First part: waves, history and applications

BCAM and UPV/EHU courses 2011-2012:
Advanced aspects in applied mathematics
Topics on numerics for wave propagation
There appears to be no single precise definition of what exactly constitutes a wave. Various restrictive definitions can be given, but to cover the whole range of wave phenomena it seems preferable to be guided by the intuitive view that **a wave is any recognizable signal that is transferred from one part of the medium to another with a recognizable velocity of propagation**. The signal may be any feature of the disturbance, such as a maximum or an abrupt change in some quantity, provided that it can be clearly recognized and its location at any time can be determined. The signal may distort, change its magnitude and change its velocity provided it is still recognizable. This may seem a little vague, but it turns out to be perfectly adequate and any attempt to be more precise appears to be too restrictive; different features are important in different types of waves."
Waves in the literature


“After all, our hearts beat, our lungs oscillate, we shiver when we are cold, we sometimes snore, we can hear and speak because our eardrums and our larynges vibrate. The light waves which permit us to see entail vibration. We move by oscillating our legs. We cannot even say vibration properly without the tip of the tongue oscillating. And the matter does not end there - far from it. Even the atoms of which we are constituted vibrate.

...if we are prepared to stretch the definition of vibration a little, it quickly becomes apparent that many of the events of everyday life have an extraordinarily cyclic quality. It is a curiously shaky world we live in.

It is no exaggeration to say that it is unlikely that there is any branch of science in which vibration does not play an important role.”
“Is the ocean composed of water or of waves or of both? Some of my fellow passengers on the Atlantic were emphatically of the opinion that it is composed of waves; but I think the ordinary unprejudiced answer would be that it is composed of water. At least if we declare our belief that the nature of the ocean is aqueous, it is not likely that anyone will challenge us and assert that on the contrary its nature is undulatory, or that it is a dualism part aqueous and part undulatory.”
Ubiquity of waves: acoustic and light waves

“The Book of Nature is written in the language of mathematics.” (Galileo Galilei, 1564-1642)

1. Eigenmode of a drum
2. Harmonics of a violin
3. Interferometry of a violin
4. Reflection and refraction of light rays in the lab
5. Reflection and refraction of light rays in a glass
Ubiquity of waves: sea, shallow water and shock waves

“Profound study of nature is the most fertile source of mathematical discoveries.” (Joseph Fourier, 1768-1830)

6 Atmospheric shock wave during a volcano eruption, Saychev Peak, Matua Island, 2009

7 Aircraft sonic boom

8 Impact of the shock wave on the water surface Iowa battleship

9 Shock wave when shooting

10 Small waves in shallow waters

11 Sea waves and surfing
Ubiquity of waves: seismic, mechanical (material) and pressure waves

“Mathematical analysis is as extensive as nature itself.” (Joseph Fourier, 1768-1830)

12 Seismic waves
13 Distorted railway line after Canterbury earthquake, New Zealand, 2010
14 Cracks in the street after earthquake

15 Mechanical waves in the water
16 Atmospheric pressure waves
Ubiquity of waves: electromagnetic waves, radio, radar, wireless...

Mathematics is the alphabet with which the universe was created. (Galileo Galilei, 1564-1642)
Euclid (300BC) was a Greek mathematician, often referred to as the father of geometry. Although many of the results in Elements were originated by earlier mathematicians, one of Euclid’s accomplishments was to present them in a coherent framework, including a system of rigorous mathematical proofs that remains the basis of mathematics 23 centuries later.

The book Catoptrics attributed to Euclid concerns the mathematical theory of mirrors. He describes the laws of reflection of light in plane and spherical concave mirrors:

- The incident ray, the reflected ray and the normal to the reflection surface at the point of the incidence lie in the same plane.
- The angle which the incident ray makes with the normal is equal to the angle which the reflected ray makes to the same normal.
- The reflected ray and the incident ray are on the opposite sides of the normal.

The first practical catoptric telescope (or Newtonian reflector) was built by Newton as a solution to the problem of chromatic aberration exhibited in telescopes using lenses as objectives (dioptic telescopes).

Optics is the earliest surviving Greek treatise on perspective:

Vision is caused by rays emanating from the eye. The eye sees objects that are within its visual cone. The visual cone is made up of straight lines.
Heron de Alexandria (10-80AD) formulated the principle of the shortest path of light:

A ray of light propagates within the same medium following the shortest possible path.

Heron’s idea to find the shortest path needed by a ray to travel from \( P \) to \( Q \) by reflecting on \( d \):

- **CONSTRUCTION** of the incidence point \( R \):
  - Find the symmetric \( P' \) of \( P \) with respect to \( d \)
  - Join \( P' \) to \( Q \) and set \( R := P'Q \cap d \)
  - \( R \) is the incidence point between the ray of light and \( d \).
  - Consequence: the incidence angle \( \alpha = \) the reflection angle \( \beta \).

- **OPTIMALITY** of \( R \): \( PR + RQ \) is the shortest of all possible paths \( PR' + R'Q \).
  By symmetry, \( PR = P'R \) and \( PR' = P'R' \). In the triangle \( R'P'Q \), \( P'Q < P'R' + R'Q \):
Galileo Galilei (1564-1642) was an Italian physicist, mathematician, astronomer and philosopher who played a major role in the Scientific Revolution. He is considered the father of modern observational astronomy and of modern science (by Einstein and Hawking).

In 1581, when studying medicine, he noticed a swinging chandelier, which air currents shifted about to swing in larger and smaller arcs. It seemed, by comparison with his heartbeat, that the chandelier took the same amount of time to swing back and forth, no matter how far it was swinging. When he returned home, he set up two pendulums of equal length and swung one with a large sweep and the other with a small sweep and found that they kept time together.

As teacher of geometry, mechanics and astronomy at the University of Padua between 1592-1610, he made significant discoveries in pure fundamental science (kinematics and astronomy) and in applied science (an improvement of the telescope). He is the first scientist looking the sky using a telescope. He found that Jupiter has satellites.

In 1615, cardinal Bellarmine asked a physical proof of the heliocentric Copernican theory against the geocentric one of Ptolemy. Galileo considered his theory of the tides to provide the required proof of the motion of the Earth around Sun and wrote Dialogue on the Two Chief World Systems. For Galileo, the tides were caused the Earth’s rotation on its axis and revolution around the Sun. Eppur si muove! Galileo has problems with the church, is taken in front of the Pope, forced to say that Earth does not move and died in prison.

“The universe...is written in mathematical language and the letters are triangles, circles and other geometrical figures.”
Johannes Kepler (1571-1630) was a German mathematician, astronomer and astrologer.

Astronomia nova is a book published in 1609, containing results on the motion of Mars. One of the greatest books on astronomy, Astronomia nova provided strong arguments for heliocentrism and contributed valuable insight into the movement of the planets, including the first mention of their elliptical path.

Epitome of Copernican Astronomy, published in three parts between 1618 and 1621, contains the three laws of planetary motion:

- The orbit of every planet is an ellipse with the Sun at one of the two foci.
- A line joining a planet and the Sun sweeps out equal areas during equal intervals of time.
- The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit.
Christian Huygens (1629-1695) was a Dutch mathematician, astronomer, physicist and horologist. His work includes studies by telescope on the nature of the rings of Saturn, coupled oscillations and centrifugal force.

In 1690, he published *Treatise on light* proposing the Huygens-Fresnel principle stating that every point to which a luminous disturbance reaches becomes a source of a spherical wave. He was able to derive the laws of reflection and refraction, but could not explain the diffraction effects.

He designed more accurate clocks than the ones available at that time, suitable for sea navigation. His invention of the pendulum clock, patented in 1657, was a breakthrough in timekeeping.

In 1673, he published his mathematical analysis of pendulums, *Horologium Oscillatorium sive de motu pendulorum*. He analysed the tautochrone problem by finding the cycloid shape of the curve down which a mass will slide under the influence of gravity in the same amount of time regardless of its starting point.

