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Abstract

This is the second of a series of three papers analyzing different types of rectangular waveguide discontinuities by using a fully automatic $hp$-adaptive finite element method.

In this paper, a fully automatic energy-norm based $hp$-adaptive Finite Element (FE) strategy applied to a number of relevant waveguide structures, is presented. The methodology produces exponential convergence rates in terms of the energy-norm error of the solution against the problem size (number of degrees of freedom).

Extensive numerical results demonstrate the suitability of the $hp$-method for solving different rectangular waveguide discontinuities. These results illustrate the flexibility, reliability, and high-accuracy of the method.

The self-adaptive $hp$-FEM provides similar (sometimes more) accurate results than those obtained with semi-analytical techniques such as the Mode Matching method, for problems where semi-analytical methods can be applied. At the same time, the $hp$-FEM provides the flexibility of modeling more complex waveguide
structures and including the effects of dielectrics, metallic screws, round corners, etc., which cannot be easily considered when using semi-analytical techniques.

**Key words:** Finite Element Method, $hp$-adaptivity, Energy-norm, Rectangular Waveguides, Waveguide Discontinuities, S-Parameters

1 Introduction

As it was mentioned in the first paper on this series [1], the accurate analysis and characterization of “waveguide discontinuities” is an important issue in microwave engineering (see e.g., [2], [3]). Waveguide discontinuities, i.e., the interruption in the translational symmetry of the waveguide, may be an unavoidable result of mechanical defects or electrical transition in waveguide systems, or they may be deliberately introduced in the waveguide to perform a certain electrical function. Specifically, discontinuities in rectangular waveguide technology are very common in the communication systems working in the upper microwave and millimeter wave frequency bands. In many cases, the rectangular waveguide discontinuities can be analyzed in two-dimensions (2D) because of the invariant nature of the geometry along one direction. This is the case of the so called H-plane and E-plane rectangular waveguide discontinuities, which are the target of this work. It is worth noting that a large number of structures and devices fit into this category.

In this paper, a fully automatic energy-norm based $hp$-adaptive Finite Element (FE) strategy [4,5], which has been extended for electromagnetic applications [6,7,8,9,10,11,12,5] is applied to a number of relevant waveguide structures. The adaptive methodology has a number of advantages that makes it suitable for the analysis of complex structures (containing several waveguide sections, discontinuities, complex geometries, dielectrics, etc.) in contrast to other semi-analytical and numerical techniques. Namely:

- It automatically resolves different types of singularities i.e., different types

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2 This work has been initiated during a stay of the first author at ICES supported by the Secretaría de Estado de Educación y Universidades of Ministerio de Educación, Cultura y Deporte of Spain. The authors wants also to acknowledge the support of Ministerio de Educación y Ciencia of Spain under project TEC2004-06252/TCM.
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of discontinuities.

- It efficiently deals with high frequencies, that is, it delivers a low dispersion error [13,14].
- It provides high-accuracy results, so the S-parameters (see comments below) [1] can be accurately computed.
- It allows for modeling of complex (non-uniform) geometries.

The waveguide theory and a detailed analysis of rectangular waveguide discontinuities (including the finite element variational formulations used) were presented in the first part of the series [1]. Extensive numerical results presented in this part illustrate the flexibility, reliability, and high-accuracy simulations obtained with this methodology, providing more accurate results than with semi-analytical (in particular, Mode-Matching —MM— [15], [16, Chapter 9]) techniques. The adaptive methodology is shown to produce exponential convergence rates in terms of the energy-norm error of the solution against the problem size (number of degrees of freedom). Thus, the electromagnetic field is accurately known (with a user pre-specified degree of accuracy) inside the structure. The high accuracy is essential in the microwave engineering design aiming at finding optimum location and size of tuning elements (e.g., screws, dielectric posts, etc.) as well as for an a posteriori analysis of such structures.

The presented numerical results include computation of the scattering parameters of the structure, which are widely used in microwave engineering for the characterization of microwave devices. The notion of the scattering parameters (or S-parameters), and their computation using a finite element solution, have been explained in the first paper of the series [1]. As the quantities of interest for the microwave engineer are mainly the S-parameters, a goal-oriented approach (in terms of the S-parameters) may be also desirable. In the third paper of the series [17], results obtained using hp energy-norm adaptivity are compared against those using a goal-oriented hp-adaptivity approach [18,19,20]. Results show that both methods are suitable for simulation of waveguide discontinuities.

The organization of the paper is as follows. The hp finite element discretization and automatic adaptivity strategy are briefly described in Section 2.1 and 2.2, respectively. The refinement strategy is based on the minimization of the projection based interpolation error, which is defined in Section 2.2.1. The steps of the mesh optimization algorithm are described in Section 2.2.2. Extensive numerical results, both for the E-plane and H-plane simulations, are shown in Section 3. Finally, some conclusions are given in Section 4.
2 $hp$ Finite Elements and Automatic Adaptivity

In order to solve the presented electromagnetic problems, a numerical technique that provides low discretization errors and, simultaneously, solves the discretized problem without prohibitive computational cost, is needed. In this context, an adaptive $hp$-Finite Element Method satisfies both properties.

2.1 $hp$-Finite Elements (FE)

Each finite element is characterized by its size $h$ and order of approximation $p$. In the $h$-adaptive version of FE method, element size $h$ may vary from element to element, while order of approximation $p$ is fixed (usually $p=1,2$). In the $p$-adaptive version of the FE method, $p$ may vary locally, while $h$ remains constant throughout the adaptive procedure. Finally, a true $hp$-adaptive version of FE method allows for varying both $h$ and $p$ locally.

The $hp$-FE method used in this paper utilizes edge (Nédélec) elements of variable order of approximation. FE spaces associated to those elements have been carefully constructed (see [21] for details) so in combination with the projection based interpolation operators (defined below), the commutativity of the de Rham diagram is guaranteed. This commutativity property is essential for showing convergence and stability of the FE method for Electromagnetics [21].

The main motivation for the use of $hp$-FEM is given by the following result: “an optimal sequence of $hp$-grids can achieve exponential convergence for elliptic problems with a piecewise analytic solution, whereas $h$- or $p$-FEM converge at best algebraically” (see [22,23,24,25,26,27,28]).

