Water wave radiation by a submerged rough disc

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A thin circular body is submerged below the free surface of deep water. The problem is reduced to a hypersingular integral equation over the boundary of the body. Using a perturbation method, the problem is then reformulated by a sequence of simpler hypersingular equations over a flat disc making it well suited for an efficiently previously used solution method. The first order approximation is computed and the hydrodynamic force due to heaving radiation motion are presented in terms of the added mass and damping coefficients for a polynomial cap and for a rough disc, modelled by a superposition of sinusoidal surfaces defined by randomly generated parameters. The solution exhibits larger maxima associated with smaller volume of submergence of the body. A slight shift of the damping coefficient maxima to lower frequencies is noticed for the caps. Rough discs with similar statistical properties exhibit different behaviours. Thus, it is the exact specific form of the rough disc that dictates the hydrodynamic force.

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1. Introduction

The study of the interaction of water waves with a disc could be separated into two cases: when the disc floats and when it is submerged. The first case is known as the dock problem. The circular dock problem have been studied by several authors [1–7]. Dock problems can be reduced to the solution of a boundary integral equation for the velocity potential $\phi$; this equation is a Fredholm integral equation of the second kind.

The second case was treated more recently and presents notable characteristics, like the occurrence of resonant frequencies where the hydrodynamic force assumes local maxima. Yu and Chwang [8] have used matched eigenfunction expansions for studying the scattering, by a horizontal disc, in water of finite depth. Martin and Farina [9] have presented a method for axisymmetric motions of a horizontal disc in deep water. They transformed the governing hypersingular integral equation for $[\phi]$ into a one dimensional Fredholm integral equation of the second kind for a new unknown function; the new equation is a generalisation of Love's integral equation, common in the theory of electrostatics of a circular-plate capacitor [10]. Numerical results of the heaving added mass and damping were presented. Farina [11] extended this work by considering the effects of taking the disc very close to the free surface and relating the hydrodynamic force to resonant frequencies. Both numerical and asymptotic methods have been used.

The three-dimensional scattering by a thin disc, in deep water was investigated by Farina and Martin [12]. The authors solved the governing hypersingular integral equation numerically using an expansion–collocation method. Similarly to the radiation problem, they found that the scattering problem presents a strong dependence on the frequency when the plate is close to the free surface. Relationships between the scattering cross-section and the peaks in the added mass have been presented.

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Yu [13] uses analytical, numerical, and semi-empirical methods and summarises the functional performance of a submerged and essentially horizontal plate for offshore wave control. The authors focus on the hydrodynamics force and on the reflection and transmission coefficients.

Roy and Ghosh [14] solve Laplace’s equation using the method of separation of variables in order to calculate forces on a circular thin disc vertically submerged in shallow water. Morison’s equation is used for the determination of the wave force.

In this work, we consider a problem, that, to the best of our knowledge, was still not investigated. We assume that the submerged disc is perturbed out of its original plane, so the disc could be denominated wrinkled or rough. A similar problem in acoustics has been studied by Jansson [15], where the scattering of an acoustic wave from a thin circular disc was investigated by an integral equation method where the disc is modelled as part of an infinite interface between two half-spaces; this interface is then perturbed. However, this approach causes the behaviour of the solution near the edge of the disc to produce singularities at the edge of the disc.

In a related mathematical problem, Beom and Earmme [16] investigated non-flat axisymmetric cracks by representing the potential as integrals in terms of Bessel functions. This kind of formulation is motivated by methods of solutions where the disc is flat. However, there will be points on the disc surface for which this representation will diverge.

The idea presented here follows those by Martin [9] where he considers the potential flow past a wrinkled disc. First we reduce the exact boundary-value problem to a hypersingular integral equation of the potential discontinuity across the disc and project, also exactly, this equation on a flat disc. Introducing perturbation expansions in terms of a small parameter $\epsilon$, we obtain a sequence of integral equations with the same kernel where the hypersingular part are evaluated analytically in a certain space of functions which are given in terms of Legendre polynomials. Apart from offering an efficient and tested computational method of solution [12] this approach carry other advantages. This leads to regular perturbations since we work on the disc surface only, not within the fluid. Physically, the potential discontinuity across the disc is not expected to be much different for the perturbed and unperturbed disc. Other motivation for this work is the modelling of more complicated bodies which relates to more important physical applications. As the calculations are based on two-dimensional integral equations, we do not assume symmetries either on the disc surface or on the motions performed by the disc itself. It is important to mention that the method could also be applied to scattering problems.

The approach presented here, that is, formulate and exact boundary integral equation, project exactly on a reference surface and then introduce a regular perturbation expansion can potentially be applied to a wide range of wave propagation problems. Thus, whenever one can solve the governing boundary integral equation for the reference surface, this method will be interesting. As examples, we point out wave propagation in viscous fluids by means of integral formulation of the Stokes equation [17,18], crack problems in elasticity [19] and viscous acoustic waves [20].

We apply the method to circular caps and random rough discs. The circular caps are given by cubic and octic polynomials. For modelling the rough disc we take a step in a direction to avoid known limitations in other models. Randomly generated rough surfaces are as close to reality as possible. However the absence of an analytical expression prevent a more qualitative analysis of being carried out. On the other hand, a deterministic rough surface avoids this disadvantage and could speed up computations. The shortcoming is its intrinsic limitations to represent real surfaces. In this work, we follow an alternative representation, proposed by Jansson [15] in a acoustics problem, aiming at preserving the advantages of both approaches. Thus, the rough disc is represented by a sum of deterministic functions that are randomly shifted with respect to each other.