Huygens was the first to derive the formula for the period of an ideal mathematical pendulum,

\[ T = 2\pi \sqrt{\frac{l}{g}}, \]

where \( T \) is the period, \( l \) the length of the pendulum and \( g \) the gravitational acceleration.
J'ai donc montré de quelle façon l'on peut concevoir que la lumière s'étend successivement par des ondes sphériques, et comment il est possible que cette extension se fasse avec une aussi grande vitesse, que les expériences et les observations célestes la demandent. Où il faut encore remarquer que, quoique les parties de l'éther soient supposées dans un continu mouvement (car il y a bien des raisons pour cela), la propagation successive des ondes n'en saurait être empêchée, parce qu'elle ne consiste point dans le transport de ces parties, mais seulement dans un petit ébranlement, qu'elles ne peuvent s'empêcher de communiquer à celles qui les environnent, nonobstant tout le mouvement qui les agite et fait changer de place entre elles.

Mais il faut considérer encore plus particulièrement l'origine de ces ondes et la manière dont elles s'étendent. Et premièrement, il s'ensuit de ce qui a été dit de la production de la lumière, que chaque petit endroit d'un corps lumineux, comme le Soleil, une chandelle, ou un charbon ardent, engendre ses ondes, dont cet endroit est le centre. Ainsi dans la flammé d'une chandelle (Fig. 4), étant distingués les points A, B, C, les cercles concentriques décrits autour de chacun de ces points représentent les ondes qui en proviennent. Et il en faut concevoir de même autour de chaque point de la surface et d'une partie du dedans de cette flamme.

Fig. 4.

Mais comme les percussions au centre de ces ondes n'ont point de suite réglée, aussi ne faut-il pas s'imaginer que les ondes mêmes s'entremêlent par des distances égales; et si ces distances paraissent telles dans cette figure, c'est plutôt pour marquer le progrès d'une même onde en des temps égaux, que pour en représenter plusieurs venues d'un même centre.

Il ne faut pas au reste que cette prodigieuse
History of waves: Bernoulli family

Jakob Bernoulli (1655-1705) and Johann Bernoulli (1667-1748) were Swiss mathematicians known for their contributions to infinitesimal calculus. They were among the first mathematicians to study, understand and apply calculus to various physical problems. The book Analyse des Infiniment Petits pour l’Intelligence des Lignes Courbes written by Guillaume de l’Hôpital, which mainly consisted of the work of Johann Bernoulli (e.g. l’Hôpital rule).

Daniel Bernoulli (1700-1782), son of Johann Bernoulli. His chief work is Hydrodynamica (1738), in which all the results are consequences of the conservation of energy (Bernoulli’s principle). It was known that a moving body transforms kinetic energy into potential energy when it gains height. Daniel realized that also a moving fluid exchanges kinetic energy for pressure. Mathematically this law is written as

$$\frac{1}{2} \rho |u|^2 + P = \text{constant},$$

where $P$ is the pressure, $\rho$ is the density and $u$ is the velocity of the fluid.

He won 10 times the Grand Prize of the Paris Academy, for topics on Newton’s theory of tides (jointly with Leonard Euler in 1740), magnetism, ocean currents.

Daniel Bernoulli also wrote some papers on vibrating strings. The history of spectral modeling of sound begins with him, who first believed that any acoustic vibration could be expressed as a superposition of sinusoidal vibrations. He showed this for several identical masses interconnected to form a discrete approximation to an ideal string.
Isaac Newton (1642-1727) was an English physicist, mathematician, astronomer, natural philosopher, alchemist and theologian.

In his book *Philosophiae Naturalis Principia Mathematica* (1687) described:

- **Newton’s law of universal gravitation.** Two points of masses $m_1$ and $m_2$ are attracted with a force $F$ directly proportional to the product of their masses and inversely proportional to the square of the distance $d$ between them: $F = Gm_1m_2/d^2$, with $G$ the gravitational constant.

- **Newton’s three laws of motion.** • The velocity of a body remains constant unless the body is acted by external forces. • The acceleration $a$ of a body is parallel and directly proportional to the force $F$ and inversely proportional to the mass $m$, i.e. $F = ma$. • The mutual forces of action and reaction between two bodies are equal, opposite and collinear.

In the Preface of Principia, Newton says:

“I wish we could derive the rest of phenomena of nature by the same kind of reasoning from mechanical principles; for I am induced by many reasons to suspect that they may all depend upon certain forces by which the particles of bodies, by some causes hitherto unknown, are either mutually impelled towards each other, and cohere in regular figures, or are repelled and recede from each other.”

In the General Scholium of Book III of Principia, Newton says:

“This most beautiful system of the sun, planets and comets could only proceed from the counsel and dominion of an intelligent and powerful being.”
In 1704, he published a record of experiments and deductions made from them entitled *Opticks* written in English (not in Latin as *Principia*). He considered light to be made up of extremely subtle corpuscles and ordinary matter to be made of grosser corpuscles.

*Opticks* is based on his lectures on Optics between 1670 and 1672, when he investigated the dispersion of light (called *inflexion of light*), demonstrating that a prism could decompose white light into a spectrum of colours and that a lens and a second prism could recompose the multicoloured spectrum into white light.

**Multi-prism dispersion theory.** Newton’s contribution to *prismatic dispersion* became central to the design of the *tunable lasers* more than 275 years later (with applications in astronomy, atomic separation, medicine, spectroscopy).

**Newton’s theory of colour.** The coloured light does not change its properties by separating out a coloured beam and shining it on various objects.

**Newtonian telescope.** From his work on optics, he concluded that the lens of any refracting telescope would suffer from the *chromatic aberration*, i.e. the dispersion of light into colours. He constructed the first known functional reflecting telescope (1672) using a mirror as objective.
Ibn al-Haytham or Alhazen (965-1040) was a Muslim scientist. His Book of Optics (1021) has been ranked as one of the most influential books in the history of physics.

- He was the first in proving that rays of light travel in straight lines,
- in reducing reflected and refracted light rays into vertical and horizontal walls.
- He is credited with the invention of the camera obscura and pinhole camera.
- Wrote on the atmospheric refraction due to morning and evening twilight,
- on the dispersion of light into its constituent colours,
- on the finite speed of light,
- on the slower movement of light in denser bodies.
- Is the first to describe accurately the various parts of the eye and give a scientific explanation of the process of vision.

The law of refraction of light was first discovered by Ibn Sahl (940-1000), Iranian mathematician, physicist and engineer, in his treatise On burning mirrors and lenses sets (984), presenting his understanding of how curved mirrors and lenses bend and focus light.
History of waves: Snell and Descartes

Willebrord Snel van Royen (1580-1626) - Dutch astronomer and mathematician. His name has been associated for several centuries to the discovery in 1621 of the law of refraction of light.

René Descartes (1596-1650) - French philosopher, mathematician and writer, who spent most of his life in Netherlands, considered the father of modern philosophy.

- The Cartesian coordinate system was named after him.
- He is considered the father of analytical geometry.
- He invented the convention of representing unknowns in equations by $x$, $y$ and $z$ and the coefficients by $a$, $b$ and $c$. Also the standard notation of superscripts for powers.
- His work is the basis for the infinitesimal calculus developed by Newton and Leibniz.
- He unifies calculus and geometry by writing equations for straight lines or circles.
- In Principles of Philosophy (1644), he discovered an early form of the law of conservation of mechanical momentum.

Contributions to Optics:

- law of refraction (Descartes's or Snell's law) - the angle subtended at the eye by the edge of the rainbow and the ray passing from the sun through the rainbow’s centre is 42°.
- law of reflection in the appendix La Dioptrique of his Le discours de la méthode pour bien conduire sa raison, et chercher la vérité dans les sciences (1637).
Pierre de Fermat (1601-1665) - French (of Basque origin) lawyer at the Parlement of Toulouse and an amateur mathematician, well-known for early developments leading to infinitesimal calculus. In particular, he is recognized for:

- In *Methodus ad disquirendam maximam et minima* (1638) and in *De tangentibus linearum curvarum* (1638), Fermat developed an original method for determining maxima, minima and tangents to various curves, equivalent to differentiation, and a technique to find centers of gravity for various plane and solid figures, which led to his further work on quadrature.

