Dispersive smoothing for the Euler-Korteweg model

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Abstract

The Euler-Korteweg system consists in a quasi-linear, dispersive perturbation of the Euler equations. The Cauchy problem has been studied in any dimension \( d \geq 1 \) by Benzoni-Danchin-Descombes, who obtained local well-posedness results when the velocity is in \( H^s \) for \( s > d/2 + 1 \). They noticed that one may expect to find some gain of regularity due to the dispersive effects, but there was no proof so far. Our aim here is to give such results in any dimension under their local existence assumptions. In the simpler case of dimension 1 we obtain unconditional Kato-smoothing (local gain of 1/2 derivative). In higher dimension a few additional hypotheses must be done to get smoothing and we briefly discuss the pertinency of these restrictions.

1 Introduction

Dispersive smoothing is by now a rather classical topic. Its first observation originates with the seminal work of Kato on the Korteweg-de-Vries equation [Kat83] (and almost at the same time Faminskii-Kruzhkov [KF83]). There has been since various generalizations and refinements of these results, for the Korteweg de Vries equation (Kenig-Ponce-Vega [KPV91]) as well as for very general dispersive equations (Constantin-Saut [CS88]). Seemingly, the first result of dispersive smoothing for the Schrödinger equation with fully variable coefficients was obtained by Doi [Doi96] (different properties, of microlocal nature, were also obtained by Craig-Kappeler-Strauss [CKS95]), who used geometric assumptions such as non trapping of bicharacteristics and flatness of the coefficients at infinity that proved to be (in some sense) sharp. More recently, the method of Doi was successfully generalized by Kenig-Ponce-Vega for the quasi-linear Schrödinger equation [KPV04] and Alazard-Burq-Zuilly [ABZ] for (a convenient reformulation of) the 1D water waves equations.

We treat here the Euler-Korteweg equations for capillary fluids, that read

\[
\begin{aligned}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t u + (u \cdot \nabla) u + \nabla g(\rho) &= \nabla (K(\rho) \Delta \rho + \frac{1}{2} K'(\rho) |\nabla \rho|^2),
\end{aligned}
\]

(EK)

where the right hand side of the second equation modelizes capillarity forces. It is a dispersive perturbation of the classical Euler equations.

The analysis of the Cauchy problem for (EK) with general \( K \) was initiated by Benzoni-Danchin-Descombes in [BGDD06] in dimension 1, they later obtained in

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[BGDD07] the local well-posedness in any dimension \( d \) for \((\nabla \rho, u)|_{t=0} \in H^s, \ s > d/2 + 1\). The dispersive nature of this system is clarified by introducing the new unknown \( \zeta = R(\rho) \), where \( R \) is a primitive of the application \( \rho \to \sqrt{K(\rho)/\rho} \). The system satisfied by \( \zeta, u \) is then artificially supplemented by an equation on the unknown \( w := \partial_x \zeta \) obtained by differentiating in \( x \) the transport equation on \( \zeta \).

Consider first the simpler case of dimension 1; the equations on \((\rho, u)\) are

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho u) &= 0, \\
\partial_t u + (u \cdot \partial_x) u + \partial_x g(\rho) &= \partial_x (K(\rho)\partial_x^2 \rho + \frac{1}{2} K'(\rho)|\partial_x \rho|^2),
\end{align*}
\]

and the extended system reads

\[
\begin{align*}
\partial_t \zeta + u \partial_x \zeta + a(\zeta) \partial_x u &= 0, \\
\partial_t u + u \partial_x u - w \partial_x w - \partial_x (a(\zeta) \partial_x w) &= -p'(\zeta)w, \\
\partial_t w + u \partial_x w + w \partial_x u + \partial_x (a(\zeta) \partial_x u) &= 0,
\end{align*}
\]

where \( g' = (p \circ R^{-1})' \), \( a = \sqrt{\rho K(\rho)} \).

One may identify the last two equations in \((EEK)\) as the real and imaginary parts of a quasi-linear Schrödinger equation on \( z = u + iw \).

\[
\partial_t z + z \partial_x z + i \partial_x (a(\zeta) \partial_x z) = -p'(\zeta)\text{Re}(z). \tag{SB}
\]

The term \( z \partial_x z \) will have to be treated with some care. Indeed if we write the equation \((SB)\) as

\[
\partial_t z + u \partial_x z + iw \partial_x z + i \partial_x (a \partial_x z) = -p'(\zeta)\text{Re}(z),
\]

we see that (at least formally by Fourier modes analysis) \( iv \partial_x z \) is a source of spectral instability. More precisely, if \( v \) is independent of \( t \), a criterion due to Mizohata [Miz81] requires that \( \int_0^X v(x)dx \) remains uniformly bounded in \( X \) for an \( L^2 \) estimate to stand. It is noticeable that this criterion is satisfied here since \( v \) is the derivative of \( \zeta(\rho) \), which is bounded as soon as \( \nabla \rho \in H^s, s > 1/2 \) (as a matter of fact, the local existence theorem in [BGDD06] requires \( s > 1 + 1/2 \)).

In higher dimension a different, but crucial, feature appears for the reformulated system

\[
\begin{align*}
\partial_t \zeta + u \cdot \nabla \zeta + a(\zeta) \text{div} u &= 0, \\
\partial_t z + (u \cdot \nabla) z + i(\nabla z) \cdot w + i \text{div}(a \text{div} z) &= -p'(\zeta)\text{Re}(z).
\end{align*}
\]

The second equation is no more a quasilinear Schrödinger equation, but a degenerate quasilinear Schrödinger equation. This fact prevents smoothing for general initial data and some (technical) cancellations of the one-dimensional case are not true anymore. An other important, but more standard feature, is the apparition of geometric assumptions such as non trapping of the bicharacteristics and some flatness at infinity of the symbol \( \xi a(x)\xi^t \).

Smoothing properties of the Euler equations are interesting for several reasons: most notably, this effect may allow to derive well-posedness results at lower level of regularity. On a more physical level it is still not known whether small perturbations of traveling waves solutions of \((EK)\) exist globally or if they may blow up, the blow up criterion of [BGDD05] involving derivatives of too high order. The slight gain of regularity proved here may be a way to close the a priori estimates.

To prove our results, we use the gauge methods of [BGDD07] and follow closely
the approach of Alazard-Burq-Zuilly in [ABZ], which consists in reducing the qua-
linear problem to a linear one by the mean of para-differential calculus. The ideas
developed by Kenig-Ponce-Vega [KPV04] for the quasi-linear Schrödinger equation
are also important when the dimension is larger than 1.

The scheme of proof can be shortly described as follows: we have $z$ an $H^s$ solution,
the equations are reformulated as equations on some $Z_s$ which is exactly $s$ times less
regular than $z$. A symbol $p$ is then constructed such that

$$\frac{d}{dt} \langle T_p Z_s, Z_s \rangle \geq \|Z_s \varphi\|_{H^{1/2}} - \|Z_s\|_{L^2},$$

with $\varphi$ having some decay at infinity. This estimate which is true for smooth solutions
then implies the local gain of $1/2$ derivative by density arguments.

Our paper is organized as follows:

- In section 2 we recall the essential results on para-differential calculus that we
  use in the rest of the paper.
- In section 3 we treat the one-dimensional case by para-linearizing the equations.
- Section 4 extends the results of section 3 under some supplementary assump-
tions when $\nabla \rho, u$ belong to Sobolev spaces.
- Since many physically pertinent solutions of the Euler-Korteweg system do not
cancel at infinity, we extend our results to perturbations of such solutions in
section 5. We shortly discuss the necessity of ‘flatness at infinity’ and prove
the smoothing effect for the linearized extended system near a traveling profile
with weaker assumptions than in the general case.
- In the appendix we give for completeness the (relatively) standard arguments
for the construction of para-differential operators that are essential tools of the
previous sections, and the proof of a weighted Garding inequality.

2 Handtool in para-differential calculus

We refer to the lecture notes of G. Métivier [Mét08], notably section 6.4 for symbols
$s(x, \xi)$ that are not smooth functions of $\xi$ at 0.

**Definition 2.1.** Let $\psi(\eta, \xi)$ be a smooth non-negative function. We say that it is
an admissible truncature function when there exists $0 < \varepsilon_1 < \varepsilon_2 < 1$ such that

$$\begin{cases}
\psi(\eta, \xi) = 1, & |\eta| \leq \varepsilon_1 (1 + |\xi|), \\
\psi(\eta, \xi) = 0, & |\eta| \geq \varepsilon_2 (1 + |\xi|),
\end{cases}$$

and for any multi-indices $(\alpha, \beta)$ there exists a constant $C_{\alpha, \beta} > 0$ such that $|\partial_\eta^\alpha \partial_\xi^\beta \psi| \leq C(1 + |\xi|)^{-|\alpha|-|\beta|}$.

For $r \geq 0$, $m \in \mathbb{R}$, the space of symbols $\Gamma^m_r$ is the set of functions $a(x, \xi) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$, $C^\infty$ in $\xi \neq 0$, $W^{r, \infty}$ in $x$ for any $\xi \neq 0$ such that for any multi-index $\alpha$ and any $|\xi| \geq 1$, $\|\partial_\xi^\alpha a(\cdot, \xi)\|_{W^{r, \infty}} \leq C_{r}(1 + |\xi|)^{m-|\alpha|}$. 

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Remark 2.2. In particular the Sobolev embeddings ensure that if \( a(x) \in H^s(\mathbb{R}^d) \), \( s > r + d/2 \), \( a \in \Gamma^r \).