Next, the fully automatic $hp$-adaptive strategy is presented. Given a problem and a discretization tolerance error, the objective is to generate automatically (without any user interaction) an $hp$-grid that does not exceed the discretization error tolerance and, at the same time, it employs a minimum number of degrees of freedom (d.o.f.), by orchestrating an optimal distribution of element size $h$ and polynomial order of approximation $p$. By doing so, it is possible to achieve exponential convergence rates in terms of the error vs. the number of d.o.f.
2.2 Fully Automatic hp-Adaptivity

The self-adaptive strategy iterates along the following steps. First, a given (coarse) hp-mesh is globally refined both in $h$ and $p$ to yield a fine mesh, i.e., each element is broken into four element sons (eight in 3D), and the order of approximation is raised uniformly by one. Then, the problem of interest is solved on the fine mesh. The difference between the fine and coarse grid solutions is used to guide optimal refinements over the coarse grid. More precisely, the next optimal coarse mesh is then determined by minimizing the projection based interpolation error of the fine mesh solution with respect to the optimally refined coarse mesh (see [4,29] for details).

The adaptive strategy is very general, and it applies to $H^1$-, $\mathbf{H}(\text{curl})$-, and $\mathbf{H}(\text{div})$-conforming discretizations. Moreover, since the mesh optimization process is based on minimizing the interpolation error rather than the residual, the algorithm is problem independent and it can also be applied to nonlinear and eigenvalue problems.

The $hp$ self-adaptive strategy incorporates also a two-grid iterative solver, which allows to solve the fine grid problems efficiently. Indeed, it has been shown in [30,31] that it is sufficient a partially converged fine grid solution to guide optimal $hp$-refinements. Thus, only few two-grid solver iterations are needed (below ten per grid).

In the remainder of this section, the projection based interpolation operator [32,33], which is the main ingredient of the mesh optimization algorithm, is presented first. Then, the mesh optimization algorithm is briefly described.

2.2.1 The projection based interpolation operator

The idea of projection based interpolation operator is based on three properties.

- **Locality**: Determination of element interpolant of a function should involve the values (and derivatives) of the interpolated function in the element only.
- **Conformity**: The union of element interpolants should be globally conforming.
- **Optimality**: The interpolation error should behave asymptotically, both in $h$ and $p$, in the same way as the actual approximation error.

The $H^1$-conforming projection based interpolation operator is presented first. Let $u \in H^{1+\epsilon}(K)$ with $\epsilon > 0$. Locality and conformity imply that the inter-
polant $w = \Pi u$ should match the interpolated function $u$ at vertexes:

$$w|_{\text{vert}} = u|_{\text{vert}}$$  \hspace{1cm} (1)

With the vertex values fixed, we project over each edge, i.e.:

$$w := \arg \min_{v : (v-u)|_{\text{vert}}=0} \| v - u \|_{\text{edge}}$$  \hspace{1cm} (2)

This definition preserves locality and conformity. It also preserves optimality provided that the optimal edge norm is selected, which is dictated by the problem being solved and the Trace Theorem (see [21] for details). For example, in 1D, the optimal edge norm is the $H^1_0$-norm. In 2D, the $H^{1/2}$-seminorm, and in 3D, the $L^2$-norm, should be used.

Using the same argument, once vertex and edge values are fixed, projection over the interior of the element (faces in 3D) is performed. Thus, the projection based interpolation operator for 2D $H^1$-problems is formally defined as:

$$w(v) := u(v) \quad \text{for each vertex } v$$

$$|w - u|_{1/2,e} \rightarrow \min \quad \text{for each edge } e$$

$$|w - u|_{1,K} \rightarrow \min \quad \text{in the interior of element } K$$  \hspace{1cm} (3)

For a definition of projection based interpolation operator for 3D $H^1$-problems, see [34].

Similarly, a projection based interpolation operator can be defined for elements in $H(\text{curl})$, which is the space of interest for the electromagnetic field. Given $E^3$ in $H(\text{curl})$, the projection based interpolator $\Pi^{\text{curl}}$ specialized to the 2D case (for the 3D see [32,35]), is denoted by $E^p = \Pi^{\text{curl}}E$, where $E^p$ is given by:

$$\| E^p_t - E_t \|_{1/2,e} \rightarrow \min \quad \text{for each edge } e$$

$$\| \nabla \times E^p - \nabla \times E \|_{0,K} \rightarrow \min$$

$$(E^p - E, \nabla \phi)_{0,K} = 0, \quad \text{for every “bubble” function, in the interior of element } K$$  \hspace{1cm} (4)

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$^3$ $E$ is used here to abstractly denote an element in $H(\text{curl})$. In this paper, discretizations in $H(\text{curl})$ are used for the magnetic field on the H-plane and the electric field on the E-plane of the structures.
Here, the bubble functions come from an appropriate polynomial space mapped by the gradient operator onto the subspace of fields $E$ with zero curl and tangential trace on the element boundary.

A similar operator can be defined for $H(\text{div})$ problems (see [21]).

Finally, it is important to mention that the de Rham diagram equipped with these projection based interpolation operators commutes (see [21,12,32] for details), which is critical for proving stability and convergence properties of the FEM for Maxwell equations.

2.2.2 The mesh optimization algorithm

The mesh optimization algorithm in 2D follows the next steps.

• Step 0: Compute an estimate of the approximation error on the coarse grid.
  The approximation error on the coarse grid is estimated by simply computing the norm of the difference between the coarse and the fine grid solutions. If the difference (relative to the fine grid solution norm) is smaller than a requested error tolerance, then the fine mesh solution is delivered as the final solution, and the process stops.

• Step 1: For each edge in the coarse grid, compute the error decrease rate for the $p$ refinement, and all possible $h$-refinements.
  Let $E_{h/2,p+1}$ denote the fine grid solution. Then, the error decrease rate is computed as:
  \[
  \text{Error decrease } (\hat{h}p) = \frac{\|E_{h/2,p+1} - \Pi_{hp}^{\text{curl}}E\| - \|E_{h/2,p+1} - \Pi_{hp}^{\text{curl}}E\|}{(p_1 + p_2 - p)},
  \]
  where $\hat{h}p = (\hat{h}, \hat{p})$ is such that $\hat{h} \in \{h, h/2\}$. If $\hat{h} = h$, then $\hat{p} = p + 1$. If $\hat{h} = h/2$, then $\hat{p} = (p_1, p_2)$, where $p_1 + p_2 - p > 0$, $\max\{p_1, p_2\} \leq p + 1$.