- research in number theory (e.g. Fermat’s Last Theorem, which he described in a note at the margin of a copy of Diophantus *Arithmetica*).

- Shortest distance? In 1657, Marin Cureau de la Chambre (1594-1669) wrote a book entitled *Light*, in which explained the equality between the incidence and refraction by the fact that light takes the shortest distance. However, refraction violated this law of shortest distance. De la Chambre attributed this contradiction to all that pesky material in the medium preventing light from having liberty to move on the shortest distance.

- Fermat’s principle or principle of least time (used to derive Snell’s law in 1657) was the first variational principle enunciated in physics since Heron de Alexandria described the principle of least distance. It was stated by Fermat in a letter to Cureau de la Chambre in 1662, entitled *Analysis of refraction* and *Syntesis of refraction*. Unable to accept the idea that light travels faster in a denser medium as Descartes stated, Fermat affirmed that

A ray of light propagating between two points takes the path that can be traversed in the least time.

Fermat found the problem too difficult to be analyzed (I admit that this problem is not one of the easiest). The proof of Fermat’s principle was performed by Leibniz (1684).
It seems that Mother Nature is an optimizer herself. Nature finds the shortest time path for a beam of light traveling through air, water and glass. Because light travels slower in glass, it avoids spending too much time in the glass and takes a more distant path through the air where it travels faster (than it does in glass). This is Fermat’s principle of optics. It is the same strategy a life guard follows to save a drowning man. If the drowning man is located on a diagonal path across the water, she will run as fast as she can along the beach and only enters the water at a point where she will get to the drowning man the quickest. She knows that she swims a lot slower than she can run. She is an optimizer.

It is a truth that has been drawn from experience, and which no one has ever contested, that the reflected ray comes off of an opaque body in the same proportion that it fell upon it...This is not a privilege of light; for not only do bodies reflect thus, but also sound and heat...This is clear for sound, because an echo, which is nothing other than a reflected sound, can only be heard in the location makes an equal angle with the first impression that the sound makes on a body...This being assumed, we must seek out the reason why reflections are made with equal angles... Indeed, could anything be said that would be more in conformity with reason than when they certify that the equality of angles occurring in reflection is made in accordance with the laws that nature maintains in all of her movements? For as she employs the shortest means in all her actions, she moves things through the shortest space: whence it comes that all bodies go directly to their centers, and that weights descend downwards, and lighter bodies ascend in straight lines, because these lines are the shortest of all, in such a way that it is necessary according to this rule that reflection be made along the shortest lines. Now it is assured that these lines make equal angles, and that if, impossibly, the angles were not equal, these lines would not be the shortest.
Fermat’s principle

You are on the beach and see someone drowning and if you are a good swimmer you rush to help him. On which path you get there as fast as possible?

Possibilities:

- the shortest path, but you know that you run much faster on the beach than swim in the water.
- You go farther on the beach and jump in the water closer to the drowning man. If you go on the beach much far away you loose time.
- the fastest path. There is a best point to which you run and after that you swim. It depends on how fast you can run on the beach and swim in the water.

REFRACTION: Light takes the fastest path. It travels faster in the air than in the water.
“The principle upon which you build your proof, namely that nature always acts by the shortest and simplest ways, is a moral principle, not a physical one, which is not and cannot be the the cause of any effect of nature...it is not by this principle that it acts, but by the secret force and virtute which lies in every thing...And it cannot be, otherwise we would be assuming some kind of awareness in nature.” (Clerselier’s letter to Fermat, 1662)

Interpretation. Nature acts without forethought, without choice, it does not look ahead and it is never faced with choices. It does not pick its way among several possibilities, taking into account their consequences, far or near into the future; at any time, it finds just one door open and it goes through that door. (Ekeland, The best of all possible worlds)

“Nature has obscure and hidden ways, which I have never tried to penetrate. I had only offered it some slight geometrical help in the matter of refraction, in case it had needed it...I heartily abandon you my pretended conquest in physics, provided you leave me in possession of my geometrical problem, all pure and in abstracto, by which one can find the path of a moving object which crosses two different mediums, and which tries to end its motion as soon as possible.” (Fermat's letter to Clerselier, 1662)

Interpretation. Fermat associated a mathematical problem with the physical phenomenon of refraction. Clerselier objected that there is no reasonable meaning to be attached to the model: things cannot actually work that way, it cannot be that light has both the desire to travel fastest and the means to compute the quickest path. Fermat answered that light propagates as if it had both that desire and these means, and while the mathematical problem may not be an accurate description of what is happening at some deeper level of reality, it is good enough to make predictions which turn out to be in agreement with experiments. The model should be kept as an working tool for scientists until it is discarded for a better one, and the question of why it works and what it means should be left to philosophers to worry about. (Ekeland, The best of all possible worlds)
Who optimizes?

God has created the world with a definite purpose in mind, namely to make it as perfect as possible:

“This is the best of all possible worlds...God has chosen the most perfect world, that is, the one which is at the same time the simplest in hypotheses and the richest in phenomena, as might be a line in geometry whose construction is easy and whose properties and effects are extremely remarkable and widespread.” (Leibniz, Discourse on Metaphysics, 1686)

Einstein-Bohr controversy about the foundations of quantum mechanics:

“God doesn’t play dice with the world.” (Einstein, Einstein and the Poet, 1943)

“I don’t know, all I am saying is that, using quantum mechanics and probability theory, I can make very accurate predictions.” (Bohr)

“The least action principle and with it all the minimum principles that one encounters in mechanics simply express that, in every case, whatever happens the circumstances determine uniquely.” (Mach, 1883)
Gottfried Wilhelm Leibniz (1646-1716) was a German mathematician and philosopher.

- He is the first to explicitly employ the notion of function to denote any of several geometric concepts derived from a curve, such as abscissa, ordinate or tangent.

- Leibniz was the first to see that the coefficients of a system of linear equations could be arranged into a matrix and to find the solution of the system by Gaussian elimination.

- Leibniz is credited, along with Newton, with the invention of infinitesimal calculus (containing both differential and integral calculus). A critical breakthrough occurred in 1675, when he employed integral calculus for the first time to find the area under the graph of a function. The product rule in differential calculus is called Leibniz’s rule. The theorem of differentiation under the integral is called the Leibniz’s integral rule.

- Several celebrated mathematicians also agreed with Fermat’s principle, particularly Leibniz, who gave the problem an elegant mathematical analysis (1684).
The proof by Leibniz


Problem: explain the law of refraction of light between two media in which the velocities are \( v_1 \) and \( v_2 \).

- Let \( A, B \) be two given points, one in each medium.
- We find the angles \( \alpha_1 \) and \( \alpha_2 \) s.t. the light travels from \( A \) to \( B \) in minimal time.
- The definition of the time function \( T(x) := \frac{\sqrt{a^2+x^2}}{v_1} + \frac{\sqrt{b^2+(l-x)^2}}{v_2} \).

Leibniz idea: Find \( x \) s.t. \( T'(x) = 0 \), where

\[
T'(x) = \frac{1}{v_1} \frac{x}{\sqrt{a^2+x^2}} - \frac{1}{v_2} \frac{l-x}{\sqrt{b^2+(l-x)^2}}.
\]

Observe that \( \sin(\alpha_1) = \frac{x}{\sqrt{a^2+x^2}} \) and \( \sin(\alpha_2) = \frac{l-x}{\sqrt{b^2+(l-x)^2}} \).