Definition 2.3. Let \( \theta \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^+) \) be such that \( \forall |\eta| \leq 1, \theta(\eta) = 0, \forall |\eta| \geq 2, \theta(\eta) = 1 \). For \( a \in \Gamma^r \) we define the para-differential operator \( T_a \) by

\[
T_a u = \int_{\mathbb{R}^d} \psi(\xi - \eta) \theta(\eta) \hat{u}(\xi - \eta) \hat{u}(\eta) d\eta
\]

Proposition 2.4. If \( a \in \Gamma^r \), \( r \geq 0 \), then \( T_a \) is continuous \( H^{s+m} \rightarrow H^s \) for \( s \in \mathbb{R} \), and

\[
\|T_a\|_{H^{s+m} \rightarrow H^s} \lesssim \sup_{|\xi| \geq 1, |\alpha| \leq d/2+1} (1 + |\xi|)^{\alpha - m} \|\partial_\xi^\alpha a(\cdot, \xi)\|_{L^\infty}
\]

Corollary 2.6. Let \( s > d/2 \), \( a \in H^s \), \( a \in H^{s+1} \), we have \( au = T_a u + Q(u, a) \), with \( \|Q(u, a)\|_{H^{s+1}} \leq C\|u\|_{H^s}\|a\|_{H^{s+1}} \).

Proof. The previous proposition gives \( au = T_a u + T_a a + R(a, u) \) where \( R \) already satisfies the estimate. But since \( u \in L^\infty \) we also have \( T_a a \in H^{s+1} \) which satisfies the expected estimate. \( \square \)

We denote by \([A, B]\) the commutator \( AB - BA \) of two operators.

Proposition 2.7. (functional calculus)
If \( (a, b) \in \Gamma^m \times \Gamma^m \), \( r, s \geq 1 \), we have the composition rule \( T_a T_b = T_a b + R \), where \( a^\sharp b = \sum_{|\alpha| < \min(r, s)} \frac{1}{i|\alpha|} \partial_\xi^\alpha a \partial_\xi^\alpha b \) and \( R \) is continuous \( H^{t+m+n} \rightarrow H^{t+\min(r, s)} \). In particular

- \([T_a, T_b]\) is continuous \( H^{t+m+n-1} \rightarrow H^t \).
- If moreover \( r, s \geq 2 \), \([T_a, T_b] = T_c + R\) where \( R \) is continuous \( H^{t+m+n-2} \rightarrow H^t \) and \( c \) is defined by

\[
c = \frac{1}{i} \{a, b\} := \frac{1}{i} \left( \sum \partial_\xi^\alpha a \partial_\xi^\alpha b - \partial_\xi^\alpha a \partial_\xi^\alpha b \right).
\]

Combining the Sobolev embeddings with the Proposition 2.7, we get the following corollary.

Corollary 2.8. If \( (a, b) \in H^s \times H^r \), \( s, r > d/2 + 1 \), \( T_a T_b = T_{ab} + R \), where \( R \) is continuous \( H^t \rightarrow H^{t+1} \).
If \( \nabla a \in H^s, b \in \Gamma^m \), \( r \geq 2 \), then \([T_a T_b] = T_{\{a, b\}} + R\) where \( R \) is continuous \( H^{t+m} \rightarrow H^{t+2} \).
Proposition 2.9. (elliptic estimate)
Let \( a \in \Gamma^m_1 \) be a symbol satisfying
\[ \text{Rea}(x, \xi) \geq c(1 + |\xi|)^m, \]
there are constants \( C_1, C_2 \) such that for all \( u \in H^m \),
\[ \|u\|_{H^s} \leq C_1 \|Ta u\|_{H^{s-m}} + C_2 \|u\|_{L^2} \]

3 The one-dimensional case
In this section we prove Kato smoothing in dimension 1. Several technical points are simplified in this case, in particular we do not need further assumption than the existence of \((\rho, u)\) solution of \( EK1D \).

3.1 Para-linearization of the Euler-Korteweg equations
Since \( \zeta = R(\rho) \) is a smooth diffeomorphism, according to Prop. 5.2 it is equivalent to prove the regularization for the extended variables \((w, u)\), or for \( z = u + iw \). We start with (SB) and consider
\[ z \in C_t H^s \cap C_t^1 H^{s-2}, \zeta \in C_t H^{s+1} \cap C_t^1 H^{s-1} \]
with \( s > 1 + 1/2 \) such that
\[ \partial_t z + u \partial_x z + iv \partial_x z + i \partial_x (a \partial_x z) = -p'(\zeta) \text{Re}(z), \quad z = u + iv, \]

The Cauchy theory of the Euler-Korteweg system (for example see [BGDD07] proposition 4.3) ensures that the \( H^s \) norm of \( z \) is controlled by the \( H^s \) norm of the initial data \((u_0, w_0)\).

According to corollary 2.6 we have
\[ u \partial_x z = T_u \partial_x z + R_1, \quad R_1 \in C_t H^s, \]
\[ v \partial_x z = T_v \partial_x z + R_2, \quad R_2 \in C_t H^s, \]
\[ a \partial_x z = T_a \partial_x z + R_3, \quad R_3 \in C_t H^{s+1}, \]

hence \( \partial_x (a \partial_x z) = \partial_x (T_a \partial_x z) + R_4, \quad R_4 \in C_t H^s. \)

Thus (SB) implies
\[ \partial_t z + T_u \partial_x z + T_{iv} \partial_x z + i \partial_x (T_a \partial_x z) = R, \quad R \in C_t H^s. \]

In order to reduce the analysis to the study of a Schrödinger equation satisfied by some distribution \( Z_s \in L^2 \), it is natural to multiply the equation by \( T_{|\xi|^r} \). Moreover we want to cancel out the ‘bad’ term \( T_{iv} \partial_x z \). Thus (following [BGDD06]) we look for a multiplier which has the form \( T_{|\xi|^r \varphi_s(x)} \) where \( \varphi_s \), to be determined, does not vanish and belongs to \( W^{r, \infty} \) for some \( r \geq 2 \).

According to Proposition 2.7, the commutators
\[ [T_{|\xi|^r \varphi_s(x)}, T_u \partial_x] \text{ and } [T_{|\xi|^r \varphi_s(x)}, T_v \partial_x], \]

define continuous operators $H^s \to L^2$, moreover
\[ [T_{\xi}^s\varphi_s(x), i\partial_x T_u \partial_x] = T_{\xi}^s\varphi_s(x) - |\xi|^2 a(x) + T', \]
where $T'$ is continuous $H^s \to L^2$ and $\{\cdot, \cdot\}$ is the Poisson bracket
\[ \{\xi^s \phi(x), -|\xi|^2 a(x)\} = \partial_x^s (|\xi|^s \phi_a) \partial_x (-|\xi|^2 a(x)) - \partial_x (|\xi|^s \phi_a) \partial_x^s (-|\xi|^2 a(x)) \]
In order to cancel out the main order term $T_{\xi}^s\varphi_s v i\partial_x z = T_{-|\xi|^s \xi \varphi_s v z}$, it is therefore sufficient to have $-s \varphi \partial_x a + 2a \partial_x \varphi = \varphi v$. The choice $\varphi = \sqrt{a^{s/2}}$ gives
\[ 2a \partial_x (a^{s/2} \sqrt{\rho}) = sa^{s/2} \sqrt{\rho} \partial_x a + 2a^{s/2+1} \partial_x \rho \]
\[ = s \varphi_a \partial_x a + a^{s/2} \frac{\partial \rho}{\sqrt{\rho}} \]
\[ = s \varphi_a \partial_x a + \phi_s v. \]
We define $Z_s = T_{\varphi_s}^s$ and $\rho$. According to the commutator estimates and the previous computations the equation on $Z_s$ is
\[ \partial_t Z_s + T_u \partial_x Z_s + i \partial_x (T_u \partial_x Z_s) = R, \quad R \in C^1 L^2, \]
where the norm of $R$ is controlled by
\[ \|(u, w)\|_{H^s} \lesssim \|(u_0, w_0)\|_{H^s}. \]
By ellipticity of $T_{\varphi_s}^s$, it is clear that a gain of derivative for $z/\langle x \rangle^{(1+\varepsilon)/2}$ is equivalent to a gain of derivatives for $Z_s/\langle x \rangle^{(1+\varepsilon)/2}$ (note that it relies on the fact $\langle x \rangle^{(1+\varepsilon)/2}$ is smooth to make it commute with $T_{\varphi_s}^s$), thus we have proved the following.

