AN OPTIMAL MASS TRANSFER PROBLEM IN THE LIMIT CASE

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Abstract. In this paper, we address the singular optimal mass transfer problem in $\mathbb{B}(0, R)$. Based on the newly developed methodology of canonical dual transformation, the non-convex variational problem can be converted into an algebraic problem, which can be solved completely. As a matter of fact, the uniqueness of the solution of the nonlinear elliptic equation does not hold since the divergence equation has many solutions in multi-dimensional case. According to the dual curve for the algebraic equation, a triality result is discussed in detail. As applications, we shall show several typical engineering models with specific terms in 1D and 2D domains by numerical simulations. Moreover, the limit case is clearly observed theoretically and practically by the canonical duality method.

Keywords Optimal mass transfer, nonconvex nonsmooth problem, divergence equation, canonical dual transformation, triality theory

Mathematics Subject Classification 35J20, 35J60, 49K20, 80A20

1. Motivation and problems

Mass transfer is the net movement of mass from one location to another by the action of driving forces, such as pressure gradient(pressure diffusion), temperature gradient(thermal diffusion), etc. In our physical world, when a system contains more components with various concentration from point to point, a natural tendency for mass to be transferred occurred in order to minimize any concentration difference within the system. This transfer phenomenon is governed by Fick's First Law: $\mathbf{F}(x) = -\mathbf{D}(x)\nabla C(x)$, which means, diffusion flux $\mathbf{F}$ from higher concentration to lower concentration is proportional to the gradient of the concentration of the substance $\nabla C(x)$ and the diffusivity of the substance in the medium $\mathbf{D}(x) = \mathrm{diag}(d_1(x), \cdots, d_n(x))$. Mass transfer is widely used for different processes and mechanisms. Some common examples include the diffusive and convective transport of chemical species, purification of blood in the kidneys and livers, separation of chemical components in distillation columns, controlling haze in the atmosphere by artificial precipitation, etc.

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The research of mass transfer dates back to the classic paper of Monge(1781), "Mémoire sur la théorie des déblais et des remblais", which is related to the most economical way of moving soil from one area to the other. The theory received a boost in the 1940's when Kantorovich generalized the transfer problem to the Kantorovich dual problem, and showed how to deal with it using his newly developed method of linear programming[11][12]. The basic Monge-Kantorovich problem is to find a mapping $\mathbf{s}$ to minimize the cost functional

$$C[\mathbf{r}] := \int_{\mathbb{R}^n} |x - \mathbf{r}(x)| d\mu^+(x)$$

among the 1-1 mappings $\mathbf{r} : \mathbb{R}^n \to \mathbb{R}^n$ that push forward μ^+ into μ^-, where μ^+ and μ^- are two nonnegative Radon measures on $\mathbb{R}^n$. During the past few decades, Monge-Kantorovich problem has been the subject of active inquiry, since it covers the domains of optimization theory, probability theory, partial differential equations, allocation mechanism in economics and membrane filtration in biology, etc.

Let $U = \mathbb{B}(0, R)$ denote the open ball in $\mathbb{R}^n$ with center 0 and large radius R, and

$$\mathcal{A} := \left\{ u \in W_0^{1,\infty}(U) \Big| |Du| \leq 1 \right\},$$

where $W_0^{1,\infty}(U)$ is a Sobolev space[1]. We are interested in the singular mass transfer problem

$$(\mathcal{P}^{(k)}) : \min_{u_k \in \mathcal{A}} \left\{ I^{(k)}(u_k) := \int_U \left(W^{(k)}(Du_k) - u_k f \right) dx \right\}, \tag{1}$$

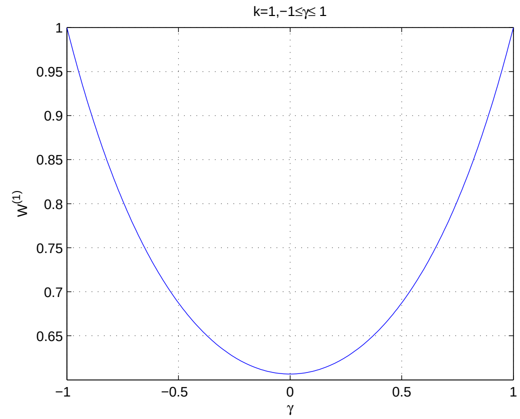
where u_k is the potential function, $f \in L^1(U)$. Furthermore, $f = f^+ - f^-$ satisfies the normalized balance condition

$$\int_U f^+ dx = \int_U f^- dx = 1.$$

And the exponential form $W^{(k)}$ is defined as

$$W^{(k)}(\gamma) := \frac{1}{k} \mathrm{e}^{\frac{k}{2}(|\gamma|^2 - 1)}.$$

In this paper, we investigate the analytic solutions for the non-convex non-smooth problem (1) through canonical duality approach. During the last few years, considerable effort has been taken to illustrate these problems from the theoretical point of view, focusing mainly on finding minimizers for a non-convex strain energy functional with a double-well potential[10][13]. Through applying canonical duality theory, the authors characterized the local energy extrema and the global energy minimizer for both hard device and soft device. At the same time, numerical experiments illustrated the important fact that smooth analytic solutions of a nonlinear mixed boundary value problem might not be minimizers of the associated potential variational problem. At the moment, there is lots of literature devoted to research of the 1D double-well potential problem, and readers can refer to [3][7][8][9] for more details. It is evident that when $|\gamma| \in [0, 1]$, then $\lim_{k \to \infty} W^{(k)}(\gamma) = 0$. While if $|\gamma| > 1$, then $\lim_{k \to \infty} W^{(k)}(\gamma) = +\infty$. In addition, the local extrema of W for both 1D and 2D are clearly displayed in Figure 1-2. Due to nonlinearity in mass transfer studies, identification of global minimizers



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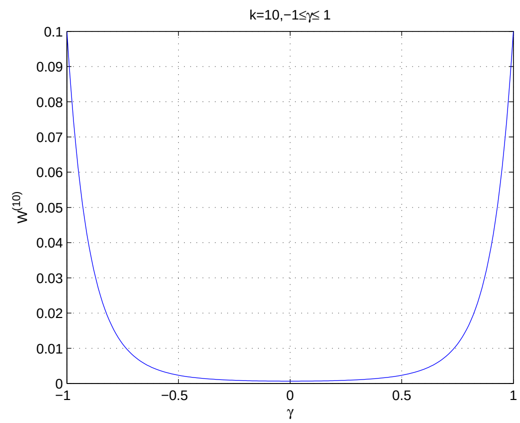

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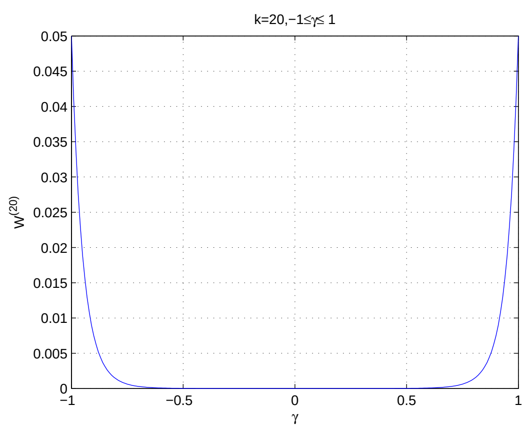

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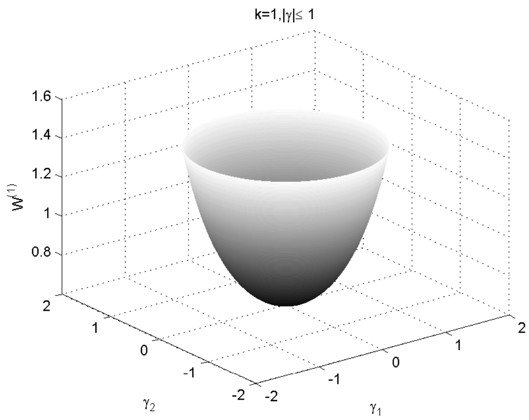
FIGURE 1. $W^{(k)}(\gamma)$ with respect to different k in 1-D



