Pi - \partial_t a\,. \tag{6}$$

Then System (3) finally becomes

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho - \operatorname{div}(\kappa \nabla \rho) = 0, \\ \partial_t u + (u + \nabla b) \cdot \nabla u + \lambda \nabla \pi = h, \\ \operatorname{div} u = 0, \end{cases} \tag{7}$$

where κ, b, λ are defined above (see (4) and (5)) and

$$h(\rho, u) = \rho^{-1} \operatorname{div}(v \otimes \nabla a). \tag{8}$$

Let us just verify Equation $(7)_2$. Observe that $(3)_2$ gives

$$\partial_t(\rho u) + \operatorname{div}(\rho v \otimes u) + \partial_t(\rho \nabla b) + \operatorname{div}(v \otimes \rho \nabla b) + \nabla \Pi = 0. \tag{9}$$

It is easy to find that

$$\partial_t(\rho\nabla b)+\nabla\Pi=-\partial_t\nabla a+\nabla\Pi=\nabla\pi \quad \text{and} \quad \mathrm{div}\,(v\otimes\rho\nabla b)=-\mathrm{div}\,(v\otimes\nabla a).$$

Thus by view of Equation $(3)_1$, Equation (9) can be rewritten as

$$\rho\partial_t u+\rho v\cdot\nabla u+\nabla\pi=\mathrm{div}\,(v\otimes\nabla a). \tag{10}$$

We thus multiply (10) by $\lambda=\rho^{-1}$ to get Equation $(7)_2$.

In System (7), although the "velocity" u is divergence-free, one encounters a (quasilinear) parabolic equation for the density ρ and the "source" term h involves two derivatives of ρ. Note that if simply $\kappa\equiv 0$, then $a\equiv b\equiv 0$ and hence System (7) becomes the so-called density-dependent Euler equations

$$\begin{cases} \partial_t\rho+v\cdot\nabla\rho=0,\\ \partial_t v+v\cdot\nabla v+\lambda\nabla\pi=0,\\ \mathrm{div}\,v=0.\end{cases} \tag{11}$$

In the above (11), due to the null heat conduction, ρ satisfies a transport equation, h vanishes and the velocity v itself is solenoidal. As early as in 1980, Beirão da Veiga and Valli [9, 10] investigated (11). We also cite the book [3] as a good survey of the boundary-value problems for nonhomogeneous fluids. By use of an energy identity, Danchin [11] studied System (11) in the framework of nonhomogeneous Besov space $B^s_{p,r}(\mathbb{R}^d)$ which can be embedded in $C^{0,1}$. Recently in [12], Danchin and the first author treated the end point case $B^s_{\infty,r}$, and studied the lifespan of the solutions in the case of space dimension $d=2$.

If the fluid is viscous, that is to say there is an additional viscous stress tensor within the momentum equation $(3)_2$, then System (3) becomes the low Mach number limit system of the full Navier-Stokes system. See [2, 13, 19] and references therein for some relevant results.

However, to our knowledge, there are few well-posedness results for the inviscid low Mach-number limit system (3). Notice that System (3) can also be viewed as a *nonhomogeneous system in the presence of diffusion*, which describes an inviscid fluid consisting of two components (say, water and salt), both incompressible, with a mass diffusion effect between them (the so-called Fick's law):

$$\mathrm{div}\,(v+\kappa\nabla\ln\rho)=0.$$

In this case, ρ and v are considered to be the mean density and the *mean-mass* velocity of the mixture respectively, κ denotes the positive diffusion coefficient, and $\nabla\Pi$, as usual, denotes some unknown pressure. For more physical backgrounds of this model, see [16]. One can also refer to Beirão da Veiga *et al.* [6] for an existence-uniqueness result of *classical* solutions.

In this paper, we will study the well-posedness of the Cauchy problem for System (3) in the framework of *general Besov spaces* $B^s_{p,r}(\mathbb{R}^d)$ (with $p\in[2,4]$ and in any space dimension $d\geq 2$) which can be embedded into the class of globally Lipschitz functions.

Similarly as in [11, 12], the analysis will be based on an intensive use of the para-differential calculus and some (newly-developed) commutator estimates.

Moreover, refined a priori estimates lead us to state a continuation criterion (in the same spirit of the well-known result of [5] for the homogeneous incompressible Euler equations), and to find a lower bound for the lifespan of the solutions in terms of the initial data only.

We refer to Section 2 for more details on our working hypothesis and discussions on the obtained results. Let us just say here that the restriction $p\in[2,4]$, which is due to the analysis of the pressure term $\nabla\pi$, can be somehow relaxed. For instance, for *finite-energy* initial data (ρ_0,u_0), well-posedness for system (7) can be recovered for any $1<p\leq+\infty$. We refer to [15] for an analysis in this direction: the endpoint case $B^1_{\infty,1}$ is permitted there and in dimension $d=2$ the lower bound for the lifespan is refined such that the solutions tend to be globally defined for initial densities which are small perturbations of a constant state.

Our paper is organized in the following way.

In next section we will present our main local-in-time well-posedness result Theorem 2.1. We will also state a continuation criterion and a lower bound for the lifespan in Theorems 2.4 and 2.5 respectively.

Section 3 is devoted to the tools from Fourier analysis.

In Section 4 we will tackle the proof of Theorem 2.1: we will give some fundamental commutator estimates (see Lemma 4.1) and product estimates (Lemma 4.3) in the *time-dependent* Besov spaces.

Section 5 is devoted to the proof of Theorem 2.4 and Theorem 2.5.

Finally, in the appendix we will give the complete proof of Lemma 4.1.

2 Main results

Let us focus on System (7) to introduce our main results. In view of Equation $(7)_1$, of parabolic type, by *maximum principle* we can assume that the density ρ (if it exists on the time interval $[0,T]$) has the same positive upper and lower bounds as the initial density ρ_0:

$$0<\rho_* \leq \rho(t,x) \leq \rho^*, \quad \forall\, t \in [0,T],\ x \in \mathbb{R}^d.$$

Correspondingly, the coefficients κ and λ can always be bounded from above and below, which ensures that the pressure π satisfies an *elliptic* equation in divergence form: applying operator "div" to Equation $(7)_2$ gives the following:

$$\operatorname{div}(\lambda\,\nabla\pi) \,=\, \operatorname{div}(h \,-\, v\cdot\nabla u)\,, \qquad \lambda \geq \lambda_* \,=\, (\rho^*)^{-1} \,>\, 0\,. \tag{12}$$

For system (7) there is no gain of regularity for the velocity u: we then suppose the initial divergence-free "velocity" field u_0 to belong to some space $B^s_{p,r}$ which can be continuously embedded in $C^{0,1}$, i.e. the triplet $(s,p,r) \in \mathbb{R}\times[1,+\infty]^2$ has to satisfy the following condition:

$$s \,>\, 1+\frac{d}{p}\,, \qquad\qquad \text{or} \qquad\qquad s \,=\, 1+\frac{d}{p}\,,\ \ r\,=\,1\,. \tag{13}$$

This requires the "source" term h to belong to $L^1([0,T];B^s_{p,r})$ which, by view of definition (8) of h, asks at least $\nabla^2\rho \in L^1([0,T];B^s_{p,r})$. Keeping in mind that ρ satisfies the parabolic type equation, we expect to gain two orders of regularity (in space) when taking the average in time. We thus have to assume the initial inhomogeneity ρ_0-1 to be in the same space $B^s_{p,r}$ as above. However in general, we only get $\nabla^2\rho$ in the time-dependent Besov space $\widetilde{L}^1_T(B^s_{p,r})$, which is a little bit larger than $L^1_T(B^s_{p,r})$ (see Definition 3.2). Therefore in the whole paper we will deal rather with the spaces $\widetilde{L}^\infty_T(B^s_{p,r})$ and $\widetilde{L}^1_T(B^s_{p,r})$ (first introduce in [8] by Chemin and Lerner); in particular, in Section 4 we will give new commutator estimates and product estimates in these *time-dependent* Besov spaces, which imply a priori estimates for System (7).

On the other hand, in order to control the *low frequencies* for $\nabla\pi$, one has to make sure that $h-v\cdot\nabla u \in L^2$. Indeed, for Equation (12) above, the a priori estimate

$$\lambda_*\|\nabla\pi\|_{L^q} \leq C\|h-v\cdot\nabla u\|_{L^q}$$

holds *independently* of λ only when $q=2$ (see Lemma 2 of [11]). Hence the fact that h is composed of quadratic forms entails that p has to verify

$$p \in [2,4]. \tag{14}$$

To conclude, we have the following theorem, whose proof will be shown in Section 4.

Theorem 2.1. *Let the triplet $(s,p,r)\in\mathbb{R}^3$ satisfy conditions (13) and (14). Let us take an initial density state ρ_0 and an initial velocity field u_0 such that*

$$0<\rho_*\leq\rho_0\leq\rho^*\,, \quad \operatorname{div} u_0=0, \quad \|\rho_0-1\|_{B^s_{p,r}} \,+\, \|u_0\|_{B^s_{p,r}} \,\leq\, M\,, \tag{15}$$

for some positive constants ρ_*, ρ^* and M. Then there exist a positive time T (depending only on ρ_*,ρ^*,M,d,s,p,r) and a unique solution $(\rho,u,\nabla\pi)$ to System (7) such that $(\varrho,u,\nabla\pi):=(\rho-1,u,\nabla\pi)$ belongs to the space $E^s_{p,r}(T)$, defined as the set of triplet $(\varrho,u,\nabla\pi)$ such that

$$\left\{\begin{array}{c} \varrho\in\widetilde{C}([0,T];B^s_{p,r})\cap\widetilde{L}^1([0,T];B^{s+2}_{p,r})\,, \quad \rho_*\leq\varrho+1\leq\rho^*, \\ u\in\widetilde{C}([0,T];B^s_{p,r})\,, \\ \nabla\pi\in\widetilde{L}^1([0,T];B^s_{p,r})\cap L^1([0,T];L^2)\,, \end{array}\right. \tag{16}$$

with $\widetilde{C}_w([0,T];B^s_{p,r})$ if $r=+\infty$ (see also Definition 3.2).