**Lemma 3.1.** The proof of Kato smoothing for $(u, v)$ amounts to prove
\[ \|Z_s/\langle x \rangle^{(1+\varepsilon)/2}\|_{L^2_{t,loc} H^{1/2}} \leq C \|Z_s\|_{C^1 L^2}, \]
for $Z_s$ solution of
\[ \partial_t Z_s + T_u \partial_x Z_s + i \partial_x (T_u \partial_x Z_s) = R, \quad R \in C^1 L^2. \]

### 3.2 Smoothing effect on $Z_s$

Doi’s method [Doi96] is based on estimates for
\[ \frac{d}{dt} \langle T_{\xi} Z_s, Z_s \rangle = \frac{d}{dt} \int_{\mathbb{R}} T_{\xi} Z_s \overline{Z_s} dx, \]
where $\psi(t, x, \xi)$ is a symbol chosen such that this derivative is bounded from below (up to negligible terms) by $\|f Z_s\|_{H^{1/2}}$, for some $f$ decaying fast enough. Here we have
\[ \frac{d}{dt} \langle T_{\xi} Z_s, Z_s \rangle = \langle \partial_t (T_{\xi} Z_s, Z_s) + (T_{\xi} Z_s, \partial_t Z_s) \]
\[ = \langle -T_{\xi} T_u \partial_x Z_s + iT_{\xi} \partial_x T_u \partial_x Z_s, Z_s \rangle \]
\[ = \langle -[T_{\xi}, T_u \partial_x] Z_s - iT_{\xi} \partial_x T_u \partial_x Z_s, Z_s \rangle \]
\[ = \langle T_{\xi} [\xi^s a] Z_s, Z_s \rangle + R, \]
where $R \in L^1_{loc,t} L^2$ (with a control by $\|Z_s\|_{C^1 L^2}$).
Proposition 3.2. For any \( \varepsilon > 0 \), there exists a symbol \( p \in \Gamma^0_{\infty} \) such that

\[
\{ip, |\xi|^2 a\} \geq c|\xi|/\langle x \rangle^{1+\varepsilon}.
\]

The construction of \( p \) is classical and postponed to the appendix. Assume that such a \( p \) exists. By integrating from 0 to \( T \) and applying the weighted Gårding inequality of appendix A.2 we find

\[
\mathcal{T}_p Z_s(T), Z_s(T) - \mathcal{T}_p Z_s(0), Z_s(0) \geq \|Z_s/\langle x \rangle^{(1+\varepsilon)/2}\|_{H^{1/2}} - C \int_0^T \|Z_s\|^2_{L^2} dt
\]

\[
\Rightarrow \|Z_s/\langle x \rangle^{(1+\varepsilon)/2}\|_{H^{1/2}} \lesssim \|Z_s\|_{C_2 L^2} \lesssim \|Z_s\|_{C_2 H^{1/2}}.
\]

This estimate implies by using regularized initial data the following:

Proposition 3.3. If \( Z_0 \in L^2 \) then for any \( \varepsilon > 0 \) \( Z_s/\langle x \rangle^{(1+\varepsilon)/2} \in L^1_{loc, t} H^{1/2}_{x} \).

4 Multi-dimensional case

In the case of several space variables, the degeneracy of the system may prevent the dispersive smoothing effect. To illustrate this fact one may look at the linearized constant coefficient system

\[
\partial_t z + u \nabla z + i\nabla(\text{div} z) = -g'w.
\]

If \( z \) is divergence free this becomes a transport equation, for which there cannot be any smoothing. On the other hand, if \( z \) is potential, the equation satisfied by \( z \) is

\[
\partial_t z + u \nabla z + i\Delta z = -g'w,
\]

and by adapting the (classical) proof of Constantin and Saut [CS88] in the case of the Schrödinger equation one easily obtains the Kato 1/2 smoothing. Based on this simple example, we focus on the case where \( z \) is potential, and we prove the same 1/2 Kato smoothing under a few standard additional assumptions (namely flatness at infinity and non trapping of the bicharacteristics). Recall the multi-dimensional extended Euler-Korteweg system reads for \( z \)

\[
\partial_t z + (u \cdot \nabla )z + i(\nabla z) \cdot w + i\nabla (\text{div} z) = -p'(|\zeta|)w,
\]

where \( \nabla z \) is for the \( j \)-th column \( (\partial_i z_j)_{1 \leq i \leq d} \) (note that this reformulation uses the fact that \( w \) is potential). As previously, we reduce the problem to an \( L^2 \) estimate by using the new quantity \( Z_s = T_{\varphi_s}|\xi|^2 z, \varphi_s \) being chosen so that commutators may control the “bad term” \( i\nabla z \cdot w \). In fact, in the irrotational case we will find out that the previous choice \( \varphi_s = \sqrt{\rho a^{s/2}} := \varphi_0 a^{s/2} \) still works. Our result reads as follows.

Theorem 4.1. Under the following assumptions:

- \( u_0 \) is irrotational,
- the hamiltonian \( a(x,0)|\xi|^2 \) has no trapped bicharacteristic,
\[ a(x,t) \in C_t W^{2,\infty} \cap C_t^1 W^{1,\infty}, \] |\partial_t a| + |\partial_t \nabla a| + |\nabla a| \leq C/|x|^{1+\varepsilon}, \]
then there exists \( T > 0 \) such that
\[ \|(u, w)/(1 + |x|)^{1+\varepsilon}\|_{L^2([0,T];(H^{s+1/2})^2)} \lesssim \|u_0, w_0\|_{H^s}. \]

**Remark 4.2.**
- In our proofs we will use the fact that if \( u_0 \) is irrotational then \( z = u + iw \) is irrotational, this follows from the fact that \( w \) is a gradient, and the implication \( \text{curl} u_0 = 0 \Rightarrow \text{curl} u = 0 \) (Corollary 4.1 in [BGDD07]).
- Since \( w = \sqrt{K(\rho)/\rho} \nabla \rho \), the same theorem holds if one replaces \( w \), resp \( w_0 \), by \( \nabla \rho \), resp. \( \nabla \rho_0 \) (see for product rules in Sobolev spaces Prop. 5.2).

### 4.1 Para-linearization

We use the convention that \( R \) is a generic harmless term in \( L_{\text{loc}}^1 L_{\text{loc}}^2 \), controlled by \( \|z\|_{C_t H^s \cap C_t^1 H^{s-2}} \).

If \( v : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \), we write of \( T_v z = \sum_{j=1}^d T_{v_j} z_j \), and for \( A : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d^2} \), \( T_A z := \left( \sum_{j=1}^d T_{A_j} z_{j}\right)_{i=1}^d \). The scalar rules of para-differential calculus extend straightforwardly to vectors.

Following the existence theorem of [BGDD07], we assume that \( z \in C_t H^s \cap C_t^1 H^{s-2} \), \( \rho \in L^\infty \), has a positive lower bound and \( \nabla \rho \in H^s \) for some \( s > 1 + d/2 \). In particular we have as in the previous section
\[
\begin{align*}
  u \nabla z &= T_{u\nabla z} + R_1, \quad R_1 \in C_t H^s, \\
v \nabla z &= T_{v\nabla z} + R_2, \quad R_2 \in C_t H^s, \\
a \partial_j z &= T_{a\partial_j z} + R_3, \quad R_3 \in C_t H^{s+1}, \text{ and for } A : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d^2}, \quad \partial_k(a \partial_j z) = T_k(a \partial_j z) + R_4, \quad R_4 \in C_t H^s.
\end{align*}
\]

We first exploit the curl-free assumption: let \( Q \) be the projector on curl-free vector fields, its symbol is \( \xi \xi' / |\xi|^2 \), and it satifies \( \nabla \text{div} Q = \Delta Q \). This implies
\[
\nabla(\text{div} vz) = a \nabla \text{div} z + (\nabla a) \text{div} z = \text{div}(a \nabla z) + (\nabla a) \text{div} z - \nabla z \cdot \nabla a. \quad (3)
\]
As in the previous section we take \( \varphi_\pm = \sqrt{\rho}a^{s/2} \) and we multiply (2) by \( T_{\varphi_\pm} z \). This gives
\[
\begin{align*}
  T_{\varphi_\pm} \xi \cdot \partial_t z + T_{\varphi_\pm} \xi \cdot (u \cdot \nabla z) + iT_{\varphi_\pm} \xi \cdot (|\nabla z| \cdot w) + T_{\varphi_\pm} \xi \cdot (i \text{div}(a \nabla z)) \\
  + iT_{\varphi_\pm} \xi \cdot (|\nabla a| \text{div} - \nabla z \cdot \nabla a) &= R, \\
  \text{hence } T_z + (u \nabla) z + i(\nabla z) \cdot w + i(\nabla a) \text{div} z - \nabla a \cdot \nabla z \\
  &= R + [i \text{div} a, T_{\varphi_\pm} z].
\end{align*}
\]
Using the rules of para-differential calculus we have
\[
[i \text{div} a, T_{\varphi_\pm} z] = [T_{-ia|\xi|^2}, T_{\varphi_\pm} z] + R = T_{-a|\xi|^2, \varphi_\pm} z + R,
\]
But
\[
\{ -a|\xi|^2, \varphi_\pm \} = -2|\xi|^4 a \xi \cdot \nabla \varphi_\pm + s|\xi|^4 \varphi_\pm \nabla a \cdot \xi \\
= -\varphi_\pm |\xi|^4 \xi \cdot w,
\]
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so that

$$[\text{div}\nabla, T_\varphi|z|]z = iT_w \cdot \nabla Z_s + R = i\nabla Z_s \cdot w + R,$$

since \(w\) is curl-free. The para-linearized equation on \(Z_s\) is thus

$$\partial_t Z_s + T_u \nabla Z_s + i\text{div}(T_u \nabla Z_s) + iT_{\nabla a}\text{div}Z_s - T_\xi\nabla a Z_s = R,$$

where

$$T_\xi\nabla a Z_s := \left(\sum_{j=1}^d -iT_{\partial_{x_j} a \partial_x Z_j})_{1 \leq i \leq d} = i(T_{\nabla a} (\nabla Z_s)^t\right),$$

**Remark 4.3.** The fact that we replaced \(\nabla\text{div}\) by \(\text{div}\nabla\) has allowed the commutator to cancel the commutator \([\text{div}\nabla, T_\varphi|z|]z\) to cancel the 'bad' term \(i\nabla z\nabla w\). Without the curl free assumption there is a remaining term \(i\nabla z\nabla w - i\text{div}(z)w\).