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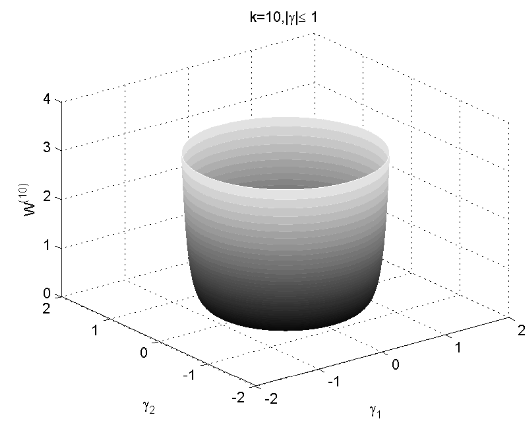

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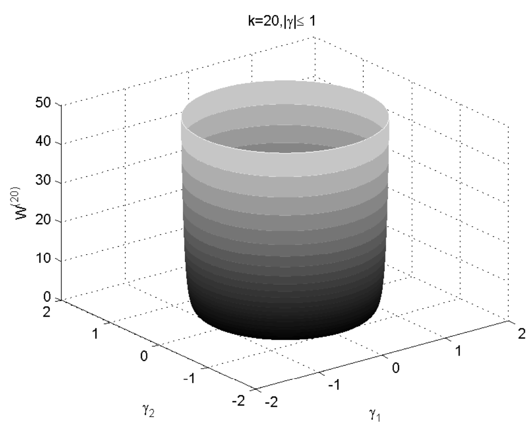

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FIGURE 2. $W^{(k)}(\gamma)$ with respect to different k in 2-D

of the variational problem $(\mathcal{P}^{(k)})$ is fundamentally difficult through traditional direct approaches and relaxation method[4][5]. The purpose of this paper is to apply the newly developed theory to the non-convex variational problem $(\mathcal{P}^{(k)})$ in the limit case $k \to \infty$.

The rest of the paper is organized as follows. In Section 2, we apply the canonical dual transformation to establish perfect dual problems and a pure complementary energy principle for $(\mathcal{P}^{(k)})$. The triality theory provides global and local extremality conditions for the non-convex problem. And complete solution sets for $(\mathcal{P}^{(k)})$ are given and the existence of smooth solutions of the corresponding Dirichlet problems is also discussed. Section 3 uses several numerical simulations in 1D and 2D domains to illustrate the theoretical results. A few remarks will conclude this paper.

2. Technique of Canonical Dual Transformation and Main Results

The corresponding Gâteaux derivative θ of $W^{(k)}$ with respect to γ is given by(see Figure 3-4)

$$\theta = (\theta_1, \cdots, \theta_n) = (\mathrm{e}^{\frac{k}{2}(|\gamma|^2-1)}\gamma_1, \cdots, \mathrm{e}^{\frac{k}{2}(|\gamma|^2-1)}\gamma_n).$$

Lemma 2.1. *By applying the variational method, we obtain the Euler-Lagrange equation for $(\mathcal{P}^{(k)})$,*

$$\begin{cases} \mathrm{div}\ \theta + f = 0 \quad x \in U, \\ \theta(0) = 0. \end{cases} \tag{2}$$

Proof. Indeed, for $\forall \phi \in \mathcal{A}, \forall \mu > 0$,

$$\left\langle D_u I^{(k)}(u_k), \phi \right\rangle_{L^1, L^\infty} = \lim_{\mu \to 0^+} \frac{I^{(k)}(u_k + \mu\phi) - I^{(k)}(u_k)}{\mu}$$

$$= \lim_{\mu \to 0^+} \int_U \frac{\frac{1}{k}\mathrm{e}^{\frac{k}{2}(|D(u_k+\mu\phi)|^2-1)} - \frac{1}{k}\mathrm{e}^{\frac{k}{2}(|Du_k|^2-1)}}{\mu} - \phi f dx$$

$$= \int_U \frac{1}{k}\mathrm{e}^{\frac{k}{2}(|Du_k|^2-1)} \lim_{\mu \to 0^+} \frac{\mathrm{e}^{\frac{k}{2}(\mu^2|D\phi|^2+2\mu Du_k\cdot D\phi)} - 1}{\frac{k}{2}(\mu^2|D\phi|^2 + 2\mu Du_k \cdot D\phi)} \cdot \frac{\frac{k}{2}(\mu^2|D\phi|^2 + 2\mu Du_k \cdot D\phi)}{\mu} - \phi f dx$$

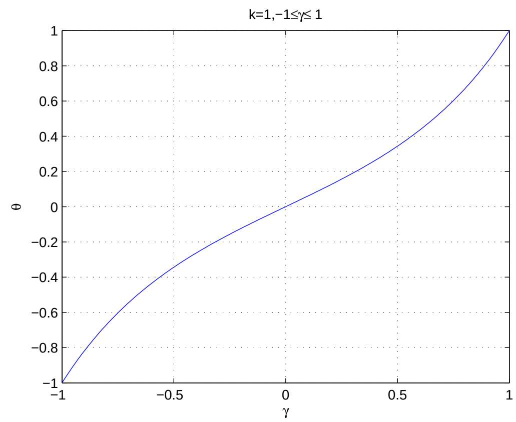
$$= \int_U \frac{1}{k}\mathrm{e}^{\frac{k}{2}(|Du_k|^2-1)} \cdot kDu_k \cdot D\phi - \phi f dx$$

$$= \int_U -\mathrm{div}(\mathrm{e}^{\frac{k}{2}(|Du_k|^2-1)} Du_k)\phi - \phi f dx.$$

□

In the following, we define a nonlinear geometric mapping

$$\Lambda^{(k)}(u_k) := \frac{k}{2}(|Du_k|^2 - 1).$$



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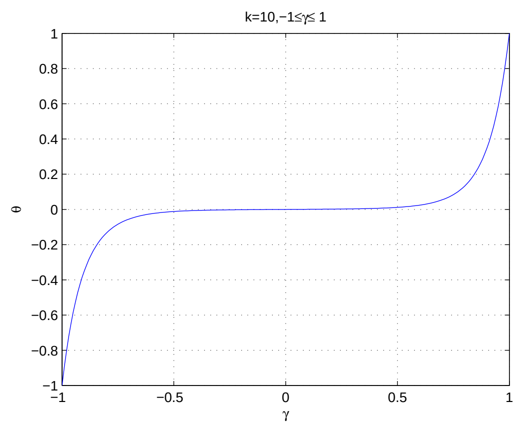

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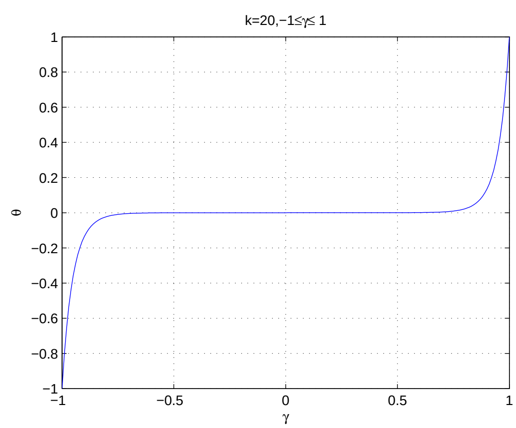

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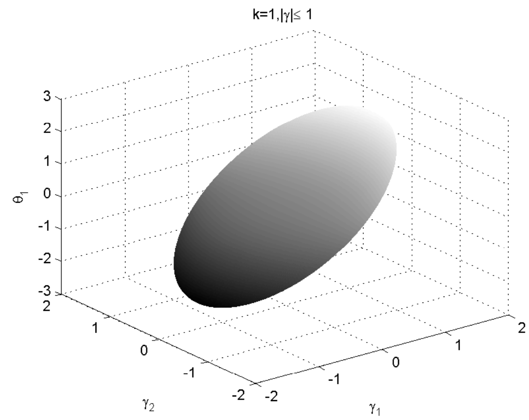
FIGURE 3. θ with respect to different k in 1-D