Then \( T'(x) = 0 \) is equivalent to Snell's law: \( \frac{\sin(\alpha_1)}{v_1} = \frac{\sin(\alpha_2)}{v_2} \).

The fact that \( T''(x) > 0 \) shows that \( x \) is really a minimum for \( T \):

\[
T''(x) = \frac{1}{v_1} \frac{a^2}{(a^2+x^2)^{3/2}} + \frac{1}{v_2} \frac{b^2}{(b^2+(l-x)^2)^{3/2}} > 0.
\]

Obs1. The Snell's law can be written as: \( \sin(\alpha_2) = v_2 \frac{\sin(\alpha_1)}{v_1} \).

Obs2. Total reflection. When \( v_1 < v_2 \), sometimes there is no \( \alpha_2 \) for \( \alpha_1 \in (0, \pi/2) \) s.t.

\[
\left| v_2 \frac{\sin(\alpha_1)}{v_1} \right| > 1.
\]
The principle of least action is the basic variational principle for dynamical systems, whose true trajectories are found by computing the action (i.e. a functional depending of all possible trajectories) and selecting the trajectory minimizing the action. The formulation of this principle is often given to Pierre-Louis Moreau de Maupertuis (1698-1759), who wrote about it:

Maupertuis, Accord de différentes lois de la nature qui avaient jusqu’ici paru incompatibles, 1744.

Maupertuis, Le lois de mouvement et du repos, déduites d’un principe de métaphysique, 1746.

“After meditating deeply on this topic, it occurred to me that light, upon passing from one medium to another, has to make a choice, whether to follow the path of shortest distance (the straight line) or the path of least time. But why should it prefer time over space? Light cannot travel both paths at once, yet how does it decide to take one path over another? Rather than taking either of these paths per se, light takes the path that offers a real advantage:

Light takes the path that minimizes its action.

The action depends on the speed of the body and on the distance it travels. However, it is neither the speed nor the distance taken separately; rather, it is proportional to the sum of the distances traveled multiplied each by the speed at which they were traveled. This action is the true expense of Nature, which she manages to make as small as possible in the motion of light.”

The original definition of action by Maupertuis is the function

\[ A(x) := v_1 \sqrt{a^2 + x^2} + v_2 \sqrt{b^2 + (l - x)^2}, \]

whose minimization gives again Snell’s law.
A burning mirror is a large convex lens that can concentrate the Sun’s rays onto a small area, heating up it and resulting in ignition of the exposed surface. The technology of making burning mirrors is known since antiquity.

- Vases filled with water used to start fires were known in the ancient world. The water remained cool even though the light passing through it would set materials on fire.
- Burning lenses were used to cauterise wounds.
- The perpetual sacred fire in the classic temples as the Olympic torch had to be pure and to come directly from the gods. For this they used the sun’s rays focused with mirrors or lenses.
- The Greek mathematician Diocles (240-180BC) wrote the book On burning mirrors.
- Archimedes (287-212 BC) has used burning mirrors (or more likely a large number of hexagonal mirrors) as a weapon in 212 BC, when Syracuse was besieged by Marcus Claudius Marcellus. The Roman fleet was incinerated. Archimedes died during the Siege of Syracuse killed by a Roman soldier.
- Today use: solar furnaces (the one in Odeillo in France can reach temperatures up to 3500°C), parabolic reflectors (the largest one at the Ben-Gurion National Solar Energy Center in Israel).
Patrick (Chevalier) d’Arcy (1723-1779) was an Irish noble that studied mathematics with Alexis Clairaut (1713-1765) in Paris. In 1752 he published his first article examined by d’Alembert, criticizing the least article principle of Maupertuis.

Consider a concave mirror $AB$ of center $C$. Take two points $f$ and $F$ symmetrically located with respect to the center. The light goes from $f$ to $F$ by reflecting at $M$.

$fMF$ is the maximum or the minimum of all the lines $fEF$, with $E$ an arbitrary point of the mirror, depending on the curvature of the mirror.

To see this, we compare the concave mirror with an ellipse of foci $f$ and $F$, which can be internal ($\alpha\beta$) or external ($AB$) to the mirror. For any point $E$ on the mirror,

- if the ellipse is internal, $fEF \geq fMF$ and $M$ minimizes the action on $\alpha\beta$
- if the ellipse is external, $fEF \leq fMF$ and $M$ maximizes the action on $AB$.

In this way, the least action principle becomes:

Light takes the path that is a critical point of the action.
Robert Hooke (1635-1703) was an English natural philosopher, architect and mathematician.

He was employed as assistant of Robert Boyle, for whom he built the vacuum pumps used in Boyle’s gas law experiments. He built some of the earliest Gregorian telescopes and observed the rotations of Mars and Jupiter. He was the first to suggest that matter expands when heated and that air is made of small particles separated by relatively large distances.

**Hooke’s law** of elasticity $F = -kx$ states that the extension of a spring is directly proportional with the load applied to it. Here

- $x$ is the **displacement** of the spring’s end from its equilibrium position
- $F$ is the **restoring force** exerted by the spring on that end
- $k$ is the so-called **spring constant**.

This law was described by Hooke, firstly by the anagram ceiiinossttuuv published in 1660, whose solution was given in his book *De potentia restitutiva* in 1678 as *Ut tensio, sic vis* meaning *As the extension, so the force*. Hooke’s work on elasticity culminated, for practical purposes, with the development of the balance spring watch.
“About two years since I printed this Theory in an Anagram at the end of my Book of the Descriptions of Helioscopes, viz. ceiiinosssttuv, id est, Ut tensio sic vis. That is, the power of any spring is in the same proportion with the tension thereof. That is, if one power stretch or bend it one space, two will bend it two, and three will bend it three, and so forward."

“It is very evident that the rule or law of nature in every springing body is that the force or power thereof to restore itself to its natural position is always proportionate to the distance or space it is removed therefrom, whether, it be by rarefaction, or separation of its parts the one from the other, or by a condensation, or crowding of those, parts nearer together. Nor is it observable in these bodies only, but in all other springy bodies whatsoever, whether Metal, Wood, Stones, baked Earths, Hair, Horns, Silk, Bones, Sinews, Glass, and the like. Respect being had to the particular figures of the bend bodies and the advantageous or disadvantageous ways of bending them."

A simple harmonic oscillator is an oscillator that is neither driven nor damped. It consists of a mass \( m \), which experiences a single force \( F \), which pulls the mass in the direction of the point \( x = 0 \) and depends only on the mass position \( x \) and a constant \( k \). Newton’s second law for the system is

\[
F = ma = m \frac{d^2x}{dt^2} = -kx.
\]

The total energy is conserved in time \( E(t) := \frac{m}{2} |\dot{x}(t)|^2 + \frac{k}{2} |x(t)|^2 \). The general solution is \( x(t) = A \cos(\omega t + \alpha) \), with \( \omega^2 = k/m \).
The acoustic origins of the harmonic analysis

- The interpretation of sound as vibration goes back to antiquity.
- In XVII century, Galileo Galilei and Marin Marsenue popularized the correspondence between pitch and frequency that Giovanni Battista Benedetti, Galileo's father Vincenzo and Isaac Beeckman had earlier articulated.
- Accordingly, a musical tone is a periodic succession of pulses transmitted by the air to the eardrum.

"The length of strings is not the direct and immediate reason behind the forms of musical intervals, nor is their tension, nor their thickness, but rather the ratio of the numbers of vibrations and impacts of air waves on our eardrum, which likewise vibrates according to the same measure of time. This point established, we may perhaps assign a very congruous reason why it comes about that among sounds different in pitch, some pairs are received in our sensorium with great delight, others with less, and some strike us with great irritation."

- Mersenne nonetheless described one the basic facts of Rameau's later theory: under quiet condition and with proper experience of the musicians one could hear several (4) tones at a time.
- Joseph Sauveur obtained the additional modes of vibration by plucking a monochord after placing a light obstacle at a simple fraction of the length of the string. To his surprise, the string did not move appreciably for a sequence of equidistant points which he called nodes. The number of nodes determined the order of the overtone, which Sauveur called harmonic since it was harmonious with the fundamental.