• Step 2: For each edge in the coarse mesh, choose between $p$ and $h$ refinement, and determine the guaranteed edge error decrease rate.
  The optimal refinement is found by comparing the error decrease corresponding to the $p$-refinement with all competitive $h$-refinements. Competitive $h$-refinements are those that result in the same increase in the number of degrees-of-freedom (d.o.f.) as the $p$-refinement, i.e., $\hat{h} = h/2$ and $p_1 + p_2 - p = 1$.
  Next, the guaranteed rate with which the interpolation error must decrease over the edge is determined. That is, for each edge the maximum of the error decrease rates for the $p$-refined edge and all possible $h$-refined edges is computed.
• Step 3: Select edges to be refined.
  
  Given the guaranteed rate for each edge in the mesh, the maximum rate for all elements is calculated

\[ \text{guaranteed rate}_{\text{max}} = \max_e (\text{edge } e \text{ guaranteed rate}) . \]

All edges that produce a rate within 1/3 of the maximum guaranteed rate, are selected for a refinement. The factor 1/3 is somehow arbitrary.

• Step 4: Perform the requested \( h \)-refinements enforcing the 1-irregularity rule of the mesh.
  
  A loop through elements of the coarse grid is performed. If at least one edge of the element is to be broken, the element is refined accordingly. As in \([4,29]\), element isotropy flags are computed. Isotropic \( h \)-refinement are enforced if the error function within the element changes comparably in both element directions.

  After this step, the topology of the new coarse mesh has been determined, and it remains only to establish the optimal distribution of orders of approximation for the involuntarily \( h \)-refined edges, and for the interior nodes of the element i.e., those nodes that are not located on the boundary of the element. For the interior nodes, the starting point for the minimization procedure will be based on the order of approximation \( p \) for the adjacent edges\(^4\).

• Step 5: Determine the optimal orders of approximation \( p \) for the refined edges and elements.
  
  This step consists basically of \( p \)-adaptivity over a given grid with the fine grid as a reference solution. Unfortunately, due to the possible presence of involuntary edge \( h \)-refinements and too low \( p \) for \( h \)-refined elements, the interpolation error of the coarse grid after step 4 may actually be larger than the interpolation error for the original coarse mesh. Thus, extra technical details are considered in order to guarantee interpolation error decrease. These details are quite involving, and are described in \([12]\).

Some remarks on the mesh optimization algorithm follow:

• A similar but yet more involved mesh optimization algorithm has been implemented for 3D problems, although the 3D electromagnetic version is still under development.
• The main difference between the fully automatic \( hp \)-adaptive strategy for

\(^4\) For triangles, the initial order of approximation will be equal to the maximum of the three edges of the element. For quadrilaterals, we have a horizontal and a vertical order of approximation \( p = (p_h, p_v) \). In this case, the starting point for the minimization procedure will be the maximum of the two horizontal edges for \( p_h \) and the maximum of the two vertical edges for \( p_v \).
elliptic and electromagnetic problems resides in the definition of the projection based interpolation operator.

- A similar algorithm can be implemented for H(div) problems.

3 Numerical Results

In the following, a number of rectangular H-plane and E-plane waveguide discontinuities, as well as more complex structures obtained by combining several discontinuities, are analyzed. The analysis of all these structures is performed by using the fully automatic hp-adaptive FE strategy presented above.

TE\textsubscript{10} mode excitation has been used in all the structures. Also, the ratio of the broad dimension \(a\) to the narrow dimension \(b\) of the rectangular waveguide sections is considered to be \(a/b = 2\). The results correspond to a given frequency which is chosen to be in the middle of the monomode region, i.e., \(k = 1.5K_{c10}\). Exceptionally, the structure analyzed in Section 3.2.4 is solved for a large number of frequencies within a given frequency region, in order to characterize its frequency response. The lengths of the waveguide sections that connect the discontinuity to the ports of the structure is typically around one wavelength for the H-plane structures, and around half a wavelength for the E-plane structures. This is enough for the first absorbing boundary condition used at the ports to perform correctly.

Typically, quite coarse meshes are used as initial grids in order to assess the robustness of the hp strategy in the context of real engineering analysis in which the initial mesh has to be as coarse as possible in order to simplify the mesh generation process. The convergence history is always shown using a log scale for the energy error (in percent of the energy norm) in the ordinate axis and a scale corresponding to \(N^{1/3}_{\text{dof}}\) (being \(N_{\text{dof}}\) the number of degrees of freedom in the mesh) in the abscissas axis. Thus, according to [27] and references therein, an straight line should appear in the plot showing the theoretical exponential convergence that can be achieved with an optimal hp adaptivity strategy. Note that the abscissas scale corresponds to \(N^{1/3}_{\text{dof}}\) while abscissas axis tics should be read as \(N_{\text{dof}}\) in the plots.

The scattering parameters obtained using the hp-FEM are compared with values computed with the Mode Matching (MM) method (see e.g., [15], [16, Chapter 9]). The MM method can be considered as a semi-analytic method. It consists of the decomposition of the domain of the problem into several simple domains, typically with translational symmetry, in which, an analytical modal expansion can be performed. Imposing the tangential continuity of the field and orthogonality of the modes yields a system of equations in which the
unknowns are the coefficients of the modal expansions.

The FEM scattering parameter results delivered from the $hp$ adaptivity are more accurate than MM results. In this context, it is important to point out that MM results are typically considered as a reference for the engineering analysis of discontinuities in rectangular waveguide technology, as for the structures shown below. In addition, the $hp$-FE technology allows for modeling of more complex structures which cannot be solved using the MM.

3.1 $H$-plane discontinuities

The analysis of several H-plane discontinuities is considered next. The boundary condition of the metallic conductors represents a Neumann boundary condition for the H-plane formulation.