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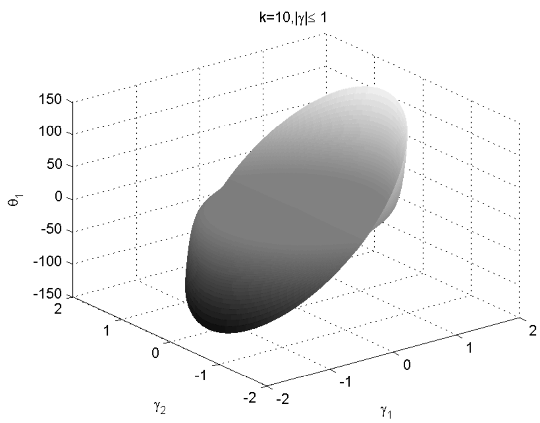

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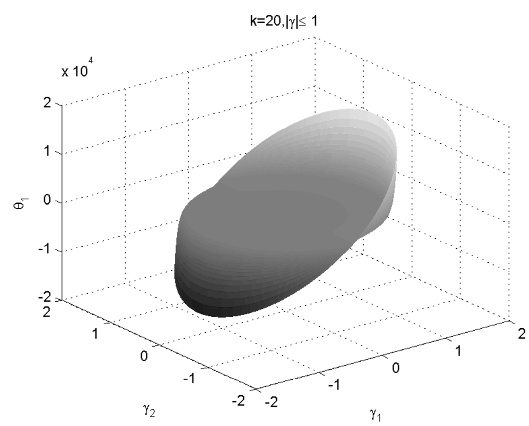

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FIGURE 4. θ_1 with respect to different k in 2-D

Accordingly, we define ξ_k by $\xi_k := \Lambda^{(k)}(u_k)$. It is clear that

$$\xi_k \in \Big\{\phi \in L^\infty(U)\Big|\phi \le 0\Big\}.$$

Next we define the canonical energy by

$$U^{(k)}(\xi_k) := \frac{1}{k}\mathrm{e}^{\xi_k},$$

which is a convex function with respect to ξ_k. The Gâteaux derivative of $U^{(k)}$ with respect to ξ_k,

$$\zeta_k := DU^{(k)}(\xi_k) = \frac{1}{k}\mathrm{e}^{\xi_k}$$

is well defined, invertible and belongs to the domain

$$\mathcal{E}^{(k)} := \Big\{\phi \in L^\infty(U)\Big|0 \le \phi \le \frac{1}{k}\Big\}.$$

With the above notations, we define the complementary energy function $U_*^{(k)}$ by the Legendre transformation

$$U_*^{(k)}(\zeta_k) := \xi_k\zeta_k - U^{(k)}(\xi_k) = \zeta_k(\ln(k\zeta_k) - 1).$$

Replacing $U^{(k)}(\Lambda^{(k)}(u_k))$ in $(\mathcal{P})$ by $\Lambda^{(k)}(u_k)\zeta_k - U_*^{(k)}(\zeta_k)$, we obtain the Gao-Strang total complementary energy $\Xi^{(k)}(u_k, \zeta_k)$ in the form

$$\Xi^{(k)}(u_k,\zeta_k) := \int_U \{\Lambda^{(k)}(u_k)\zeta_k - U_*^{(k)}(\zeta_k) - fw\}dx. \tag{3}$$

For our purpose, we introduce the following criticality condition.

Definition 2.2. $(\bar{u}, \bar{\zeta}) \in \mathcal{A} \times \mathcal{E}^{(k)}$ *is said to be a critical point of* $\Xi^{(k)}(u,\zeta)$ *if and only if*

$$D_u\Xi^{(k)}(\bar{u},\bar{\zeta}) = 0, \tag{4}$$

$$D_\zeta\Xi^{(k)}(\bar{u},\bar{\zeta}) = 0, \tag{5}$$

where D_u, D_ζ *denote the partial Gâteaux derivatives on* $\mathcal{A}$ *and* $\mathcal{E}$, *respectively.*

Lemma 2.3. *For a fixed* ζ_k, (4) *leads to the equilibrium equation*

$$\mathrm{div}(k\zeta_k D\bar{u}_k) + f = 0 \ \textit{in}\ U. \tag{6}$$

Proof. Indeed, for $\forall\phi \in \mathcal{A}$, $\forall\mu > 0$,

$$\begin{aligned}
\left\langle D_u \Xi^{(k)}(u_k, \zeta_k), \phi \right\rangle_{L^1, L^\infty} &= \lim_{\mu\to 0^+} \frac{\Xi^{(k)}(u_k + \mu\phi, \zeta_k) - \Xi^{(k)}(u_k, \zeta_k)}{\mu} \\
&= \lim_{\mu\to 0^+} \int_U \frac{\Lambda^{(k)}(u_k + \mu\phi) - \Lambda^{(k)}(u_k)}{\mu} \zeta_k dx - \int_U f\phi dx \\
&= \lim_{\mu\to 0^+} \int_U \frac{k(|Du_k + \mu D\phi)|^2 - |Du_k|^2)}{2\mu} \zeta_k dx - \int_U f\phi dx \\
&= \int_U kDu_k \cdot D\phi\zeta_k dx - \int_U f\phi dx \\
&= \int_U -\text{div}(k\zeta_k Du_k)\phi dx - \int_U f\phi dx.
\end{aligned}$$

□

Lemma 2.4. *While for a fixed $u_k \in \mathcal{A}$, (5) is consistent with the constitutive law*

$$\Lambda^{(k)}(u_k) = DU_*^{(k)}(\bar{\zeta}_k). \tag{7}$$

Proof. Indeed, for $\forall\psi \in \mathcal{A}$, $\forall\mu > 0$,

$$\begin{aligned}
\left\langle D_\zeta \Xi^{(k)}(u_k, \zeta_k), \psi \right\rangle_{L^1, L^\infty} &= \lim_{\mu\to 0^+} \frac{\Xi^{(k)}(u_k, \zeta_k + \mu\psi) - \Xi^{(k)}(u, \zeta_k)}{\mu} \\
&= \lim_{\mu\to 0^+} \int_U \left(\Lambda^{(k)}(u_k)\psi - \frac{U_*^{(k)}(\zeta_k + \mu\psi) - U_*^{(k)}(\zeta_k)}{\mu}\right) dx \\
&= \int_U \left(\Lambda^{(k)}(u_k) - DU_*^{(k)}(\zeta_k)\right)\psi dx.
\end{aligned}$$

□

Consequently, we know that the critical point $(\bar{u}_k, \bar{\zeta}_k)$ solves (6). Next we consider the pure complementary energy functional

$$I_d^{(k)}(\zeta_k) := \Xi^{(k)}(\bar{u}_k, \zeta_k), \tag{8}$$

where $\bar{u}_k$ is a solution of (6).

