Asymptotics and stabilization for dynamic models of nonlinear beams

Dedicated to Jüri Engelbrecht with friendship and admiration

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Received 15 December 2009, accepted 25 March 2010

Abstract. We prove that the von Kármán model for vibrating beams can be obtained as a singular limit of a modified Mindlin–Timoshenko system when the modulus of elasticity in shear $k$ tends to infinity, provided a regularizing term through a fourth-order dispersive operator is added. We also show that the energy of solutions for this modified Mindlin–Timoshenko system decays exponentially, uniformly with respect to the parameter $k$, when suitable damping terms are added. As $k \to \infty$, one deduces the uniform exponential decay of the energy of the von Kármán model.

Key words: vibrating beams, Mindlin–Timoshenko system, von Kármán system, singular limit, uniform stabilization.

The Mindlin–Timoshenko system of equations is a mathematical model for describing the transverse vibrations of beams. It is more accurate than the Euler–Bernoulli theory, since it also takes transverse shear effects into account. It is used, for example, to model aircraft wings (see, for instance, [3]).

For a beam of length $L > 0$, this one-dimensional nonlinear system reads as

$$
\frac{\rho h^3}{12} \phi_{tt} - D \phi_{xx} + k (\phi + \psi_x) = 0 \quad \text{in} \ Q,
$$

$$
\rho \psi_{tt} - k (\phi + \psi_x)_x - Eh \left[ \psi_x \left( \eta_x + \frac{1}{2} \psi_x^2 \right) \right]_x = 0 \quad \text{in} \ Q, \quad (1)
$$

$$
\rho \eta_{tt} - Eh \left( \eta_x + \frac{1}{2} \psi_x^2 \right)_x = 0 \quad \text{in} \ Q,
$$

where $Q = (0, L) \times (0, T)$, with $(0, L)$ being the segment occupied by the beam, and $T$ a given positive time.

In system (1), subscripts mean partial derivatives. The unknowns $\phi = \phi(x, t)$, $\psi = \psi(x, t)$, and $\eta = \eta(x, t)$ represent, respectively, the angle of rotation, the vertical displacement, and the longitudinal displacement at time $t$ of the cross section located $x$ units from the end-point $x = 0$. The constant $h > 0$ represents the thickness of the beam which, in this model, is considered to be small and uniform with respect to $x$. The constant $\rho$ is the mass density per unit volume of the beam and $D = Eh^3/12 \left( 1 - \mu^2 \right)$ is called the modulus of flexural rigidity, where $E$ is Young’s modulus and $\mu$ is Poisson’s ratio, $0 < \mu < 1/2$. The

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The component of the Mindlin–Timoshenko system. Note, however, that our results need the fourth-order regularizing term in that is, in the absence of the term tends to infinity have been recently investigated in a number of different situations. For the linear case, that we shall make precise below.

These results are proved under suitable boundary and initial conditions. The issue of the dependence of the decay rate or, more generally, the stabilizability of intensive research. In the present paper this issue is addressed in the context of nonlinear models for the transverse shear effects are neglected, and one obtains the so-called von Kármán system (see [8]):

\[
\begin{align*}
\rho h \psi_{tt} - \frac{\rho h^3}{12} \psi_{xxxx} + D \psi_{xxxx} &= -E h \left[ \psi_x \left( \eta_x + \frac{1}{2} \psi_x^2 \right) \right]_x = 0 \quad \text{in } Q, \\
\rho h \eta_{tt} - E h \left( \eta_x + \frac{1}{2} \psi_x^2 \right)_x &= 0 \quad \text{in } Q.
\end{align*}
\]

Note that neglecting the shear effects of the beam is formally equivalent to considering the modulus \( k \to \infty \) in system (1), since \( k \) is inversely proportional to the shear angle. There is also an extensive literature about this model, addressing problems of the existence, uniqueness, and asymptotic behaviour in time when some damping effects are considered, as well as some other important properties (see [6,8] and references therein). These models are also relevant when analysing the effect of the microstructure on the dispersive properties of materials (see [4]).