3.1.1 $H$-plane waveguide section

The first structure shown in Fig. 1 is a simple rectangular waveguide section. This structure is chosen as a first verification of the code, specifically for the boundary conditions at the ports and the control of the dispersion error. Since there is no discontinuity in the translational symmetry for this structure, it may be analyzed by means of either the H-plane or E-plane formulations. The results shown in this section correspond to the H-plane analysis. Also, because there is no discontinuity, the scattering parameters of the structure are known to be $S_{11} = S_{22} = 0$ and $S_{21} = S_{12} = \exp(-j\beta_{10}l)$, where $l$ denotes the waveguide section length. In this case, $l$ is equal to 2 wavelengths and, thus, $S_{21} = S_{12} = \exp(-j4\pi) = 1$. The field solution is also known: it corresponds to the field TE$_{10}$ mode inside of the waveguide section.

The solution is smooth: a half-sine type variation in the $y$ direction (the $\pm \xi$ local axis of the waveguide) and constant in amplitude and phase variation as $\exp(-j\beta_{10}x)$ along the $x$ direction (the $\pm \zeta$ local axis of the waveguide). Thus, the $hp$-adaptive strategy is expected to deliver an increase in the polynomial order of approximation $p$. The initial mesh used for the analysis is shown in Fig. 2 together with some intermediate meshes. The colors indicate, according to the scale on the right, the order $p$ of the elements. It is important to note that the order corresponds to the $H^1$ Lagrange multiplier and that the field of $H$(curl) is of order $p - 1$. As an example, the green color of the initial mesh of Fig. 2(a) indicate that all elements are of order 3 for the Lagrange multiplier and order 2 for the magnetic field. It is observed that order $p$ is increased until the maximum $p$ ($p = 9$) is reached in the fine grid; from this moment, $h$ refinement is selected until the specified error criterion is satisfied.
The convergence history for the exact error and the estimated error is plotted in Fig. 4, showing the quality of the error estimation and the exponential behavior of the error. The exponential convergence is deduced from the observation of a straight line in the plot for the log type ordinates axis and the $N_{\text{dof}}^{1/3}$ scale in the abscissas axis set in the figure. It is worth noting that the slope change in the convergence history corresponds to the moment when the maximum $p$ is reached, so $h$ refinements are forced. In other words, this slowdown in convergence would not have occurred if higher order elements were allowed. A plot of the field in the structure, specifically, $|H_y|$, is shown in Fig. 3 where it is clearly observed the sine and the zero variations along the $y$ and $x$ axis, respectively. Note that the the zero variation along the $x$ axis is because we are referring to $|H_y|$ and not $H_y$. A constant magnitude in the direction of propagation means that there is only one wave in the waveguide and, thus, $|H_y| = |H^0_y| |\sin(\pi \xi/a)|$, so it does not depend on the $\zeta \equiv \pm x$ direction. If a discontinuity had generated a reflected wave, a stationary wave pattern would have been observed at the input waveguide (as it is seen in the examples below).

The scattering parameters have been computed at each iteration step of the $hp$ strategy. Due to the reciprocity and symmetry of the structure, the scattering behavior of the discontinuity is characterized by performing one analysis (exciting any of the two ports). The results for the first iterations are shown in Tab. 1. A fast convergence is observed.

Table 1
Scattering parameters for the H-plane waveguide section

| Iter. | $|S_{11}|$    | $|S_{21}|$    | $\arg(S_{21})$ |
|-------|--------------|--------------|----------------|
|   1   | 1.0302e-02   | 0.9991183    | 10.1204°       |
|   2   | 5.9652e-04   | 0.9994050    | 0.9797°        |
|   3   | 4.6872e-07   | 0.9999995    | 0.0402°        |
|   4   | 2.7117e-07   | 0.9999997    | 0.0013°        |
| Analytic | 0.0 | 1.0 | 0.0° |
Figure 1. H-plane waveguide section
Figure 2. Initial mesh and some $hp$ meshes for the H-plane waveguide section

Figure 3. Magnitude of $H_y$, i.e., $|H_y|$, corresponding to the H-plane waveguide section
Figure 4. Convergence history for the H-plane waveguide section (energy norm error for the magnetic field solution)
3.1.2  H-plane right angle bend

An H-plane 90° bend is analyzed next. The structure is shown in Fig. 5. The structure is analyzed by exciting port 1 (on the left). The bend is obviously a common part of microwave circuits. The initial mesh used for the analysis is shown in Fig. 6. Despite the coarseness of this mesh, the $hp$-strategy achieves an energy error lower than 1% error after 5 iterations. The convergence history (up to an error as low as 0.01%) is shown in Fig. 9. The final mesh is shown in Fig. 8. It is observed how the mesh is refined around the corner. The $hp$-strategy in this case tends also to increase the $p$. Actually, all elements of the final mesh (except those near the corner) have reached the maximum $p$ order. This is the right strategy as the solution of the problem is smooth (the boundary condition at the conductors for the H-plane formulation is of homogeneous Neumann type\(^5\)). The $h$-refinement around the corner is precisely due to the fact that the maximum $p$ has been reached and, in order, to reduce the error in this region, the elements must be made smaller.

A plot of the field in the structure, specifically, $|H_y|$, is shown in Fig. 7. The $y$-component corresponds to the local $\xi$ component at the excitation port and the local $\zeta$ component at the transmitted port. Notice the figure the stationary wave pattern in the input waveguide (between the excitation port and the bend) because of the combination of the two waves propagating in opposite directions (the excited wave and the reflected wave at the bend). No stationary wave is observed in the output waveguide as there is only one wave propagating outward the transmitted port. As in the previous case, $S_{21} = S_{12}$ and $S_{22} = S_{11}$. The results for $S_{11}$ and $S_{21}$ are shown (for some of the $hp$ meshes) in Tab. 2. The scattering parameters computed with the $hp$-FEM method are compared with those obtained with a MM technique. Only four significant digits are shown in the table as the MM results are presumed to have no more than 4 digits of accuracy. Observe the very good agreement of the $hp$-FEM results with those provided by MM; better than 1% after the second iteration. After the fourth/fifth iteration, the FEM results seem to be more accurate than those provided by the MM, as implied by the convergence pattern shown in the table.