Numerical solutions of order $\epsilon$ are obtained and the added mass and damping coefficients are presented for the heaving radiation problem.

2. Formulation

A Cartesian coordinate system is chosen, in which $z$ is directed vertically downwards into the fluid. We take the mean free surface lying at $z = 0$. We assume the presence of a submerged body into the fluid with a smooth, closed and bounded surface $S$. We suppose that the motions of the fluid are of small-amplitude, time-harmonic, that the fluid is incompressible and inviscid, and that the motion is irrotational. We denote $\phi$ as the potential flow and $[\phi]$ as the discontinuity in $\phi$ across $S$. Thus, the time dependent velocity potential is $\text{Re}\{\phi(x, z, t)\}e^{-i\omega t}$, where $\omega$ is the angular frequency.

The conditions to be satisfied by $\phi$ are Laplace’s equation

$$
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \phi = 0 \quad \text{in the fluid}
$$

along with the free-surface condition

$$
K \phi + \frac{\partial \phi}{\partial z} = 0 \quad \text{on } z = 0,
$$

where $K = \omega^2 / g$; $g$ being the acceleration due to gravity.

On the surface of the body, the normal velocity is prescribed by

$$
\frac{\partial \phi}{\partial n} = V \quad \text{on } S,
$$

where $V$ is a given function and $\frac{\partial}{\partial n}$ denotes normal differentiation.
Additionally, \( \phi \) must satisfy a radiation condition:

\[
r^{1/2} \left( \frac{\partial \phi}{\partial r} - iK\phi \right) \to 0 \quad \text{when} \quad r = (x^2 + y^2)^{1/2} \to \infty.
\]

The points \( P, Q \) denote points in the fluid and the points \( p, q \) denote points on the submerged body. The free surface Green function for this problem is given by

\[
G(P, Q) \equiv G(\xi, \eta, \zeta; x, y, z) = G_0(R, z - \xi) + G_1(R, z + \xi),
\]

where \( R = ((x - \xi)^2 + (y - \eta)^2)^{1/2} \), \( G_0(R, z - \xi) = (R^2 + (z - \xi)^2)^{-1/2} \) and

\[
G_1(R, z + \xi) = \int_0^\infty e^{-k(z+\xi)}J_0(kR)\frac{K}{k-R}\,dk.
\]

Here \( J_0 \) is the Bessel function of order zero. The path integral defining \( G_1 \) above runs below the singularity \( K \). \( G \) satisfies the free surface condition, the Laplace equation, and has a weak singularity at \( P = Q \).

For any harmonic function \( \phi \), satisfying \( \phi = 0(r^{-1}) \) as \( r \to \infty \), we have from Green’s second identity, the following integral representation.

\[
\phi(P) = \frac{1}{4\pi} \int_{Q} \left( \phi(q) \frac{\partial P}{\partial n_q} - G(P, q) \frac{\partial \phi}{\partial n_q} \right) \,dS_q,
\]

where \( \frac{\partial}{\partial n_q} \) denotes normal differentiation at \( q \) on \( S \).

Now, for a thin body with surface \( \Omega \), denote the two sides of \( \Omega \) by \( \Omega^+ \) and \( \Omega^- \) and define the discontinuity in \( \phi \) across \( \Omega \) by

\[
[\phi] = \lim_{q \to q^+} \phi(Q) - \lim_{q \to q^-} \phi(Q),
\]

where \( q \in \Omega, q^- \in \Omega^- \), \( q^+ \in \Omega^+ \) and \( Q \) is a point in the fluid. Thus, Eq. (4) reduces to

\[
\phi(P) = \frac{1}{4\pi} \int_{\partial\Omega} [\phi(q)] \frac{\partial}{\partial n_q} G(P, q) \,dS_p,
\]

where \( n_q = n_q^+ \) denotes now the normal unit vector at \( q \) on \( \Omega^+ \). Applying boundary condition (1)-(5) gives

\[
\frac{1}{4\pi} \int_{\partial\Omega} [\phi(q)] \frac{\partial}{\partial n_q} G(p, q) \,dS_q = V(p), \quad p \in \Omega.
\]

where the integral must be interpreted in the Hadamard’s finite-part sense. Eq. (6) is the governing hypersingular integral equation for \([\phi]\); this is to be solved subject to the edge condition

\[
[\phi] = 0 \quad \text{in} \quad \partial\Omega.
\]

Now let

\[
\Omega : z = F(x, y) + \frac{b}{2}, \quad (x, y) \in D,
\]

where \( D \) is the unit disc in the xy-plane and \( \frac{b}{2} \) is the depth to which the body is submerged. Let \( p, q \in \Omega \) such that \( p = (\xi, \eta, \zeta), q = (x, y, z) \). The normal vector to \( \Omega \) is then given by

\[
N = \left( \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, 1 \right)
\]

and a unit normal vector is therefore, expressed by \( n = N/|N| \). Using the notation

\[
w(x, y) = [\phi(q)],
\]

it can be shown by a direct calculation that formula (5) becomes

\[
\phi(\xi, \eta, \zeta) = \frac{1}{4\pi} \int_D w(x, y) \frac{N \cdot R_F}{R_F} \,dS + \frac{1}{4\pi} \int_D w(x, y)(\nabla G_1 \cdot N) \,dS,
\]

where \( R_F = (\xi - x, y - \eta, \zeta - F(x, y)) \), \( R_F = |R_F| \) and \( dS = dx\,dy \).