Lemma 2.5. *Actually, the pure complementary energy functional can be rewritten as*

$$I_d^{(k)}(\zeta_k) = -\frac{1}{2} \int_U \left(\frac{|\theta|^2}{k\zeta_k} + k\zeta_k + 2\zeta_k(\ln(k\zeta_k) - 1) \right) dx. \tag{9}$$

Proof. By applying Green's formula, we have

$$
\begin{aligned}
\Xi^{(k)}(u_k,\zeta_k) &= \int_U \Big\{ \Big(\frac{k}{2}(|Du_k|^2-1\Big)\zeta_k - U_*^{(k)}(\zeta_k) - fu_k \Big\} dx \\
&= \int_U \Big\{ k\zeta_k|Du_k|^2 - fu_k \} dx \\
&\quad - \textstyle\int_U \Big\{ \frac{k}{2}(|Du_k|^2-1)\zeta_k + k\zeta_k + \zeta_k(\ln(k\zeta_k)-1)\Big\} dx \\
&= \underbrace{-\int_U \Big\{ \mathrm{div}(k\zeta_k Du_k) + f \Big\} u_k dx}_{(I)} \qquad (10)\\
&\quad \underbrace{-\frac{1}{2}\int_U \Big\{ k\zeta_k|Du_k|^2 + k\zeta_k + 2\zeta_k(\ln(k\zeta_k)-1)\Big\} dx.}_{(II)}
\end{aligned}
$$

Since $\bar{u}_k$ is a solution of (6), then the first part (I) disappears. Keeping in mind the definition of θ, we reach the conclusion immediately. □

Now we establish the dual variational problem

$$
(\mathcal{P}_d^{(k)}) : \max_{\zeta_k \in \mathcal{E}^{(k)}} \Big\{ I_d^{(k)}(\zeta_k) = -\frac{1}{2}\int_U \Big(\frac{|\theta|^2}{k\zeta_k} + k\zeta_k + 2\zeta_k(\ln(k\zeta_k)-1)\Big) dx \Big\}. \qquad (11)
$$

The variation of $I_d^{(k)}$ with respect to ζ_k leads to the dual algebraic equation (DAE), namely,

$$
|\theta|^2 = k{\zeta_k}^2(2\ln(k\zeta_k)+k). \qquad (12)
$$

Let $\lambda_k = k\zeta_k$, then the above (DAE) can be rewritten as(see Figure 5-7)

$$
|\theta|^2 = \lambda_k^2 \ln(\mathrm{e}\lambda_k^{\frac{2}{k}}). \qquad (13)
$$

In particular, when $k \to \infty$, $\lambda_k \to \lambda$, then (DAE) becomes

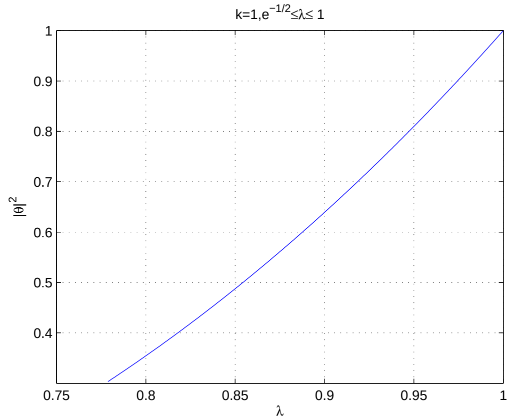
$$
|\theta|^2 = \lambda^2, \lambda > 0. \qquad (14)
$$

Remark 2.6. *From (13) and (14), we know that $|\theta|^2$ is monotonously increasing with respect to $\lambda_k > \mathrm{e}^{-\frac{k}{2}}$ and $\lambda > 0$. Consequently, there exists a unique positive root λ_i and λ for (12) and (13), respectively. Moreover, in the limit case, λ corresponds to the Monge-Kantorovich cost of optimally rearranging the probability measure $d\mu^+ = f^+dx$ to $d\mu^- = f^-dy$[4].*

Lemma 2.7. *Actually, when $k \geq 3$, we have the following approximation for (13),*

$$
|\theta|^2 = (1-\frac{2}{k})\lambda_k^2 + \frac{2}{k}\lambda_k^3 + R_k(\lambda_k),
$$

where $|R_k(\lambda_k)| \leq \frac{1}{k}$ for any $\lambda_k \in [\mathrm{e}^{-\frac{k}{2}}, 1]$.



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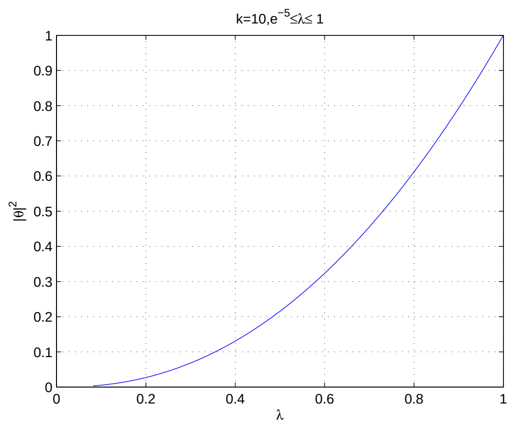

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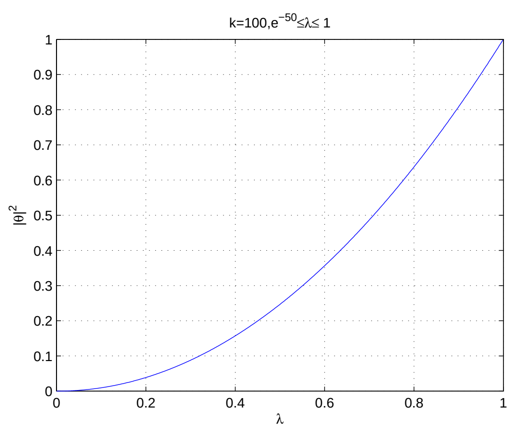

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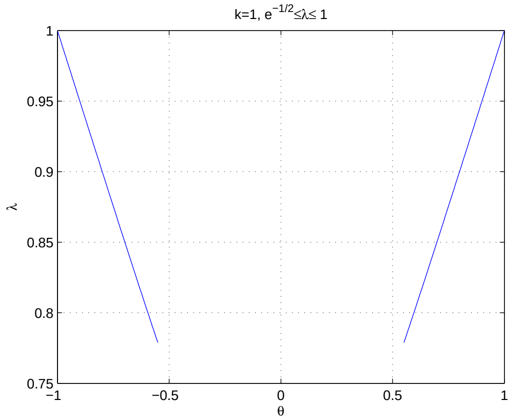
FIGURE 5. $|\theta|^2$ with respect to $\lambda > 0$ due to different k



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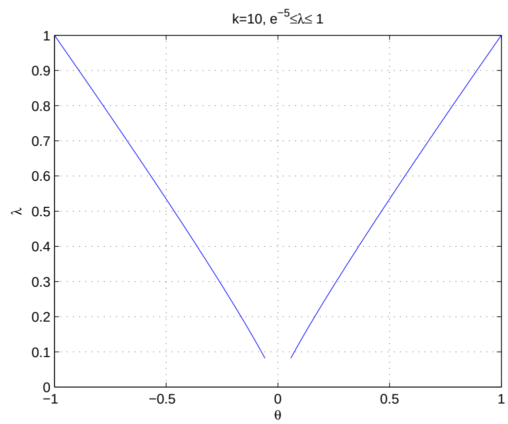

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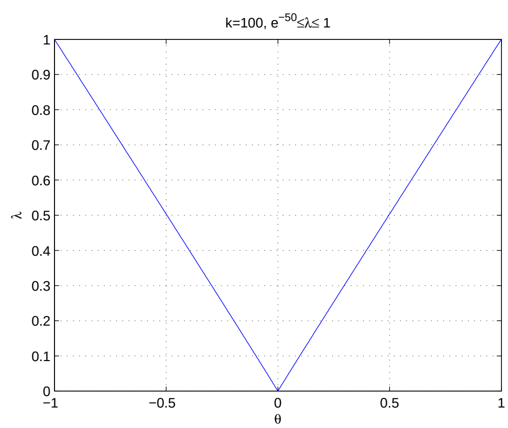

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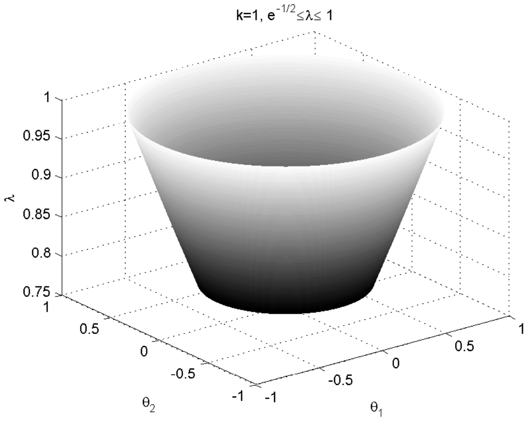
FIGURE 6. θ with respect to $\lambda > 0$ due to different k in 1D