To summarize we have obtained

**Lemma 4.4.** The proof of Kato smoothing for \((u, v)\) amounts to prove

$$\|Z_s/\langle x\rangle^{(1+\varepsilon)/2}\|_{L^2_t L^2_x H^{1/2}} \leq C\|Z_s\|_{C_t L^2},$$

for \(Z_s\) solution of

$$\partial_t Z_s + T_u \cdot \nabla Z_s + i\text{div}(T_u \nabla Z_s) + iT_{\nabla a}\text{div}Z_s - T_\xi\nabla a Z_s = R,$$

with \(R \in C_t L^2\) whose norm is controlled by \(\|(u_0, w_0)\|_{H^s}\).

### 4.2 The smoothing effect

Let \(p \in \Gamma^0\), a scalar symbol to be determined later, independent of time. Following Doi we differentiate with respect to \(t\) the integral \(\int Z_s^* T_p Z_s := \langle T_p Z_s, Z_s \rangle\).

$$\frac{d}{dt}\langle T_p Z_s, Z_s \rangle = \langle T_p \partial_t Z_s, Z_s \rangle + \langle T_p Z_s, \partial_t Z_s \rangle$$

$$= \left<T_p \left(- (T_u \cdot \nabla)Z_s - i\text{div}(a\nabla Z_s) + T_\xi(a\nabla a_{\xi})Z_s\right), Z_s\right>$$

$$+ \left<T_p Z_s, -(T_u \cdot \nabla)Z_s - i\text{div}(a\nabla Z_s) + T_\xi(a\nabla a_{\xi})Z_s\right>$$

Now using that \(\text{div}(a\nabla \cdot)\) is self-adjoint and \(T_{iu\xi}^* = -T_{iu\xi} + Q\), with \(Q\) of order 0 we find

$$\frac{d}{dt}\langle T_p Z_s, Z_s \rangle = \langle -[T_p, \text{div}\nabla]Z_s, Z_s \rangle + \langle -[T_p, T_u \cdot \nabla]Z_s, Z_s \rangle$$

$$+ \langle (T_p T_\xi(a\nabla a_{\xi}))^* T_p Z_s, Z_s \rangle + I, \quad (4)$$

where \(I \in L^1_{t, loc}\) is controlled by \(\|Z_s\|_{C_t L^2_x} \lesssim \|(u_0, w_0)\|_{H^s}\).

We will generically call such terms \(I\) even though they may change from a line to another.

The commutator \([T_p, T_u \cdot \nabla]\) is an operator of order 0, thus \(-[T_p, T_u \cdot \nabla]Z_s, Z_s\) can
be included in I. There is no such cancellation for the second line of (4), since it is easily checked that \( (T_{(\xi \cdot )\nabla a - \nabla a\xi^t})^* = T_{(\xi \cdot )\nabla a - \nabla a\xi^t} + Q \), with \( Q \) of order 0. On the other hand we have

\[
\text{Re} \left( T_p(T_{(\xi \cdot )\nabla a - \nabla a\xi^t} Z_s, Z_s) \right) = \text{Re} i \sum_{k,l} (T_p(T_{\partial a\partial l}Z_k - T_{\partial a\partial l}Z_k, Z_l) - \text{Im} \left( \sum_{k,l} -\langle T_pZ_k, T_{\partial a\partial l}Z_l \rangle - \langle T_{\partial a\partial l}Z_k, T_pZ_l \rangle \right) + I.
\]

If \( p \) is real we obtain by reordering the sum

\[
\text{Re} \left( T_p(T_{(\xi \cdot )\nabla a - \nabla a\xi^t} Z_s, Z_s) \right) = \text{Im} \left( \sum_{k,l} \langle T_pZ_k, T_{\partial a\partial l}Z_k \rangle + \langle T_{\partial a\partial l}Z_k, T_pZ_l \rangle \right) + I
\]

Injecting these estimates in (4) gives

\[
\frac{d}{dt} \langle T_pZ_s, Z_s \rangle = \langle [T_p, i\nabla \text{div}]Z_s, Z_s \rangle + I.
\]

We may now paralinearize the last term: \( i\text{div}(a\nabla \cdot ) = -T_{a(\xi x)} + Q_1 \), with \( Q_1 \) of order 1, thus

\[
\langle [T_p, i\text{div}(a\nabla \cdot )]Z_s, Z_s \rangle = \langle [T_p, T_{a(\xi x)}]Z_s, Z_s \rangle + I = \langle T_{(p,a)|\xi|^2} Z_s, Z_s \rangle
\]

There is a huge gain here since we can now decouple the equations

\[
\frac{d}{dt} \langle T_pZ_j, Z_j \rangle = \langle T_{(p,a)|\xi|^2} Z_j, Z_j \rangle + I.
\]

We may now apply a para-differential version of Doi’s operator construction.

**Lemma 4.5.** Under the assumption of Theorem 4.1, there exists a real valued symbol \( p \in \Gamma^0_1 \) and some constant \( c, C > 0 \) such that for \( t \) small enough

\[
\{p, a(t)|\xi|^2\} := \sum_{j=1}^d \langle \partial_{\xi_j} p \rangle \partial_{x_j} (a|\xi|^2) - \langle \partial_{x_j} p \rangle \partial_{\xi_j} (a|\xi|^2) \geq \frac{|\xi|}{(x)^{1+\varepsilon}} - C
\]

The (sketch of) construction of this operator is made in appendix A.3. The application of the weighted Gårding inequality (Prop. A.2) combined with Lemma 4.5 readily gives

\[
\int_0^T \langle T_pZ_s, Z_s \rangle = \int_0^T \langle T_{(p,a)|\xi|^2} Z_s, Z_s \rangle + \int_0^T I \geq c' \|Z_s/(x)^{(1+\varepsilon)/2}\|_{H^{1/2}}^2 - C\|Z_s\|_{L^2}^2 + \int_0^T I \Rightarrow \|Z_s/(x)^{(1+\varepsilon)/2}\|_{L^2([0,T];H^{1/2})} \lesssim \|Z_0\|_{L^2}^2 \lesssim \|(u_0, w_0)\|_{H^s}^2.
\]
5 Solutions with nonzero endstates

The authors of [BGDD07] did not restrict their analysis to solutions \((u, w)\) vanishing at infinity, indeed there exists special traveling waves solutions that \(u\) has different endstates (they call such solutions *capillary profiles*). Those solutions are physically pertinent (they can correspond to a change of state) and thus should be included in the present analysis. We will work under the regularity assumptions of theorem 1 in [BGDD07] supplemented by the assumptions of our Theorem 4.

**Theorem 5.1.** Let \((\rho, u)\) be a solution to \((E)\) such that

\[
(\nabla^2 \rho, \nabla u) \in C([0, T]; H^{s+3}),
\]

and \(u\) is irrotational. We denote \((\rho_0, u_0) := (\rho(t = 0), u(t = 0))\). Let \((\rho_0, u_0)\) be an initial data such that

\[
(\rho_0, u_0) - (\rho_0, u_0) \in H^{s+1} \times H^s, \quad s > 1 + d/2.
\]

We assume that \(\rho_0([0, T] \times \mathbb{R}^d) \subset K, \) with \(K\) a compact set of \(\mathbb{R}\) on which \(a\) and \(g\) are smooth.

\[
\begin{aligned}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t u + (u \cdot \nabla) u + \nabla g(\rho) &= \nabla (K(\rho) \Delta \rho + \frac{1}{2} K'(\rho) |\nabla \rho|^2), \\
(\rho, u)|_{t=0} &= (\rho_0, u_0).
\end{aligned}
\]

Let \((\rho, u) \in (\rho_0, u_0) + C([0, T]; H^{s+1} \times H^s)\) be the solution provided by the Theorem 1.1 in [BGDD07]. Then, assuming that \((\rho - \rho_0, u - u_0)\) satisfies the assumptions of theorem (4.1), the solution of (5) satisfies moreover \((\nabla \rho, u) - (\nabla \rho_0, u_0) \in L^1_{loc,t}(H^{s+1/2} \times H^{s+1/2})\).

Key ingredients for this result are standard multiplication and composition estimates in Sobolev spaces, that we recall here for the ease of the reader. Proofs can be found in the appendix B of [BGDD07].

**Proposition 5.2.** In every case, we assume \(s \geq 0\).

Product rule:
For \(k \in \mathbb{N}\), there exists \(C(s, k, d)\) such that

\[
\|uv\|_{H^s} \leq C(\|u\|_{L^\infty} \|v\|_{H^s} + \|\nabla^k u\|_{H^{s-k}} \|v\|_{L^\infty}).
\]

Composition rules:
Let \(F \in W^{\sigma+1, \infty}(I; \mathbb{R})\), where \(\sigma\) is the smallest integer such that \(\sigma \geq s\), and assume \(F(0) = 0\). Then if \(\text{Im}(v) \subset I\)

\[
\|F(u)\|_{H^s} \leq C(1 + \|u\|_{L^\infty})^\sigma \|F'\|_{W^{\sigma, \infty}} \|v\|_{H^s}.
\]

Without assuming \(F(0) = 0\) and for \(m \in \mathbb{N}\)

\[
\|\nabla^m F(v)\|_{H^s} \leq C(1 + \|v\|_{L^\infty})^{m+\sigma} \|F'\|_{W^{m+\sigma, \infty}} \|D^m v\|_{H^s}.
\]
If \( \text{Im}(v) \subseteq I \), \( \text{Im}(w) \subseteq I \)

\[
\|F(w) - F(v)\|_{H^s} \leq C \left( \|F\|_{L^\infty} \|v - w\|_{H^s} + (1 + \|v\|_{L^\infty} + \|w\|_{L^s})^{\sigma + k + 1} \|F''\|_{W^{\sigma + k}}(\|\nabla^k v\|_{H^{s-k}} + \|\nabla^k (w - v)\|_{H^{s-k}}) \right).
\]

We may now prove Theorem 5.1.