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\(^5\) The same domain but with Dirichlet boundary conditions corresponds to the E-plane bend which is analyzed in Section 3.2.1.
Figure 5. H-plane 90 degrees bend
Figure 6. Initial mesh for the H-plane 90° bend

Figure 7. Magnitude of $H_y$, i.e., $|H_y|$, corresponding to the H-plane 90° bend
Figure 8. Final $hp$ mesh for the H-plane $90^\circ$ bend

Figure 9. Convergence history for the H-plane $90^\circ$ bend
Table 2
Scattering parameters for the H-plane 90° bend

| Iter. | $|S_{11}|$ | $|S_{21}|$ | arg($S_{11}$) | arg($S_{21}$) |
|-------|------------|------------|--------------|--------------|
| Iter. 1 | 0.5465     | 0.8372     | 10.049°      | 112.615°     |
| Iter. 2 | 0.4459     | 0.8951     | 16.948°      | 101.604°     |
| Iter. 3 | 0.4148     | 0.9099     | 5.639°       | 95.665°      |
| Iter. 4 | 0.4156     | 0.9096     | 5.619°       | 95.617°      |
| MM    | 0.4161     | 0.9093     | 5.345°       | 95.345°      |
3.1.3 *H-plane symmetric inductive iris*

The structure is shown in Fig. 10. The discontinuity consists of the narrowing of the broad dimension of the waveguide along a certain length. This region is referred to as an *iris*\(^6\). Because the iris is centered with respect to the waveguide broad dimension, it is called a *symmetric iris*. Finally, the term inductive is used because the discontinuity scattering behavior with respect to the planes of the discontinuity is equivalent to an inductance. The character of the discontinuity, inductive or capacitive, can be deduced by observing which field lines (electric or magnetic) of the waveguide mode (the TE\(_{10}\) in this case) are “cut” by the discontinuity [2]. It is clear from Fig. 10 that only the magnetic field lines are cut in this case (the electric field of the TE\(_{10}\) is perpendicular to the H-plane of the waveguide). Thus, the symmetric iris of Fig. 10 is of the inductive type. An analogous structure, but of capacitive type, is considered in Section 3.2.3.

The initial mesh used for the analysis is shown in Fig. 11. The analysis is made by exciting port 1 (at the left). The convergence history (up to an error as low as 0.02\%) is shown in Fig. 12. Notice the exponential convergence of the method.

A plot of the field in the structure, specifically, \(|H_y|\), is shown in Fig. 13. The \(y\)-component corresponds to the \(\pm \xi\) component of the field modes in the waveguide. Observe the stationary wave pattern at the input port due to the wave reflected from the discontinuity and a singular behavior of the field at the re-entrant corners. The magnitude of the fields is higher at the left corners. This is “caught” by the *hp* strategy that refines around the left corners during the first few iterations, and once the error around the left corners is controlled (comparable to other regions of the structure), it starts to “see” the error corresponding to the region around the right corners (see figures 14 and 15).

As in the previous case, \(S_{21} = S_{12}\) and \(S_{22} = S_{11}\). The results for \(S_{11}\) and \(S_{21}\) are shown (for some of the *hp* meshes) in Tab. 3. Only the results of the first iterations are shown as the *hp* FEM results for the consecutive meshes are presumed to be more accurate than those of MM. Observations analogous to those mentioned in the previous case may be made here.

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\(^6\) The term *iris* is used in this context to refer to an aperture that connects two waveguide sections.
Figure 10. H-plane symmetric inductive iris ($l/a = 0.6$, $t/a = 0.2$)
Figure 11. Initial mesh for the H-plane symmetric inductive iris

Figure 12. Convergence history for the H-plane inductive iris
Figure 13. Magnitude of $H_y$ for the H-plane inductive iris

Figure 14. 11th mesh for the H-plane symmetric inductive iris showing the refinements around the left corners
Figure 15. 19th mesh for the H-plane symmetric inductive iris showing the refinements around the left and also right corners

Table 3
Scattering parameters for the H-plane symmetric inductive iris

|    | |S11| |S21| arg(S11) | arg(S21) |
|----|------------------|------------------|
| Iter. 2 | 0.7333 | 0.6799 | -157.92° | -27.243° |
| Iter. 4 | 0.7401 | 0.6725 | -156.59° | -26.325° |
| Iter. 6 | 0.7386 | 0.6741 | -156.48° | -26.588° |
| Iter. 8 | 0.7414 | 0.6711 | -156.46° | -26.287° |
| MM   | 0.7417 | 0.6708 | -156.51° | -26.259° |

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3.1.4 H-plane zero thickness septum

The structure, shown in Fig. 16, consists of an obstacle (the septum) placed at the center of the waveguide. The obstacle exhibits a translational symmetry along the narrow dimension of the waveguide (η local axis of the waveguide ports). Thus, it can be analyzed using the H-plane formulation. Also, as for the previous structure, the scattering behavior is basically inductive since the only field lines that are cut by the septum are those of the magnetic field.

The initial mesh used for the analysis is shown in Fig. 17. The analysis is made by exciting port 1 (on the left). The convergence history is shown in Fig. 18. It is observed that the error convergence (except for the first couple of meshes, due to a deliberate coarseness of the initial mesh) behaves as predicted by the theory and a straight line is obtained in the plot (which means exponential convergence).

An example of one of the hp meshes provided by the adaptivity, specifically, the mesh of the 7th iteration, is shown in Fig. 19. Again, as predicted by the theory, it is observed the h-refinements towards the corners where there is a singular behavior of the field and the p-refinements in the regions where the field variation is smooth.

A plot of \(|H_y|\) is shown in Fig. 20. The y-component corresponds, as in the previous case, to the ±ξ component of the field modes in the waveguide. Observe the stationary wave pattern at the input port due to the wave reflected from the discontinuity and the singular behavior of the field at the septum corners.