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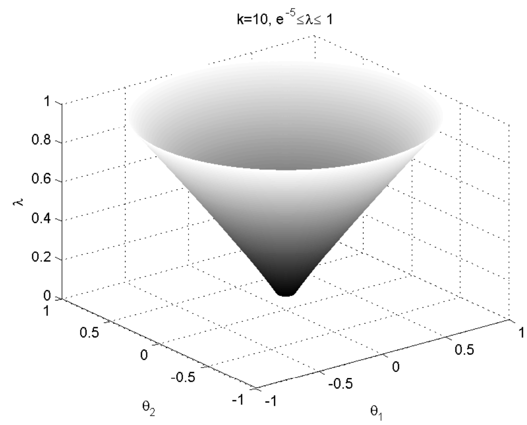

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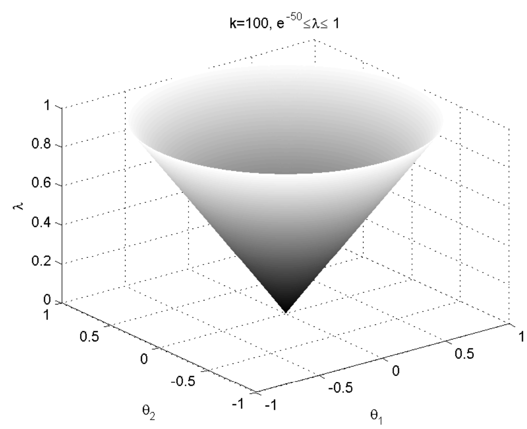

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FIGURE 7. θ with respect to $\lambda > 0$ due to different k in 2D

Proof. Since $\lambda_k \in [e^{-\frac{k}{2}}, 1]$, we can rewrite (13) by using Taylor's Formula for $\ln \lambda_k$ at 1,

$$\theta^2 = \lambda_k^2\Big(1 + \frac{2}{k}(\lambda_k - 1) - \frac{1}{k\eta^2}(\lambda_k - 1)^2\Big) = (1 - \frac{2}{k})\lambda_k^2 + \frac{2}{k}\lambda_k^3 - \frac{1}{k}\frac{\lambda_k^2}{\eta_k^2}(\lambda_k - 1)^2,$$

where $\eta_k \in (\lambda_k, 1)$. It is evident that

$$|\frac{1}{k}\frac{\lambda_k^2}{\eta_k^2}(\lambda_k - 1)^2| \leq |\frac{1}{k}\frac{\lambda_k^2}{\lambda_k^2}(\lambda_k - 1)^2| \leq \frac{1}{k}.$$

This concludes our proof. □

Remark 2.8. *On the basis of Lemma 2.7, we can control the error as necessary as we need for large k. So it is convenient to treat the limit case $k \to \infty$ in our numerical simulation.*

By comparing (2) with (6), we deduce that, for $i, j = 1, \dots, n$, in order to give an integral form of the solution u_k, the following compatibility condition has to be satisfied

$$\Phi_{\lambda_k}(\theta_i, \theta_j) \triangleq \begin{vmatrix} \partial_{x_i} & \partial_{x_j} \\ \theta_i\lambda_k^{-1} & \theta_j\lambda_k^{-1} \end{vmatrix} = 0. \tag{15}$$

Let us define the subregion $\mathcal{S}$ as

$$\mathcal{S} \triangleq \Big\{x \in U \mid \Phi_{\lambda_k}(\theta_i, \theta_j) = 0,\ i, j = 1, \dots, n\Big\}. \tag{16}$$

Evidently, the compatibility condition (15) guarantees the path independency of the integral for $\theta\lambda_k^{-1}$ in $\mathcal{S}$. As a result, the analytical solutions of BVP (2) can be given by the path integral in $\mathcal{S}$,

$$u(x) = \int_{x_0}^{x} \theta\lambda_k^{-1} d\mathbf{s} + u(x_0), \tag{17}$$

where $x,\ x_0 \in \mathcal{S}$. Summerizing the above discussion, we obtain the theorem below.

Theorem 2.9. *For a given $f(x)$ such that $\theta(x)$ is determined by BVP (2), then DAE (13) has a unique real root $\bar{\lambda}_k > 0$. And the function defined in $\mathcal{S}$ by*

$$\bar{u}_k(x) = \int_{x_0}^{x} \theta(\mathbf{s})\bar{\lambda}_k^{-1}(\mathbf{s}) d\mathbf{s} + u(x_0) \tag{18}$$

is the solution of BVP (2). Furthermore,

$$I^{(k)}(\bar{u}_k) = I_d^{(k)}(\bar{\zeta}_k). \tag{19}$$

Proof. Identity (19) is obtained by direct calculation of $I^{(k)}(u)$ and $I_d^{(k)}(\zeta)$ in (1) and (9),

$$I^{(k)}(\bar{u}_k) = \Xi^{(k)}(\bar{u}_k, \bar{\zeta}_k) = I_d^{(k)}(\bar{\zeta}_k).$$

□

Remark 2.10. *Theorem 2.9 demonstrates that the pure complementary energy functional $I_d^{(k)}(\zeta_k)$ is canonically dual to the total potential energy functional $I^{(k)}(u_k)$. In effect, the identity (19) indicates there is no duality gap between the primal problem $(\mathcal{P}^{(k)})$ and the dual problem $(\mathcal{P}_d^{(k)})$.*

In the following, we apply the triality theory to obtain the extremality conditions for these critical points.

Theorem 2.11. *Suppose that f is given and $\theta(x)$ satisfies the divergence equation (2). Then, for $|\theta(x)|^2 > 0$, $\forall x \in \mathcal{S}$, then DAE (12) has a unique positive root $\bar{\zeta}_k(x) > 0$, which is a global maximizer of $I_d^{(k)}$ over $\mathcal{E}^{(k)}$, and the corresponding solution $\bar{u}_k(x)$ in the form of (18) is a global minimizer of $I^{(k)}$ over $\mathcal{A}$,*

$$I^{(k)}(\bar{u}_k) = \min_{u_k \in \mathcal{A}} I^{(k)}(u_k) = \max_{\zeta_k \in \mathcal{E}^{(k)}} I_d^{(k)}(\zeta_k) = I_d^{(k)}(\bar{\zeta}_k). \tag{20}$$

Proof. First, we recall the second variation formula for both $I^{(k)}(u)$ and $I_d^{(k)}(\zeta)$. On the one hand, for $\forall \varsigma \in \mathcal{A}_s := \Big\{u \in \mathcal{A} \Big| Du \neq 0\Big\}$,

$$\begin{aligned} \delta^2 I^{(k)}(u_k)\varsigma &= \int_U \tfrac{d^2}{dt^2}\Big\{W(D(u_k + t\varsigma))\Big\}\Big|_{t=0} dx \\ &= \int_U \mathrm{e}^{\frac{k}{2}(|Du_k|^2-1)}\Big(k(Du_k \cdot D\varsigma)^2 + |D\varsigma|^2\Big)dx. \end{aligned} \tag{21}$$

On the other hand, for $\forall \eta \in \mathcal{E}_s^{(k)} := \Big\{\zeta \in \mathcal{E}^{(k)} \Big| \zeta \neq 0\Big\}$,

$$\begin{aligned} \delta^2 I_d^{(k)}(\zeta_k)\eta &= -\tfrac{1}{2}\int_U \tfrac{d^2}{dt^2}\Big\{\tfrac{|\theta|^2}{k(\zeta_k + t\eta)} + 2(\zeta_k + t\eta)\Big(\ln(k(\zeta_k + t\eta)) - 1\Big)\Big\}\Big|_{t=0} dx \\ &= -\int_U \Big\{\tfrac{|\theta|^2\eta^2}{k\zeta_k^3} + \tfrac{\eta^2}{\zeta_k}\Big\}dx. \end{aligned} \tag{22}$$

From (21) and (22), one knows immediately that

$$\delta^2 I^{(k)}(u_k)\varsigma > 0, \quad \delta^2 I_d^{(k)}(\zeta_k)\eta < 0.$$