**Proof of theorem 5.1.** We focus on the extended system

\[
\partial_t \tilde{z} + (u \cdot \nabla) \tilde{z} + i(\nabla z) \cdot w + i \nabla a(\text{div} \tilde{z}) = -p'(\zeta) w.
\]

Set \( \tilde{u} = u - u_s \), \( \tilde{w} = w - w_s \), \( \tilde{z} = z - z_s \), then \( \tilde{z} \) satisfies

\[
\partial_t \tilde{z} + u \cdot \nabla \tilde{z} + i(\nabla \tilde{z}) \cdot w + i \nabla a(\text{div} \tilde{z}) = -p' w + p' w_s - \tilde{u} \cdot \nabla \tilde{z} - i \nabla \tilde{z} \cdot \tilde{w} + i \nabla (a - a_0) \text{div} \tilde{z}
\]

\[
\quad \quad \quad \quad \quad \quad =: R
\]

with \( a = a(\zeta) \).

We are mostly reduced to the argument of the previous section. Indeed it is easy to see that the previous analysis can still be applied for the left hand term (note that the construction of \( \psi \) essentially relies on the fact that \( a \) is bounded away from zero, in particular the limits of \( u \) at \( \pm \infty \) do not matter, and the gauge transformation only relies on the relation between \( a(\zeta) \) and \( w \)). If we use as previously the function \( \tilde{Z}_s = T_{\rho_s(\zeta)} \tilde{z} \), the only new terms in

\[
\frac{d}{dt} \left< T_p \tilde{Z}_s, \tilde{Z}_s \right>,
\]

are \( \left< T_p R, \tilde{Z}_s \right> \) and \( \left< T_p \tilde{Z}_s, R \right> \). In order to include these in \( I \in L^1_{loc,t} \), it suffices to check that \( R \in C_t H^s \).

Since \( \nabla \tilde{z} \in C_t H^{s+1} \) and \( \tilde{z} \in C_t H^s \) the rules of product in Sobolev spaces imply

\( -\tilde{u} \cdot \nabla \tilde{z} - i \nabla \tilde{z} \cdot \tilde{w} \in C_t H^s \).

Now using the rules of composition we also have \( a - a_0 \in C_t H^{s+1} \), which implies

\( i \nabla ((a - a_0) \text{div} \tilde{z}) \in C_t H^s \).

Similarly, noting that \( -p' w + p' w_s = \nabla (-p(\zeta) + p(\zeta)) \) we have \( -p(\zeta) + p(\zeta) \in C_t H^{s+1} \) (we see here that the assumption on the decay of the perturbation is essential even for \( \tilde{\rho} \)), and thus \( -p' w + p' w_s \in C_t H^s \). This allow to ‘neglect’ these new terms, and the rest of the proof goes as in section 4.

**Remark 5.3.** The careful reader may have noted that we do not need in fact

\( (\nabla^2 \rho, \nabla w) \in C([0, T]; H^{s+3}) \).

This is in fact needed for the existence theorem in [BGDD07], but the Kato-smoothing only requires \( (\nabla^2 \rho, \nabla w) \in C([0, T]; H^{s+1}) \).

The decay assumption for the derivatives of \( a \) are somehow not satisfying in our frame because the Euler-Korteweg system admits special planar traveling waves solutions which only depend on \( x \cdot n - ct \) (\( n \) is the direction of propagation, \( c \) is the speed, see [BGDDJ05]). In particular, for \( t \) fixed they are constant on any
affine hyperplan orthogonal to \( n \) and \( \nabla a(x \cdot n) \) can obviously not decay as \( 1/(x)^{1+\varepsilon} \).

Though we do not have a general result, we will give some insights indicating that smoothing may occur also for perturbations of traveling waves. We restrict ourselves to the extended system linearized near a smooth traveling wave \((\zeta, u, w)\), of initial value \((\zeta_0, u_0, w_0)\).

\[
\begin{align*}
\partial_t z + u \cdot \nabla z + i \nabla x \cdot w + i \nabla (a \nabla z) &= -p'w - p''\rho w - z \cdot \nabla u \\
\quad - i \nabla x \cdot w - i \nabla (a' \rho \nabla z),
\end{align*}
\]

where we denote generically \( f = f(u, w, \zeta) \) and \( f_0 = f(u_0, w_0, \zeta_0) \). Up to a linear change of basis, we may assume that \( n = (1 \ 0 \ \cdots \ 0) \) and we will use the notation \( x = (x_1, x') \).

**Proposition 5.4.** The bicharacteristics

\[
\begin{align*}
\dot{X}(t, x_0, \xi_0) &= 2a_0(X)\Xi, \ t \geq 0, \\
\dot{\Xi}(t, x_0, \xi_0) &= -|\Xi|^2 \nabla a_0(X), \ t \geq 0, \\
X(0, x_0, \xi_0) &= x_0, \ \Xi(0, x_0, \xi_0) = \xi_0,
\end{align*}
\]

are not trapped, i.e. \(|X(t, x_0, \xi_0)| \rightarrow |t| \rightarrow \infty \infty \). More precisely they satisfy the following uniform non trapping property:

\[
\forall (x_0, \xi_0), \exists r > 0: \forall (x_1, x', \xi) \in |x_0, 1 - r, x_0, 1 + r| \times \mathbb{R}^{d-1} \times B(\xi_0, r) :
\forall K > 0, \exists T_K > 0: \forall |t| \geq T_K : |X(t, x, \xi) - x| \geq K.
\]

**Proof.** We first check ‘simple’ non trapping. Recall that the bicharacteristics are integral curves of the hamiltonian \( a_0(x)|\xi|^2 \), whose conservation readily implies

\[
C/ \sup a_0 \leq |\Xi|^2 \leq C/ \inf a_0.
\]

Since \( a_0 = a_0(x_1) \), we have

\[
-|\Xi|^2 \nabla a_0(X) = -|\Xi|^2 \begin{pmatrix} a_0' \\ 0 \\ \vdots \\ 0 \end{pmatrix}
\]

As a consequence, \( \Xi_j(t) = \xi_{0,j} \) and \( (X_j)_{2 \leq j \leq d} \) are monotone of slope larger than \( \inf(a_0)|\xi_{0,j}| \). They remain bounded in \( t \) iff

\[
\Xi_2 = \cdots = \Xi_n = 0.
\]

But in this case we have \( a_0(X)|\Xi|^2 = a_0(X_1)|\Xi_1|^2 \), the conservation of this quantity implies that \( \Xi_1 \) is uniformly bounded away from 0. We deduce that \( X_1(t) \) is monotone and (the modulus of) its slope has a lower bound, thus \( |X_1(t)| \rightarrow t \rightarrow \infty \infty \) and there can be no trapped bicharacteristic.

Since we have no decay assumption uniform non trapping can not be deduced from the ‘simple’ non trapping. We give here a direct proof. Two cases must be distinguished:
• There exists $j \geq 2$ such that $\xi_{0,j} \neq 0$. For $r$ small enough,

$$\forall (x, \xi) \in B((x_0, \xi_0), r), |\xi_j| \geq |\xi_{0,j}|/2,$$

which directly implies $|X_j(s, x, \xi) - x_j| \geq |\xi_{0,j}| \inf(a_0)$, and thus the (local) uniform non-trapping.

• For $j \geq 2$, $\xi_{0,j} = 0$. We denote $\xi' = (\xi_2, \cdots, \xi_d)$. Then $a_0(X) |\Xi|^2 = a_0(X) (|\xi_1|^2 + |\xi_2|^2 + \cdots + |\xi_j|^2) = a_0(x) |\xi|^2$. Take $r$ small enough such that $\forall (x, \xi) \in B((x_0, \xi_0), r), |\xi'|^2 < \inf(a_0)/(2 \sup(a_0)) |\xi_j|^2$, this gives

$$|\Xi|^2 + |\xi'|^2 \geq \frac{\inf(a_0)}{\sup(a_0)} |\xi|^2 \Rightarrow |\Xi|^2 \geq \frac{\inf(a_0)}{2 \sup(a_0)} |\xi| |\xi_j|^2.$$

As a consequence $\Xi$ has a constant sign, and up to decreasing $r$ so that $|\xi_1| \geq |\xi_{0,1}|/2$

$$|X_1 - x_1| \geq \frac{\inf(a_0)^{3/2}}{\sqrt{8} \sup(a_0)} |\xi_1|,$$

which implies again (local) uniform non-trapping.

\[\square\]

Property 5.4 indicates that there is (in some sense) dispersion, and thus smoothing should occur. We now prove this under a special assumption.