The results for \(S_{11}\) and \(S_{21}\) corresponding to some of the iterations of the hp adaptivity are shown in Tab. 4. As in the previous cases, \(S_{21} = S_{12}\) and \(S_{22} = S_{11}\) due to reciprocity and symmetry. Only the results up to the 10th iteration are shown, since the error in the scattering parameters obtained from the hp meshes (for iterations higher than the 10th) is expected to be lower than the one of the MM results.
Figure 16. H-plane zero thickness septum (l/a = 0.1)
Figure 17. Initial mesh for the H-plane zero thickness septum

Figure 18. Convergence history for the H-plane zero thickness septum
Figure 19. 7th mesh for the zero thickness septum

Figure 20. Magnitude of $H_y$ corresponding to the H-plane zero thickness septum showing a stationary wave pattern at the input waveguide and singular behavior of the field at the septum corners
Table 4
Scattering parameters for the H-plane zero thickness septum

| Iter.  | $|S_{11}|$ | $|S_{21}|$ | arg($S_{11}$) | arg($S_{21}$) |
|--------|-----------|-----------|----------------|----------------|
| Iter. 1 | 0.7383    | 0.6743    | 206.84°        | -53.120°       |
| Iter. 4 | 0.7466    | 0.6653    | 205.58°        | -53.801°       |
| Iter. 7 | 0.7740    | 0.6332    | 208.64°        | -51.486°       |
| Iter. 10 | 0.7785    | 0.6277    | 209.03°        | -50.962°       |
| MM     | 0.7788    | 0.6273    | 209.03°        | -50.939°       |
3.1.5 \textit{H-plane zero length septum}

This discontinuity is also a septum but it is placed transverse to the wave propagation. The dimension of the septum along the propagation direction is considered zero. A representation of the structure is shown in Fig. 21.

The analysis is made by exciting port 1 (on the left) and considering the coarse initial mesh of Fig. 22. As in the other septum discontinuity, the error convergence (see Fig. 23) behaves as predicted by the theory (exponential convergence) and a straight line is obtained in the plot. Fig. 24 shows, as an example, the mesh corresponding to the 7th iteration where the refinement around the septum corners can be seen. This is what is expected from the field solution. The $y$ component of the solution (its magnitude) is shown in Fig. 25. Again, a stationary wave pattern is observed in the input waveguide section.

Finally, the results for $S_{11}$ and $S_{21}$ corresponding to some of the iterations of the $hp$ adaptivity are shown in Tab. 5. Only results up to the 11th iteration are shown, since the error in the scattering parameters obtained from the $hp$ meshes is expected to be lower (for iterations higher than the 11th) than the one of the MM results.

| Iter. | $|S_{11}|$ | $|S_{21}|$ | $\arg(S_{11})$ | $\arg(S_{21})$ |
|-------|----------|----------|----------------|----------------|
| 2     | 0.7200   | 0.6939   | 50.520°        | -37.987°       |
| 5     | 0.7817   | 0.6236   | 56.432°        | -33.550°       |
| 8     | 0.7883   | 0.6153   | 57.042°        | -32.962°       |
| 11    | 0.7896   | 0.6136   | 57.165°        | -32.840°       |
| MM    | 0.7897   | 0.6135   | 57.171°        | -32.829°       |

Table 5
Scattering parameters for the H-plane zero length septum
Figure 21. H-plane zero length septum \((t/a = 0.1)\)
Figure 22. Initial mesh for the H-plane zero length septum

Figure 23. Convergence history for the H-plane zero length septum
Figure 24. 7th mesh for the zero length septum

Figure 25. Magnitude of $H_y$ corresponding to the H-plane zero length septum showing a stationary wave pattern at the input waveguide and singular behavior of the field at the septum corners
3.2 E-plane discontinuities

The analysis of several E-plane discontinuities is considered next. In contrast to the H-plane formulation, the boundary condition of the metallic conductors is of the Dirichlet type for the E-plane formulation.

3.2.1 E-plane right angle bend

This discontinuity (Fig. 26) is as the one of Section 3.1.2, a 90 degrees bend. However, the plane of the bend in this case is the E-plane. The domain shape is the same as for the H-plane bend (actually, the initial mesh is also the same, see Fig. 6—) but, this time, the homogeneous Dirichlet boundary condition at the conductors are employed. This, apparently, produces a field singularity that occurs at the corner. The \( \text{hp} \)-strategy behaves as expected and, in contrast to the H-plane bend case, an \( h \)-refinement toward the singularity is observed while increasing the \( p \) backward. One of the meshes obtained by the \( \text{hp} \)-adaptivity procedure is shown in Fig. 27.

Plots of the field component magnitudes \(|E_y|\) and \(|E_x|\) are shown in Fig. 28. A stationary wave pattern is observed in the input waveguide (between the excitation port and the bend). The \( y \)-component corresponds to the component along the local \( -\eta \) axis of the excitation port and the component along the \( \zeta \) local axis of the transmitted port. Since the \( \text{TE}_{10} \) does not have \( \zeta \) component, the \( E_y \) component is null (numerically null provided that the port is far enough from the discontinuity) at the transmitted port (port 2). Analogously, the \( E_x \) component is null at the incident port.

The convergence history is shown in Fig. 29. Except for the peak around the third \( hp \) iteration (due to the coarseness of the initial mesh), the error shows an exponential decay. With respect to the convergence of the scattering parameters, Tab. 6 shows their values (in magnitude and phase) for some of the iterations. The MM results are shown for comparison purposes. A good agreement is observed. It is worth noting again that only the results up to the 10th iteration are shown, since the error in the scattering parameters obtained from the \( hp \) meshes (for iterations higher than the 10th) is lower than the one coming from the MM results.
Figure 26. E-plane 90 degrees bend
Figure 27. hp mesh of 11th iteration of the E-plane 90° bend

Table 6
Scattering parameters for the E-plane 90° bend

| Iteration | $|S_{11}|$ | $|S_{21}|$ | arg($S_{11}$) | arg($S_{21}$) |
|-----------|----------|----------|---------------|---------------|
| Iter. 1   | 0.5542   | 0.8323   | -47.810°      | -137.81°      |
| Iter. 4   | 0.5387   | 0.8425   | -46.794°      | -136.94°      |
| Iter. 7   | 0.5487   | 0.8360   | -48.590°      | -138.45°      |
| Iter. 10  | 0.5499   | 0.8352   | -48.558°      | -138.49°      |
| MM        | 0.5507   | 0.8347   | -48.462°      | -138.46°      |
Figure 28. Magnitudes of $E_y$ and $E_x$ corresponding to the E-plane 90° bend showing a singular behavior of the field at the corner
Figure 29. Convergence history for the E-plane 90° bend
3.2.2  E-plane right angle bend with a round corner

This structure is also a 90° bend in the E-plane but with a round corner (see Fig. 30). In practice, depending on the mechanical process used to build the bends, the bends may either have sharp corners or round corners (as in this case). Although there is no field singularity because of the roundness of the corner, there is a high variation of the fields around the corner, and the adaptivity behaves analogously to the case of a sharp corner (in the pre-asymptotic regime, as a high variation in the fields is “seen” as a singularity).