Our goal now is to clarify and understand the governing equation (6). In order to do this, consider the following definitions and notations.

\[
F_1 = \frac{\partial F}{\partial x}, \quad F_2 = \frac{\partial F}{\partial y}
\]
with \( F_1^0 \) and \( F_2^0 \) being the corresponding functions at \((\xi, \eta)\). Let also \( \Lambda = \frac{F(x,y)-F(\xi,\eta)}{R} \) and \( \bar{\Lambda} = \frac{F(x,y)+F(\xi,\eta)}{R} \) and define the angle \( \Theta \) by \( x - \xi = R \cos \Theta \) and \( y - \eta = R \sin \Theta \).

Projecting onto \( D \), we can rewrite (6) as

\[
\frac{1}{4\pi} \int_D Hw(q)dA + \frac{1}{4\pi} \int_D Ww(q)dA = V(p), \quad p \in D.
\]

(10)

where (see [21])

\[
H(\xi, \eta; x, y) = \frac{1}{R^3} \left( \frac{1 + F_1 F_1^0 + F_2 F_2^0}{(1 + \Lambda^2)^2} - 3 \frac{(F_1 \cos \Theta + F_2 \sin \Theta - 1)(F_1^0 \cos \Theta + F_2^0 \sin \Theta - 1)}{(1 + \Lambda^2)^2} \right)
\]

(11)

and

\[
W = \frac{\partial^2 G_1}{\partial n_q \partial n_p} \bigg|_D = \int_0^\infty e^{-k \lambda R} e^{-k b} \mathcal{K} \frac{k + K}{k - K} dk,
\]

(12)

where

\[
\mathcal{K} = F_1 F_1^0 \frac{k}{2R} (2 \sin^2 \Theta J_1(kR) + k R \cos^2 \Theta (J_0(kR) - J_2(kR)))
\]

\[
+ F_2 F_2^0 \frac{k}{2R} (2 \cos^2 \Theta J_1(kR) + k R \sin^2 \Theta (J_0(kR) - J_2(kR)))
\]

\[
+ (F_2 F_0^0 + F_1 F_1^0) \frac{k}{2R} \cos \Theta \sin \Theta (k R (J_0(kR) - J_2(kR)) - 2 J_1(kR))
\]

\[
+ (F_1^0 - F_1) k^2 \cos \Theta J_1(kR) + (F_2^0 - F_2) k^2 \sin \Theta J_1(kR) + k^2 1 J_0(kR).
\]

(13)

Eq. (10) is the governing equation for the problem of any submerged non-planar circular disc \( \Omega \) in water of infinite depth. Its solution gives the jump in the velocity potential \( \phi \) across \( \Omega \). With this information, one can evaluate \( \phi \) at any point \( P \) in the fluid by using (8). Eq. (10) could be solved numerically, although not by the semi-analytical expansion–collocation method proposed by Farina e Martin [12] for the solution of hypersingular integral equations on a disc. Alternatively, an approximation to the solution could be obtained by a boundary perturbation method. We present such a method next. This method follows the one proposed by Martin [21] for treating the problem of a wrinkled disc in an unbounded fluid.

3. Radiation problem and perturbation method

We now assume that

\[
V(p) = n_3, \quad n_3 = \frac{1}{\sqrt{F_1^2 + F_2^2 + 1}},
\]

(14)

where \( n_3 \) is the vertical component of the unit normal vector to the disc. This corresponds to the disc performing heave (vertical) oscillations. Thus the problem stated in Section 2 becomes a radiation problem.

In order to consider a perturbation of the flat disc, we introduce the function \( f \) such that

\[
F(x, y) = \epsilon f(x, y),
\]

(15)

where \( \epsilon \) is a small parameter and \( f \) is independent of \( \epsilon \). In [21] it is shown that

\[
H = \frac{1}{R^3} (1 + \epsilon^2 K_2 + O(\epsilon^4)),
\]

where

\[
K_2 = f_1 f_1^0 + f_2 f_2^0 - \frac{3}{2} \lambda^2 - 3(f_1 \cos \Theta + f_2 \sin \Theta - \lambda)(f_1^0 \cos \Theta + f_2^0 \sin \Theta),
\]

\[
\lambda = \left( f(x, y) - f(x, y) \right) / R \text{ and } f_1, f_1^0 \text{ are defined similarly to } F_1, F_1^0; \text{ see the comments after (9)}.
\]

In order to get a similar expression for \( W \), substitute (15) in (12), giving

\[
W = W_0 + \epsilon W_1 + \epsilon^2 W_2,
\]

(16)

where

\[
W_0 = \int_0^\infty e^{-k(\epsilon f(x, y) + f(\xi, \eta) + b)} k^2 J_0(kR) \frac{k + K}{k - K} dk,
\]

(17)

\[
W_1 = \left( f_1^0 - f_1 \right) \cos \Theta + \left( f_2^0 - f_2 \right) \sin \Theta \int_0^\infty e^{-k(\epsilon f(x, y) + f(\xi, \eta) + b)} k^2 J_1(kR) dk,
\]

