Courses 2022-23

CLASSICAL TOOLS IN DIFFERENTIAL TOPOLOGY

PABLO PORTILLA CUADRADO

10 – 21 October 2022
Monday to Friday (10 sessions)
10:00 – 12:00 (a total of 20 hours)

The course aims to be a comprehensive introduction to some of the most used tools in differential topology with a particular emphasis on transversality and Morse theory. Next, we give a brief motivation to these two concepts.

Transversality. Think of a parabola tangent to a line in R2. Intuition tells us that this situation is rather atypical (or unstable). Indeed, whenever we perturb either the parabola or the line, one of two things might happen: they will be separated or they will intersect in two points. On the contrary, a small perturbation of the latter two situations does not produce a qualitative change (we say that they are stable). This phenomenon is made rigorous through transversality.

This concept is ubiquitous in geometry and in particular it is necessary for the understanding of most of the arguments used in differential topology. In particular, one needs to master transversality in order to deal with more refined and important theories such us intersection theory or degree theory.

Morse theory. A line admits a smooth function that takes arbitrarily large values, while a function defined on a circle, this being compact, must reach a maximum and a minimum. Morse theory deals with this phenomenon. More concretely it treats the problem of understanding the topology of a manifold by studying the smooth functions defined on it. In the end, Morse theory says that any manifold can be recovered from a ball by attaching to it very specific simple pieces (handles). Moreover, the number of such pieces and where they are attached is governed by the isolated critical points of a generic enough (Morse) function.

This theory has a vast amount of applications. For example, one can recover classical results like the (smooth version of the) Jordan curve theorem or the classification of manifolds of dimension 2. But it has been used as well as a powerful tool in proving some other results that are considered milestones in differential topology: for instance, the h-cobordism theorem by Smale.

General perspective. One can look at transversality and Morse theory from a more modern point of view and think of them as finite dimensional models for Floer homology theory. A mini-course on Floer homology theory will be given by Javier Fernández de Bobadilla at Universidad Complutense de Madrid right after this course.

Inaugural lecture:
Sympletic wars part II: Flexibility strikes bark – Francisco Presas (ICMAT)

Complementary lecture:
Morse Homology – María Pe Pereira (UCM)
Characteristics classes – Jose Seade (UNAM)

Pre-course: for those who need to develop a background, upon request there will be a pre-course *Guided reading in algebraic topology* given by Eki Gonzalez (BCAM).

CONTENTS

First part: transversality, fixed points and degree theory
The course will start with an introductory lesson recalling concepts that the audience is supposed to be familiar with. In particular: manifolds (with boundary), Implicit/Inverse Function theorems, homotopies and vector fields. Then we will dedicate the rest of this first part to concepts related (but not restricted) to transversality. This part of the course will have as bibliography the book "Diferential topology" by V. Guillemin and A. Pollack.

The program of maximums is the following. (Consider points 1 to to 4 as minimums of this first part).

1. Preliminary concepts.
2. Sard's theorem: regular values are dense. Partitions of unity: gluing local information. Embedding theorem: every manifold fits in some Euclidean space.
3. Transvesarlity. Transversality is stable. Transversality is generic. Transversality near on the boundary.
4. Tubular neighborhood theorem.
5. Lefschetz number. Fixed point theorem.
6. Index of a vector field. Poincare-Hopf index theorem.
7. Degree of a map. Hopf's degree theorem.

Second part: Morse theory towards h-cobordism.
This second part will be dedicated to the systematic study of Morse theory. If time permits, the course will end with a sketch of a proof of the h-cobordism theorem. This part of the course will have as bibliography the book "An introduction to Morse theory" by Y. Matsumoto. (Consider points 1 to to 4 as minimums of this second part).

1. Gluing manifolds along the boundary. Gluing diffeomorphisms.
2. Morse Lemma. Morse functions are generic.
3. Gradient-like vector fields.
4. Handle decomposition of a manifold induced by a Morse function.
5. Handlebody calculus: handle sliding and handle canceling.
6. Whitney's trick. h-cobordism theorem.

***Registration is free, but inscription is required before 5th October 2022:** So as to inscribe go to https://forms.gle/vhwHTtEwepT5tSYM6 and fill the registration form.