Galilei, Discorsi e dimostrazioni matematiche: Intorno à due nuoue scienze attenenti alla mecanica i movimenti locali, 1638.

Marsenne, Harmonie universelle, 1636.

Sauveur, Système général des intervalles des sons et son application à tous les systèmes et à tous les instrumens de musique, 1701.

Saveur, Sur l'application des sons harmoniques aux jeux d’orgues, 1702.
Harmonic oscillations occur in Isaac Newton’s derivation of the velocity of sound.

Brook Taylor gave the first theory of the fundamental mode of a vibrating string. He first showed that the resultant of the tensions acting on the extremities of an element of the string was proportional to the curvature of this element and directed along its normal.

In 1727, Johann Bernoulli pioneered the study of a weightless string loaded with equidistant, discrete masses.

In 1742, Daniel Bernoulli solved the more difficult problem of vibrating elastic clamped bands, finding that the sounds of two different modes were heard simultaneously. For the first time, Daniel Bernoulli gave a theoretical justification for the superposition of modes already assumed by Sauveur in the case of vibrating strings. The argument was necessarily more physical than mathematical, since no PDEs could yet be written. Clearly, it is the hearing of the sounds of several modes that prompted Bernoulli to imagine superposition.

“Both sounds exist at once and are very distinctly perceived...This is no wonder, since neither oscillation helps or hinder the other; indeed, when the band is curved by reason of one oscillation, it may always be considered as straight in respect to another oscillation, since the oscillations are virtually infinitely small. Therefore oscillations of any kind may occur, whether the band be destitute of all other oscillation or executing others at the same time. In free bands, whose oscillations we shall now examine, I have often perceived three or four sounds at the same time.”

Taylor, De motu nervi tensii, 1713.

Bernoulli, De vibrationibus et sono laminarum elasticarum commentationes physicomathematicae, 1751
Jean-Baptiste le Rond d’Alembert (1717-1783) was a French mathematician, mechanician, physicist, philosopher and music theorist.

D’Alembert’s first exposure to music theory was in 1749 when he was asked to review an article submitted to the academy by Jean-Philippe Rameau. D’Alembert wrote a glowing review praising the author’s deductive character as an ideal scientific model. He saw in Rameau’s music theory support for his own scientific ideas.

In 1752, d’Alembert attempted a fully comprehensive survey of Rameau’s works in his *Eléments de musique théorique et pratique suivant les principes de M. Rameau*, helping to popularize the work of the composer and advertise his own theories.

In 1747, he publishes *Recherches sur la courbe que forme une corde tendue mise en vibration*, in which he obtained the PDE of a vibrating string simply by combining *Taylor’s expression of the restoring force* with *Newton’s acceleration law*.

Derivation of the 1 − d wave equation from from Hooke’s law:

Imagine an array of $n$ weights of mass $m$ interconnected with massless springs of length $h$. The equation of of the displacement $y_k$ of the $k$-th mass is given by:

$$\frac{d^2 y_k(t)}{dt^2} = \left(\frac{na}{L}\right)^2 (y_{k+1}(t) - 2y_k(t) + y_{k-1}(t)), \quad k = 1, 2, \cdots, n-1.$$

- $L$ = the total length of the string and $h := L/n$.
- $x_k = kL/n$, the position of the $k$-th mass.
- $a = L\sqrt{k/m}$, $k=$the tension or spring constant.

D’Alembert observed that when $n \to \infty$ or $h \to 0$ the finite difference converges to $\frac{\partial^2 y}{\partial x^2}$. 
In their works on the $1 - d$ wave equation (only one space variable) aimed to explain the behavior of the vibrating string, L. Euler, d'Alembert and D. Bernoulli invented the concept of \textbf{partial differential equation (PDE)}. 

Previously, the displacement of the vibrating string was studied as function of the time or of the distance between a point on the string and one of the endpoints of the string.

One of the first approximations of the vibrating string was to consider it as a collagen of beads: the string was considered to be composed by $n$ equal masses separated by the same distance and interconnected by pieces of elastic, flexible and without load (weight) thread.

To treat the continuous string, the number of masses was considered to tend to infinity and their size and weight to decrease so that the total weight of the increasing quantity of beads approximated the one of the continuous string.

The subtleties of passing to the limit were ignored.
History of waves: d’Alembert

Nowadays, the analysis of the most existing undulatory phenomena passes through hyperbolic PDEs and systems of PDEs whose prototype is the $1 - d$ wave equation describing the vibrations of a flexible beam:

$$y_{tt} - a^2 y_{xx} = 0, \quad y_{tt} = a^2 y_{xx} \text{ or } \frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}, \quad x \in (0, L).$$

In practice, this PDE should be completed with initial $(y(x, 0) = y^0(x) \text{ and } y_t(x, 0) = y^1(x))$ and boundary $(y(0, t) = y(L, t) = 0)$ conditions to guarantee the uniqueness of its solution. D’Alembert considered the case $y^1 = 0$.

D’Alembert looked for solutions in the form $y(x, t) = f(at + x) + g(at - x)$.

- from the boundary conditions $\Rightarrow f(at) + g(at) = 0$ and $f(at + L) + g(at - L) = 0$ or $f(at + L) = f(at - L), \forall t$. Thus, $f$ is periodic of period $2L$ in the variable $x + at$.
- $f$ is an odd function since $\frac{\partial y}{\partial t} = 0$ for all $x \in [0, L]$.

$$y(x, t) = f(at + x) - f(at - x).$$

For d’Alembert, $f$ should be odd, periodic and have second-order derivatives. He admitted that his method did not allowed the initial condition of a plucked string, i.e. a triangular shape as initial position.
Leonhard Euler (1707-1783) was a pioneering Swiss mathematician and physicist.

His father, Paul Euler, was a friend of the Bernoulli family. He took particular mathematics classes from Johann Bernoulli, who quickly discovered his incredible talent for mathematics.

In 1726, Euler completed a dissertation on the propagation of sound with the title De Sono.

He worked in almost all areas of mathematics: geometry, infinitesimal calculus, trigonometry, algebra, number theory and graph theory.

He is also known for his work in mechanics, fluid dynamics, optics and astronomy.

He introduced much of the modern mathematical terminology and notation:

- He was the first to write $f(x)$ to denote the function $f$ applied to the argument $x$.
- The modern notation for the trigonometric functions.
- The letter $e$ for the base of the natural logarithm (known as Euler’s number).
- The Greek letter $\Sigma$ for summations.
- The letter $i$ to denote the imaginary unit.
- The use of the Greek letter $\pi$ to denote the ratio of a circle’s circumference to its diameter was popularized by Euler.

He made important discoveries in infinitesimal calculus:

- **Power series.** $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2 / 6$.

- **Euler’s formula for complex exponentials.** $\exp(i\varphi) = \cos(\varphi) + i\sin(\varphi)$.

- **Calculus of variations.** His best-known result: the Euler-Lagrange equation: $q$ is a stationary point of $J(q) = \int_a^b L(t, q(t), q'(t)) \, dt$ iff $L_x(t, q(t), q'(t)) - \frac{d}{dt} L_v(t, q(t), q'(t)) = 0$, where $L = L(t, x, v)$. 
Euler made important contributions in optics. He disagreed with Newton’s corpuscular theory of light in *Opticks*, which was then the prevailing theory. In his papers on optics ensured that the wave theory of light proposed by Christian Huygens would become the dominant mode of thought, at least until the development of the quantum theory of light.

After reading d’Alembert’s memoir, Euler published a theory based on the wave equation and the d’Alembert formula, in which he tolerated any continuous curve with piecewise continuous slope and curvature.