When one assumes the linear filament of the beam to remain orthogonal to the deformed middle surface, and asymptotic behaviour in time when some damping effects are considered, as well as some other important properties (see [6,8] and references therein). These models are also relevant when analysing the effect of the microstructure on the dispersive properties of materials (see [4]).

The connections between these two systems and the asymptotic limit problem of passing to the limit as \( k \) tends to infinity have been recently investigated in a number of different situations. For the linear case, that is, in the absence of the term \( \psi_x \left( \eta_x + \frac{1}{2} \psi_x^2 \right)_x \), and without the equation for \( \eta \), it was proved in [8]...
Theorem 1. If \( \{ \phi_0, \psi_0, \eta_0, \eta_1 \} \) is a weak solution of problem (3), (6), (7) has a unique weak solution in the class
\[
\{ \phi, \psi, \eta \} \in C^0 \left( [0, \infty) ; H_0^1 (0, L) \times H_0^2 (0, L) \times V \right) \cap C^1 \left( [0, \infty) ; [L^2 (0, L)]^2 \times H \right).
\]

Moreover, the energy \( E_k(t) \), given by
\[
E_k(t) = \frac{1}{2} \left( \frac{\rho h^3}{12} |\phi_k(t)|^2 + \rho h |\psi_k(t)|^2 + \rho h |\eta_k(t)|^2 + D|\phi_k(t)|^2 \\
+ k |\phi(t) + \psi_k(t)|^2 + Eh |\eta_k(t) + \frac{1}{2} |\psi_k(t)|^2 |^2 + \frac{\sigma}{k} |\psi_{xxk}(t)|^2 \right),
\]
satisfies
\[
E_k(t) = E_k(0), \quad \forall t \geq 0.
\]
Theorem 2. Let \( \{ \phi^k, \psi^k, \eta^k \} \) be the unique solution of (3), (6), (7) with data \( \{ \phi_0, \phi_1, \psi_0, \psi_1, \eta_0, \eta_1 \} \in \mathcal{D} \) satisfying
\[
\phi_0 + \psi_{0x} = 0 \quad \text{in} \quad (0, L).
\]

Then, letting \( k \to \infty \), one gets
\[
\{ \phi^k, \psi^k, \eta^k \} \to \{ -\psi_\ast, \psi_\ast, \eta_\ast \} \quad \text{weak* in} \quad L^\infty \left( 0, T; [H^1_0(0,L)]^2 \times V \right),
\]
where \( \{ \psi_\ast, \eta_\ast \} \) solves the von Kármán system
\[
\begin{align*}
\rho h \psi_{tt} - \frac{\rho h^3}{12} \psi_{xxt} + D \psi_{xxxx} - E h \left[ \psi_x \left( \eta_x + \frac{1}{2} \psi_x^2 \right) \right]_x &= 0 \quad \text{in} \quad Q, \\
\rho h \eta_{tt} - E h \left( \eta_x + \frac{1}{2} \psi_x^2 \right)_x &= 0 \quad \text{in} \quad Q, \\
\psi(0, \cdot) &= \psi(L, \cdot) = \psi_x(0, \cdot) = \psi_x(L, \cdot) = \eta_x(0, \cdot) = \eta_x(L, \cdot) = 0 \quad \text{on} \quad (0, T), \\
\psi(\cdot, 0) &= \psi_0, \quad \psi_x(\cdot, 0) = \psi_1 + \frac{h^2}{12} \phi_1x \\
\eta(\cdot, 0) &= \eta_0, \quad \eta_x(\cdot, 0) = \eta_1
\end{align*}
\]

