LAB1: HARMONIC OSCILLATORS

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Problem 1. Consider the harmonic oscillator equation:

$$x''(t)+\omega^2 x(t)=0,\quad t\in[0,T],\ x(0)=x^0,\ x'(0)=x^1. \tag{0.1}$$

Set $T=4\omega\pi$, $\omega=1$ and the following four initial data:

$$I)\ x^0=1,\ x^1=0;\quad II)\ x^0=0,\ x^1=1;\quad III)\ x^0=1,\ x^1=1;\quad IV)\ x^0=1,\ x^1=-1. \tag{0.2}$$

Task 1. Compute the explicit solution and plot it using Matlab.

Consider a general system of the form

$$x'(t)=f(x(t),v(t)),\quad v'(t)=g(x(t),v(t)),\ t\in(0,T),\quad x(0)=x^0,\ v(0)=v^0. \tag{0.3}$$

To solve system (0.3) with Matlab we can proceed as follows:

Step 1. Introduce the following function and save it as funct.m

```
function der=funct(t,X)
der=[f(X(1),X(2));g(X(1),X(2))];
```

Step 2. The following Matlab program solves problem (0.3) using the ode45 function:

```
X0 = [x^0, v^0]; % vector of initial data
T = 4π; dt = T/1000;% the time step
divt = [0 : dt : T];
[t0, sol] = ode45(@funct, divt, x0);
```

Task 2. Solve problem (0.1) by using the above program. Plot the true solution $x(t)$ and the numerical one $sol(:,1)$ in the same figure. Plot $sol(:,1)$ against $sol(:,2)$ in the horizontal/vertical direction.

The following program draws the phase portrait (x,v) of system (0.3) for $x\in[a,b]$ and $v\in[c,d]$.

```
x = [a : h_x : b]; v = [c : h_v : d]; % the division of the intervals [a,b] and [c,d];
[X, V] = meshgrid(x, v);
Xdot = f(X, V);
Vdot = g(X, V);
quiver(X, V, Xdot, Vdot)
```

Task 3: Draw the phase portrait of problem (0.1) for $a=b=c=d=2$ and $h_x=h_v=0.2$.

Problem 2. Consider the pendulum equation:

$$x''(t)+\omega^2\sin(x(t))=0,\quad t\in[0,T],\ x(0)=x^0,\ x'(0)=x^1. \tag{0.4}$$

Set $T=4\pi$, $\omega=1$ and the following four initial data:

$$I)\ x^0=\pi/10,\ x^1=0;\quad II)\ x^0=\pi/2,\ x^1=0;\quad III)\ x^0=\pi,\ x^1=0;\quad IV)\ x^0=3\pi/2,\ x^1=0. \tag{0.5}$$

Repeat Tasks 2 and 3 in the previous problem.

Problem 3. Consider the damped harmonic oscillator:

$$x''(t)+2\beta x'(t)+x(t)=0,\quad t\in[0,T],\ x(0)=x^0,\ x'(0)=x^1. \tag{0.6}$$

Set $T=4\pi$, $x^0=1$, $x^1=0$ and the following four values of β:

$$I)\ \beta=0.05;\quad II)\ \beta=0.5;\quad III)\ \beta=1;\quad IV)\ \beta=1.5. \tag{0.7}$$

Repeat Tasks 1,2,3 in Problem 1.

Problem 4. Consider the damped harmonic oscillator with driven force:

$$x''(t) + 2\beta x'(t) + x(t) = A\cos(\omega t), \quad t \in [0,T],\ x(0) = x^0,\ x'(0) = x^1. \tag{0.8}$$

Set $T = 1/\beta$, $x^0 = 1$, $x^1 = 0$, $A = 1$ and the following four values of β:

$$I)\ \beta = 1; \quad II)\ \beta = 1/10; \quad III)\ \beta = 1/50; \quad IV)\ \beta = 1/200. \tag{0.9}$$

Repeat Task 2 of Problem 1 for all the following six values of ω: $\omega = 1/2$, $\omega = 1/4$, $\omega = (1+\omega_d)/2$, $\omega = \omega_d$, $\omega = (\omega_d + \omega_r)/2$ and ω_r, where $\omega_d := \sqrt{1-\beta^2}$ and $\omega_r := \sqrt{1-2\beta^2}$ are the damping and resonance frequencies.