Euler, *De vibratione chordarum exercitatio*, *Nova acta eruditorum*, 1749.

- The non-existence of the partial derivatives entering the wave equation did not worry him as long as the relation between solution and initial data had a well-defined geometric meaning.
- He later prided himself of covering the case of the plucked string, for which the initial shape is triangular.
- We may retrospectively judge that d’Alembert abusively restricted the solutions to be analytic, while Euler tolerated solutions for which the differentials in the wave equation did not acquire a well-defined meaning until modern distribution theory.
- In 1753 Daniel Bernoulli published a lengthy, polite, but angry reaction to d’Alembert’s and Euler’s contributions to the problem of vibrating string. In his opinion, he had indicated the true physical solution years earlier and the sophisticated mathematics of his competitors had only obscured the subject.

I saw at once that one could admit this multitude of curves for the vibrating string according to d’Alembert and Euler only in a sense altogether improper. I do not less admire the calculations of Messrs. d’Alembert and Euler, which certainly include what is most profound and most advanced in all of analysis, but which show at the same time that an abstract analysis, if heeded without any synthetic examination of the question proposed, is more likely to surprise than enlighten. It seems to me that giving attention to the nature of the vibrations or strings suffices to foresee without any calculation all that these great geometers have found by the most difficult and abstract calculations that the analytic mind has yet conceived.
Bernoulli argued that any sonorous body could vibrate in a series of simple modes with a well-defined frequency of oscillation. As he had earlier indicated, these modes could be superposed to produce more complex vibrations. The rest of his argumentation depended on two assertions:

- d'Alembert's and Euler's supposedly new solutions are nothing but mixtures of simple modes, since in an infinite series of the form $f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/L)$ there is a sufficiently large number of constants $a_n$ so that the series can adapt to any curve.
- The aggregation of these partial modes in a single formula is incompatible with the physical character of the decomposition.

Bernoulli did not have a direct mathematical proof of the first point. Also without rigorous proof, he stated that any possible movement of the vibrating string takes the form $y(x, t) = \sum_{n=1}^{\infty} a_n \cos(n\pi at/L) \sin(n\pi x/L)$. Instead, he showed that with this superposition the basic periodicity properties of the solutions of d'Alembert and Euler can be reproduced.

In his reaction to Bernoulli’s memoirs, Euler praised his colleague for having best developed the physical part of the problem of vibrating strings, but denied the generality and superiority of the multi-modes solution. His arguments were of two kinds. Firstly, he rejected Bernoulli’s claim that the superposition of simple modes always preserved the structure of the partial modes:

When the number of terms becomes infinite, it seems doubtful that the curve is composed of an infinite number of sine curves: The infinity seems to destroy the nature of the composition.

To illustrate this point, Euler gave the example below to indicate that Fourier synthesis is able to produce a much greater variety of functions than appears at a first glance:

$$f(x) = \sum_{n=1}^{\infty} \alpha^n \sin(n\pi x/L) = \frac{1}{2i} \sum_{n=1}^{\infty} \alpha^n \left( \exp(i n\pi x/L) - \exp(-i n\pi x/L) \right) = \frac{\alpha \sin(\pi x/L)}{1 - 2\alpha \cos(\pi x/L)}$$
Euler to Bernoulli: If Mr. Bernoulli’s consideration provided all the curves that may occur in the motion of strings, it would certainly be infinitely preferable to our method, which could then only be regarded as an extremely thorny detour to reach a solution so easy to find.

Euler to Bernoulli on periodicity: All the curves comprised in Bernoulli’s equation, even though the number of terms is increased to infinity, have certain characters that distinguish them from all other curves. For if we take a negative abscissa $x$, the ordinate also becomes negative and equal to that which corresponds to the positive abscissa $x$; similarly, the ordinate that corresponds to the abscissa $L + x$ is negative and equal to that which corresponds to the abscissa $x$. Therefore, if the curve that has been given to the string at the beginning does not have these properties, it is certain that it is not comprised in the said equation. Now no algebraic curve can have these properties, which must therefore all be excluded from this equation; there is no doubt that an infinite number of transcendental curves must also be excluded.

He transferred to the limit of a sequence of functions the analyticity properties of the general term of this sequence.

Euler said that any function representable as trigonometric series must be periodic and odd, whereas in the d’Alembert formula $y(x, t) = f(x + at) + f(at - x)$, $f$ could be arbitrary. Moreover, he observed the possibility of performing a periodic odd extension of $f$, but this could lead to a lack of continuity, so that he considered that Bernoulli solutions are a subclass of his solutions.

D’Alembert objected that the solutions of Bernoulli and Euler were twice differentiable.

This controversy was maintained for 10 years without conclusions.
Joseph-Louis Lagrange (1736-1813) was a mathematician and astronomer born in Turin who lived part of his life in Prussia and part in France. He made significant contributions to all fields of analysis, number theory, classical and celestial mechanics.

- was one of the creators of the calculus of variations, deriving the Euler-Lagrange equations for extrema of functionals. He also extended the method to take into account possible constraints, arriving at the method of Lagrange multipliers.
- Lagrange invented the method of solving differential equations known as variation of constants
- In calculus, Lagrange developed a novel approach to interpolation and Taylor series.

Based on the explicit solutions given by Euler to the system of ODEs

\[
\ddot{y}_k(t) = \alpha^2 (y_{k+1}(t) - 2y_k(t) + y_{k-1}(t)), \quad k = 1, N, \quad y_0(t) = y_{N+1}(t) = 0, \quad y_k(0) = y_k^0, \quad y_k'(0) = y_k^1
\]

as

\[
y_k(t) = \sum_{n=1}^{N} (\alpha_n \cos(\omega_n t) + \beta_n \sin(\omega_n t)) \sin(nk\pi/(N + 1)), \quad \text{with} \quad \omega_n = 2\alpha \sin(n\pi/(2(N + 1))),
\]

after passing to the limit as \( N \to \infty \) in the above system (some of the computations not being rigorous), he obtained the formula:

\[
y(x, t) = \left( \frac{2}{L} \int_0^L y^0(z) \sum_{k=1}^{\infty} \sin(k\pi z/L) \, dz \right) \sin(k\pi x/L) \cos(k\pi at/L)
\]

\[
+ \left( \frac{2}{L} \int_0^L y^1(z) \sum_{k=1}^{\infty} \frac{1}{k\pi a/L} \sin(k\pi z/L) \, dz \right) \sin(k\pi x/L) \sin(k\pi at/L).
\]
Because his aim was to obtain the same result as Euler, he did not observed that he obtained the expression of the Fourier coefficients.

D'Alembert objection on Lagrange methodology to pass to the limit by approximating \( \sin(k\pi/2N) \sim k\pi/2N \) as \( N \to \infty \) when \( k \sim N \).

In 1760, Lagrange consider the wave equation and pretends to have obtained d'Alembert formula without using derivatives of the initial data, \( y^0 \) and \( y^1 \).

There, then, is the theory of this great geometer (Euler) placed beyond all doubt and established upon direct and clear principles that rest in no way on the law of continuity (i.e. analyticity) which Mr. d'Alembert requires; there, moreover, is how it can happen that the same formula that has served to support and prove the theory of Mr. Bernoulli on the mixture of isochronous (i.e. harmonic) vibrations when the number of moving bodies is finite shows us the insufficiency of this theory when the number of these bodies becomes infinite. Indeed the change that this formula undergoes in passing from one case to the other is such that the simple motions which made up the absolute motions of the whole system destroy each other for the most part, and those which remain are so disfigured and altered as to become absolutely unrecognizable. It is truly annoying that so ingenious a theory...is shown false in the principal case, to which all the small reciprocal motions occurring in nature may be related.