Then (20) is concluded. ☐

3. Numerical simulations for typical examples in 1D and 2D

3.1. **Optimal transfer problem in 1D.** Let

$$f = \frac{1}{4}\cos x, \ \ x \in U := (-2\pi, 2\pi).$$

It is easy to check that

$$\int_{-2\pi}^{2\pi} f^+ dx = \int_{-2\pi}^{2\pi} f^- dx = 1.$$

In this case, the primal variational problem is as follows,

$$(\mathcal{P}^{(k)}) : \min_{u_k \in \mathcal{A}} \Big\{I^{(k)}(u_k) = \int_{-2\pi}^{2\pi} \Big(W^{(k)}(Du_k) - \frac{1}{4}u_k \cos x\Big)dx\Big\}.$$

For (2), there exists a unique solution in the form of

$$\theta(x) = -\frac{1}{4}\sin x.$$

Actually, for large k, we use the approximation in Lemma 2.7,

$$\theta^2 \approx (1-\frac{2}{k})\lambda_k^2 + \frac{2}{k}\lambda_k^3, \quad \lambda_k > 0. \tag{23}$$

We mainly consider $k = 10$, $k = 1000$. Actually, from (23), we have

$$\lambda_{10} \approx -\frac{4}{3} + \frac{8\cdot\sqrt[3]{2}(1+i\sqrt{3})}{3\omega_{10}(\theta^2)} + \frac{(1-i\sqrt{3})\omega_{10}(\theta^2)}{6\cdot\sqrt[3]{2}},$$

$$\lambda_{1000} \approx -\frac{499}{3} + \frac{249001(1+i\sqrt{3})}{6\omega_{1000}(\theta^2)} + \frac{(1-i\sqrt{3})\omega_{1000}(\theta^2)}{6},$$

where

$$\omega_{10}(\theta^2) := \sqrt[3]{128 - 135\theta^2 + 3\sqrt{15}\sqrt{-256\theta^2 + 135\theta^4}},$$

$$\omega_{1000}(\theta^2) := \sqrt[3]{124251499 - 6750\theta^2 + 30\sqrt{15}\sqrt{-124251499\theta^2 + 3375\theta^4}}.$$

The figures for λ_k, $\frac{\theta}{\lambda_k}$, u_k, $k = 10, 1000$ are presented in Figure 8-9. It is worth noticing that for u_k, the figures between $(-\pi, \pi)$ can be translated upwards or downwards due to different values of $u_k(0)$. Now we consider the limit case $k \to \infty$. Identity (14) indicates that $\lambda = |\theta|$. In this case,

$$\frac{\theta(x)}{\lambda(x)} = \begin{cases} -1, & x \in (-2\pi, -\pi), \\ 1, & x \in (-\pi, 0), \\ -1, & x \in (0, \pi), \\ 1, & x \in (\pi, 2\pi). \end{cases}$$

If we let $u(-2\pi) = u(2\pi) = 0$, then the global minimizer is represented as

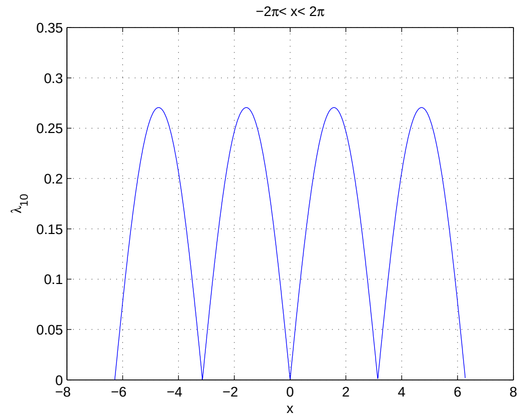
$$u(x) = \begin{cases} \int_{-2\pi}^{x} \frac{\theta(t)}{\lambda(t)} dt = -\int_{-2\pi}^{x} dt = -x - 2\pi, & x \in (-2\pi, -\pi), \\ \int_{0}^{x} \frac{\theta(t)}{\lambda(t)} dt + u(0) = \int_{0}^{x} dt + u(0) = x + u(0), & x \in (-\pi, 0), \\ \int_{0}^{x} \frac{\theta(t)}{\lambda(t)} dt + u(0) = -\int_{0}^{x} dt + u(0) = -x + u(0), & x \in (0, \pi), \\ \int_{2\pi}^{x} \frac{\theta(t)}{\lambda(t)} dt = \int_{2\pi}^{x} dt = x - 2\pi, & x \in (\pi, 2\pi). \end{cases}$$

Similar results can be deduced for a variety of functions f, such as the piecewise continuous impulse function

$$f = \begin{cases} -1, & x \in (-1, 0), \\ 1, & x \in (0, 1). \end{cases}$$

In this case,

$$\theta = \begin{cases} x, & x \in (-1, 0), \\ -x, & x \in (0, 1). \end{cases}$$



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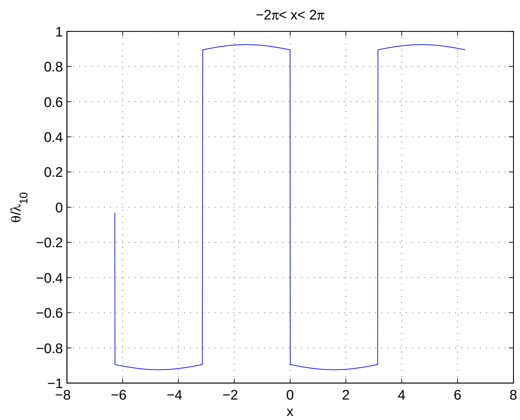

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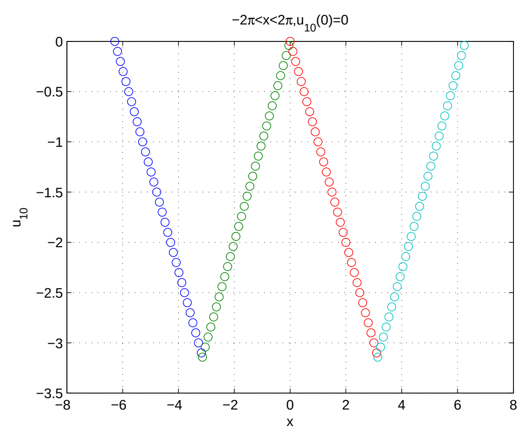

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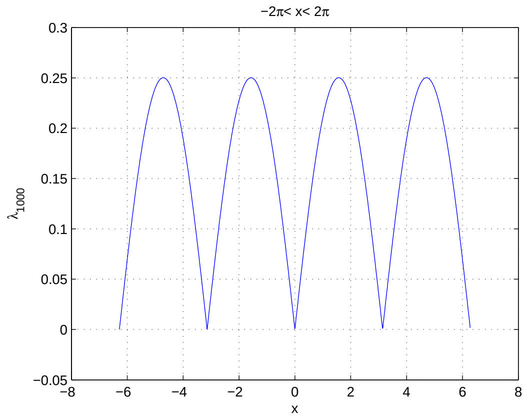
FIGURE 8. $k = 10$



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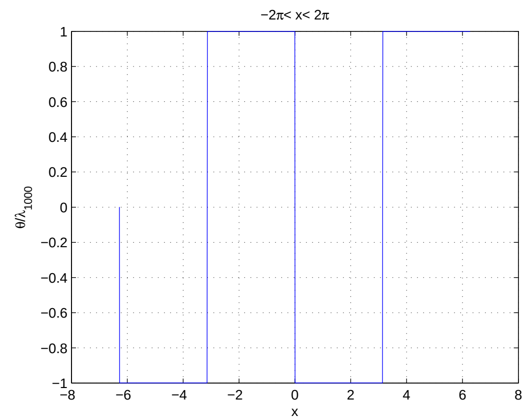

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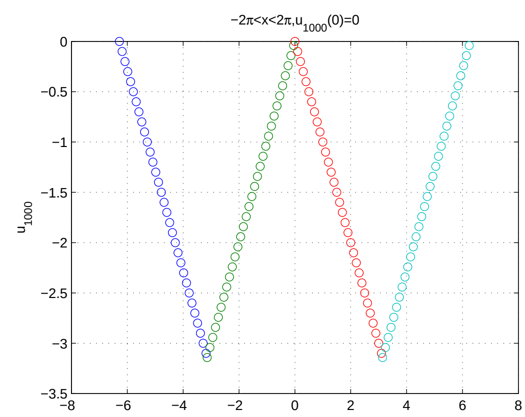

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FIGURE 9. $k = 1000$

And the corresponding λ, $\frac{\theta}{\lambda}$ and u in the limit case $k \to \infty$ are presented in Figure 10. The global minimizers u_k for $(\mathcal{P}^{(k)})$ can always be approximated by piecewise linear functions.