(18)
and
\[
W_2 = \left[ \frac{\sin^2 \Theta}{R} f_1^0 + \frac{\cos^2 \Theta}{R} f_2^0 - \frac{(f_2^0 + f_1^0) \sin(2\Theta)}{2R} \right] \int_0^\infty e^{-k\epsilon(f(x,y) + f(\xi,\eta) + b)} kj_1(kR) \frac{k + K}{k - K} \, dk \\
+ \left[ \frac{\cos^2 \Theta f_1^0 + \sin^2 \Theta f_2^0 - \frac{(f_2^0 + f_1^0) \sin(2\Theta)}{2}}{2} \right] \frac{1}{2} \int_0^\infty e^{-k\epsilon(f(x,y) + f(\xi,\eta) + b)} k^2 (u_0(kR) - j_2(kR)) \frac{k + K}{k - K} \, dk.
\]

Expanding \( e^{-k\epsilon(f(x,y) + f(\xi,\eta) + b)} \) in Taylor's series, we obtain
\[
W_0 = W_{00} + \epsilon W_{01} + \epsilon^2 W_{02}, \\
W_1 = W_{10} + \epsilon W_{11} + \epsilon^2 W_{12}, \\
W_2 = W_{20} + \epsilon W_{21} + \epsilon^2 W_{22},
\]
where
\[
W_{00} = \int_0^\infty \frac{k + K}{k - K} e^{-k\epsilon k^2 j_0(kR)} \, dk
\]
and the functions \( W_{01}, \ldots, W_{22} \) are given in Appendix A.

Substituting (15) in (14) and expanding in Taylor series, we get
\[
n_3 = 1 + \frac{1}{2} \left( f_1^2 + f_2^2 \right) \epsilon^2 + O(\epsilon^3).
\]

Similarly, for \( w \), assume
\[
w = w_0 + \epsilon w_1 + \epsilon^2 w_2 + \cdots.
\]

Now, substituting (16) and (22) in (10), with \( V \) given by (21), we obtain
\[
\frac{1}{4\pi} \int_D \frac{w}{R^3} \, dA + \frac{1}{4\pi} \int_D W_{00} w_0 \, dA = 1,
\]
\[
\frac{1}{4\pi} \int_D \frac{w_1}{R^3} \, dA + \frac{1}{4\pi} \int_D W_{01} w_0 \, dA = -\frac{1}{4\pi} \int_D (W_{10} + W_{01}) w_0 \, dA,
\]
\[
\frac{1}{4\pi} \int_D \frac{w_2}{R^3} \, dA + \frac{1}{4\pi} \int_D W_{02} w_0 \, dA = -\frac{1}{4\pi} \int_D \frac{K_2 w_0}{R} \, dA - \frac{1}{4\pi} \int_D (W_{20} + W_{11} + W_{20}) w_0 \, dA
- \frac{1}{4\pi} \int_D (W_{01} + W_{10}) w_1 \, dA + \frac{1}{2} \left( f_1^2 + f_2^2 \right).
\]

Note that Eq. (23) appears in [9, Eq. 4.1] and in [12, Eq. 17]. Thus, the first order equation of the present perturbation method recovers the governing equation for the plane disc: this corresponds to the problem of a horizontal and plane circular disc performing heave oscillations.

By defining the integral operators
\[
H_{ij} = \int_D W_{ij} \, dA \quad \forall i, j \in \{0, 1, 2\},
\]
\[
H = \int_D \frac{w}{R^3} \, dA,
\]
\[
K_2 = \int_D \frac{K_2 w}{R^3} \, dA,
\]
we can write Eqs. (23)–(25) in a more compact form as
\[
(H + H_{00}) w_0 = 1,
\]
\[
(H + H_{00}) w_1 = -(H_{10} + H_{01}) w_0,
\]
\[
(H + H_{00}) w_2 = -(K_2 + H_{02} + H_{11} + H_{20}) w_0 - (H_{01} + H_{10}) w_1 + \frac{1}{2} \left( f_1^2 + f_2^2 \right).
\]

Eqs. (26)–(28) form a sequence of integral equations that approach the solution of the governing equation (10). Note that the simple structure of these equations offers an alternative to the solution of the problem: in order to solve it, one has just to invert the integral operator \( H_{00} + H \). Note further that the function \( f \) is only present in the right-hand side of the equations. This means that all the information about the specific geometry of the plate is in these terms of the equations. Thus, it is possible to pre-solve the problem for any perturbation of the disc by inverting the operator mentioned above. This can be done efficiently by the numerical method presented in Section 4.
3.1. Hydrodynamic coefficients

The hydrodynamic force on the heaving disc is given by an integral of \([\phi]\) over the plate and decomposed in the added mass \(A\) and damping \(B\) coefficients as [22]

\[
A + iB = - \int_D [\phi] n_3 \, dS.
\]

Projecting on the disc, we have

\[
A + iB = - \int_D [\phi] n_3 |N| \, dS = - \int_D [\phi] \, dS.
\]

Using (7) and (22) in the above equation gives the hydrodynamic coefficients in the form of expansions:

\[
A = A_0 + \epsilon A_1 + \epsilon^2 A_2 + \cdots \tag{29}
\]

and

\[
B = B_0 + \epsilon B_1 + \epsilon^2 B_2 + \cdots. \tag{30}
\]

4. Alternative expressions and numerical method

In this section we show how to compute a solution of the problem formulated in the section above.