Remark 2.2. Let us state briefly here the corresponding well-posedness result for the original system (3). By view of the change of variables (6), we have $u=\mathcal{P}v$, $\nabla b=\mathcal{Q}v$, where $\mathcal{P}$ denotes the Leray projector over divergence-free vector fields and $\mathcal{Q} = \mathrm{Id} - \mathcal{P}$: $\widehat{\mathcal{Q}u}(\xi) = -(\xi/|\xi|^2)\,\xi\cdot\widehat{u}(\xi)$. Assume Conditions (13) and (14), and the initial datum (ρ_0, v_0) such that

$$0<\rho_*\leq\rho_0\leq\rho^*\,,\quad \nabla b(\rho_0)=\mathcal{Q}v_0,\qquad \|\rho_0-1\|_{B^s_{p,r}}+\|\mathcal{P}v_0\|_{B^s_{p,r}}\leq M\,.$$

Then, there exist a positive time T and a unique solution $(\rho, v, \nabla\Pi)$ to System (3) such that

$$\begin{cases} \rho-1\in\widetilde{C}([0,T];B^s_{p,r})\cap\widetilde{L}^1([0,T];B^{s+2}_{p,r})\,,\\ v\in\widetilde{C}([0,T];B^{s-1}_{p,r})\cap\widetilde{L}^2([0,T];B^s_{p,r})\,,\quad \mathcal{P}v\in\widetilde{C}([0,T];B^s_{p,r})\,,\\ \nabla\Pi\in\widetilde{L}^1([0,T];B^s_{p,r})\,,\end{cases}$$

with $\widetilde{C}_w([0,T];B^s_{p,r})$ if $r=+\infty$.

One notices from above that the initial velocity v_0 needs not to be in $B^s_{p,r}$ whereas the velocity $v(t)$ will belong to it for almost every $t\in[0,T]$. On the other side, this local existence result would hold if, initially, $\rho_0-1\in B^{s+1}_{p,r}$, $v_0\in B^s_{p,r}$, $\rho_0\in[\rho_*,\rho^*]$ and $\nabla b(\rho_0)=\mathcal{Q}v_0$.

Let us also point out that since $\partial_t\rho\notin L^1([0,T];L^2)$ in general, we do not know whether $\nabla\Pi\in L^1([0,T];L^2)$ (recall also definition (6)). Hence it seems not convenient to deal with System (3) directly since the low frequences of $\nabla\Pi$ can not be controlled *a priori*.

Remark 2.3. If we assume an additional *smallness* hypothesis over the initial inhomogeneity aside from (15), which ensures that the pressure satisfies a Laplace equation (up to a perturbation term), then Condition (14) imposed on p is not necessary. Theorem 2.1 still holds true, except for the fact $\nabla\pi\in L^1([0,T];L^2)$.

Next, one can get a Beale-Kato-Majda type continuation criterion (see [5] for the original version) for solutions to System (7). Notice that, according to the solutions space $E^s_{p,r}$ defined by (16), in order to bound the $\widetilde{L}^1_T(B^s_{p,r})$-norm of the *nonlinear* terms in h and $v\cdot\nabla u$, one requires $(u,\nabla\rho)\in L^\infty_T(L^\infty)$, $(\nabla u,\nabla^2\rho)\in L^2_T(L^\infty)$, etc. Hence, by a refined a priori estimate (see Subsection 5.1) we get the following statement.

Theorem 2.4. [Continuation Criterion] *Let the triplet $(s,p,r)\in\mathbb{R}\times[1,+\infty]^2$ satisfy conditions* (13) *and* (14). *Let $(\rho,u,\nabla\pi)$ be a solution of* (7) *on $[0,T[\,\times\mathbb{R}^d$ such that:*

$$(\rho-1,u,\nabla\pi)\in\Big(\widetilde{C}([0,T[\,;B^s_{p,r})\cap\widetilde{L}^1_{loc}([0,T[\,;B^{s+2}_{p,r})\Big)\times\widetilde{C}([0,T[\,;B^s_{p,r})\times\widetilde{L}^1_{loc}([0,T[\,;B^s_{p,r})$$

and, for some $\sigma>0$,

$$\sup_{t\in[0,T[}\|(\nabla\rho(t),u(t))\|_{L^\infty}+\int_0^T\Big(\big\|(\nabla^2\rho,\nabla u)\big\|^2_{L^\infty}+\|\nabla\pi(t)\|_{B^{-\sigma}_{p,\infty}\cap L^\infty}\Big)\,dt<+\infty\,.$$

Then $(\rho,u,\nabla\Pi)$ could be continued beyond T (if T is finite) into a solution of (7) *with the same regularity.*

Even in the two-dimensional case, it's hard to expect global-in-time well-posedness for this system: the parabolic equation $(7)_1$ allows to improve regularity for the density term, but such a gain is (roughly speaking) deleted by the nonlinear term in the momentum equation $(7)_2$. However, similar as in [12], we manage to establish an *explicit* lower bound for the lifespan of the solution, in *any* dimension $d\geq2$. The proof will be the matter of Section 5.2.

Theorem 2.5. *Under the hypotheses of Theorem 2.1, there exist positive constants L, $\ell>6$ (depending only on d, p, r, ρ_* and ρ^*) such that the lifespan of the solution to System* (7) *given by Theorem 2.1 is bounded from below by the quantity*

$$\frac{L}{1+\|u_0\|_{B^s_{p,r}}+\|\varrho_0\|^\ell_{B^s_{p,r}}}\,. \tag{17}$$

Remark 2.6. Thanks to Theorem 2.4, the lifespan is independent of the regularity. Therefore, the $B^s_{p,r}$-norm in (17) can be replaced by the (weaker) $B^{1+d/4}_{4,1}$ norm.

Remark 2.7. The lower bound (17) might be improved by scaling. Indeed, System (7) is invariant under the following transformation:

$$(\rho^\varepsilon\,,\,u^\varepsilon\,,\,\nabla\pi^\varepsilon)(t\,,\,x)\;:=\;(\rho\,,\,\varepsilon^{-1}\,u\,,\,\varepsilon^{-2}\,\nabla\pi)(\varepsilon^{-2}\,t\,,\varepsilon^{-1}\,x).$$

Set $\varepsilon^2=\|u_0\|_{B^{1+d/4}_{4,1}}$, then, by (17), the lifespan of the solution $(\rho^\varepsilon,u^\varepsilon,\nabla\pi^\varepsilon)$ is bigger (up to a constant factor) than the quantity $(1+\varepsilon^{-1}\|\rho_0-1\|^\ell_{B^{1+d/4}_{4,1}})^{-1}$. Correspondingly the lifespan of $(\rho,u,\nabla\pi)$ is bigger than or equal to

$$\frac{\varepsilon^{-2}L}{1+\varepsilon^{-1}\|\rho_0-1\|^\ell_{B^{1+d/4}_{4,1}}}.$$

We change the word here that in the sequel, C always denotes some "harmless" constant (may vary from time to time) depending only on d,s,p,r,ρ_*,ρ^*, unless otherwise defined. Notation $A\lesssim B$ means $A\le CB$ and $A\sim B$ says A equals to B, up to a constant factor. For notational convenience, the notation ϱ always represent $\rho-1$, unless otherwise specified.

3 An overview on Fourier analysis techniques

Our results mostly rely on Fourier analysis methods which are based on a nonhomogeneous dyadic partition of unity with respect to Fourier variable, the so-called Littlewood-Paley decomposition. Unless otherwise specified, all the results which are presented in this section are proved in [4], Chap. 2.

In order to define a Littlewood-Paley decomposition, fix a smooth radial function χ supported in (say) the ball $B(0,\frac43)$, equals to 1 in a neighborhood of $B(0,\frac34)$ and such that χ is nonincreasing over $\mathbb{R}_+$. Set $\varphi(\xi)=\chi(\frac{\xi}{2})-\chi(\xi)$. The *dyadic blocks* $(\Delta_j)_{j\in\mathbb{Z}}$ are defined by[1]

$$\Delta_j:=0\;\text{ if }\;j\le-2,\quad\Delta_{-1}:=\chi(D)\quad\text{and}\quad\Delta_j:=\varphi(2^{-j}D)\;\text{ if }\;j\ge0.$$

We also introduce the following low frequency cut-off:

$$S_ju:=\chi(2^{-j}D)=\sum_{j'\le j-1}\Delta_{j'}\quad\text{for}\quad j\ge0,\quad S_ju\equiv0\quad\text{for}\quad j\le0.$$

One can now define what a Besov space $B^s_{p,r}$ is:

Definition 3.1. Let u be a tempered distribution, s a real number, and $1\le p,r\le\infty$. We set

$$\|u\|_{B^s_{p,r}}:=\Big(\sum_j2^{rjs}\|\Delta_ju\|^r_{L^p}\Big)^{\frac1r}\quad\text{if}\quad r<\infty\quad\text{and}\quad\|u\|_{B^s_{p,\infty}}:=\sup_j\big(2^{js}\|\Delta_ju\|_{L^p}\big).$$

We then define the space $B^s_{p,r}$ as the subset of distributions $u\in\mathcal{S}'$ such that $\|u\|_{B^s_{p,r}}$ is finite.

When solving evolutionary PDEs, it is natural to use spaces of type $L^\rho_T(X)=L^\rho(0,T;X)$ with X denoting some Banach space. In our case, X will be a Besov space so that we will have to localize the equations by Littlewood-Paley decomposition. This will provide us with estimates of the Lebesgue norm of each dyadic block *before* performing integration in time. This leads to the following definition for the so-called Chemin-Lerner Spaces, introduced for the first time in paper [8].

Definition 3.2. For $s\in\mathbb{R}$, $(q,p,r)\in[1,+\infty]^3$ and $T\in[0,+\infty]$, we set

$$\|u\|_{\widetilde{L}^q_T(B^s_{p,r})}=\Big\|\Big(2^{js}\|\Delta_ju(t)\|_{L^q_T(L^p)}\Big)_{j\ge-1}\Big\|_{\ell^r}.$$

We also set $\widetilde{C}_T(B^s_{p,r})=\widetilde{L}^\infty_T(B^s_{p,r})\cap C([0,T];B^s_{p,r})$.

Remark 3.3. From the above definition, it is easy to see $B^s_{2,2}\equiv H^s$ for all $s\in\mathbb{R}$ and $B^s_{p,r}\hookrightarrow C^{0,1}$ under the hypothesis (13). More generally, one has the continuous embedding $B^k_{p,1}\hookrightarrow W^{k,p}\hookrightarrow B^k_{p,\infty}$ and a time-dependent version

$$L^1_t(B^s_{p,1})\;\hookleftarrow\;\widetilde{L}^1_t(B^{s+\epsilon/2}_{p,1})\;\hookleftarrow\;\widetilde{L}^1_t(B^{s+\epsilon}_{p,\infty}),\;\forall\epsilon>0.$$

[1]Throughout we agree that $f(D)$ stands for the pseudo-differential operator $u\mapsto\mathcal{F}^{-1}(f(\xi)\mathcal{F}u(\xi))$.

The following fundamental lemma (referred in what follows as *Bernstein's inequalities*) describes the way derivatives act on spectrally localized functions.