The analysis is made by exciting port 1 (on the left) and considering the coarse initial mesh of Fig. 31. Fig. 32 shows a sample mesh corresponding to the 10th iteration. A refinement pattern similar to the one of Fig. 27 can be observed. The exponential convergence history is shown in Fig. 34.

Plots of the field component magnitudes $|E_y|$ and $|E_x|$ are shown in Fig. 33. Comments analogous to those made on the E-plane bend with a sharp are valid for this case as well. Tab. 7 shows the values (in magnitude and phase) of $S_{11}$ and $S_{21}$ for the first few iterations. A fast convergence is observed and seven digits are needed in order to be able to observe the convergence of the scattering parameters. No MM results are shown for this case. The analysis of this structure by MM requires the use of special functions and somehow differs of what it is usually referred to as the MM method.

| Iter. | $|S_{11}|$ | $|S_{21}|$ | arg($S_{11}$)  | arg($S_{21}$) |
|-------|----------|----------|----------------|----------------|
| 1     | 0.5835441 | 0.8120788 | -86.29061°    | -176.29024°    |
| 2     | 0.5835960 | 0.8120416 | -86.27239°    | -176.27328°    |
| 3     | 0.5836238 | 0.8120215 | -86.26833°    | -176.26855°    |
| 4     | 0.5836237 | 0.8120217 | -86.26894°    | -176.26880°    |
Figure 30. E-plane 90 degrees bend with round corner ($r/b = 0.2$)
Figure 31. Initial mesh for the E-plane 90° bend with round corner

Figure 32. 10th hp mesh for the E-plane 90° bend with round corner
Figure 33. Magnitudes of $E_y$ and $E_x$ corresponding to the E-plane 90°
Figure 34. Convergence history for the E-plane $90^\circ$ bend with round corner
3.2.3 E-plane capacitive symmetric iris

This discontinuity (Fig. 35) is due to a symmetric iris (as the one of Section 3.1.3), but in the E-plane. Thus, the FEM domain is identical to the one used for the H-plane inductive symmetric iris, but with different boundary conditions on the conductors boundaries (of Dirichlet type for this case). The analysis is made by exciting port 1 (on the left).

The initial mesh is shown in Fig. 36(a). Fig. 36(b) shows a sample mesh corresponding to the 4th iteration. A refinement pattern around the corners of the iris is observed due to the presence of field singularities at those locations. The convergence history (up to an error as low as 0.1%) is shown in Fig. 37. Exponential convergence is again observed.

Plots of the field component magnitudes $|E_y|$ and $|E_x|$ are shown in Fig. 38. A stationary wave pattern is observed in the input waveguide (between the excitation port and the iris) as well as a singular behavior of the field at the corners of the iris. The $y$-component of the field in the structure corresponds to the $\mp \eta$ component of the waveguide modes. Analogously, the $x$-component corresponds to the $\pm \zeta$ component of the waveguide. Thus, the $E_x$ component of the field is generated at the discontinuity, and it is only significant close to it.

The results for $S_{11}$ and $S_{21}$ corresponding to some of the iterations of the $hp$ adaptivity are shown in Tab. 8. The equalities $S_{21} = S_{12}$ and $S_{22} = S_{11}$ hold due to reciprocity and symmetry of the structure. Only the results of the first iterations are shown as the $hp$ FEM results for the consecutive meshes are presumed to be more accurate than those of MM.

Table 8
Scattering parameters for the E-plane capacitive symmetric iris

|         | $|S_{11}|$ | $|S_{21}|$ | $\arg(S_{11})$ | $\arg(S_{21})$ |
|---------|-----------|-----------|----------------|----------------|
| Iter. 1 | 0.3180    | 0.9481    | -163.31°       | -53.24°        |
| Iter. 2 | 0.3070    | 0.9517    | -162.71°       | -52.60°        |
| Iter. 5 | 0.3013    | 0.9535    | -162.37°       | -52.25°        |
| Iter. 6 | 0.3009    | 0.9537    | -162.35°       | -52.22°        |
| MM      | 0.3008    | 0.9537    | -162.36°       | -52.23°        |
Figure 35. E-plane capacitive symmetric iris ($d/b = 0.6, t/b = 0.2$)
Figure 36. Initial mesh and mesh of the 4th iteration for the E-plane capacitive symmetric iris
Figure 37. Convergence history for the E-plane capacitive symmetric iris
Figure 38. Magnitude of $E_y$ and $E_x$ corresponding to the E-plane capacitive symmetric iris.
3.2.4 E-plane double stub section

The structure is shown in Fig. 39. It consists of a main waveguide going from port 1 to port 2 and two waveguides loading it (referred to as stubs). The stubs are closed at their ends causing a total reflection of the energy at their inputs. Thus, the load impedance that they present to the main waveguide is purely imaginary. The behavior of this structure may be roughly explained as follows. The load of each stub is like a discontinuity in the waveguide, producing a reflected wave (and also a transmitted wave) with a given phase. As there are two stubs, i.e., two discontinuities, the contributions from the two stubs may (totally or partially) add or cancel, depending on the relative phase between the corresponding waves. The relative phase depends (for given stubs dimensions) on the electrical distance $\theta_d$ between them. As the electrical distance depends on the frequency for a given physical distance $d$, i.e., $\theta_d = \beta_0 d$, the frequency response can be adjusted for several applications. For example, the double stub section can be designed to work as a phase shifter, i.e., causing an extra shift in the phase of the wave at the transmitted port for a given frequency band. This is done by designing the double stub in such a way that there is an adding interference at the transmitted port of the two waves generated by the stubs (with a given phase). Another usual application is an impedance matching network, i.e., the double stub is designed to compensate the reflection present at a given port, due to, e.g., a change in the height of the waveguide, in a given frequency band. This is done by adjusting the design so there is a cancellation of the two reflected waves at the stubs junctions ($180^\circ$ out of phase with respect to each other).