The main problem, the representation of a function as a trigonometric series, was solved later on by Fourier.
During this controversy on the resolution of the wave equation, they continued to analyze problems related to musical instruments: vibration of physical structures and hydrodynamics related to the propagation of sound in the air:

- In 1762 Euler tackled the problem of the vibrating string of variable thickness, stimulated by the following statement of Jean-Philippe Rameau: The consonance of the musical sound is due to the fact that the tones composing any sound are harmonics of the fundamental one, i.e. their frequencies are integer multiples of the fundamental frequency.

- When considering the equation $y_{tt} - c^2(x)y_{xx} = 0$, Euler does not find a general solution. Only in one particular case: when the string is constituted by two pieces of lengths $a$, $b$ and thickness $m$, $n$, the frequency $\omega$ can be found by solving the equation $m \tan(\omega a/m) + n \tan(\omega b/n) = 0$.

- In 1763 D’Alembert also analyzes the vibrating string of variable thickness and introduces the method of separation of variables:

  $y_{tt} - c^2(x)y_{xx} = 0$ and $y(x, t) = h(t)g(x) \Rightarrow \frac{h''(t)}{h(t)} = c^2(x) \frac{g''(x)}{g(x)}$

  i.e. $h''(t) + \lambda^2 h(t) = 0$ and $g''(x) + \lambda^2 c^2(x)g(x) = 0$.

  This is a great contribution to the spectral problems.

- Euler analyzed the equation of the heavy vibrating string $y_{tt} - c^2y_{xx} = g$ and obtained the solution

  $y(x, t) = \phi(x + ct) + \psi(x - ct) - gx(x - L)/(2c^2)$,

  i.e. the general solution of the non-homogeneous problem = general solution of the homogeneous one + particular solution of the non-homogeneous problem.
Euler also tackled the problem of vibrations of rectangular and circular membranes, but the complete theory (in arbitrary domains) was developed by Denis Poisson in 1829.

Euler worked on the propagation of sound in the air and hydrodynamics. The air is a compressible fluid and the theory of propagation of sound is part of the fluid mechanics, but also of the elasticity, since the air is also an elastic medium.

In 1739, D. Bernoulli initiated the analysis of sounds emitted by musical instruments and confirmed many of his theoretical results by experiments.

Euler, by considering the sound emitted by bells and the vibration of beams obtained fourth-order equations in space, but their study did not progress in the XVIII century, only in 1909 by the contribution of Walther Ritz (1878-1909).
Joseph Fourier (1768-1830) was a French mathematician and physicist best known for initiating the investigation of Fourier series and their applications to problems of heat transfer and vibrations.

In 1822 Fourier presented his work on heat flow *Théorie analytique de la chaleur* (The Analytic Theory of heat) in which he based his reasoning on Newton’s law of cooling: the flow of heat between two adjacent molecules is proportional to the extremely small difference of their temperatures. Main contributions:

- In mathematics, Fourier claimed that any function of a variable, whether continuous or discontinuous, can be expanded in a trigonometric series of sines. One can imagine that the heat that comes at every instant from the source thus divides itself into distinct portions, that it propagates according to one of the said elementary laws and that all these partial motions occur without troubling each other.

- Though this result is not correct, Fourier’s observation that some discontinuous functions are the sum of infinite series was a breakthrough. The question of determining when a Fourier series converges has been fundamental for centuries.

- Fourier faced the problem of developing a constant into a sum of cosines, i.e.

  \[
  1 = \sum_{k=0}^{\infty} c_k \cos((2k + 1)x), \quad x \in [-\pi/2, \pi/2].
  \]

As he was unaware of the orthogonality of simple modes, he relied on his algebraic prowess to solve the infinite system of linear equations obtained by identifying all the derivatives of the sum of cosines at the origin with the derivatives of the function to be developed. In the present case, this gives:

\[
\sum_{k=0}^{\infty} c_k = 1 \quad \text{and} \quad \sum_{k=1}^{\infty} (2k + 1)^2 n c_k = 0, \quad \forall n \geq 1 \Rightarrow c_k = \frac{4}{\pi} \frac{(-1)^k}{2k + 1}.
\]
First nontrivial convergence result of series

\[ S_n(y) = \sum_{k=0}^{n-1} \frac{(-1)^k}{(2k + 1)} \cos(2k + 1)y \]  

(100)

to get

\[ S'_n(y) = \sum_{k=0}^{n-1} (-1)^{k+1} \sin(2k + 1)y = (-1)^n \frac{\sin 2ny}{\cos y}, \]  

(101)

and

\[ S_n(x) = S_n(0) + (-1)^n I_n(x), \quad \text{with} \quad I_n(x) = \int_0^x \frac{\sin 2ny}{\cos y} \, dy. \]  

(102)

Fourier repeatedly integrated the latter integral by parts to obtain a series whose every term contained a negative power of \( n \), and hence concluded that it vanished in the limit of infinite \( n \). In another section, he showed that the integral remaining after \( p \) partial integrations was majored by a quantity of the order \( n^{-p} \), which made \( I_\infty(x) = 0 \) a rigorous result. Together with Leibniz’s well-known result \( S_\infty(0) = \pi/4 \), this implies the desired identity (99).121

Fourier thus obtained the first convergence proof for a trigonometric series that was not trivially convergent. In the following section of his draft, he showed...
Fourier’s next challenge was to extend to any indefinitely differentiable function the algebraic method of elimination he had used to develop a constant into a series of cosines. Some time must have elapsed before he managed to do so, for the draft of 1805/1806 does not include it. In the sine case, the development reads

\[ f(x) = \sum_{k=1}^{\infty} a_k \sin kx, \quad (103) \]

and the linear system to be solved is

\[ \sum_{k=1}^{\infty} (-1)^n k^{2n+1} a_k = f^{(2n+1)}(0), \quad \text{with } n = 0, 1, 2, \ldots \quad (104) \]

After a few pages of difficult calculations, Fourier obtained

\[ a_k = \frac{2}{\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^{k+n+1} \frac{1}{k^{2n+1} m!} f^{(m+2n)}(0). \quad (105) \]

Using Taylor’s formula, he rewrote this as

\[ a_k = \frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^{k+n+1} k^{-2n-1} f^{(2n)}(\pi), \quad (106) \]

in which he recognized the result of the repeated integration by parts of \(^{123}\)

\[ a_k = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin kx \, dx. \quad (107) \]
Fourier has obtained the following trigonometric expansion of any function \( f \) defined over \([0, 2\pi]\):

\[
f(x) = \frac{1}{2\pi} \int_{0}^{2\pi} f(y) \, dy + \frac{1}{\pi} \sum_{k=1}^{\infty} \left( \cos(kx) \int_{0}^{2\pi} f(y) \cos(ky) \, dy + \sin(kx) \int_{0}^{2\pi} f(y) \sin(ky) \, dy \right).
\]

By the method of approximation, I had obtained the development of a function in sines and cosines of multiple arcs. Having next resolved the question of an infinity of bodies that communicate heat to each other, I recognized that this development had to apply to an arbitrary function as well.

Orthogonality of eigenfunctions: Seeking to verify the same theorem a third time, I used the procedure which consists in multiplying by \( \sin(rx) \) the two sides of a sine development and integrating from \( x = 0 \) to \( x = 2\pi \). Fourier was clearly unaware of Euler’s anterior use of the same strategy.
In the oral presentation of his theory at the French Academy on December 21, 1807, Fourier ascribed the same degree of physical reality to the harmonics of a vibrating string and to the partial modes of heat propagation:

The system of initial temperatures can be such that the ratios originally established among them persist without any alteration during the whole process of cooling. This singular state that enjoys the property of subsisting once it is formed can be compared to the figure that a sonorous string takes when it yields the principal sound. It can take diverse analogous forms, the ones corresponding to the subordinate sounds (harmonics) in the case of the elastic string. Consequently, for every solid there is an infinite number of simple modes according to which heat can propagate and dissipate without change of the initial distribution law...Whatever be the manner in which the different points of the body have been heated, the initial and arbitrary system (of temperatures) can be decomposed into several simple and durable states similar to those I just described. Each of these states subsists independently of all the others and undergoes no other changes than those which would still occur if it were alone. The relevant decomposition is not a purely rational and analytical result; it occurs effectively and results from the physical properties of heat. Indeed the speed with which the temperatures decrease in each simple system is not the same for every system...To be true, these properties are not always as sensible as the isochronism of pendulums and the multiple resonance of vibrating strings are; but they can be established by observation and they became manifest in all of my experiments.
A problematic reception

In 1807, Fourier submitted to the French Academy a long memoir entitled Théorie de la propagation de la chaleur dans les solides. This memoir contained the various results he had reached so far, rearranged in an order differing from the chronological order of discovery: the discretized bar and the discretized annulus, the general equations for a continuous body, the lamina, Fourier’s theorem by elimination, the same theorem by orthogonality of simple modes, the continuous annulus, the transition from the discrete to the continuous annulus, the sphere, the cylinder, the cube, and experiments.