Lemma 3.4. *Let $0<r<R$. There exists a constant C such that, for any $k\in\mathbb{Z}^+$, $\lambda\in\mathbb{R}^+$, $(p,q)\in[1,\infty]^2$ with $p\leq q$ and any function u of L^p, one has*

$$\operatorname{Supp}\widehat{u}\subset B(0,\lambda R)\Longrightarrow \|\nabla^k u\|_{L^q}\leq C^{k+1}\lambda^{k+d(\frac{1}{p}-\frac{1}{q})}\|u\|_{L^p};$$
$$\operatorname{Supp}\widehat{u}\subset\{\xi\in\mathbb{R}^N\,/\,r\lambda\leq|\xi|\leq R\lambda\}\Longrightarrow C^{-k-1}\lambda^k\|u\|_{L^p}\leq\|\nabla^k u\|_{L^p}\leq C^{k+1}\lambda^k\|u\|_{L^p}.$$

Lemma 3.4 implies the following embedding result immediately, as a generalization of Remark 3.3:

Corollary 3.5. *Space $B^{s_1}_{p_1,r_1}$ is continuously embedded in Space $B^{s_2}_{p_2,r_2}$ whenever $1\leq p_1\leq p_2\leq\infty$ and*

$$s_2<s_1-d/p_1+d/p_2\quad\text{or}\quad s_2=s_1-d/p_1+d/p_2\quad\text{and}\quad 1\leq r_1\leq r_2\leq\infty.$$

Let us now recall the so-called Bony's decomposition introduced in [7] for the products. Formally, any product of two tempered distributions u and v, may be decomposed into

$$uv=T_uv+T_vu+R(u,v) \tag{18}$$

with

$$T_uv:=\sum_j S_{j-1}u\Delta_jv\quad\text{and}\quad R(u,v):=\sum_j\sum_{|j'-j|\leq1}\Delta_ju\,\Delta_{j'}v.$$

The above operator T is called "paraproduct" operator whereas R is called "remainder" operator.

We have the following classical estimates in Besov spaces for the products:

Proposition 3.6. *Let $s,s_1,s_2\in\mathbb{R}$, $1\leq r,r_1,r_2,p\leq\infty$ with $\frac{1}{r}\leq\min\{1,\frac{1}{r_1}+\frac{1}{r_2}\}$.*

- *For the paraproduct one has*

$$\|T_uv\|_{B^s_{p,r}}\lesssim\|u\|_{L^\infty}\|v\|_{B^s_{p,r}}.$$

- *For the remainder, if $s_1+s_2+d\min\{0,1-\frac{2}{p}\}>0$, then*

$$\|R(u,v)\|_{B^{s_1+s_2-\frac{d}{p}}_{p,r}}\lesssim\|u\|_{B^{s_1}_{p,r_1}}\|v\|_{B^{s_2}_{p,r_2}}.$$

- *If $s>0$, one has the following product estimate*

$$\|uv\|_{\widetilde{L}^q_T(B^s_{p,r})}\leq C\left(\|u\|_{L^{q_1}_T(L^\infty)}\|v\|_{\widetilde{L}^{q_2}_T(B^s_{p,r})}+\|u\|_{\widetilde{L}^{q_3}_T(B^s_{p,r})}\|v\|_{L^{q_4}_T(L^\infty)}\right),\quad\frac{1}{q}:=\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{q_3}+\frac{1}{q_4}.$$

One also needs the following commutator estimate for the transport terms in the equations:

Proposition 3.7. *Let $s>-d\min\{1/p,1/p'\}$, $(p,r)\in[1,\infty]^2$ such that $r=1$ if $s=1+d/p$. Then we have*

$$\int_0^t\left\|2^{js}\,\|[\varphi,\Delta_j]\nabla\psi\|_{L^p}\right\|_{\ell^r}d\tau\;\leq\;C\int_0^t\|\nabla\varphi(t)\|_{B^{\frac{d}{p}}_{p,1}\cap B^{s-1}_{p,r}}\|\nabla\psi\|_{B^{s-1}_{p,r}}\,d\tau.$$

The following results pertain to the composition of functions in Besov spaces: they will be needed for estimating functions depending on the density. We refer to Chap. 2 of [4] and to [11] for their proofs.

Proposition 3.8. *Let I be an open interval of $\mathbb{R}$ and $F:I\to\mathbb{R}$ a smooth function. Then for any compact subset $J\subset I$, $s>0$, $(q,p,r)\in[1,+\infty]^3$ and any function a valued in J, we have*

$$\|\nabla(F(a))\|_{\widetilde{L}^q_T(B^{s-1}_{p,r})}\leq C\|\nabla a\|_{\widetilde{L}^q_T(B^{s-1}_{p,r})}.$$

If furthermore $F(0)=0$, then $\|F(a)\|_{\widetilde{L}^q_T(B^s_{p,r})}\leq C\,\|a\|_{\widetilde{L}^q_T(B^s_{p,r})}$.

Finally, we shall make an extensive use of energy estimates (see [11] for the proof) for the following elliptic equation satisfied by the pressure Π:

$$-\operatorname{div}(\lambda\nabla\Pi)=\operatorname{div}F\quad\text{in}\quad\mathbb{R}^d,\quad\lambda=\lambda(x)\geq\lambda_*>0,$$

Lemma 3.9. *For all vector field F with coefficients in L^2, there exists a tempered distribution Π, unique up to constant functions, such that*

$$\lambda_*\|\nabla\Pi\|_{L^2}\leq\|F\|_{L^2}.$$

4 Proof of the local well-posedness

In this section we aim to prove the well-posedness result Theorem 2.1 for System (7). The a priori estimates we will establish in Subsection 4.1 will be the basis throughout the following context.

4.1 Linearized equations

In this subsection we will build a priori estimates for the linearized equations associated to System (7). Firstly, one gives some commutator and product estimates in the *time-dependent* Besov spaces, which entails the a priori estimate for the density immediately.

Lemma 4.1. *Let* $s, s_i, \sigma_i \in \mathbb{R}$, $i = 1, 2$, $(p, r) \in [1, +\infty]^2$.

- *If* $s > 0$, *then*

$$\int_0^t \left\| 2^{js} \left\| [\varphi, \Delta_j] \nabla\psi \right\|_{L^p} \right\|_{\ell^r} d\tau \ \leq\ C \int_0^t \Big(\|\nabla\varphi\|_{L^\infty} \|\psi\|_{B^s_{p,r}} + \|\nabla\varphi\|_{B^{s-1}_{p,r}} \|\nabla\psi\|_{L^\infty} \Big)\, d\tau. \tag{19}$$

- *If* $s > 0$ *and*

$$s + 1 = \theta s_1 + (1-\theta) s_2 = \eta\sigma_1 + (1-\eta)\sigma_2, \quad \text{with } \theta, \eta \in (0, 1],$$

then for any $\varepsilon > 0$, *one has*

$$\left\| 2^{js} \int_0^t \left\| \nabla\left([\varphi, \Delta_j]\, \nabla\psi \right) \right\|_{L^p} d\tau \right\|_{\ell^r} \leq \frac{C\,\theta}{\varepsilon^{(1-\theta)/\theta}} \int_0^t \|\nabla\varphi\|_{L^\infty}^{1/\theta} \|\psi\|_{B^{s_1}_{p,r}}\, d\tau + \\ + \frac{C\,\eta}{\varepsilon^{(1-\eta)/\eta}} \int_0^t \|\nabla\psi\|_{L^\infty}^{1/\eta} \|\nabla\varphi\|_{B^{\sigma_1 - 1}_{p,r}}\, d\tau + (1-\theta)\varepsilon \|\psi\|_{\widetilde{L}^1_t(B^{s_2}_{p,r})} + (1-\eta)\,\varepsilon\, \|\nabla\varphi\|_{\widetilde{L}^1_t(B^{\sigma_2 - 1}_{p,r})}. \tag{20}$$

The proof of (19) is classical while the proof of (20) can be found in the appendix. Let's just give a remark which will be used for estimating the density:

Remark 4.2. If $s \geq d/p$ with $r = 1$ when $s = d/p$, then (20) with $\theta = 1/2$, $\eta = 1$ becomes

$$\left\| 2^{js} \int_0^t \left\| \nabla\left([\varphi, \Delta_j]\, \nabla\psi \right) \right\|_{L^p} d\tau \right\|_{\ell^r} \leq C_\varepsilon \int_0^t \|\nabla\varphi\|^2_{B^s_{p,r}} \|\psi\|_{B^s_{p,r}} + \varepsilon\, \|\psi\|_{\widetilde{L}^1_t(B^{s+2}_{p,r})}.$$

Indeed, one only has to use the interpolation inequality of the embedding $B^{d/p+2}_{p,1} \hookrightarrow C^{0,1} \hookrightarrow B^{d/p}_{p,1}$ (noticing also $B^{d/p}_{p,1} \hookleftarrow B^s_{p,r}$, $L^1_t(B^{d/p+2}_{p,1}) \hookleftarrow \widetilde{L}^1_t(B^{s+2}_{p,r})$) and Young's inequality.

Next lemma is in the same spirit of (20), by view of Proposition 3.6 which gives estimates for the product of two functions. The proof is quite similar (and easier) and hence omitted.

Lemma 4.3. *Let* $s > 0$, $s_i, \sigma_i \in \mathbb{R}$, $i = 1, 2$, $(p, r) \in [1, +\infty]^2$, *such that* $s = \theta s_1 + (1-\theta) s_2 = \eta\sigma_1 + (1-\eta)\sigma_2$ *with* $\theta, \eta \in (0, 1]$. *Then the following holds:*

$$\|f\,g\|_{\widetilde{L}^1_t(B^s_{p,r})} \leq \frac{C\,\theta}{\varepsilon^{(1-\theta)/\theta}} \int_0^t \|f\|_{L^\infty}^{1/\theta} \|g\|_{B^{s_1}_{p,r}}\, d\tau + \frac{C\,\eta}{\varepsilon^{(1-\eta)/\eta}} \int_0^t \|g\|_{L^\infty}^{1/\eta} \|f\|_{B^{\sigma_1}_{p,r}}\, d\tau + \\ + (1-\theta)\,\varepsilon\, \|f\|_{\widetilde{L}^1_t(B^{s_2}_{p,r})} + (1-\eta)\,\varepsilon\, \|g\|_{\widetilde{L}^1_t(B^{\sigma_2}_{p,r})}, \forall \varepsilon > 0.$$

Let us come to the linearized density equation

$$\partial_t\varrho + u \cdot \nabla\varrho - \mathrm{div}\,(\kappa\,\nabla\varrho) = f\,, \quad \varrho_{|t=0} = \varrho_0, \tag{21}$$

for which one has the following a priori estimate:

Proposition 4.4. *Let the triple* (s, p, r) *verify*

$$s \geq \frac{d}{p}\,, \ \ p \in (1, +\infty)\,, \ \ r \in [1, +\infty]\,, \qquad \text{with} \qquad r = 1 \quad \text{if} \quad s = 1 + \frac{d}{p} \ \text{ or } \ \frac{d}{p}. \tag{22}$$

Let u,κ,f be smooth such that $\nabla u\in B^{s-1}_{p,r}$, $\kappa\geq\kappa_*>0$, $\nabla\kappa\in B^s_{p,r}$ and $f\in\widetilde{L}^1_{T_0}(B^s_{p,r})$. Then there exists a positive constant C_1 (depending only on κ_*,d,s,p,r) such that, for every smooth solution ϱ of (21), the following estimate holds true for every $t\in[0,T_0]$:

$$\|\varrho\|_{\widetilde{L}^\infty_t(B^s_{p,r})}+\|\varrho\|_{\widetilde{L}^1_t(B^{s+2}_{p,r})}\;\leq\;C_1\,e^{C_1K(t)}\left(\|\varrho_0\|_{B^s_{p,r}}+\|f\|_{\widetilde{L}^1_t(B^s_{p,r})}\right),$$

where we have defined $K(0)=0$ and

$$K'(t)\;:=\;1+\|\nabla u\|_{B^{d/p}_{p,1}\cap B^{s-1}_{p,r}}+\|\nabla\kappa\|^2_{B^s_{p,r}}\,.$$

The proof is quite standard (see eg. the proof of Proposition 4.1 in [13]): we apply the operator Δ_j to the equation, we integrate first in space and then in time; then we use the commutator estimates and Gronwall's Inequality to get the result. Let's just sketch the proof.