**Proposition 5.5.** Assume that there exists $\alpha > 0$ such that

$$2 \sqrt{a_0(x_1)} - \frac{a_0'(x_1)}{\sqrt{a_0(x_1)}} x_1 \geq \alpha.$$

Then the solution $(u, w)$ of the linearized system (6) satisfies

$$\forall \varepsilon > 0, \exists T > 0 : \|\{(u, w)/\langle x \rangle^{1+\varepsilon}\|_{L^1([-T,T])}((H^{s+1/2})^2) \leq C\|\{(u_0, w_0)\|_{(H^s)^2}.$$

**Proof.** The proof is essentially similar to that of theorem 5.1, thus we will only detail the original points. By the same arguments it is reduced to the smoothing for $Z_s = T_{\varphi_{\varepsilon}}\langle x \rangle^{\varepsilon} = T_{\varphi_{\varepsilon}}\langle x \rangle^{\varepsilon}(u + iw)$, which satisfies the equation

$$\frac{d}{dt} \langle T_p Z_s, Z_s \rangle = \langle T_p(-i \nabla (a \text{div} Z_s)), Z_s \rangle + \langle T_p Z_s, -i \nabla (a \text{div} Z_s) \rangle + I,$$

where $I \in L^1_{\text{loc},L^2}$ has a norm controlled by $\|Z\|_{L^2}$. It is sufficient to construct $p$ such that $\{a|\xi|^2, p\} \geq |\xi|/\langle x \rangle^{1+\varepsilon} - C$. If there exists $q$ such that $\partial_j^2 \partial_j q \leq C\langle x \rangle \langle \xi \rangle^{-1/2}$ and $\{a|\xi|^2, q\} \geq |\xi|$ the method to deduce $p$ from $q$ can be directly applied (this step is indeed independent of the assumptions on $a_0$). Thus we only prove that such a $q$ exists.

We take $q$ of the form $f(x_1 - ct)\langle x, \xi \rangle/|\xi|$. Then

$$\{a|\xi|^2, q\} = 2f(x_1 - ct)\overline{a(x_1 - ct)}|\xi| + 2a_0(1 - a_0)\langle x_1 - ct \rangle \langle x, \xi \rangle/|\xi|$$

$$-a'(x_1 - ct)|\xi|f(x_1 - ct) + a'(x_1 - ct)\langle x_1 - ct \rangle |\xi| \langle x, \xi \rangle \xi_1$$

$$= |\xi|(2af - x_1 a'f) + \frac{\xi_1 \langle x, \xi \rangle}{|\xi|} (a'f + 2f'a)$$

$$\xi_1 \langle x, \xi \rangle$$
The term $\frac{\xi(x,\xi)}{|\xi|} (a'f + 2f'q)$ is not bounded in $x'$, thus it must be cancelled. The only way to do so is by taking $f = c/\sqrt{a}$. Fixing $c = 1$, we have

$$\{ a|\xi|^2, q \} = |\xi| (2af - x_1 a') = |\xi| \left( 2\sqrt{a} - x_1 \frac{a'}{\sqrt{a}} \right) \geq |\xi| \left( \alpha - ct \frac{a'}{\sqrt{a}} \right) \geq \frac{a|\xi|}{2},$$

for $t$ small enough.

\[ \square \]

**Remark 5.6.** The condition

$$2\sqrt{a_0(x_1)} - \frac{a_0'(x_1)}{\sqrt{a_0(x_1)}} x_1 \geq \alpha \tag{7}$$

may seem very artificial, it is however satisfied by several traveling waves. Obviously if (7) is true for $a_0(-b)$ instead of $a_0$, it then suffices to perform a coordinate translation to apply Prop 5.5, thus the result only depend on the shape of $a_0$ and not its graph. The most obvious profiles satisfying this assumption are those such that $a_0$ is increasing and then decreasing. Using

$$2\sqrt{a_0(x_1)} - \frac{a_0'(x_1)}{\sqrt{a_0(x_1)}} x_1 = 2\sqrt{a_0(0)} + \int_0^{x_1} \frac{a_0'(y)}{\sqrt{a_0(y)}} - \frac{a_0'(x_1)}{\sqrt{a_0(x_1)}} dy$$

we obtain another example when $\sqrt{a_0}$ is increasing with an inflexion point at $x_1 = 0$ (see [BGDDJ05] for various examples of traveling profiles in dimension 1).

### A Appendix

#### A.1 Construction of $p$ (one-dimensional case)

**Proposition A.1.** For any $\varepsilon > 0$, there exists $p \in \Gamma_{\infty}^0$, $c > 0$ such that

$$\{|\xi|^2 a, ip\} \geq c|\xi|/\langle x \rangle^{1+\delta}.$$  

**Proof.** The obvious candidate is $q(x, \xi) = x \frac{\xi}{|\xi|}$ since $\{ |\xi|^2 a, i x \frac{\xi}{|\xi|} \} = 2\xi \frac{\xi}{|\xi|} a = 2a|\xi|$, but it is not in $\Gamma_{\infty}^0$ since it is not bounded in $x$. Thus it is necessary to use convenient truncatures of this function.

Let $\varphi \in C^\infty(\mathbb{R}; [0, 1])$ be such that $\varphi = 0, x \leq 1, \varphi = 1, x \geq 2$. For $\varepsilon > 0$ (supposed small, specified later) we define $\varphi_+ = \varphi(x/\varepsilon)$, $\varphi_- = \varphi(-x/\varepsilon)$, $\varphi_0 = 1 - \varphi_+ - \varphi_-$. If $\psi_j = \varphi_j(q/\langle x \rangle)$, we define the symbol $p$ by

$$p(x, \xi) = \frac{q}{\langle x \rangle} \psi_0 + (f(|q|) + 2\varepsilon) (\psi_+ - \psi_-),$$

where $f$ is a primitive of $1/\langle x \rangle^{1+\varepsilon}$. 
Using the fact that $\xi/|\xi|$ is constant outside 0 we find

$$\{\xi^2 a, \iota p\} = \partial_\xi a(\psi_0 \xi/\langle x \rangle |\xi|) + q \partial_\xi (\psi_0 /\langle x \rangle)$$

$$+ 2a\xi \partial_\xi (f(q) + 2\varepsilon) \psi_+ + 2a\xi (f(q) + 2\varepsilon) \partial_\xi \psi_+$$

$$- 2a\xi \partial_\xi (f(q) + 2\varepsilon) \psi_- - 2a\xi (f(q) + 2\varepsilon) \partial_\xi \psi_-$$

$$= I_1 + I_2 + I_3 + I_4 + I_5,$$

where the $I_j$’s are handled as follows:

**First term:** $I_1 = 2a|\xi|\psi_0/\langle x \rangle + 2a\xi q \partial_\xi (\psi_0 /\langle x \rangle)$, with

$$q \partial_\xi (\psi_0 /\langle x \rangle) = -\psi_0 q x /\langle x \rangle^3 + q /\langle x \rangle \psi_0 \partial_\xi (\langle x \rangle \xi /|\xi|) = -\psi_0 q x /\langle x \rangle^3 + q \psi_0 /\langle x \rangle^4 |\xi|/|\xi|,$$

thus

$$I_1 = 2a|\xi|\psi_0 /\langle x \rangle + 2a\xi (-\psi_0 q x /\langle x \rangle^3 + q \psi_0 /\langle x \rangle^4 |\xi|/|\xi|).$$

**Second term:** $I_2 = 2a|\xi| f(|q|) \psi_+ = 2a|\xi| /\langle x \rangle^{1+\delta} \psi_+.$

**Third term:** $I_3 = 2a\xi (f(|q|) + 2\varepsilon) \psi_+ /\langle x \rangle^3.$

**Fourth term:** $I_4 = 2a|\xi| f(|q|) \psi_- = 2a|\xi| /\langle x \rangle^{1+\delta} \psi_-.$

**Fifth term:**

$$I_5 = -2a\xi (f(|q|) + 2\varepsilon) \psi_- /\langle x \rangle^3 = 2a\xi (f(|q|) + 2\varepsilon) \psi_+(-x, -\xi).$$

so that

$$\{\xi^2 a, \iota p\} = 2a|\xi| \left( \psi_0 /\langle x \rangle (1 - \text{sign}(\xi) \frac{q x}{\langle x \rangle^2}) + \psi_+ /\langle x \rangle^{1+\delta} + \psi_- /\langle x \rangle^{1+\delta} \right)$$

$$+ 2a|\xi| \left( q \psi_0 /\langle x \rangle^4 + (2\varepsilon + f(|q|))(\psi_+ + \psi_+(-x, -\xi)) /\langle x \rangle^3 \right).$$

If $\psi_0 (q/\langle x \rangle) \neq 0$ we have $|q|/\langle x \rangle \leq \varepsilon$, therefore $\psi_0 /\langle x \rangle (1 - \text{sign}(\xi) \frac{q x}{\langle x \rangle^2}) \geq \psi_0 /\langle 2 \langle x \rangle \rangle$ for $\varepsilon$ small enough.

Using that $\psi_0 = \psi_+ - \psi_+(-x, -\xi)$ and the support condition on $\psi_0$ we obtain

$$q \psi_0 /\langle x \rangle^4 \geq \varepsilon /\langle x \rangle^3 (-\psi_+ - \psi_+(-x, -\xi)),$$

which implies

$$2a|\xi| \left( q \psi_0 /\langle x \rangle^4 + (2\varepsilon + f(|q|))(\psi_+ + \psi_+(-x, -\xi)) /\langle x \rangle^3 \right) \geq 0.$$

We finally deduce the inequality

$$\{\xi^2 a, \iota p\} \geq 2a|\xi| \left( \frac{\psi_0}{2 \langle x \rangle} + \frac{\psi_+}{\langle x \rangle^{1+\delta}} + \frac{\psi_-}{\langle x \rangle^{1+\delta}} \right) \geq a|\xi| /\langle x \rangle^{1+\delta}.$$
A.2 A weighted Gårding inequality for non smooth symbols

This result is a direct adaptation of a version for an operator of \( \Gamma^{1/2}_\rho \) in [ABZ].