4.1. Alternative expressions for \(W\)

The integrands of the integral equations (26)--(28) involve the regular part of the free surface Green function, that is, \(G_1\), and its derivatives. The numerical implementation of these functions are not trivial. Specifically, these integrands present path integrals that involve Bessel functions. Nevertheless, we can express these integrals in terms of Bessel functions and Struve functions which are suitable for more efficient numerical calculation. According to [23] (see also [24,25]), we have

\[
G_1 = \int_0^\infty \frac{k + K}{k - K} e^{-k(r+x)} J_0(kR) \, dk
\]

\[
= K \left[ (X^2 + Y^2)^{-1/2} - \pi e^{-Y} (H_0(X) + Y_0(X)) - 2 \int_0^Y e^{iY} (X^2 + r^2)^{-1/2} \, dr \right] - 2\pi iK e^{-Y} J_0(X), \tag{31}
\]

where \(X = KR, Y = K(z + \xi)\), \(H_0\) is the Struve function of order 0 and \(J_0\) and \(Y_0\) denote the Bessel functions of the first and second kind, respectively. Expression (31) is suitable for numerical calculation; this has been used in several computer codes for water wave analysis. See for instance [26].

Using (31), it can be shown that the integrands \(W_{00}, \ldots, W_{22}\), originally written as ((40)--(46)) in Appendix A, admit similar representations. For example,

\[
W_{00} = 2K^2 (R^2 + b^2)^{-1/2} + (2Kb - 1)(R^2 + b^2)^{-3/2} + 3b^2 (R^2 + b^2)^{-5/2}
\]

\[
- \pi K^3 e^{-Kb} (H_0(KR) + Y_0(KR)) - 2K^3 e^{-Kb} \int_0^K e^{i((KR)^2 + r^2)^{-1/2} \, dr} + 2\pi iK^3 e^{-Kb} J_0(KR) \tag{32}
\]

is an alternative expression for \(W_{00}\), which allows more efficient numerical computation than (40) does. The expression (32) does not involve path integrals whose calculation are computationally expensive. Furthermore, the Struve and Bessel functions present in this alternative term are efficiently computed by approximating orthogonal polynomials; see [27]. Integrals such as the one in (32) can be efficiently computed; see [23,24]. Similar alternative expressions for the \(W_{01}, W_{02}, W_{10}\) and \(W_{20}\) are shown in Appendix B.

4.2. Expansion–collocation method

The governing equations (26)--(28), obtained by the perturbation method in Section 3, can be written in the same form, which is

\[
(H + H_{00})u = g, \tag{33}
\]

where \(g\) is a known function, which can involve solutions of lower order problems. As a particular case, the plane disc equation (26) has an axisymmetric solution and can be solved by reducing it to a non singular one dimensional Fredholm integral equation of the second kind [9, Eq. 7.6]. A simple numerical method can be used for this equation; for instance a Nyström method combined with the Gauss–Legendre quadrature rule, as employed by Martin and Farina [9]. However,
as the solutions of Eqs. (27) and (28) are not axisymmetric, we need a more general method of solution. We employ the expansion–collocation method used by Farina and Martin [12] for solving an equation of the form of (33). In fact, this method does not require that $V = 1$. This forcing could be any function of two variables; for instance, this could represent an incident wave and in this way, the problem would be a scattering one. In order to describe the expansion–collocation method, introduce cylindrical polar coordinates $(r, \theta, z)$, so that $x = r \cos \theta$ and $y = r \sin \theta$. Then, the disc is given by

$$D = \{(r, \theta, z) : 0 \leq r < 1, -\pi \leq \theta < \pi, z = b/2\}. \quad (34)$$

If we write $\xi = s \cos \alpha$, $\eta = s \sin \alpha$, we have

$$R^3 = [r^2 + s^2 - 2rs \cos(\theta - \alpha)]^{3/2}. \quad (35)$$

Hence we can write (33) as

$$\frac{1}{4\pi} \int_D u(s, \alpha) \left\{ \frac{1}{R^3} + W_{00}(r, \theta; s, \alpha, b, K) \right\} s \, ds \, d\alpha = g(r, \theta), \quad (r, \theta) \in D. \quad (35)$$

We shall expand $u$ using the basis functions $B^m_k$, defined by

$$B^m_k(r, \theta) = P^m_{m+2k+1} \left( \sqrt{1 - r^2} \right) e^{ik\theta}, \quad k, m = 0, 1, \ldots,

where $P^m_n$ is an associated Legendre function. The radial part of these basis functions can also be expressed in terms of Gegenbauer polynomials.

The functions $[B^m_k]$ are orthogonal over the unit disc with respect to the weight $(1 - r^2)^{-1/2}$.