3.2. **Optimal transfer problem in 2D.** Let f be a piecewise function defined in $\mathbb{B}(0, \sqrt{\frac{2}{\pi}})$, $(r := \sqrt{x^2+y^2} \in (0, \sqrt{\frac{2}{\pi}}))$

$$f(r) = \begin{cases} -1, & r \in (0, \sqrt{\frac{1}{\pi}}), \\ 1, & r \in (\sqrt{\frac{1}{\pi}}, \sqrt{\frac{2}{\pi}}). \end{cases}$$

It is easy to check that

$$\int_U f^+ dxdy = \int_U f^- dxdy = 1.$$

In this case, there exists a solution θ for (2),

$$\theta(x,y) = \begin{cases} \frac{1}{2}(x,y), & r \in (0, \sqrt{\frac{1}{\pi}}), \\ -\frac{1}{2}(x,y), & r \in (\sqrt{\frac{1}{\pi}}, \sqrt{\frac{2}{\pi}}). \end{cases}$$

Actually, for large k, we use the approximation in Lemma 2.7,

$$|\theta|^2 \approx (1 - \frac{2}{k})\lambda_k^2 + \frac{2}{k}\lambda_k^3, \quad \lambda_k > 0. \tag{24}$$

We mainly consider $k = 10$, $k = 1000$. Actually, from (24), we have

$$\lambda_{10} \approx -\frac{4}{3} + \frac{8 \cdot \sqrt[3]{2}(1+i\sqrt{3})}{3\omega_{10}(|\theta|^2)} + \frac{(1-i\sqrt{3})\omega_{10}(|\theta|^2)}{6 \cdot \sqrt[3]{2}},$$

$$\lambda_{1000} \approx -\frac{499}{3} + \frac{249001(1+i\sqrt{3})}{6\omega_{1000}(|\theta|^2)} + \frac{(1-i\sqrt{3})\omega_{1000}(|\theta|^2)}{6},$$

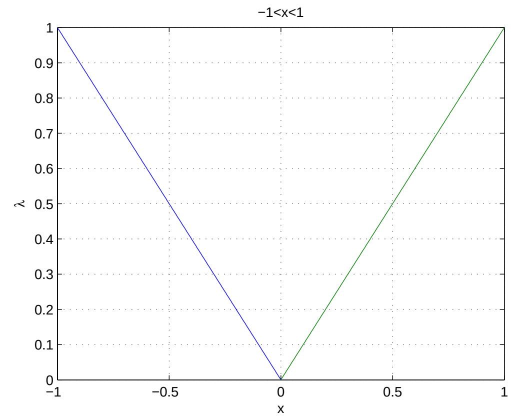
where

$$\omega_{10}(|\theta|^2) := \sqrt[3]{128 - 135|\theta|^2 + 3\sqrt{15}\sqrt{-256|\theta|^2 + 135|\theta|^4}},$$

$$\omega_{1000}(|\theta|^2) := \sqrt[3]{124251499 - 6750|\theta|^2 + 30\sqrt{15}\sqrt{-124251499|\theta|^2 + 3375|\theta|^4}}.$$

It is easy to check the compatibility condition (15) holds in $\mathbb{B}(0, \sqrt{\frac{2}{\pi}})$. The figures for λ_k, $\frac{\theta_1}{\lambda_k}$, u_k, $k = 10, 1000$ are presented in Figure 11-14. It is worth noticing that for u_k, the figures between $r \in (0, \sqrt{\frac{1}{\pi}})$ can be translated upwards or downwards due to different values of $u_k(0)$. Now we consider the limit case $k \to \infty$. Identity (14) indicates that $\lambda = \frac{r}{2}$. In this case,

$$\frac{\theta_1(x,y)}{\lambda(x,y)} = \begin{cases} \frac{x}{\sqrt{x^2+y^2}}, & x \in (0, \sqrt{\frac{1}{\pi}}), \\ -\frac{x}{\sqrt{x^2+y^2}}, & x \in (\sqrt{\frac{1}{\pi}}, \sqrt{\frac{2}{\pi}}). \end{cases}$$



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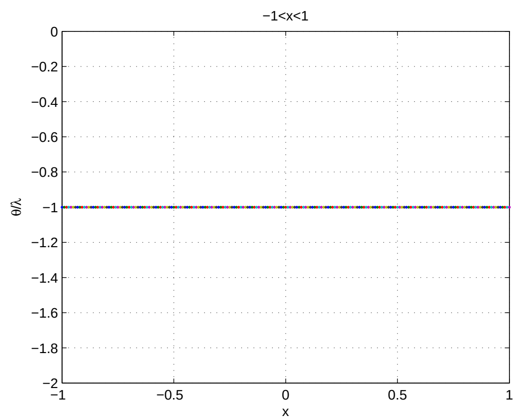

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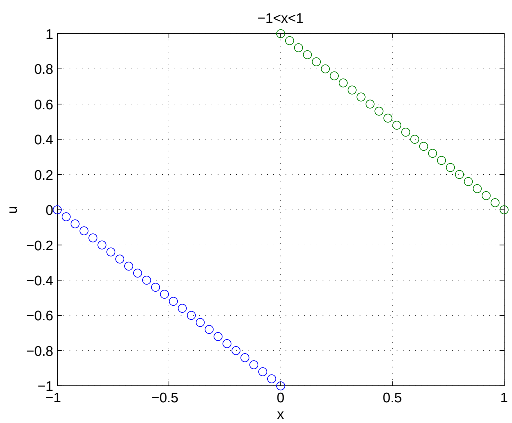

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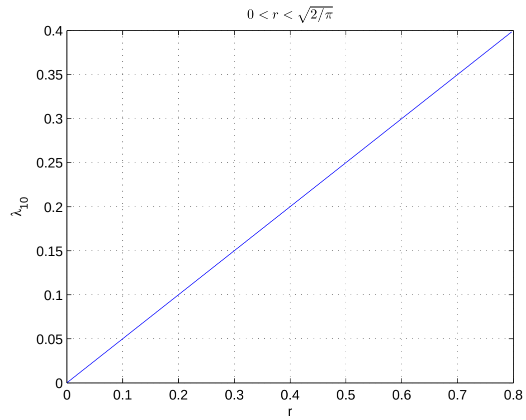
FIGURE 10. f is an impulse function



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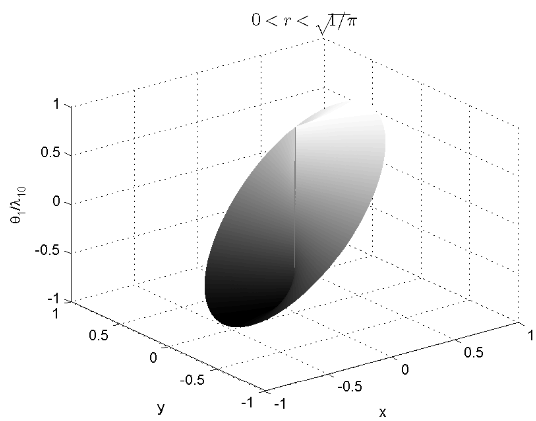