One also has that the exponential decay for the energy
\[
E(t) = \frac{1}{2} \left( \rho h [\psi_t(t)]^2 + |\eta_t(t)|^2 \right) + \frac{\rho h^3}{12} [\psi_{xx}(t)]^2 + D |\psi_{xxx}(t)|^2 + E h \left| \eta_x(t) + \frac{1}{2} |\psi_x(t)|^2 \right|^2,
\]
associated to the solution of the von Kármán system
\[
\begin{align*}
\rho h \psi_{tt} - \frac{\rho h^3}{12} \psi_{xxt} + D \psi_{xxxx} - E h \left[ \psi_x \left( \eta_x + \frac{1}{2} \psi_x^2 \right) \right]_x + \beta \psi_t - \alpha \psi_{xxt} &= 0 \quad \text{in} \quad Q, \\
\rho h \eta_{tt} - E h \left( \eta_x + \frac{1}{2} \psi_x^2 \right)_x + \gamma \eta_t &= 0 \quad \text{in} \quad Q, \\
\psi(0, \cdot) &= \psi(L, \cdot) = \psi_x(0, \cdot) = \psi_x(L, \cdot) = \eta_x(0, \cdot) = \eta_x(L, \cdot) = 0 \quad \text{on} \quad (0, T), \\
\{ \psi(\cdot, 0), \psi_x(\cdot, 0), \eta(\cdot, 0), \eta_x(\cdot, 0) \} &= \{ \psi_0, \psi_1, \eta_0, \eta_1 \}
\end{align*}
\]

can be obtained as a limit (as \( k \to \infty \)) of the uniform stabilization of the perturbed Mindlin–Timoshenko system
\[
\begin{align*}
\frac{\rho h^3}{12} \phi_{tt} - D \phi_{xt} + k(\phi + \psi_t) + \alpha \phi_t &= 0 \quad \text{in} \quad Q, \\
\rho h \psi_{tt} - k(\phi + \psi_x)_x - E h \left[ \psi_x \left( \eta_x + \frac{1}{2} \psi_x^2 \right) \right]_x + \frac{\sigma}{k} \psi_{xxxx} + \beta \psi_t &= 0 \quad \text{in} \quad Q, \\
\rho h \eta_{tt} - E h \left( \eta_x + \frac{1}{2} \psi_x^2 \right)_x + \gamma \eta_t &= 0 \quad \text{in} \quad Q, \\
\phi(0, \cdot) &= \phi(L, \cdot) = \psi(0, \cdot) = \psi(L, \cdot) = \eta_x(0, \cdot) = \eta_x(L, \cdot) = 0 \quad \text{on} \quad (0, T), \\
\{ \phi(\cdot, 0), \psi(\cdot, 0), \eta(\cdot, 0) \} &= \{ \phi_0, \psi_0, \eta_0 \} \quad \text{in} \quad (0, L), \\
\{ \phi(\cdot, 0), \psi(\cdot, 0), \eta(\cdot, 0) \} &= \{ \phi_1, \psi_1, \eta_1 \} \quad \text{in} \quad (0, L),
\end{align*}
\]

where \( \alpha, \beta, \) and \( \gamma \) are positive constants.

System (13) can be obtained as a limit, when \( k \to \infty \), of system (14).
Notice that the energy of system (14) is given by (9) and it is dissipated according to the law
\[
\frac{dE_k(t)}{dt} = - \left( \alpha |\phi(t)|^2 + \beta |\psi(t)|^2 + \gamma |\eta(t)|^2 \right).
\] (15)
This energy decays exponentially (as \( t \to \infty \)) uniformly with respect to \( k \):

**Theorem 3.** Let \( \{\phi, \psi, \eta\} \) be the global solution of system (14) for data \( \{\phi_0, \phi_1, \psi_0, \psi_1, \eta_0, \eta_1\} \in \mathcal{H} \) satisfying (11). Then there exists a constant \( \omega > 0 \) such that
\[
E_k(t) \leq 4E_k(0) e^{-\frac{\omega}{2}t}, \forall t \geq 0.
\] (16)

As a consequence of inequality (16), letting \( k \to \infty \), one recovers the exponential decay of the energy \( E(t) \), associated to system (13). This is in agreement with the results obtained in [9].

The interested reader is referred to [2] for further details on this topic and, in particular, for the proofs of the results presented here.

**ACKNOWLEDGEMENTS**

The first author was partially supported by INCTMat, FAPESQ-PB, CNPq (Brazil) grants 308150/2008-2 and 620108/2008-8. The third author was supported by grant MTM2008-03541 of the MICINN (Spain) and the ERC Advanced Grant FPT-246775 NUMERIWAVES.

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Mittelineaarse talade dünaamiliste mudelite asümpootika ja stabiliseerimine

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