The next formula, due to Krenk [28] is essential in the construction of the method:

$$\frac{1}{4\pi} \int_D \frac{1}{R^3} B^m_k(s, \alpha) s \, ds \, d\alpha = C^m_k \frac{B^m_k(r, \theta)}{\sqrt{1 - r^2}}, \quad (36)$$

where

$$C^m_k = -\frac{\pi}{4} \frac{(2k + 1)!!}{(2m + 2k + 1)!!} \left[ P^m_{m+2k+1}(0) \right]^2. \quad (36)$$

Eq. (36) allows us to evaluate the hypersingular integrals analytically.² To exploit (36), we expand $[\phi]$ in terms of the functions $B^m_k$. For brevity, we write

$$[\phi] = w \approx \sum_{k,m} a^m_k B^m_k := \sum_{N_1}^{N_2} \sum_{m=0}^{N_2} a^m_k B^m_k. \quad (37)$$

Substituting (37) in the integral equation (35) and then evaluating the hypersingular integrals analytically using (36), we obtain

$$\sum_{k,m} a^m_k \left\{ c^m_k B^m_k(r, \theta) + \frac{1}{4\pi} \int_D B^m_k(s, \alpha) W_{00}(r, \theta; s, \alpha, d, K) s \, ds \, d\alpha \right\} = g(r, \theta), \quad (r, \theta) \in D. \quad (38)$$

It remains to determine the unknown coefficients $a^m_k$. We use a collocation method, in which evaluation of (38) at $(N_1 + 1)(N_2 + 1)$ points on the disc gives a linear system for the coefficients $a^m_k$. For a discussion on the choice of the collocation points on a disc and other numerical issues on the collocation–expansion method, including its analogue for two-dimensional wave problems, see [12]. Next, we will present numerical results using this method.

5. Numerical results

In this section we will show the approximate solutions of problems involving two classes of submerged thin bodies. First, we consider circular caps where their surfaces are given by a polynomial function of $r$. Then, rough discs are analysed. Their surfaces are modelled by randomly generated sinusoidal functions.

The numerical solution is correct up to order $\epsilon$. More precisely, the Eqs. (26) and (27) are solved and we express the added mass and damping coefficients by the approximations $A = A_0 + \epsilon A_1$ and $B = B_0 + \epsilon B_1$, respectively.

We have used $N_1 = N_2 = 7$ in the experiments. This value provided numerical evidence of convergence of the expansions employed in the method. In all experiments described below, we use $\epsilon = 0.01$.

---

² Another consequence of formula (36) is that the functions $B^m_k(r, \theta)/\sqrt{1 - r^2}$ can be seen as eigenfunctions of the integral operator $\tilde{\mathcal{H}}$ defined by

$$\tilde{\mathcal{H}} v(r, \theta) = \frac{1}{4\pi} \int_D \frac{v(s, \alpha)}{\sqrt{1 - s^2}} \, ds \, d\alpha.$$
5.1. Circular caps

Let \( f \) be given by the function

\[
    f = \alpha r^p - \beta,
\]

where \( r = \sqrt{x^2 + y^2} \). Thus, the submerged body is given by the surface

\[
    \Omega : z = \epsilon (\alpha r^p - \beta) + \frac{b}{2}, \quad (x, y) \in D.
\]

The shape of \( \Omega \) resembles that of a cap or a bowl. The parameters \( \alpha \) and \( \beta \) control how close this body is to the free surface and \( \alpha \) defines the body’s concavity. Specifically, in case \( \alpha < 0 \), the body is concave up and if \( \alpha > 0 \), \( \Omega \) is concave down. The distance between \( \Omega \) and the mean free surface \( \eta_0 : z = 0 \), is

\[
    \text{dist}(\Omega, \eta_0) = \min_{(x,y) \in D} |\Omega(x,y)| = \frac{b}{2} + \epsilon \min_{[0,1]} f(r).
\]

For the submergence depth equal to 0.4, that is \( b = 0.8 \), we will show results for cubic and octic caps represented by the third and eighth degree polynomials \(-9r^3, -39r^3, -39r^8\), which are concave up and \( 9r^3 - 9, 39r^3 - 39, 39r^8 - 39 \), which are concave down. Fig. 1 shows the geometries of two of the cubic caps considered here. In Fig. 2, the hydrodynamic coefficients are plotted as functions of \( K \), for a circular disc and for the concave up and concave down cubic caps, \(-9r^3 \) and \( 9r^3 - 9 \), respectively. Note that the peaks in the curves are higher for the caps compared to the flat plate. Additionally, the maxima are larger for the concave down cap and the added mass of the concave up cap crosses that of the concave down cap near \( K = 1.5 \). As the distance of the plates to the free surface decreases, the maxima increase their values, as can be seen in Fig. 3. We note that the maxima in the damping coefficients are shifted to smaller wavenumbers. Additionally the higher peaks are also narrower. This is a characteristic observed for shallower submergence depths regimes where this behaviour becomes more clear; see for instance, [9,11]. Thus, it is remarkable that this aspect is captured already by the first order approximation solution.

Two other aspects are noticed. Again, the concave down caps (items (c) and (d)) present higher maxima. It is also interesting to notice that when the concavity is upwards (items (a) and (b)), the cubic cap has a larger maximum than the...
Fig. 3. As functions of $K$, the added mass coefficients are shown in (a) and (c) and the damping coefficients are plotted in (b) and (d). The disc is represented by the solid line. In (a) and (b), the dashed and dot-dashed lines show the results for the concave up cubic and octic cups given by $f = -39r^3$ and $f = -39r^6$, respectively. In (c) and (d), the dashed and dot-dashed lines show the results for the concave down cubic and octic cups given by $f = 39r^3 - 39$ and $f = -39r^3 - 39$, respectively.