Applying Δ_j to Equation (21) yields

$$\partial_t\varrho_j\,+\,u\cdot\nabla\varrho_j\,-\,\mathrm{div}\,(\kappa\,\nabla\varrho_j)\;=\;f_j\,+\,\mathcal{R}^1_j\,-\,\mathcal{R}^2_j\,,\tag{23}$$

where we have set $\varrho_j:=\Delta_j\varrho$, $f_j:=\Delta_jf$, $\mathcal{R}^1_j:=[u,\Delta_j]\cdot\nabla\varrho$ and $\mathcal{R}^2_j:=\mathrm{div}\,[\kappa,\Delta_j]\,\nabla\varrho$.

Hence one has

$$\frac{d}{dt}\|\varrho_j\|^p_{L^p}+C\,2^{2j}\,\|\varrho_j\|^p_{L^p}\;\leq\;C\,\|\varrho_j\|^{p-1}_{L^p}\left(\|f_j\|_{L^p}+\left\|\mathcal{R}^1_j\right\|_{L^p}+\left\|\mathcal{R}^2_j\right\|_{L^p}\right),\quad j\geq0,$$

thanks to the following Bernstein type inequality (see e.g. Appendix B of [11])

$$-\int_{\mathbb{R}^d}\mathrm{div}\,(\kappa\nabla\varrho_j)|\varrho_j|^{p-2}\varrho_j=(p-1)\int_{\mathbb{R}^d}\kappa|\nabla\varrho_j|^2|\varrho_j|^{p-2}\geq C(d,p,\kappa_*)\,2^{2j}\int_{\mathbb{R}^d}|\varrho_j|^p,\quad j\geq0,\,p\in(1,\infty).$$

Since the equality in the above holds also for $j=-1$, the following holds:

$$\begin{aligned}\|\varrho\|_{\widetilde{L}^\infty_t(B^s_{p,r})}+\|\varrho\|_{\widetilde{L}^1_t(B^{s+2}_{p,r})}\leq C\Bigg(&\|\varrho_0\|_{B^s_{p,r}}+2^{-(s+2)}\,\|\Delta_{-1}\varrho\|_{L^1_t(L^p)}+\\&+\|f\|_{\widetilde{L}^1_t(B^s_{p,r})}+\left\|2^{js}\int_0^t\left\|\mathcal{R}^1_j(\tau)\right\|_{L^p}d\tau\right\|_{\ell^r}+\left\|2^{js}\int_0^t\left\|\mathcal{R}^2_j(\tau)\right\|_{L^p}d\tau\right\|_{\ell^r}\Bigg).\end{aligned}\tag{24}$$

The low-frequency term $\Delta_{-1}\varrho$ can be easily bounded in $[0,T_0]$:

$$2^{-(s+2)}\,\|\Delta_{-1}\varrho\|_{L^1_t(L^p)}\;\leq\;C\int_0^t\|\varrho\|_{B^s_{p,r}}\,d\tau\,.$$

One applies Proposition 3.7 on the first commutator term and Remark 4.2 on the second commutator term. Finally performing Gronwall's inequality on (24) gives the conclusion.

Remark 4.5. Let us point out here that in the process of proving the uniqueness of the solutions to System (7), there is one derivative loss for the difference of two solutions. We therefore have to look for a priori estimates for the unknowns in $B^s_{p,r}$, under a weaker condition (22) on the indices (instead of (13)).

The linearized equation for the velocity reads

$$\left\{\begin{array}{c}\partial_tu+w\cdot\nabla u+\lambda\nabla\pi=h,\\ \mathrm{div}\,u=0,\\ u|_{t=0}=u_0,\end{array}\right.\tag{25}$$

where the initial datum u_0, the transport vector field w, the coefficient λ and the source term h are all smooth and decrease rapidly at infinity, such that $\lambda\geq\lambda_*>0$. We have the following a priori estimate:

Proposition 4.6. *Let*

$$s>\frac{d}{p}-\frac{d}{4},\quad p\in[2,4],\quad r\in[1,+\infty]\,,\qquad\textit{with}\qquad r=1\;\textit{ if }\;s=\frac{d}{p}\;\textit{ or }\;1+\frac{d}{p}\,.\tag{26}$$

Then the following estimates hold true:

$$\|u\|_{\widetilde{L}_t^\infty(B^s_{p,r})} \le C_2 e^{C_2 W(t)}\Big(\|u_0\|_{B^s_{p,r}} + \|h\|_{\widetilde{L}^1_t(B^s_{p,r})\cap L^1_t(L^2)}\Big), \quad W(t) = \int_0^t \|\nabla w\|_{B^{\frac{d}{p}}_{p,1}\cap B^{s-1}_{p,r}}, \tag{27}$$

$$\|\nabla\pi\|_{\widetilde{L}^1_t(B^s_{p,r})\cap L^1_t(L^2)} \le C_2\Big(\|h\|_{\widetilde{L}^1_t(B^s_{p,r})\cap L^1_t(L^2)} + W(t)\|u\|_{\widetilde{L}_t^\infty(B^s_{p,r})}\Big), \tag{28}$$

where $C_2 = C_2(t)$ is a positive time-dependent function, depending only on $d, p, s, r, \lambda_*, \lambda^*(T)$, with

$$\lambda^*(t) := \|\lambda\|_{L^\infty_t(L^\infty)} + \|\nabla\lambda\|_{\widetilde{L}^\infty_t(B^{\frac{d}{p}}_{p,1}\cap B^{s-1}_{p,r})}.$$

The proof is similar as in [11] and let us also just sketch it.
Firstly, Proposition 3.7 entails the following estimate for u :

$$\|u(t)\|_{\widetilde{L}_t^\infty(B^s_{p,r})} \le \Big(\|u_0\|_{B^s_{p,r}} + \int_0^t W'\|u\|_{B^s_{p,r}} + \|h - \lambda\nabla\pi\|_{\widetilde{L}^1_t(B^s_{p,r})}\Big). \tag{29}$$

By product estimates in Proposition 3.6, we have (noticing $\lambda\nabla\pi \equiv (\Delta_{-1}\lambda)\nabla\pi + ((Id-\Delta_{-1})\lambda)\nabla\pi)$

$$\|\lambda\nabla\pi\|_{\widetilde{L}^1_t(B^s_{p,r})} \le C\lambda^*\|\nabla\pi\|_{\widetilde{L}^1_t(B^s_{p,r})}, \text{ if } s > -\min\{\frac{d}{p}, \frac{d}{p'}\}, \text{ with } r = 1 \text{ if } s = \frac{d}{p}.$$

Thanks to $\operatorname{div} u \equiv 0$, π satisfies the following elliptic equation:

$$\operatorname{div}(\lambda\nabla\pi) = \operatorname{div}(h - w\cdot\nabla u) = \operatorname{div}(h - u\cdot\nabla w + u\operatorname{div} w).$$

Similar as to get (24), one finds

$$\|\nabla\pi\|_{\widetilde{L}^1_t(B^s_{p,r})} \lesssim \|\nabla\Delta_{-1}\pi\|_{L^1_t(L^p)} + \|\operatorname{div}(h - w\cdot\nabla u)\|_{\widetilde{L}^1_t(B^{s-1}_{p,r})} + \|2^{j(s-1)}\|\operatorname{div}[\lambda,\Delta_j]\nabla\pi\|_{L^1_t(L^p)}\|_{\ell^r}. \tag{30}$$

To bound the above commutator term, one applies Proposition 3.7 to it; then one uses the following interpolation inequality (with some $\epsilon, \eta \in (0,1)$)

$$\|\nabla\pi\|_{L^1_t(B^{s-1}_{p,r})} \lesssim \|\nabla\pi\|_{\widetilde{L}^1_t(B^{s+\epsilon-1}_{p,r})} \lesssim \|\nabla\pi\|^{1-\eta}_{\widetilde{L}^1_t(B^{\frac{d}{p}-\frac{d}{2}}_{p,\infty})}\|\nabla\pi\|^\eta_{\widetilde{L}^1_t(B^s_{p,r})} \lesssim \|\nabla\pi\|^{1-\eta}_{L^1_t(L^2)}\|\nabla\pi\|^\eta_{\widetilde{L}^1_t(B^s_{p,r})};$$

finally one has

$$\|2^{j(s-1)}\|\operatorname{div}[\lambda,\Delta_j]\nabla\pi\|_{L^1_t(L^p)}\|_{\ell^r} \le C(d,s,p,r,\varepsilon,\epsilon,\lambda^*)\|\nabla\pi\|_{L^1_t(L^2)} + \varepsilon\|\nabla\pi\|_{\widetilde{L}^1_t(B^s_{p,r})}.$$

Therefore, thanks to $\|\nabla\Delta_{-1}\pi\|_{L^p} \lesssim \|\nabla\pi\|_{L^2}$ and Lemma 3.9, it rests to dealing with

$$\|u\cdot\nabla w\|_{L^1_t(L^2)}, \quad \|u\operatorname{div} w\|_{L^1_t(L^2)} \quad \text{and} \quad \|\operatorname{div}(w\cdot\nabla u)\|_{\widetilde{L}^1_t(B^{s-1}_{p,r})}.$$

For $p \le 4$, $s > d/p - d/4$, we can easily find that

$$\|u\operatorname{div} w\|_{L^1_t(L^2)}, \|u\cdot\nabla w\|_{L^1_t(L^2)} \le \int_0^t \|u\|_{L^4}\|\nabla w\|_{L^4} \lesssim \int_0^t \|u\|_{B^s_{p,r}}\|\nabla w\|_{B^{\frac{d}{p}}_{p,\infty}}\, d\tau\,.$$

On the other hand, it is easy to decompose $\|\operatorname{div}(w\cdot\nabla u)\|_{B^{s-1}_{p,r}}$ into

$$\|T_{\partial_i w^j}\partial_j u^i + T_{\partial_j u^i}\partial_i w^j + \operatorname{div}\big(R(w^j,\partial_j u)\big)\|_{B^{s-1}_{p,r}},$$

which can be controlled, according to Proposition 3.6, by $W'(t)\|\nabla u\|_{B^{s-1}_{p,r}}$.

