Courses 2021-22

HARMONIC ANALYSIS

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January 10th – February 25th, 2022 (15 sessions)
16:00 - 18:00 (a total of 30 hours)

1. Description

This course aims to cover topics in Harmonic Analysis that we consider fundamental within the current development of the field. It is oriented towards graduate students (and possibly undergraduate students in the final year of the Mathematics' degree). The students are expected to have a knowledge on the basic properties and objects related to real variable methods of Fourier analysis.

The lectures will be held face-to-face and live-streamed, either at BCAM or at UPV/EHU facilities (according to availability), during the period 10th January 2022 - 25th February 2022, 2 x 50 minutes, twice per week, with a total of 30 hours (a small adjustment is required, one of the weeks should have three sessions instead of two). Along the lectures, exercises will be proposed to be solved by the students. In some instances, proof-guiding of the theoretical results will be also suggested as exercises. The language of the course will be English.

2. Topics for the course

The course is organised in two parts:

- The first part of the course will be a quick review of the basic background, and meant to be done very briefly in the first lectures;
- The second part, which will take the majority of the time of the course, is meant to introduce the students to a broader set of topics in harmonic analysis. The goal is to provide them with the basic tools and references one needs to understand modern research topics in harmonic analysis. The last lectures in this second part are intended to introduce the students to some selected current topics in research. Also, pending on the interest of the students, small projects may be provided.

In what follows, we provide a sequence of topics meant to be explored in the two parts the course. The second part of the course is not meant to cover the entirety of the list of subjects below, only a subset of the topics which will depend on time and feedback from the students.

2.1. First part.

(1) Hardy-Littlewood maximal function and Lebesgue differentiation:

- Vitali covering Lemma;
- Hardy-Littlewood-Wiener theorem;
- Lebesgue differentiation and differentiation bases.

(2) Convolutions and approximate identities:
- Young's inequality;
- Maximal operators of convolution type;
- Pointwise and L^p convergence of approximate identities.

(3) Real and Complex Interpolation:
- Riesz-Thorin interpolation;
- Marcinkiewicz interpolation.

(4) Fourier Transform:
- $L^1 + L^2$ Theory;

(5) Fourier series:
- Poisson summation;
- Criterias for pointwise convergence;
- Convergence in norm.

(6) The Schwartz Class and Tempered Distributions.
(7) Calder on-Zygmund decomposition.
(8) Hilbert and Riesz Transform:
- Conjugate harmonic extension;
- Strong and weak L^p boundedness;
- Convergence of Fourier integrals.

(9) Singular Integrals:
- Method of rotations;
- Singular integrals with even/odd kernels;
- Calder on-Zygmund theorem.

2.2 Second Part.

(1) Fractional Integration:
- Definition of the Riesz potentials and Fourier properties;
- Hardy-Littlewood-Sobolev theorem;
- The fractional maximal function;

(2) Littewood-Paley Theory:
- Vector-valued inequalities;
- Square functions;
- Multipliers;
- The spherical maximal function;
- Bochner-Riesz Multipliers.

(3) Smoothness spaces:
- Sobolev spaces and their embeddings;
- Littlewood-Paley theory and smoothness of functions.

(4) Hardy Spaces and BMO:
- Atomic decomposition;
- H^1 and BMO duality;
- John-Niremberg theorem;
- Singular integrals on Hardy spaces;
- Singular integrals on BMO.

(5) Singular Integrals II:

- Non-convolution type singular integrals;
- Vector-valued extensions;
- *T*(1) and *T*(*b*) theorems;

(6) Weighted inequalities:

- The A_p condition and reverse Hölder Property;
- Strong and weak type weighted inequalities;
- Rubio de Francia's extrapolation.

(7) Sparse Domination:

- De nition and properties of a sparse operator;
- Weighted inequalities via sparse domination.

(8) Oscillatory Integrals:

- Phases without critical points;
- Van der Corput Lemma;
- Stationary Phase.

(9) Fourier Restriction:

- Stein-Tomas theorem;
- Knapp Examples;
- The restriction conjecture;
- Strichartz estimates;
- Bilinear estimates.

REFERENCES:

[1] J. Duoandikoetxea, *Fourier Analysis*. Graduate Studies in Mathematics, 29, AMS, Providence, RI 2001.
[2] L. Grafakos, *Classical Fourier Analysis*. Graduate Texts in Mathematics, 249, Springer, New York, NY 2008.
[3] L. Grafakos, *Modern Fourier Analysis*. Graduate Texts in Mathematics, 250, Springer, New York, NY 2008.
[4] C. Muscalu, W. Schlag, *Classical and Multilinear Harmonic Analysis*. Cambridge University Press, New York NY 2013.
[5] E. Stein, *Harmonic Analysis*. Princeton University Press, Princeton, NJ 1993.
[6] E. Stein, *Singular Integrals and Differentiability Properties of Functions*. Princeton University Press, Princeton, NJ 1970.
[7] E. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*. Princeton University Press, Princeton, NJ 1971.
[8] T. Wolff, *Lectures on Harmonic Analysis*. American Mathematical Society, Providence, RI 2003.

***Registration is free, but inscription is required before 3rd January, 2022:** So as to inscribe go to https://forms.gle/1Xr6ndNfB4f1fT2o6 and fill the registration form.