Fig. 4. Profiles of the cubic (solid lines) and octic (dashed lines) cups cross-sections. In (a), the caps $\epsilon(-39r^3) + 0.4$ and $\epsilon(-39r^6) + 0.4$ are plotted and in (b), the caps $\epsilon(39r^3 - 39) + 0.4$ and $\epsilon(39r^6 - 39) + 0.4$ are shown.

Octic cap. For concavity downwards (items (c) and (d)), this relative aspect of the maxima is reversed. These two features could be explained by the following empiric argument, supported by additional numerical experiments which are not shown here. The raise in the hydrodynamic coefficients maxima is accompanied by a decrease in the volume of submergence defined by \( \int_D \epsilon f(x, y) + b/2 dS \). In Fig. 4, the profiles the of the cubic (solid lines) and octic cups (dashed lines) are plotted. Thus, we can clearly see that the volume of submergence is smaller for the cubic cup when the concavities are upwards. The situation is opposite when the concavities are downwards.

We can infer that, for a fixed polynomial degree $p$, the concave down caps have smaller volume of submergence than concave up caps. With this respect, we can also postulate that, to certain extent, isolated points $x$, or a set of measure zero, very close to the free surface, such that \( |x| = \text{dist}(\Omega, \eta_0) \), contribute less to the increase of the hydrodynamic force maxima than a region close to the free surface, even if this region achieves the value \( |x| = \text{dist}(\Omega, \eta_0) \) only at a single point.

We remark that similar numerical experiments were carried out for a submergence depth of $b/2 = 0.2$ and the results obtained were qualitatively the same.
5.2 Rough discs

We model a rough disc by superposition of sinusoidal functions defined by randomly generated parameters. This model was used by Jansson [15] for studying acoustic scattering from a rough disc. We use

\[ f(r, \theta) = \sum_{i=1}^{N} 2w_i \sin \left( \frac{r \cos(\theta - \psi_i) - X_i}{\lambda_i} \right) \sin \left( \frac{r \sin(\theta - \psi_i) - Y_i}{\lambda_i} \right), \]

where \( X_i, Y_i \) and \( \psi_i \) are randomly generated parameters and lie in the domains \([0, 1], [0, 1]\) and \([0, \pi]\), respectively. The parameters \( \lambda_i \) control the intensity of corrugation; smaller values of this parameters mean more roughness. The weights are normalised in order to satisfy \( \sum w_i^2 = 1 \). It is shown in [15] that ensemble average \( \langle f^2 \rangle \) is equal to 1.

We will show the results of three classes of random surfaces, A, B and C. In each class, two particular surfaces are examined. See Table 1 for the values of \( \lambda_i \) and \( N \) which define these classes. In all cases, the weights \( w_i \) are all equal and the \( \lambda_i \) are equally spaced in their ranges. What make the surfaces in a class differ from each other are the specific random parameters \( X_i, Y_i \) and \( \psi_i \) generated. In Tables 2–7, in Appendix C, we show the random generated data for each of the six surfaces considered. These are graphically represented in Fig. 5. We computed the added mass and damping...
coefficients for all the six rough discs. When $b/2 = 0.4$, the relative error in the hydrodynamics coefficients between the rough and the flat disc were within 2%, with the exception for the added mass of the disc A1. This case is shown in Fig. 6. We notice the rough disc added mass is larger until near $K = 2.5$, where the flat disc and rough disc solutions coincide. When $b/2 = 0.2$ the situation becomes more interesting; the results are presented in Fig. 7. As observed previously in a work on acoustic scattering from a rough disc [15], we see that the results vary according with the specific geometry of each rough surface.

The region where the flat and rough discs differ the most is the one close the maxima. The discs of the class A showed the larger discrepancy from the flat disc. Disc B1 virtually coincides with the flat disc. We conclude that it is not the statistical properties of the rough surface that characterise the hydrodynamics force. Rather it is the exact specific form the roughness that will define this.

6. Discussion

The boundary value problem of water wave radiation by a submerged thin non-planar surface is reduced to a hypersingular integral equation. By using a boundary perturbation method, we formulate the problem of a submerged perturbed disc in terms of sequence of hypersingular integral equations, $(\mathcal{H} + H_{00})w_n = g_n$, over a flat disc. This approach allows the simplification of the problem and the application of a efficient semi-analytical method. Numerical experiments computing the first order approximation are reported for polynomials caps and rough discs. The hydrodynamic coefficients show larger maxima for the perturbed disc when the volume of submergence decreases. The damping coefficient of the caps has maxima shifted to low frequencies accompanied by a narrower band. We observe that the solution of the rough disc problem depends on the specific form of the roughness rather than on its statistical properties. Specifically, we found that the solution is not a function of only the wavenumber, rms height, correlation function and the ensemble average, $\langle f^2 \rangle$ which is held constant, equal to one, in all rough discs considered by us. Other conclusion is that two apparently very similar rough discs could produce different radiated waves. This finding imply also the possibility that a particular pattern of radiated waves would not distinguish different radiating rough discs, if these objects are to be defined by the statistical parameters above. Thus, in order to establish the radiated field, the exact geometry of the rough disc is essential even if the surface is highly irregular.

To point out a few related topics of research, we mention that the scattering problem, where an incident wave is prescribed, can be treated in a similar way. It could be interesting to compute higher order approximations using formulations similar to the ones presented here.