To conclude, Estimate (28) holds, and so does estimation (27), by view of (29) and Gronwall's Inequality.

4.2 Proof of the existence

In this subsection we will follow the standard procedure to prove the local existence of the solution to System (7): we construct a sequence of approximate solutions which have uniform bounds and then we prove the convergence to a unique solution. In particular, in order to bound the nonlinearities, the density should be *small* when *integrated in time.* Since we admit also large initial density ρ_0, we will introduce the large linear part ρ_L of the solution ρ, so that the remainder part $\bar{\rho} := \rho - \rho_L$ is small and hence easier to handle. In the convergence part, we will first show convergence in a space with *lower regularity* (i.e. in space $E^{d/p}_{p,r}(T)$, see (16)) and then the solution is in $E^s_{p,r}$ by Fatou's property.

We will freely use the following estimates (by Propositions 3.6 and 3.8):

$$\|uv\|_{B^s_{p,r}} \lesssim \|u\|_{B^s_{p,r}}\|v\|_{B^s_{p,r}},\ \|f(\rho)\|_{B^s_{p,r}} \le C(\|\rho\|_{L^\infty})\|\varrho\|_{B^s_{p,r}} \text{ with } f(1)=0,\ s>\frac{d}{p} \text{ or } s\ge\frac{d}{p}, r=1, \tag{31}$$

and their time-dependent version

$$\|uv\|_{\widetilde{L}^q_t(B^s_{p,r})} \lesssim \|u\|_{\widetilde{L}^{q_1}_t(B^s_{p,r})}\|v\|_{\widetilde{L}^{q_2}(B^s_{p,r})},\ \|f(\rho)\|_{\widetilde{L}^q_t(B^s_{p,r})} \le C(\|\rho\|_{L^\infty_t(L^\infty)})\|\varrho\|_{\widetilde{L}^q_t(B^s_{p,r})} \quad \text{with } \frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}.$$

4.2.1 Step 1 – Construction of a sequence of approximate solutions

In this step, we take (s,p,r) such that Conditions (13) and (14) hold true. Let us introduce the approximate solution sequence $\{(\varrho^n, u^n, \nabla\pi^n)\}_{n\ge 0}$ by induction.

Without loss of generality we can assume

$$\frac{\rho_*}{2} \le S_n\rho_0, \qquad \forall n\in\mathbb{N};$$

then, first of all we set $(\varrho^0, u^0, \nabla\pi^0) := (S_0\varrho_0, S_0u_0, 0)$, which are smooth and fast decaying at infinity.

Now, we assume by induction that the triplet $(\varrho^{n-1}, u^{n-1}, \nabla\pi^{n-1})$ of smooth and fast decaying functions has been constructed. Besides, let us suppose also that there exists a sufficiently small parameter τ (to be determined later), a positive time T^* (which may depend on τ) and a positive constant C_M (which may depend on M) such that

$$\frac{\rho_*}{2} \le \rho^{n-1} := 1+\varrho^{n-1}, \quad \|\varrho^{n-1}\|_{\widetilde{L}^\infty_{T^*}(B^s_{p,r})} \le C_M, \quad \|\varrho^{n-1}\|_{\widetilde{L}^2_{T^*}(B^{s+1}_{p,r})\cap\widetilde{L}^1_{T^*}(B^{s+2}_{p,r})} \le \tau, \tag{32}$$

$$\|u^{n-1}\|_{\widetilde{L}^\infty_{T^*}(B^s_{p,r})} \le C_M,\ \|u^{n-1}\|_{L^2_{T^*}(B^s_{p,r})\cap L^1_{T^*}(B^s_{p,r})} \le \tau,\ \|\nabla\pi^{n-1}\|_{\widetilde{L}^1_{T^*}(B^s_{p,r})\cap L^1_{T^*}(L^2)} \le \tau^{1/2}. \tag{33}$$

Remark that the above estimates (32) and (33) obviously hold true for $(\varrho^0, u^0, \nabla\pi^0)$, if T^* is assumed to be small enough.

Now we define $(\varrho^n, u^n, \nabla\pi^n)$ as the unique smooth global solution of the linear system

$$\begin{cases} \partial_t\varrho^n + u^{n-1}\cdot\nabla\varrho^n - \mathrm{div}\,(\kappa^{n-1}\nabla\varrho^n) = 0, \\ \partial_t u^n + (u^{n-1}+\nabla b^{n-1})\cdot\nabla u^n + \lambda^{n-1}\nabla\pi^n = h^{n-1}, \\ \mathrm{div}\, u^n = 0, \\ (\varrho^n, u^n)|_{t=0} = (S_n\varrho_0, S_nu_0), \end{cases} \tag{34}$$

where we have set $a^{n-1}=a(\rho^{n-1})$, $b^{n-1}=b(\rho^{n-1})$, $\kappa^{n-1}=\kappa(\rho^{n-1})$, $\lambda^{n-1}=\lambda(\rho^{n-1})$ and

$$\begin{aligned} h^{n-1} &= (\rho^{n-1})^{-1}\mathrm{div}\,((u^{n-1}+\nabla b^{n-1})\otimes\nabla a^{n-1}) \\ &= (\rho^{n-1})^{-1}\Big(\Delta b^{n-1}\nabla a^{n-1} + u^{n-1}\cdot\nabla^2 a^{n-1} + \nabla b^{n-1}\cdot\nabla^2 a^{n-1}\Big). \end{aligned} \tag{35}$$

We want to show that also the triplet $(\varrho^n, u^n, \nabla\pi^n)$ verifies (32) and (33).

First of all, we apply the maximum principle to the linear parabolic equation for ϱ^n, yielding $\rho^n := 1+\varrho^n \in [\rho_*/2, \rho^*]$. Now, we introduce ϱ_L as the solution of the heat equation with the initial datum $\varrho_0 \in B^s_{p,r}$:

$$\begin{cases} \partial_t\varrho_L - \Delta\varrho_L = 0 \\ (\varrho_L)_{|t=0} = \varrho_0\,. \end{cases}$$

Then, for any positive time $T < +\infty$, there exists some constant $C_T > 0$ depending on T such that

$$\|\varrho_L\|_{\widetilde{L}^\infty_T(B^s_{p,r})} + \|\varrho_L\|_{\widetilde{L}^1_T(B^{s+2}_{p,r})} \leq C_T \|\varrho_0\|_{B^s_{p,r}}. \tag{36}$$

Furthermore, given $\tau > 0$, we can choose $T^* < +\infty$ such that one has

$$\|\varrho_L\|_{\widetilde{L}^2_{T^*}(B^{s+1}_{p,r})\cap\widetilde{L}^1_{T^*}(B^{s+2}_{p,r})} \leq \tau^2. \tag{37}$$

Indeed, by definition we have

$$\|\varrho_L\|_{\widetilde{L}^1_{T^*}(B^{s+2}_{p,r})} = \left\|\left(2^{js}\int_0^{T^*}\|2^{2j}e^{t\Delta}\Delta_j\varrho_0\|_{L^p}\,dt\right)_j\right\|_{\ell^r}.$$

The operator $e^{t\Delta}\Delta_j$ belongs to $\mathcal{L}(L^p) := \{A : L^p \to L^p \text{ linear and bounded}\}$: more precisely,

$$\left\|e^{t\Delta}\,\Delta_j\right\|_{\mathcal{L}(L^p)} \leq C\ \forall j \geq -1, \quad \text{and} \quad \left\|e^{t\Delta}\,\Delta_j\right\|_{\mathcal{L}(L^p)} \leq C\,e^{-C\,t\,2^{2j}}\ \forall j \geq 0.$$

Then, for some fixed N large enough, we infer

$$\left\|\left(\int_0^{T^*} 2^{2j}e^{-C\,t\,2^{2j}}\,dt\right)_{0\leq j\leq N}\right\|_{\ell^\infty} \leq C(1-e^{-C2^{2N}T^*});$$

From this, by decomposing ϱ_0 into low frequencies (large part) and high frequencies (small part) and choosing T^* small enough, one gathers $\|\varrho_L\|_{\widetilde{L}^1_{T^*}(B^{s+2}_{p,r})} \leq \tau^2$. The term $\|\varrho_L\|_{\widetilde{L}^2_{T^*}(B^{s+1}_{p,r})}$ can be handled in the same way or by interpolation inequality. Hence, our claim (37) is proved.

Now we define the sequence $\varrho^n_L = S_n\varrho_L$: it too solves the free heat equation, but with initial data $S_n\varrho_0$. Hence, it too satisfies (36) and (37).

We next consider the small remainder $\bar{\varrho}^n := \varrho^n - \varrho^n_L$. We claim that it fulfills, for all $n \in \mathbb{N}$,

$$\|\bar{\varrho}^n\|_{\widetilde{L}^2_{T^*}(B^{s+1}_{p,r})} \leq \|\bar{\varrho}^n\|_{\widetilde{L}^\infty_{T^*}(B^s_{p,r})} + \|\bar{\varrho}^n\|_{\widetilde{L}^1_{T^*}(B^{s+2}_{p,r})} \leq \tau^{3/2}. \tag{38}$$

In fact, $\bar{\varrho}^n = \varrho^n - \varrho^n_L$ solves

$$\begin{cases} \partial_t\bar{\varrho}^n + u^{n-1}\cdot\nabla\bar{\varrho}^n - \operatorname{div}(\kappa^{n-1}\nabla\bar{\varrho}^n) = -u^{n-1}\cdot\nabla\varrho^n_L + \operatorname{div}((\kappa^{n-1}-1)\nabla\varrho^n_L), \\ \bar{\varrho}^n|_{t=0} = 0. \end{cases} \tag{39}$$

So, if we define

$$K^{n-1}(t) := t + \left\|\nabla u^{n-1}\right\|_{L^1_t(B^{s-1}_{p,r})} + \left\|\nabla\kappa^{n-1}\right\|^2_{L^2_t(B^s_{p,r})},$$

by Proposition 4.4 we infer that

$$\|\bar{\varrho}^n\|_{\widetilde{L}^\infty_{T^*}(B^s_{p,r})\cap\widetilde{L}^1_{T^*}(B^{s+2}_{p,r})} \leq C\,e^{CK^{n-1}(T^*)}\left\|-u^{n-1}\cdot\nabla\varrho^n_L + \operatorname{div}\left((\kappa^{n-1}-1)\nabla\varrho^n_L\right)\right\|_{\widetilde{L}^1_{T^*}(B^s_{p,r})}.$$

Inductive assumptions and estimate (37) for ϱ_L help to bound the right-hand side in the above relation by $CC_M\tau^2$. Therefore, (38) is proved, and hence (32) holds for $\varrho^n = \bar{\varrho}^n + \varrho^n_L$, for sufficiently small τ.