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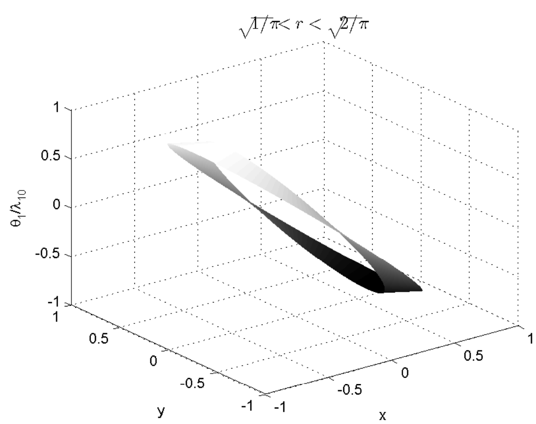

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FIGURE 11. $k = 10$

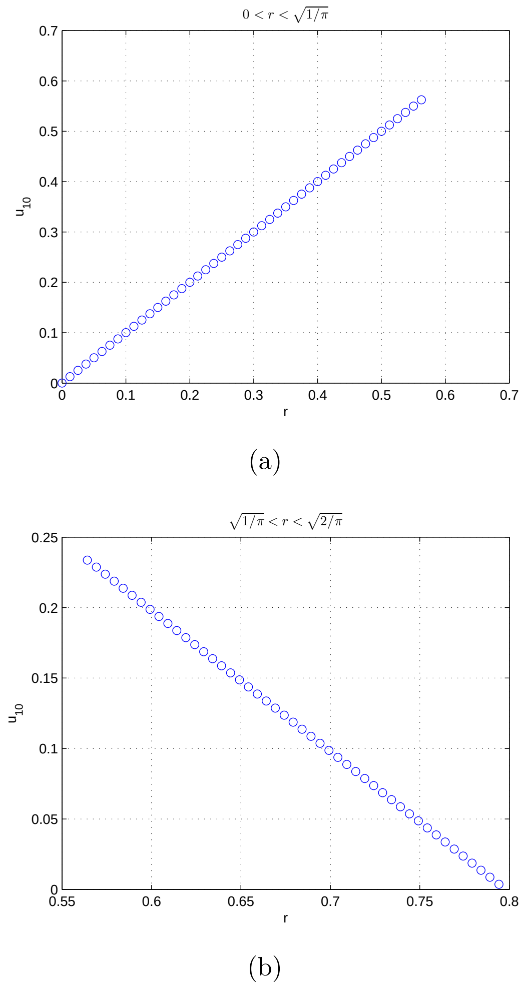


FIGURE 12. global minimizer u_{10},$u_{10}(0) = 0$

$$\frac{\theta_2(x,y)}{\lambda(x,y)} = \begin{cases} \frac{y}{\sqrt{x^2+y^2}}, & x \in (0, \sqrt{\frac{1}{\pi}}), \\ -\frac{y}{\sqrt{x^2+y^2}}, & x \in (\sqrt{\frac{1}{\pi}}, \sqrt{\frac{2}{\pi}}). \end{cases}$$

If we let $u(\sqrt{\frac{2}{\pi}}) = 0$, then the global minimizer is represented as

$$u(r) = \begin{cases} r + u(0), & r \in (0, \sqrt{\frac{1}{\pi}}), \\ \\ -r + \frac{2}{\pi}, & x \in (\sqrt{\frac{1}{\pi}}, \sqrt{\frac{2}{\pi}}). \end{cases}$$

Remark 3.1. *Compared with* [10], *the FDM method cannot be applied here since there is no uniqueness for the solutions of (2) in multi-dimensional cases(see*[2]*). It is exciting to see that, the canonical duality method helps us find the global minimizers for* $(\mathcal{P}^{(k)})$ *successfully.*

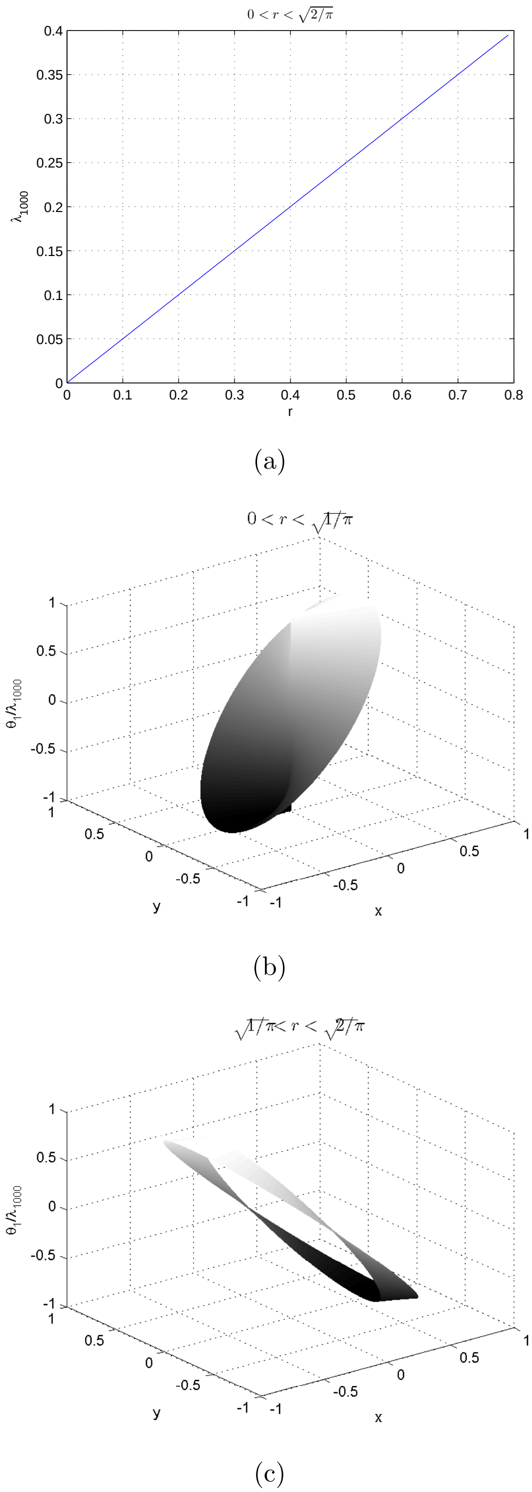
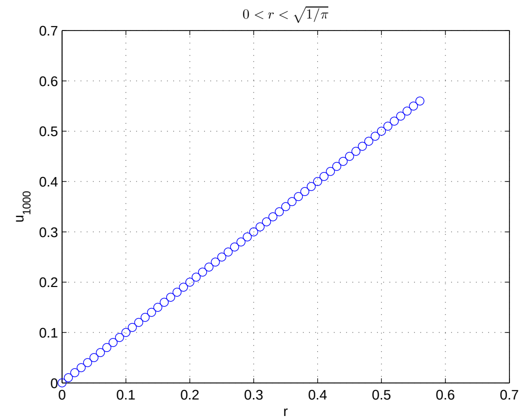


Figure 13. $k = 1000$



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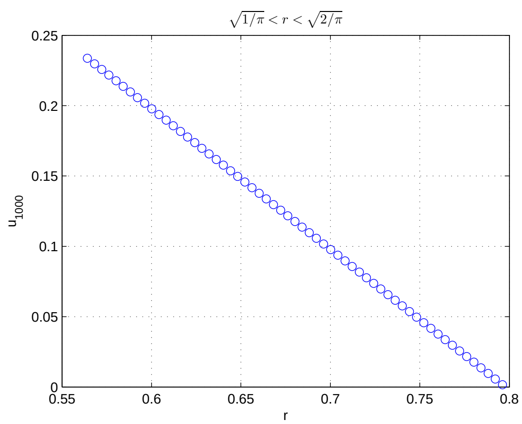

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FIGURE 14. global minimizer u_{1000},$u_{1000}(0)=0$

Remark 3.2. *For the general double potential case with* $W^{(k)}(\gamma):=\frac{1}{k}\mathrm{e}^{\frac{k}{2}(|\gamma|^2-\alpha)}$, $\alpha\geq 0$, *similar results can be deduced instantly. In this case, we can choose piecewise segment functions, in which the absolute value of segment slope is equal to* $\sqrt{\alpha}$, *as global minimizers. Further work is to be done concerned with more singular variational problems, such as optimal Lipschitz extensions, weak KAM theory, etc. (See* [6]*).*

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