In here, the double stub has been designed for the latter application, i.e., to have a null reflection around a frequency given by $k_0 = 1.39k_c$ ($k_c$ being the cut-off frequency of the TE$_{10}$ mode). Notice in Fig. 39 that the waveguide sections at the two ports are identical. Thus, if the two stubs were not present there would be no reflection ($S_{11}$ ideally null) as the structure would simply consists of a waveguide section. The reason we choose to analyze the structure in Fig. 39 (which has a little practical application) is because this is a good test case. The idea is that for the null reflection frequency, the fields in the main waveguide have to be basically the same as those in a single waveguide section. Since the stubs are identical, the structure is symmetrical ($S_{11} = S_{22}$). As in the other cases, the analysis is made by exciting port 1 (on the left).

The frequency response of the structure around the frequency corresponding to $k = 1.39K_c$ is shown in Fig. 40(a). The ordinate axis corresponds to $S_{11}$ in dB, i.e., $10 \log_{10} |S_{11}|^2$. The results have been obtained by running the

\[7\] The dB is a logarithmic unit for dimensionless magnitudes, but it is always with respect to the power (and not the field) magnitudes. Thus, when applied to the scattering parameters that relate field quantities, the “$10 \log_{10}$” factor has to be applied to the square of the scattering coefficient. For example, $S_{11} = -40$ dB means
hp-adaptivity until an energy error of 1% is achieved and computing the $S_{11}$ using the final hp-mesh. The expected low values of the reflection coefficient $S_{11}$ around $k_0/k_c = 1.39$ is observed. For a comparison, Fig. 40 presents also results obtained using a MM technique. A very good agreement is observed.

Figure 40(b) shows the frequency response over a broad frequency interval that consists of a centered band covering the 60% of the monomode frequency band. A very good agreement between the hp-FEM and MM results is observed. The only exception is around the frequency corresponding to $k_0/k_c = 1.74$. For these frequencies, a more refined mesh seems to be needed.

The convergence has been studied by running the hp-adaptivity with an energy-norm error tolerance of 0.01% (and a maximum number of iterations equal to 20). Results corresponding to five significant frequency points (symmetrically chosen around the value of $k_0/k_c = 1.39$): $k_0/k_c=1.32, 1.36, 1.42, 1.46$ are displayed in Fig. 41. For $k_0/k_c = 1.32, 1.46$ there is a high reflection of the energy at the input waveguide; for $k_0/k_c = 1.39$ there is a very low (almost null) reflection at the input; and for $k_0/k_c = 1.36, 1.42$ an intermediate situation occurs. Except for the first few iterations, the plots follow approximately a straight line, reflecting an exponential decrease of the energy-norm error. The erratic behavior of the error during the first few iterations is due to the coarseness of the initial mesh (shown in Fig. 42(a)).

The final meshes for $k_0/k_c = 1.32$ (high reflection at the input), $k_0/k_c = 1.39$ (low reflection at the input), and $k_0/k_c = 1.36$ (intermediate reflection at the input) are displayed in Figures 42 and 43. For the case of high reflection, the mesh (shown in Fig. 42(b)) displays a typical refinement pattern around the corners (junctions of the stubs with the main waveguide). The electric field for this case is plotted in Fig. 44(a). A stationary wave pattern in the input waveguide is observed due to the interference between the incident and the reflected waves. On the other hand, the mesh for the low reflection case (shown in Fig. 43(b)) displays a situation very similar to the situation of the smooth field solution inside a waveguide section (i.e., without singularities). As it was explained above, this occurs because the stubs do not load the main waveguide (it is like if they were not present in the structure). This is best understood by seeing Fig. 44(b), which displays the electric field in the structure for this case. The stationary wave pattern at the input waveguide can hardly be seen, which means that the level of the reflected wave is very low. Finally, the mesh for the intermediate case ($k_0/k_c = 1.36$) is shown in Fig. 43(a). It is observed how that the power reflected at the input waveguide is 40 dB below the power of the excitation at the input waveguide, i.e., $10^4$ lower (or equivalently, the electric and magnetic fields are $10^2$ lower).

\footnote{Note that the magnitude plotted is $\sqrt{|E_x|^2 + |E_y|^2}$ that, although does not correspond to $|E|$ or other physically meaningful magnitude, it is useful for visualizing in one plot the field in the main waveguide and in the stubs.}
effectively the mesh corresponds to an intermediate case between the meshes of Figures 42(b) and 43(b).

Figure 39. E-plane double stub structure ($bs_1/b = bs_2/b = 5.0249$, $l/b = 1.2608$)
(a) Frequency response in the band of interest

(b) Frequency response over the 60% of the monomode region

Figure 40. Frequency response of E-plane double stub section (|S_{11}| in dB)
Figure 41. Convergence history for the E-plane double stub section
Figure 42. Initial mesh and mesh corresponding to 1% energy error for the E-plane double stub section
Figure 43. Meshes corresponding to 1% energy error for the E-plane double stub section
Figure 44. Electric field $\sqrt{|E_x|^2 + |E_y|^2}$ in the E-plane double stub section

(a) High reflection at the input waveguide ($K_0/K_c = 1.32$)

(b) Low reflection at the input waveguide ($K_0/K_c = 1.39$)
4 Conclusions

An $hp$-adaptive Finite Element Method for studying the characterization of microwave rectangular waveguide discontinuities with a geometry invariant along one direction (a common situation in rectangular waveguide technology), has been presented. The assumption on the geometry of the discontinuity enables a 2D analysis in so called H-plane or E-plane of the structure.

A fully automatic $hp$-adaptive strategy based on maximizing the rate of decrease of the (projection-based) interpolation error of the fine grid solution, has been applied to a number of important engineering examples. Computation of the scattering matrix that characterize the electromagnetic behavior of the discontinuities for the microwave engineer has been implemented as a post-processing of the solution.

A wide variety of structures have been analyzed, including microwave engineering devices of medium complexity. The $hp$ adaptivity has shown to deliver exponential convergence rates for the error for both regular and singular solutions. A consistent convergence pattern makes us believe that the results are more accurate than those obtained with semi-analytical techniques. At the same time, this $hp$-methodology presents the important advantage of being a purely numerical method, which allows for modeling of complex waveguide structures that cannot be solved by using semi-analytical techniques.

5 Acknowledgment

The authors would like to thank Sergio Llorente-Romano at the Universidad Politécnica de Madrid for their helpful discussions on the MM techniques and for letting us use their MM codes that have been used to produce some of the MM results shown in this paper.
References


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