We now want to get (33), relying mainly on Proposition 4.6. In fact, product estimates (31) and the embedding result in Remark 3.3 entail (noticing also $L^4 \hookleftarrow B^{s-1}_{p,\infty}$)

$$\|h^{n-1}\|_{\widetilde{L}^1_{T^*}(B^s_{p,r})\cap L^1_{T^*}(L^2)}, \quad W^{n-1}(T^*) := \int_0^{T^*}\|\nabla u^{n-1} + \nabla^2 b^{n-1}\|_{B^{s-1}_{p,r}} \leq C\tau.$$

Thus, applying Proposition 4.6 to system (34) implies

$$\|u^n\|_{\widetilde{L}^\infty_{T^*}(B^s_{p,r})} \leq C(T^*)(\|S_n u_0\|_{B^s_{p,r}} + C\tau) \leq C_M, \quad \|\nabla\pi^n\|_{\widetilde{L}^1_{T^*}(B^s_{p,r})\cap L^1_{T^*}(L^2)} \leq C\tau + C_M C\tau.$$

Hence (33) also holds true for small τ and T^*.

4.2.2 Step 2 – Convergence of the sequence

In this step we will consider the "difference" sequence

$$(\delta\varrho^n, \delta u^n, \nabla\delta\pi^n) := (\varrho^n - \varrho^{n-1}, u^n - u^{n-1}, \nabla\pi^n - \nabla\pi^{n-1}), \qquad \forall\, n \geq 1$$

in the Banach space $E^{d/p}_{p,1}(T^*)$ (recall (16) for its definition).

First of all, by System (34), $(\delta\varrho^n, \delta u^n, \nabla\delta\pi^n)$ solves

$$\begin{cases} \partial_t\delta\varrho^n + u^{n-1}\cdot\nabla\delta\varrho^n - \mathrm{div}\,(\kappa^{n-1}\nabla\delta\varrho^n) = F^{n-1}, \\ \partial_t\delta u^n + (u^{n-1} + \nabla b^{n-1})\cdot\nabla\delta u^n + \lambda^{n-1}\nabla\delta\pi^n = H^{n-1}, \\ \mathrm{div}\,\delta u^n = 0, \\ (\delta\varrho^n, \delta u^n)|_{t=0} = (\Delta_n\varrho_0, \Delta_n u_0), \end{cases} \tag{40}$$

where

$$\begin{aligned} F^{n-1} &= -\delta u^{n-1}\cdot\nabla\varrho^{n-1} + \mathrm{div}\,((\kappa^{n-1} - \kappa^{n-2})\nabla\varrho^{n-1}), \\ H^{n-1} &= h^{n-1} - h^{n-2} - (\delta u^{n-1} + \nabla\delta b^{n-1})\cdot\nabla u^{n-1} - (\lambda^{n-1} - \lambda^{n-2})\nabla\pi^{n-1}. \end{aligned}$$

Next we apply a priori estimates given by Propositions 4.4 and 4.6, with $s = d/p$, $p \in [2,4]$ and $r = 1$, to $\delta\varrho^n$ and $(\delta u^n, \nabla\delta\pi^n)$ respectively. The use of inductive assumptions gives us

$$\|\delta\varrho^n\|_{L^\infty_{T^*}(B^{d/p}_{p,1})\cap L^1_{T^*}(B^{d/p+2}_{p,1})} \leq C\left(\|\Delta_n\varrho_0\|_{B^{d/p}_{p,1}} + \|F^{n-1}\|_{L^1_{T^*}(B^{d/p}_{p,1})}\right), \tag{41}$$

$$\|\delta u^n\|_{\widetilde{L}^\infty_{T^*}(B^{d/p}_{p,1})} + \|\nabla\delta\pi^n\|_{L^1_{T^*}(B^{d/p}_{p,1}\cap L^2)} \leq C\Big(\|\Delta_n u_0\|_{B^{d/p}_{p,1}} + \|H^{n-1}\|_{L^1_{T^*}(B^{d/p}_{p,1}\cap L^2)}\Big). \tag{42}$$

Next we use the following fact, coming from Proposition 3.8: for f smooth, $s \geq d/p$,

$$\|\delta f^m\|_{B^s_{p,1}} := \|f(\rho^m) - f(\rho^{m-1})\|_{B^s_{p,1}} \leq C(\|\varrho^m\|_{B^s_{p,1}}, \|\varrho^{m-1}\|_{B^s_{p,1}})\,\|\delta\varrho^m\|_{B^s_{p,1}}.$$

Therefore, one easily gets

$$\begin{aligned} \|F^{n-1}\|_{L^1_{T^*}(B^{d/p}_{p,1})} \leq{}& C(\|\delta\varrho^{n-1}\|_{L^2_{T^*}(B^{d/p+1}_{p,1})}\|\varrho^{n-1}\|_{L^2_{T^*}(B^{d/p+1}_{p,1})} \\ &+ \int_0^{T^*}\|\delta u^{n-1}\|_{B^{d/p}_{p,1}}\|\nabla\varrho^{n-1}\|_{B^{d/p}_{p,1}} + \|\delta\varrho^{n-1}\|_{B^{d/p}_{p,1}}\|\varrho^{n-1}\|_{B^{d/p+2}_{p,1}}), \\ \|H^{n-1}\|_{L^1_{T^*}(B^{d/p}_{p,1}\cap L^2)} \leq{}& C(\|\delta\varrho^{n-1}\|_{L^1_{T^*}(B^{d/p+2}_{p,1})} \\ &+ \|\delta\varrho^{n-1}\|_{L^2_{T^*}(B^{d/p+1}_{p,1})}\Big(\|(\varrho^{n-1}, \varrho^{n-2})\|_{L^2_{T^*}(B^{d/p+2}_{p,1})} + \|(u^{n-1}, u^{n-2})\|_{L^2_{T^*}(B^{d/p+1}_{p,1})}\Big) \\ &+ \int_0^{T^*}\|\delta\varrho^{n-1}\|_{B^{d/p}_{p,1}}\|\nabla\pi^{n-1}\|_{B^{d/p}_{p,1}} + \|\delta u^{n-1}\|_{B^{d/p}_{p,1}}(\|\varrho^{n-1}\|_{B^{d/p+2}_{p,1}} + \|u^{n-1}\|_{B^{d/p+1}_{p,1}})). \end{aligned}$$

Plugging the uniform estimate (32) into (41) to get bounds on $\|\delta\varrho^{n-1}\|_{L^1_{T^*}(B^{d/p+2}_{p,1})}$ (which appears in H^{n-1}) and then relation (42) becomes

$$\begin{aligned} \|\delta u^n\|_{\widetilde{L}^\infty_{T^*}(B^{d/p}_{p,1})} + \|\nabla\delta\pi^n\|_{\widetilde{L}^1_{T^*}(B^{d/p}_{p,1})} \leq{}& C(\|(\Delta_n u_0, \Delta_{n-1}\varrho_0)\|_{B^{d/p}_{p,1}} + \tau\|(\delta\varrho^{n-1}, \delta\varrho^{n-2})\|_{L^2_{T^*}(B^{d/p+1}_{p,1})} \\ &+ \int_0^{T^*}\|(\delta\varrho^{n-1}, \delta u^{n-1}, \delta u^{n-2})\|_{B^{d/p}_{p,1}}\Big(\|\nabla\pi^{n-1}\|_{B^{d/p}_{p,1}} + \|(\varrho^{n-1}, \varrho^{n-2})\|_{B^{d/p+2}_{p,1}} + \|u^{n-1}\|_{B^{d/p+1}_{p,1}}\Big). \end{aligned}$$

Let us now define

$$B^n(t) := \|\delta\varrho^n\|_{L^\infty_t(B^{d/p}_{p,1})} + \|\delta\varrho^n\|_{L^1_t(B^{d/p+2}_{p,1})} + \|\delta u^n\|_{L^\infty_t(B^{d/p}_{p,1})} + \|\nabla\delta\pi^n\|_{L^1_t(B^{d/p}_{p,1}\cap L^2)};$$

then, from previous inequalities we gather

$$B^n(t) \leq C\,\|(\Delta_{n-1}\varrho_0, \Delta_n\varrho_0, \Delta_n u_0)\|_{B^{d/p}_{p,1}} + \tau^{\frac{1}{2}}\left(B^{n-1}(t) + B^{n-2}(t)\right) + C\int_0^t\left(B^{n-1} + B^{n-2}\right)D(\sigma)d\sigma,$$

with $\|D(t)\|_{L^1([0,T^*])} \leq C$. Noticing

$$\|(\Delta_n\varrho_0, \Delta_n u_0)\|_{B^{d/p}_{p,1}} \leq C\, 2^{n(d/p)}\, \|(\Delta_n\varrho_0, \Delta_n u_0)\|_{L^p}\ \forall n \geq 0,$$

it follows $\sum_n B^n(t) < +\infty$ uniformly in $[0, T^*]$. Hence, we gather that the sequence $(\varrho^n, u^n, \nabla\pi^n)$ is a Cauchy sequence in the functional space $E^{d/p}_{p,1}(T^*)$. Then, it converges to some $(\varrho, u, \nabla\pi)$, which actually belongs to the space $E^s_{p,r}(T^*)$ by Fatou property. Hence, by interpolation, the convergence holds true in any intermediate space between $E^s_{p,r}(T^*)$ and $E^{d/p}_{p,1}(T^*)$, and this is enough to pass to the limit in our equations. Thus, $(\varrho, u, \nabla\pi)$ is actually a solution of System (7).

The proof of uniqueness is exactly analogous to the above convergence proof, and hence omitted.

5 Proof of Theorems 2.4 and 2.5

In this section we aim to get a continuation criterion and a lower bound of the lifespan for the local-in-time solutions given by Theorem 2.1. It is only a matter of repeating a priori estimates established previously, but in an "accurate" way (we use L^∞-norm instead of $B^{s-1}_{p,r}$-norm) for obtaining the continuation criterion, whereas in a "rough" way (we use (47), (48) below) for bounding the lifespan from below.

5.1 Proof of the continuation criterion

Theorem 2.4 actually issues easily from the following fundamental lemma.