Fourier presented three derivations of his fundamental theorem instead of selecting the most direct one. One may surmise that the derivation by elimination had given him too much sweat for being left aside. In a note added to his memoir in 1808, Fourier explained that the three proofs complemented each other. The proof by elimination established the existence of the trigonometric development, but only for indefinitely differentiable functions. The proof by orthogonality gave the form of the coefficients without establishing the existence of the development. The proof by taking the limit of the discrete problem seemed to give both the existence and the form of the development for arbitrary functions, but depended on an artificial model of heat propagation. Lastly, Fourier gave a rigorous proof of convergence in the particular case of the trigonometric series for a linear function.

The four examiners of Fourier’s memoir, Lagrange, Laplace, Monge, and Lacroix never wrote the expected report. The examiners must have thought that Fourier’s memoir should not be published in its present shape.
Summarizing...

- **Euclid (300BC):** laws of reflection
- **Heron of Alexandria (10-70):** shortest path principle
- **Galileo Galilei (1564-1642), Johannes Kepler (1571-1630), Isaac Newton (1642-1727):** Epur si muove! - laws of planetary motion
- **Christian Huygens (1629-1697):** Huygens principle of spherical waves, pendulum, equilibria.
- **Johann Bernoulli (1667-1748) and Jakob Bernoulli (1655-1705):** pendulum, equilibria
- **Alhazen (965-1040), Sahl (940-1000):** propagation and refraction of light.
- **Roijen Snell (1580-1626), René Descartes (1596-1650):** laws of reflection of light
- **Pierre de Fermat (1601-1665), Pierre Louis Moreau de Maupertuis (1698-1759), Gottfried Leibniz (1646-1716):** minimization principles associated with refraction of light.
- **Burning mirrors**
- **Robert Hooke (1635-1703):** motion of springs
- **Jean Philippe Rameau (1683-1764):** musical esthetics
- **Daniel Bernoulli (1700-1782), Leonard Euler (1707-1783), Jean le Rond D’Alembert (1717-1783) and Joseph-Louis Lagrange (1736-1813):** infinitesimal calculus, derivation of the wave equation for the motion of strings, long controversy between the two representation formulae: d’Alembert and superposition.
- **D. Bernoulli (1753), Lagrange (1759), d’Alembert (1763):** method of separation of variables.
- **Euler-Lagrange equations in calculus of variations** characterizing critical points of functionals.
- **William Rowan Hamilton (1805-1865):** Hamiltonian mechanics.
Siméon Denis Poisson (1781-1840): vibrating membranes and the method of spherical means.

Joseph Fourier (1768-1830): Fourier series

Hermann von Helmholtz (1821-1894): Helmholtz equation for the analysis of waves.

Bernhard Riemann (1826-1866): in 1858-1859 he uses Green’s formulas to obtain representation formulas for the solution of the Laplace equation; also studies shock waves in gas dynamics ⇒ Riemann problem.

Gustav Robert Kirchhoff (1824-1887): application of the Riemann’s method to the wave equation

Euler-Poisson-Darboux equation giving representation formula for 3-d waves, Jean Gaston Darboux (1842-1917).


August Kundt (1839-1894) and Ernst Chladni (1756-1827): various physical experiments to prove properties of waves in tubes, fluids and plates

Augustin Louis Cauchy (1789-1857): modern theory of ODEs

Jean-Marie Duhamel (1797-1872): variation of constants formula.

Sofia Kovalevskaya (1850-1891): Cauchy-Kovalevskaya theorem on the existence, uniqueness and analyticity of solutions of systems with analytic coefficients.

Henri Navier (1785-1836) and George Gabriel Stokes (1819-1903): Navier-Stokes equations for viscous fluids (previously derived by Stokes for perfect fluids) and having as particular case the Euler equations (inviscid Navier-Stokes equations)

Henri Hugoniot (1851-1887) and William Rankine (1820-1872): law of the velocity of propagation of shock waves.

Ernst Mach (1838-1916): shock waves for supersonic missiles.
Johannes Martinus Burgers (1895-1981): Burgers equation was analyzed in 1916, is the 1-d analogue of the Navier-Stokes equations.


Jean Leray (1906-1998): in 1934, he wrote an important paper founding the study of weak solutions of the Navier-Stokes equations.

John Scott Russell (1808-1882): in 1834, he discovered the solitons in nature.

Diederik Korteweg (1848-1941) and Hendrik de Vries (1896-1989): KdV equation for the motion of 1-d water waves presenting solitons-type solutions.

Thomas Young (1773-1829) and Gabriel Lamé (1795-1870): elasticity

lord James Clerk Maxwell (1831-1879): electromagnetism.

Gaston Floquet (1847-1920) and Felix Bloch (1905-1983): Floquet theory for waves with periodic potentials (Bloch waves).

Christian Doppler (1803-1853): Doppler effect.

Laurent Schwartz (1915-2002): theory of distributions, giving sense to weak solutions of PDEs.

Albert Einstein (1879-1955): relativity theory.

...and to the history of numerical methods...

- **Euler, Carl Runge (1856-1927), Martin Wilhelm Kutta (1867-1944):** numerical methods for ODEs.
- **Carl Friedrich Gauss (1777-1855) and Rehuel Lobatto (1797-1866):** quadrature formulas
- **Gauss, Wilhelm Jordan (1842-1899) and André-Louis Cholesky (1875-1918):** methods of resolution for linear systems.
- **Lagrange polynomials.** Although named after Joseph Louis Lagrange, it was first discovered in 1779 by Edward Waring and rediscovered in 1783 by Euler.
- **Pafnuty Lvovich Chebyshev (1821-1894), Charles Hermite (1822-1901):** numerical interpolation
- **Riemann sums.**
- **Isaac Newton and Joseph Raphson (1648-1715):** Newton-Raphson method to find zeros of functions.
- **Josiah Willard Gibbs (1839-1903) and Runge:** Gibbs, Runge phenomena
- **d’Alembert:** The obtention of the wave equation modeling the motion of the vibrating string as limit of the semi-discrete model of coupled point masses.
- Alternating method of **Karl Hermann Schwarz (1843-1921)** in 1870 to solve Laplace equation in complicated domains (union between a disc and a rectangle) ⇒ domain decomposition techniques.
- **Walther Ritz (1878-1909):** Ritz method to solve plate equation and to compute Ernst Chladni (1756-1827) figures.
- **Boris Grigoryevich Galerkin (1871-1941):** Galerkin methods.
- **Richard Courant (1888-1972):** first finite element methods.
- **Peter Lax (1926-)** and **Robert Davis Richtmyer (1910-2003):** Lax(-Richtmyer) equivalence theorem, i.e convergence=consistency+stability.
- **John von Neumann (1903-1957):** von Neumann stability analysis to check the stability of finite difference schemes of linear PDEs
- **Taylor series.**
Some related bibliography


M. Gander, G. Wanner, From Euler, Ritz and Galerkin to modern computing.

Crowell, Vibrations and waves.