**Proposition A.2.** Let \( T_c \) be a para-differential operator such that
\[
c(x, \xi) \geq K|\xi|/\langle x \rangle^{1+\delta}, \quad c \in \Gamma^1_1
\]
then
\[
\exists K_1, K_2 > 0 : \langle T_c u, u \rangle \geq K_1 \|u/\langle x \rangle^{1+\delta}\|_{H^{1/2}} - K_2 \|u\|_{L^2}^2.
\]

**Proof.** Since the symbol \( c \) is not in \( \Gamma^{1/2}_\rho \) we cannot directly use usual the sharp Gårding’s inequality. However the method consists in reducing the analysis to a simple elliptic estimate as for the Gårding’s inequality.

It suffices to check that
\[
\sum_{j=0}^{\infty} 1/2^{j(1+\delta)} \|\theta_j u\|_{H^{1/2}}^2 \leq C_1 \langle T_c u, u \rangle - C_2 \|u\|_{L^2}^2,
\]
where \( (\theta_j) \) is a sequence of functions \( C^\infty_c(\mathbb{R}; \mathbb{R}^+) \) such that
\[
\sum_{j=0}^{\infty} \theta_j^2 = 1, \quad \text{supp}(\theta_0) \subset B(0, 2), \quad \theta_j = \theta(2^{-j} \cdot), \quad \text{supp}(\theta) \subset \{1/2 \leq |x| \leq 2\}.
\]

Let us write
\[
\langle T_c u, u \rangle = \sum_{j=0}^{\infty} \langle T_c \theta_j^2 u, u \rangle.
\]

Let \( \Theta \) be a function \( C^\infty_c \) such that \( 0 \notin \text{supp}(\Theta) \), \( \Theta = 1 \) on \( \text{supp}(\theta) \). We also define \( \Theta_j = \Theta(2^{-j} \cdot) \), as well as \( \Theta_0 \in C^\infty_c \) with \( \Theta_0 = 1 \) on \( \text{supp}(\theta_0) \). Then
\[
\sum_{j=0}^{\infty} \langle T_c \theta_j^2 u, u \rangle = \sum_{j=0}^{\infty} \langle T_c \theta_j^2 u, \Theta_j u \rangle + \langle T_c \theta_j^2 u, (1 - \Theta_j) u \rangle
\]
\[
= \sum_{j=0}^{\infty} \langle T_c \theta_j u, \theta_j \Theta_j u \rangle + \langle \langle T_c, \theta_j \rangle \theta_j u, \Theta_j u \rangle
\]
\[
+ \langle \theta_j u, \theta_j T^*_c (1 - \Theta_j) u \rangle
\]
\[
= \sum_{j=0}^{\infty} \langle \Theta_j T_c \theta_j u, \Theta_j u \rangle + I_1 + I_2
\]
\[
= \sum_{j=0}^{\infty} \langle T_c \Theta_j \theta_j u, \theta_j u \rangle + I_1 + I_2 + \langle (\Theta_j T_c - T_c \Theta_j) \theta_j u, \theta_j u \rangle
\]
\[
= \sum_{j=0}^{\infty} \langle T_c \Theta_j \theta_j u, \theta_j u \rangle + I_1 + I_2 + I_3.
\]

By hypothesis we have \( c \Theta_j \geq 0 \), in particular its square root is \( r := \sqrt{c \Theta_j} \) is real positive. The rules of para-differential calculus then imply
\[
T_r(T^*_r) = T_c \Theta_j + R_j,
\]

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where $R_j$ is a continuous operator $L^2 \to H^{1/2+1/2-1} = L^2$, with a constant of continuity bounded uniformly in $j$. This gives

$$\sum_0^\infty \langle T_c \Theta_j \theta_j u, \theta_j u \rangle = \sum_0^\infty \|T_c \theta_j u\|_{L^2}^2 + \langle R_j \theta_j u, \theta_j u \rangle.$$  

The symbol $r$ satisfies $r \geq 2^{-j((1+\delta)/2)|\xi|^{1/2}}$ (for $\sqrt{c} \geq I_d$ on $[0, 1]$). In particular $r + (1 - \Theta_j)2^{-j((1+\delta)/2)|\xi|^{1/2}}$ is elliptic of order $1/2$. We deduce that

$$\|T_c \theta_j u\|_{L^2}^2 \geq c2^{-j((1+\delta)|\xi|^{1/2}} - C\|T_{(1-\Theta_j)2^{-j((1+\delta)/2)|\xi|^{1/2}}} \theta_j u\|_{L^2}^2,$$

which is (8) 'up to neglectable terms'. It remains to quantify what means neglectable. 

$$I_1 = \sum_0^\infty \langle (T_c, \theta_j) \theta_j u, \Theta_j u \rangle \lesssim \sum_0^\infty \|\theta_j u\|_{L^2}^2 + \|\Theta_j u\|_{L^2}^2 \lesssim \|u\|_{L^2}^2.$$ 

Since $\Theta_j$ is a real smooth bounded function as well as all its derivatives uniformly in $j$, the operator $T_c \Theta_j - \Theta_j$ is continuous $L^2 \to L^1$. Consequently $\Theta_j T_c - T_c \Theta_j$ is continuous $L^2 \to L^2$ and we have

$$I_3 = \sum_0^\infty \langle (\Theta_j T_c - T_c \Theta_j) \theta_j u, \theta_j u \rangle \lesssim \sum_0^\infty \|\theta_j u\|_{L^2}^2 \lesssim \|u\|_{L^2}^2.$$ 

For the control of $I_2 = \langle \theta_j u, \theta_j T_c^*(1 - \Theta_j) u \rangle$ we must use a property of ‘para-localization’: denote $d \in \Gamma^j$ the symbol of $T_c^*$, 

$$\theta_j T_d(1 - \Theta_j) u = \frac{1}{2\pi} \int e^{i(x-y)} \theta_j(x)(1 - \Theta_j(y)) \partial_y\left(\hat{d}(\zeta, \eta)\psi(\eta)\chi(\zeta, \eta)\right)u(y)dyd\eta d\zeta$$

$$= \frac{1}{2\pi} \int e^{i(x-y)} \theta_j(x)(1 - \Theta_j(y)) \partial_y\left(\hat{d}(\zeta, \eta)\psi(\eta)\chi(\zeta, \eta)\right)u(y)dyd\eta d\zeta$$

First note that by definition of $\theta_j, \Theta_j$, on the support of the integrand $|x - y| \gtrsim 2^j$. An integration by parts on the variable $\eta$ gives

$$\theta_j T_c^*(1 - \Theta_j) u = -\frac{1}{2\pi} \int e^{i(x-y)} \theta_j(x)(1 - \Theta_j(y)) \partial_y\left(\hat{d}(\zeta, \eta)\psi(\eta)\chi(\zeta, \eta)\right)u(y)dyd\eta d\zeta$$

$$= -\frac{1}{2\pi} \int e^{i(x-y)} \theta_j(x)(1 - \Theta_j(y)) \partial_y\left(\hat{d}(\zeta, \eta)\psi(\eta)\chi(\zeta, \eta)\right)u(y)dyd\eta d\zeta.$$ 

We have

$$\partial_y(\hat{d}(\zeta, \eta)\psi(\eta)\chi(\zeta, \eta)) = \psi'(\eta)\hat{d}(\zeta, \eta)\chi(\zeta, \eta) + \partial_\eta\hat{d}(\zeta, \eta)\psi(\eta)\chi(\zeta, \eta) + \hat{d}(\zeta, \eta)\psi(\eta)\partial_\eta\chi(\zeta, \eta).$$
The first term has compact support in $\eta$ because $\psi = 1$ for $|x| \geq 2$ and consequently is fastly decaying in $\eta$. The second term is of order $0$ in $\eta$ because $d$ is of order $1$, the third term is of order $0$ in $\eta$, because (by homogeneity) $\partial_\eta \chi$ is of order $-1$. We also note that $\theta_j(x)(1 - \Theta_j(y))/|x - y| \lesssim 2^{-j}$, thus $\theta_j T_\xi^* (1 - \Theta_j)u$ is in $L^2$, its norm being moreover controlled by $2^{-j}\|u\|_{L^2}$.

Finally we obtain

$$
\sum_0^\infty (\theta_j u, \theta_j T_\xi^* (1 - \Theta_j)u) \lesssim \sum_0^\infty 2^{-j} \|u\|^2_{L^2} + \|\theta_j u\|^2_{L^2} \lesssim \|u\|^2_{L^2}.
$$

The last term $\|T_{(1 - \Theta_j)2^{-j(1+\delta)/2}|\xi|^{1/2}\theta_j u}\|^2_{L^2}$ can be treated in the same way as $I_2$, or more simply by writing

$$
T_{(1 - \Theta_j)2^{-j(1+\delta)/2}|\xi|^{1/2}\theta_j u} = T_{(1 - \Theta_j)2^{-j(1+\delta)/2}|\xi|^{1/2}\theta_j \Theta_j u} = T_{\Theta_j (1 - \Theta_j)2^{-j(1+\delta)/2}|\xi|^{1/2}\Theta_j u} + R_j \Theta_j u = R_j \Theta_j u,
$$

with $R_j$ bounded from $L^2$ to $L^2$. \hfill \square

### A.3 Construction of $p$ (general case)

This section is devoted to the construction of the symbol $p$ such that

$$
\{a|\xi|^2, p\} \geq c\frac{|\xi|}{\langle x \rangle} - C.
$$

(9)

The scheme of construction is to construct a function $q(x, \xi)$ such that $|q| \lesssim \langle x \rangle$ and $\{a_0|\xi|^2, p\} \geq |\xi| - C$, this implies the existence of $p$ such that $\{a_0|\xi|^2, p\} \geq c\frac{|\xi|}{\langle x \rangle} - C$, and we finally check that $p$ satisfies (9) for $t$ small enough.