Lemma 5.1. *Let $s > 0$, $p \in (1, +\infty)$ and $r \in [1, +\infty]$. Let $(\rho, u, \nabla\pi)$ be a solution of (7) over $[0, T[\,\times\mathbb{R}^d$ such that the hypotheses in Theorem 2.4 hold true. If T is finite, then one gets*

$$\|\rho - 1\|_{\widetilde{L}^\infty_T(B^s_{p,r})\cap\widetilde{L}^1_T(B^{s+2}_{p,r})} + \|u\|_{\widetilde{L}^\infty_T(B^s_{p,r})} + \|\nabla\pi(t)\|_{\widetilde{L}^1_T(B^s_{p,r})} < +\infty\,. \tag{43}$$

In fact, to prove Theorem 2.4 from Lemma 5.1 is quite standard: once (43) and Conditions (13), (14) hold true, then there exists a positive time t_0 (thanks to Theorem 2.1) such that, for any $\widetilde{T} < T$, System (7) with initial data $\Big(\rho(\widetilde{T}), u(\widetilde{T})\Big)$ has a unique solution until the time $\widetilde{T} + t_0$. Thus, if we take, for instance, $\widetilde{T} = T - (t_0/2)$, then we get a solution until the time $T + (t_0/2)$, which is, by uniqueness, the continuation of $(\rho, u, \nabla\pi)$. Theorem 2.4 then follows.

Therefore, we focus only on Lemma 5.1: one uses L^∞-norm (instead of Besov norm) to establish a priori estimates.

Let us consider the density term. Our starting point is (21), with $f = 0$. One argues as in proving Proposition 4.4, but controls commutators $\mathcal{R}^1_j$ and $\mathcal{R}^2_j$ (see (23) for definition) by use of Commutator Estimates (19) and (20) (with $\theta = \eta = 1/2$) instead. More precisely, keeping in mind that $\kappa = \kappa(\rho)$, we arrive at

$$\int_0^t \Big\|2^{js}\,\|\mathcal{R}^1_j\|_{L^p}\Big\|_{\ell^r}\,d\tau \leq \int_0^t \Big(\|\nabla u\|_{L^\infty}\,\|\varrho\|_{B^s_{p,r}} + \|\nabla\varrho\|_{L^\infty}\,\|u\|_{B^s_{p,r}}\Big)\,d\tau\,,$$
$$\Big\|2^{js}\int_0^t \|\mathcal{R}^2_j\|_{L^p}\Big\|_{\ell^r}\,d\tau \leq \frac{C}{\varepsilon}\int_0^t \|\nabla\varrho\|^2_{L^\infty}\,\|\varrho\|_{B^s_{p,r}}\,d\tau + \varepsilon\,\|\varrho\|_{\widetilde{L}^1_t(B^{s+2}_{p,r})}\,.$$

Hence, (24) becomes

$$\|\varrho\|_{\widetilde{L}^\infty_t(B^s_{p,r})\cap\widetilde{L}^1_t(B^{s+2}_{p,r})} \lesssim \|\varrho_0\|_{B^s_{p,r}} + \int_0^t (1 + \|\nabla u\|_{L^\infty} + \|\nabla\varrho\|^2_{L^\infty})\|\varrho\|_{B^s_{p,r}} + \int_0^t \|\nabla\varrho\|_{L^\infty}\|u\|_{B^s_{p,r}}. \tag{44}$$

Let us now consider velocity field and pressure term: we use Lemma 4.1 to control the commutator $\mathcal{R}_j := [u + \nabla b(\rho)\,,\,\Delta_j]\cdot\nabla u$ and arrive at

$$\|u\|_{\widetilde{L}^\infty_t(B^s_{p,r})} \leq C\Big(\|u_0\|_{B^s_{p,r}} + \int_0^t \|(\nabla u, \nabla^2\varrho)\|_{L^\infty}\,\|u\|_{B^s_{p,r}} + \|\nabla u\|^2_{L^\infty}\,\|\varrho\|_{B^s_{p,r}}\,d\tau + \varepsilon\|\varrho\|_{\widetilde{L}^1_t(B^{s+2}_{p,r})}$$
$$+ \|h\|_{\widetilde{L}^1_t(B^s_{p,r})} + \|\lambda\,\nabla\pi\|_{\widetilde{L}^1_t(B^s_{p,r})}\Big). \tag{45}$$

Lemma 4.3 and Proposition 3.8 help us to control the non-linear term h (see (8) for the definition):

$$\|h\|_{\widetilde{L}^1_t(B^s_{p,r})} \lesssim \int_0^t \Big(\big\|\nabla^2\varrho\big\|_{L^\infty} + \|\nabla\varrho\|^2_{L^\infty}\Big)\,\|u\|_{B^s_{p,r}}\,d\tau \,+\, \Big(1+\|(\nabla\rho,u)\|_{L^\infty_t(L^\infty)}\Big)\,\|\varrho\|_{\widetilde{L}^1_t(B^{s+2}_{p,r})}$$
$$+\int_0^t \Big(\|u\|^2_{L^\infty}\,\|\nabla\varrho\|^2_{L^\infty} + \|\nabla\varrho\|^4_{L^\infty} + \|\nabla^2\varrho\|^2_{L^\infty}\Big)\,\|\varrho\|_{B^s_{p,r}}\,d\tau.$$

By decomposing λ into $\lambda(1)$ and $\lambda-\lambda(1)$, one has

$$\|\lambda(\rho)\,\nabla\pi\|_{\widetilde{L}^1_t(B^s_{p,r})} \leq C\,\left(\|\nabla\pi\|_{\widetilde{L}^1_t(B^s_{p,r})} \,+\, \int_0^t \|\nabla\pi\|_{L^\infty}\,\|\varrho\|_{B^s_{p,r}}\,d\tau\right).$$

One then uses (30) to bound $\|\nabla\pi\|_{\widetilde{L}^1_t(B^s_{p,r})}$: Lemma 4.3 help to bound the nonlinear term $\mathrm{div}\,((u+\nabla b)\cdot\nabla u) \equiv (\nabla u+\nabla^2 b):\nabla u$ by

$$C\Big(\int_0^t \big(\|\nabla u\|_{L^\infty} + \big\|\nabla^2\varrho\big\|_{L^\infty}\big)\,\|u\|_{B^s_{p,r}}d\tau \,+\, \int_0^t \|\nabla u\|^2_{L^\infty}\,\|\varrho\|_{B^s_{p,r}}\,d\tau \,+\, \varepsilon\|\varrho\|_{\widetilde{L}^1_t(B^{s+2}_{p,r})}\Big).$$

Finally, Lemma 4.1 entails the control for the commutator term:

$$\left\|2^{j(s-1)}\int_0^t \|\mathrm{div}\,([\lambda,\Delta_j]\nabla\pi)\|_{L^p}d\tau\right\|_{\ell^r} \lesssim \int_0^t \Big(\|\nabla\varrho\|_{L^\infty}\,\|\nabla\pi\|_{B^{s-1}_{p,r}} \,+\, \|\nabla\varrho\|_{B^{s-1}_{p,r}}\,\|\nabla\pi\|_{L^\infty}\Big)\,d\tau.$$

Hence, interpolation inequality for $\|\nabla\pi\|_{B^{s-1}_{p,r}}$ between $B^{-\sigma}_{p,\infty}$ and $B^s_{p,r}$ helps us to gather

$$\begin{aligned}\|\nabla\pi\|_{\widetilde{L}^1_t(B^s_{p,r})} \;\lesssim\;& C(\|\nabla\varrho\|_{L^\infty_t(L^\infty)},s,\sigma)\|\nabla\pi\|_{L^1_t(B^{-\sigma}_{p,\infty})}\\ &+\int_0^t\Big(\|\nabla u\|_{L^\infty} + \big\|\nabla^2\varrho\big\|_{L^\infty} + \|\nabla\varrho\|^2_{L^\infty} + \|\nabla\varrho\|^4_{L^\infty}\Big)\|u\|_{B^s_{p,r}}d\tau\\ &+\int_0^t\Big(\|\nabla u\|^2_{L^\infty} + \|u\|^2_{L^\infty}\|\nabla\varrho\|^2_{L^\infty} + \|\nabla^2\varrho\|^2_{L^\infty} + \|\nabla\pi\|_{L^\infty}\Big)\|\varrho\|_{B^s_{p,r}}d\tau\\ &+\Big(1+\|\nabla\varrho\|_{L^\infty_t(L^\infty)} + \|u\|_{L^\infty_t(L^\infty)}\Big)\|\varrho\|_{\widetilde{L}^1_t(B^{s+2}_{p,r})}.\end{aligned}\tag{46}$$

In the end, we discover from (45) that $\|u\|_{\widetilde{L}^\infty_t(B^s_{p,r})}$ satisfies also Inequality (46), just with an additional term $\|u_0\|_{B^s_{p,r}}$ on the right-hand side. Recalling Estimate (44) for the density, we can replace $\|\varrho\|_{\widetilde{L}^1_t(B^{s+2}_{p,r})}$ in Inequality (46) by the right-hand side of it.

Thus, we can sum up (44) and the (modified) estimate (46) for the velocity u, yielding Lemma 5.1 by Gronwall's Lemma.

5.2 Lower bounds for the lifespan of the solution

The aim of the present subsection is analyzing the lifespan of the solutions to system (7). We want to show, as carefully as possible, the dependence of the lifespan T on the *initial data*.

We will use freely the following inequalities:

$$\|ab\|_{B^{s-i}_{p,r}} \lesssim \|a\|_{B^{s-i}_{p,r}}\|b\|_{B^{s-i}_{p,r}},\quad \|a^2\|_{B^s_{p,r}} \lesssim \|a\|_{L^\infty}\|a\|_{B^s_{p,r}},\quad \|a\|_{L^\infty},\|\nabla a\|_{L^\infty} \lesssim \|a\|_{B^s_{p,r}},\quad i=0,1,\tag{47}$$

and thanks to Conditions (13) and (14),

$$\|ab\|_{L^2} \leq \|a\|_{L^4}\|b\|_{L^4} \lesssim \|a\|_{B^{s-1}_{p,r}}\|b\|_{B^{s-1}_{p,r}}.\tag{48}$$

By embedding results, without any loss of generality, throughout this subsection we will always assume $(s,p,r)=(1+d/4,4,1)$. For notation convenience, we define $R_0 := \|\varrho_0\|_{B^{1+d/4}_{4,1}}$ and $U_0 := \|u_0\|_{B^{1+d/4}_{4,1}}$,

$$R(t) \;=\; \|\varrho\|_{L^\infty_t(B^{1+d/4}_{4,1})}\,,\qquad S(t) \;:=\; \|\varrho\|_{L^1_t(B^{3+d/4}_{4,1})}\qquad\text{and}\qquad U(t) \;:=\; \|u\|_{L^\infty_t(B^{1+d/4}_{4,1})}\,.$$

Then $S'(t)\equiv\|\varrho(t)\|_{B^{s+2}_{p,1}}$ controls high regularity of the density.