The perturbation of only the boundary of a circular disc below a free surface, making the disc non-circular can result in integral equations over the unperturbed domain and be well suited for the method of solution presented in this work. A work on this topic is in progress.
Finally, the problem of interaction of waves with fractals, which can be defined in a form similar to (39), could be modelled and treated by the method in this work.

Acknowledgements

The first author (J.S. Ziebell) acknowledges financial support from CAPES and CNPq.
Appendix A. Expansion terms for $W$

\[ W_{00} = \int_0^\infty k + K \frac{k e^{-k^2}}{k - K} J_0(kR) \, dk, \]

\[ W_{01} = -(f(x, y) + f(\xi, \eta)) \int_0^\infty k + K \frac{k e^{-k^2}}{k - K} J_0(kR) \, dk, \]

\[ W_{02} = \frac{1}{2} (f(x, y) + f(\xi, \eta))^2 \int_0^\infty k + K \frac{k e^{-k^2}}{k - K} J_0(kR) \, dk, \]

\[ W_{10} = -(f_1 - f_1^0) \cos \Theta + (f_2 - f_2^0) \sin \Theta \int_0^\infty k + K \frac{k^2 e^{-k^2}}{k - K} J_1(kR) \, dk, \]

\[ W_{11} = [(f_1 - f_1^0) \cos \Theta + (f_2 - f_2^0) \sin \Theta](f(x, y) + f(\xi, \eta)) \int_0^\infty k + K \frac{k^3 e^{-k^2}}{k - K} J_1(kR) \, dk, \]

\[ W_{12} = \frac{1}{2} [(f_1 - f_1^0) \cos \Theta + (f_2 - f_2^0) \sin \Theta](f(x, y) + f(\xi, \eta))^2 \int_0^\infty k + K \frac{k^4 e^{-k^2}}{k - K} J_1(kR) \, dk, \]

\[ W_{20} = \left[ \frac{\sin^2 \Theta}{R} f_1 J_0^0 + \frac{\cos^2 \Theta}{R} f_2 J_2^0 - (f_1 J_1^0 + f_2 J_3^0) \frac{\sin(2\Theta)}{2R} \right] \]

\[ \times \int_0^\infty e^{-k^2} k_1(kR) \frac{k + K}{k - K} \, dk + \left[ \cos^2 \Theta f_1 J_0^0 \sin^2 \Theta f_2 J_2^0 - (f_1 J_1^0 + f_2 J_3^0) \frac{\sin(2\Theta)}{2} \right] \]

\[ \times \frac{1}{2} \int_0^\infty e^{-k^2} k^2 (J_0(kR) - J_2(kR)) \frac{k + K}{k - K} \, dk, \]

\[ W_{21} = (f(x, y) + f(\xi, \eta)) \left[ \left[ -\frac{\sin^2 \Theta}{R} f_1 J_0^0 - \frac{\cos^2 \Theta}{R} f_2 J_2^0 + (f_1 J_1^0 + f_2 J_3^0) \frac{\sin(2\Theta)}{2R} \right] \right. \]

\[ \times \int_0^\infty e^{-k^2} k_1(kR) \frac{k + K}{k - K} \, dk + \left[ \cos^2 \Theta f_1 J_0^0 \sin^2 \Theta f_2 J_2^0 - (f_1 J_1^0 + f_2 J_3^0) \frac{\sin(2\Theta)}{2} \right] \]

\[ \left. \times \frac{1}{2} \int_0^\infty e^{-k^2} k^2 (J_0(kR) - J_2(kR)) \frac{k + K}{k - K} \, dk \right], \]

\[ W_{22} = \frac{1}{2} (f(x, y) + f(\xi, \eta))^2 \left[ \left[ \frac{\sin^2 \Theta}{R} f_1 J_0^0 + \frac{\cos^2 \Theta}{R} f_2 J_2^0 - (f_1 J_1^0 + f_2 J_3^0) \frac{\sin(2\Theta)}{2R} \right] \right. \]

\[ \times \int_0^\infty e^{-k^2} k_1(kR) \frac{k + K}{k - K} \, dk + \left[ \cos^2 \Theta f_1 J_0^0 \sin^2 \Theta f_2 J_2^0 - (f_1 J_1^0 + f_2 J_3^0) \frac{\sin(2\Theta)}{2} \right] \]

\[ \left. \times \frac{1}{2} \int_0^\infty e^{-k^2} k^2 (J_0(kR) - J_2(kR)) \frac{k + K}{k - K} \, dk \right]. \]

Appendix B. Alternatives expressions

\[ W_{01} = (f(x, y) + f(\xi, \eta)) \left[ -2bk^3(R^2 + b^2)^{-1/2} + (2K - 2K^2 b)(R^2 + b^2)^{-3/2} \right. \]

\[ + (9b - 6Kb^2)(R^2 + b^2)^{-5/2} - 15b^3(R^2 + b^2)^{-7/2} + \pi K e^{-kb} (H_0(kR)) \]

\[ + Y_0(kR)) + 2K e^{-K^2} \int_0^{K^2} e^t ((K^2)^2 + t^2)^{-1/2} \, dt + 2\pi K e^{-K^2} J_0(kR) \right]. \]
Appendix C. Random data of the rough surfaces

In *Tables 2–7*, we see the random generated data used for constructing surfaces A1, A2, B1, B2, C1 and C2.

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where $I^*$ denotes the Gamma function.
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Table 7
Randomly generated data for surface C2.

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References