We recall our assumptions:

- $a(x, t) \in C_t H^{s+1} \cap C_t^1 H^s$, and $|\partial_t a| + |\partial_x \nabla a| + |\nabla a| \leq 1/\langle x \rangle^{1+\epsilon}$,

- The hamiltonian $a_0(x)|\xi|^2 := a(\zeta_0)|\xi|^2$ has no trapped bicharacteristics.

**Remark A.3.** If $(\rho, y)$ is a particular soliton solution, the decay assumption is satisfied by $a := a(\rho)$ only in the direction of propagation. In particular for traveling waves only depending of $x \cdot n - ct$, $a$ is constant on any affine hyperplane orthogonal to $n$, and the construction of this section does not apply.

We denote $a_0(x) = a(\zeta_0(x))$ with $\zeta_0$ the initial data. As a first step, we prove that there exists $q$ such that

$$
\begin{cases}
    \{a_0|\xi|^2, q\} \geq c|\xi| - C, \\
    \forall \alpha, \beta, \ |\partial_\xi^\alpha \partial_\xi^\beta q| \lesssim \langle x \rangle \langle \xi \rangle^{-|\beta|},
\end{cases}
$$

Let $q_1 = x \cdot \xi/|\xi|$, the decay assumption $\nabla a_0 \leq C/\langle x \rangle^{1+\epsilon}$ gives

$$
\{a_0|\xi|^2, x \cdot \xi/|\xi|\} = \sum_{j=1}^d 2a_0 \xi_j |\xi| - \partial_{x_j} a_0 |\xi|^2 (x_j/|\xi| - x \cdot \xi_j/|\xi|^3)
\leq 2a_0 |\xi| - |\xi| \nabla a_0 \cdot x + x \cdot \xi \nabla a_0 \cdot \xi/|\xi|
\geq a_0 |\xi| \text{ for } x \text{ large enough}.
$$
Say that this inequality is valid for $|x| \geq M$, and take $\psi \in C^\infty(\mathbb{R}^+)$ non decreasing, equal to 0 on $[0, M]$ and 1 on $[M + 1, \infty]$, then

$$\{a_0|\xi|^2, \psi(|x|^2)x \cdot \xi/|\xi|\} \leq \psi(|x|^2)\{a_0|\xi|^2, x \cdot \xi/|\xi|\} + x \cdot \xi/|\xi|\{a_0|\xi|^2, \psi(|x|^2)\}$$

$$\geq \psi(|x|^2)a_0|\xi| + 2a_0(x \cdot \xi)^2\psi′(|x|^2)/|\xi|$$

$$\geq \psi(|x|^2)a_0|\xi|.$$

This construction works as well if we replace $a_0$ by $a$, it will not be the case for the second part.

By bilinearity of the Poisson bracket, it remains to find a symbol whose Poisson’s bracket is positive for $x$ small. Let us define the bicharacteristics

$$\left\{ \begin{array}{l}
\dot{X}(t, x_0, \xi_0) = 2a_0(X)\Xi, \ t \geq 0,
\dot{\Xi}(t, x_0, \xi_0) = -|\Xi|^2\nabla a_0(X), \ t \geq 0,
X(0, x_0, \xi_0) = x_0, \ \Xi(0, x_0, \xi_0) = \xi_0,
\end{array} \right.$$ and assume that the bicharacteristics are not trapped, that is for any $(x_0, \xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$ we have $|X(t, x_0, \xi_0)| \to t \to +\infty + \infty$.

Remark A.4. In dimension greater than 1, non trapping is not an empty assumption. Even in the elementary case of a diagonal, asymptotically flat Laplacian, trapping may occur as we can see for instance on the symbol

$$a(x)|\xi|^2 := (\chi(x)e^{x^2+x^2} + \varphi)|\xi|^2$$

where $\chi = 1$ for $|x| \leq 5$, $\chi = 0$ for $|x| \geq 6$, $\varphi = 0$ for $|x| \leq 4$, $\varphi = 1$ for $|x| \geq 5$. It is easy to check that $X := (\cos(2ct), \sin(2ct)), \ \Xi := (−\sin(2ct), \cos(2ct))$ is a bounded solution of

$$\left\{ \begin{array}{l}
\dot{X}(t, x_0, \xi_0) = 2a_0(X)\Xi, \ t \geq 0,
\dot{\Xi}(t, x_0, \xi_0) = -|\Xi|^2\nabla a(X), \ t \geq 0,
X(0, x_0, \xi_0) = \left( \begin{array}{c}1 \\ 0 \end{array} \right), \ \Xi(0, x_0, \xi_0) = \left( \begin{array}{c}0 \\ 1 \end{array} \right),
\end{array} \right.$$ and assume that the bicharacteristics are not trapped, that is for any $(x_0, \xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$ we have $|X(t, x_0, \xi_0)| \to t \to +\infty + \infty$.

Let us define

$$q_2(x, \xi) := -\psi_1(x)\psi_2(\xi) \int_0^\infty \psi_1(X(s, x, \xi))\langle \Xi(s, x, \xi) \rangle ds$$

where $\psi_1 \in C^\infty_c$ is equal to 1 for $|x| \leq M + 1$, $\psi_2 \in C^\infty$ is equal to 1 for $|\xi| \geq 1$ and vanishes on a neighbourhood of 0. Because of the non trapping assumption $q_2$ is well defined. Moreover, by homogeneity in $\xi$ of the symbol $a(x)|\xi|^2$, we have

$$(X(s, x, \xi), \Xi(s, x, \xi)) = (X(s|\xi|, x, \xi/|\xi|), |\xi|\Xi(s|\xi|, x, \xi/|\xi|)).$$

Thus, setting $s|\xi| = s'$,

$$q_2(x, \xi) = -\left( \frac{\psi_1(x)\psi_2(\xi)}{|\xi|} \right) \int_0^\infty \psi_1(X(s', x, \xi/|\xi|))\langle |\xi|\Xi(s', x, \xi/|\xi|) \rangle ds,$$
defines a symbol\(^1\) of order 0. Its Poisson bracket with \(a|\xi|^2\) satisfies
\[
\{a_0|\xi|^2, q_2\} = - \sum_{j=1}^d a_0 2\xi_j (\partial_{x_j}\psi_1(x))\psi_2 \int_0^\infty \psi_1(X(s,x,\xi))\langle \Xi(s,x,\xi) \rangle ds \\
\quad + \sum_{j=1}^d |\xi|^2 \partial_{x_j} a_0 \psi_1 \partial_{\xi_j} \psi_2 \int_0^\infty \psi_1(X(s,x,\xi))\langle \Xi(s,x,\xi) \rangle ds \\
\quad + 2\langle \xi \rangle \psi_1^2(x) \psi_2(\xi).
\]
Since \(\partial_{x_j}\psi_1\) vanishes on \(B(0,M+1)\) and \(\partial_{\xi_j} \psi_2\) is compactly supported, we have
\[
\{a|\xi|^2, q_2\} \geq -C(|\xi|_{1,|x|\geq M+1} + 1) + \langle \xi \rangle \psi_1^2(x) \psi_2(\xi).
\]
Taking \(M = (C+1)/\inf(a_0)\), we have
\[
\{a_0|\xi|^2, Mq_1 + q_2\} \geq |\xi|_{1,|x|\geq M+1} - C + \langle \xi \rangle \psi_1^2(x) \psi_2(\xi) \geq |\xi| - C'.
\]
Then a standard argument (in fact an elementary generalization of the proof in the one-dimensional case where \(Mq_1 + q_2\) would play the role of \(q\), see also the very clear lecture notes of C. Kenig [Ken05] pp. 48 – 49) gives the existence of \(p\) such that
\[
\{a_0|\xi|^2, p\} \geq c \frac{|\xi|}{\langle x \rangle^{1+\epsilon}} - C.
\]
Now assume also that \(|\partial_t a| + |\partial_n a| \lesssim 1/\langle x \rangle^{1+\epsilon}\). This implies
\[
\{a|\xi|^2, p\} \geq c \frac{|\xi|}{\langle x \rangle^{1+\epsilon}} - C + \{ \int_0^t \partial_t a|\xi|^2, p \} \geq c \frac{|\xi|}{2\langle x \rangle^{1+\epsilon}} - 2C.
\]
for \(t\) small enough.

References


\(^1\)The smoothness of the symbol is a consequence of the uniform non trapping, i.e. for \((x,\xi)\) in a neighbourhood of \((x_0,\xi_0)\), the bicharacteristics \((X(s,x,\xi), \Xi(s,x,\xi))\) leave any compact \(K\) for \(s \geq s(K)\) independent of \((x,\xi)\). This is a non trivial but standard consequence of the non-trapping assumption combined with the decay assumption for \(\nabla a\).