From (44), we infer that

$$R(t) + S(t) \leq C\Big(R_0 + \int_0^t R(U+1)\,d\tau + \int_0^t R^3\,d\tau\Big).$$

Now, if we define

$$T_R := \sup\Big\{t > 0 \;\Big|\; \int_0^t R^3(\tau)\,d\tau \leq 2\,R_0\Big\}, \tag{49}$$

by Gronwall's Lemma, for $t \in [0, T_R]$ and for some large enough C, we get

$$R(t) + S(t) \leq C\,R_0\,\mathcal{E}(t), \text{ with } \mathcal{E}(t) := \exp\left(C\int_0^t \big(1 + U(\tau)\big)\,d\tau\right). \tag{50}$$

Next we aim to bound U, just as in the last subsection 5.1. Firstly, one applies the classical commutator estimates to the commutator $\mathcal{R}_j = [u + \nabla b, \Delta_j] \cdot \nabla u$ to arrive at

$$U(t) \leq C\Big(U_0 + \int_0^t U^2 + U\,\|\varrho\|_{B^{s+1}_{p,1}}\,d\tau + \int_0^t \|h\|_{B^s_{p,1}}\,d\tau + \int_0^t \|\lambda\,\nabla\pi\|_{B^s_{p,1}}\,d\tau\Big). \tag{51}$$

Next, since the interpolation inequality for the embeddings $B^s_{p,1} \hookrightarrow B^{s-1}_{p,1} \hookrightarrow L^2$ holds, from estimate (30) for π one derives that, for some $\delta > 1$,

$$\|\nabla\pi\|_{B^s_{p,1}} \leq C\left((1 + R^\delta)\,\|\nabla\pi\|_{L^2} + (1+R)\,\|(\nabla h, \nabla(u+\nabla b) : \nabla u)\|_{B^{s-1}_{p,1}}\right).$$

Then, estimates (47) and (48) help to bound h, $\nabla(u + \nabla b) : \nabla u$ and $\nabla\pi$ as follows:

$$\|h\|_{B^s_{p,1}},\ \|\nabla(u+\nabla b) : \nabla u\|_{B^{s-1}_{p,1}} \lesssim U\|\varrho\|_{B^{s+1}_{p,1}} + UR\|\varrho\|_{B^{s+1}_{p,1}} + R^2 S' + RS' + U^2,$$
$$\|\nabla\pi\|_{L^2} \lesssim \|(h, \nabla(u+\nabla b) : \nabla u)\|_{L^2} \lesssim UR^2 + U\|\varrho\|_{B^{s+1}_{p,1}} + U^2 + R^3 + RS' + UR.$$

Now we use interpolation to write $\|\varrho\|_{B^{s+1}_{p,1}} \lesssim R^{1/2}(S')^{1/2}$, and Young inequality to separate the term S'. Hence, from (51) and previous inequalities one infers that

$$U(t) \lesssim U_0 + \int_0^t \big(1 + R^{\delta+1}\big)\Big(U(R^2+R) + U^2(1+R^3) + R^3 + S'(1+R^2)\Big)d\tau.$$

Let us restrict now to the interval $[0, T_R]$: from (50) and the previous estimate for U we get (possibly taking a bigger C)

$$\begin{aligned} U(t) &\lesssim \mathcal{E}(t)\left(U_0 + \big(1 + R_0^{\delta+4}\big)\int_0^t \big(U + U^2 + S' + R^3\big)d\tau\right) \\ &\lesssim \mathcal{E}(t)\left(1 + U_0 + R_0^{\delta+5} + \big(1 + R_0^{\delta+4}\big)\int_0^t \big(U + U^2\big)d\tau\right). \end{aligned}$$

Now we define

$$T_U := \sup\Big\{t > 0 \;\big|\; \mathcal{E}(t) \leq 2,\ \big(1 + R_0^{\delta+4}\big)\int_0^t \big(U + U^2\big)\,d\tau \leq 2\big(1 + U_0 + R_0^{\delta+4}\big)\Big\}. \tag{52}$$

Then, for $0 \leq t \leq \min\{T_R, T_U\}$, we infer the estimate

$$U(t) \leq C\big(1 + U_0 + R_0^{\delta+5}\big).$$

Finally, by a bootstrap argument, it is easy to check that the time T, defined by (17) in Theorem 2.5, with sufficiently small L, is less than T_R and T_U. Hence Theorem 2.5 follows.

A Proof of Lemma 4.1

In the appendix we will prove Estimate (20). The following classical properties will be used freely throughout this section:

- for any $u \in \mathcal{S}'$, the equality $u = \sum_j \Delta_j u$ holds true in $\mathcal{S}'$;
- for all u and v in $\mathcal{S}'$, the sequence $(S_{j-1}u\,\Delta_j v)_{j\in\mathbb{N}}$ is spectrally supported in dyadic annuli.

First of all, let us recall an easy version of Young inequality:

$$a\,b \;\leq\; \theta\,\varepsilon^{-(1-\theta)/\theta}\,a^{1/\theta} \,+\, (1-\theta)\,\varepsilon\,b^{1/(1-\theta)}, \quad \forall\theta\in\,]0,1[\,,\varepsilon>0,\,a,b\in\mathbb{R}^+. \tag{A.1}$$

We decompose the commutator by use of Bony's paraproduct:

$$[\varphi,\Delta_j]\cdot\nabla\psi \;=\; R^1_j(\varphi,\psi) \,+\, R^2_j(\varphi,\psi) \,+\, R^3_j(\varphi,\psi) \,+\, R^4_j(\varphi,\psi) \,+\, R^5_j(\varphi,\psi)\,, \tag{A.2}$$

where, setting $\widetilde{\varphi} = (\mathrm{Id}\,-\Delta_{-1})\varphi$, we have defined

$$\begin{aligned}
R^1_j(\varphi,\psi) &:= [T_{\widetilde{\varphi}},\Delta_j]\cdot\nabla\psi\\
R^2_j(\varphi,\psi) &:= T'_{\Delta_j\nabla\psi}\widetilde{\varphi} \;=\; \sum_k S_{k+2}\Delta_j\nabla\psi\,\cdot\Delta_k\widetilde{\varphi}\\
R^3_j(\varphi,\psi) &:= -\Delta_j T_{\nabla\psi}\widetilde{\varphi}\\
R^4_j(\varphi,\psi) &:= -\Delta_j R(\widetilde{\varphi},\nabla\psi)\\
R^5_j(\varphi,\psi) &:= [\Delta_{-1}\varphi,\Delta_j]\cdot\nabla\psi\,.
\end{aligned}$$

One finds easily

$$R^1_j(\varphi,\psi) \;=\; \sum_{|\nu-j|\leq 1}\int_{\mathbb{R}^d_z} 2^{-j}\left(\int_0^1 h(z)\,z\,\cdot\nabla S_{\nu-1}\widetilde{\varphi}(x-2^{-j}\lambda z)\,\cdot\Delta_\nu\nabla\psi(x-2^{-j}z)\,d\lambda\right)dz\,.$$

This ensures that

$$\mathcal{R}^1 := \left\|\left(2^{js}\int_0^t\|\nabla R^1_j\|_{L^p}\right)_j\right\|_{\ell^r} \;\lesssim\; \left\|\left(2^{js}\int_0^t\|\nabla S_{j-1}\widetilde{\varphi}\|_{L^\infty}\,\|\Delta_j\nabla\psi\|_{L^p}\,d\tau\right)_j\right\|_{\ell^r}.$$

We apply Young inequality (A.1) to the integrand on the right hand side to get, for some constant C:

$$\mathcal{R}^1 \;\leq\; \frac{C\,\theta}{\varepsilon^{(1-\theta)/\theta}}\int_0^t\|\nabla\varphi\|^{1/\theta}_{L^\infty}\,\|\psi\|_{B^{s_1}_{p,r}}\,d\tau \,+\, (1-\theta)\,\varepsilon\,\|\psi\|_{\widetilde{L}^1_t(B^{s_2}_{p,r})}\,. \tag{A.3}$$

Let us now handle

$$\begin{aligned}
\mathcal{R}^2 &:= \left\|2^{j\,s}\int_0^t\|\nabla R^2_j\|_{L^p}\,d\tau\right\|_{\ell^r}\\
&\lesssim \left\|2^{j\,s}\int_0^t\sum_{\mu\geq j-2}\Big(\|\nabla^2 S_{\mu+2}\Delta_j\psi\|_{L^\infty}\,\|\Delta_\mu\widetilde{\varphi}\|_{L^p} + \|S_{\mu+2}\nabla\Delta_j\psi\|_{L^\infty}\,\|\nabla\Delta_\mu\widetilde{\varphi}\|_{L^p}\Big)\,d\tau\right\|_{\ell^r}\\
&\lesssim \left\|2^{j\,s}\int_0^t\sum_{\mu\geq j-2}\|\nabla\Delta_j\psi\|_{L^\infty}\,\|\nabla\Delta_\mu\widetilde{\varphi}\|_{L^p}\,d\tau\right\|_{\ell^r}\\
&\lesssim \left\|\int_0^t\|\nabla\psi\|_{L^\infty}\sum_{\mu\geq j-2}2^{(j-\mu)s}2^{\mu s}\|\nabla\Delta_\mu\widetilde{\varphi}\|_{L^p}\,d\tau\right\|_{\ell^r}.
\end{aligned}$$

We just do exactly as above (the way to obtain (A.3)): if $s>0$, then we have

$$\mathcal{R}^2 \;\leq\; \frac{C\,\eta}{\varepsilon^{(1-\eta)/\eta}}\int_0^t\|\nabla\psi\|^{1/\eta}_{L^\infty}\,\|\nabla\widetilde{\varphi}\|_{B^{\sigma_1-1}_{p,r}}\,d\tau \,+\, (1-\eta)\,\varepsilon\,\|\nabla\widetilde{\varphi}\|_{\widetilde{L}^1_t(B^{\sigma_2-1}_{p,r})}\,. \tag{A.4}$$

Moreover, since

$$\mathcal{R}^3 := \left\|2^{j\,s}\int_0^t \left\|\nabla R_j^3\right\|_{L^p}\,d\tau\right\|_{\ell^r} \leq \left\|2^{j(s+1)}\int_0^t \sum_{\mu\sim j}\|S_{\mu-1}\nabla\psi\|_{L^\infty}\,\|\Delta_\mu\widetilde{\varphi}\|_{L^p}\right\|_{\ell^r}$$

we can immediately see that (A.4) holds also for $\mathcal{R}^3$. Similarly, (A.4) follows immediately for

$$\mathcal{R}^4 := \left\|2^{j\,s}\int_0^t \left\|\nabla R_j^4\right\|_{L^p}\,d\tau\right\|_{\ell^r} \lesssim \left\|\int_0^t \|\nabla\psi\|_{L^\infty}\sum_{\mu\geq j-2}2^{(j-\mu)(s+1)}\left(2^{\mu(s+1)}\|\Delta_\mu\widetilde{\varphi}\|_{L^p}\right)\,d\tau\right\|_{\ell^r}.$$

Finally, the last term

$$\mathcal{R}^5 := \left\|2^{j\,s}\int_0^t \left\|\nabla R_j^5\right\|_{L^p}\,d\tau\right\|_{l^r}$$

can be handled as $\mathcal{R}^1$, leading us to the same estimate as (A.3) and so to the end of the proof.

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