BCAM, June 6, 2011

Complexity of control systems

Enrique Zuazua
IKERBASQUE & Basque Center for Applied Mathematics
E-48160 Derio - Basque Country- Spain
zuazua@bcamath.org http://www.bcamath.org/zuazua/

THE CLASSICAL CLASSIFICATION OF PDE's

Elliptic + parabolic + hyperbolic

But REAL MODELS are often MORE COMPLEX

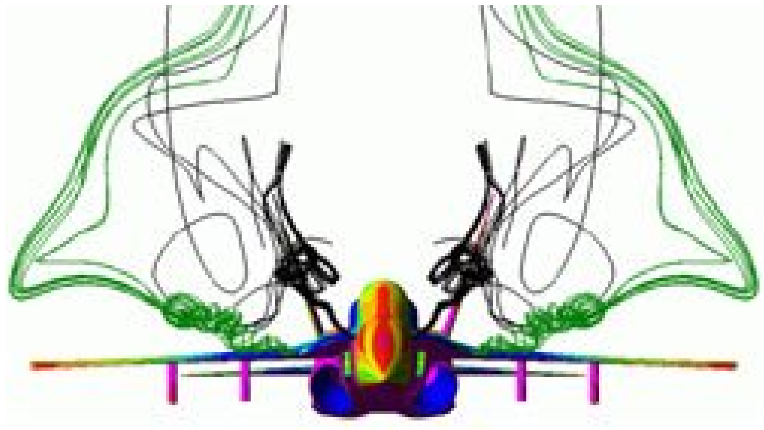
Often, reality is of a multi-physics nature and consequently, more realistic models are also of HYBRID type.

TWO CLASSES OF EXAMPLES:

- MULTI-STRUCTURES.

  interconnected structures; structures of different dimensions.

- FLUID-STRUCTURE INTERACTION.



http://www.mscsoftware.com.au/products/software/cfdrc/cfd-fastran/fsi.

Aeroelasticity: the influence of an exterior flow on a structure (aircraft in flight);

Aeroacoustics: the vibrations (noise) induced by a turbulent flow originated by a boundary layer on a solid wall (flight vehicles);

Hydroelasticity: small vibrations coupling an elastic structure and a liquid (gravity waves in the reservoir of a space vehicle);

Elastoacoustics: noise in a cavity produced by the vibrations of the surounding structure.

THE MAIN DIFFICULTY:

Solving (analytically or numerically) these interaction systems is not simply superposing two separate theorems for the corresponding type of PDE.

Interactions do produce new (unexpected?) phenomena and new analytical (and numerical tools) are required.

THE RECEIPT: To decompose the original system into simpler subsystems of specific forms. Treat each subsystem by an ad-hoc method, and put or assamble together the results to get the global one.

THE BASIC MATHEMATICAL INGREDIENT: LIE'S THEOREM

Consider the system of ODE's:

$$\begin{cases} \dot{x}(t) = (A+B)x(t) \\ x(0) = x_0, \end{cases} \tag{1}$$

where $x = x(t) \in \mathbb{R}^N$, A and B are square $N \times N$-matrices with time-independent coefficients.

For $x_0 \in \mathbb{R}^N$ there exists a unique global solution $x(t) \in C^\omega\left(\mathbb{R};\, \mathbb{R}^N\right)$. Moreover,

$$x(t) = e^{(A+B)t}x_0 = \sum_{k=1}^{\infty} \frac{(A+B)^k t^k}{k!} x_0. \tag{2}$$

Theorem 1 *(Lie's Theorem) Marius Sophus Lie (17 December 1842 – 18 February 1899)*

$$e^{A+B} = \lim_{n\to\infty} \left(e^{\frac{A}{n}} e^{\frac{B}{n}} \right)^n . \tag{3}$$

Note that, in general, A and B do not commute and therefore $e^{(A+B)t}$ IS NOT THE PRODUCT OF e^{At} and e^{Bt}. Indeed, (for $t = 1$, for instance):

$$\begin{aligned} e^{A+B} = \sum_{k=0}^{\infty} \frac{(A+B)^k}{k!} &= I + [A+B] + \frac{[A+B]^2}{2} + \cdots \\ &= I + [A+B] + \frac{A^2 + AB + BA + B^2}{2} + \cdots \end{aligned} \tag{4}$$

But

$$
\begin{aligned}
e^A e^B &= \left[\sum_{k=0}^{\infty} \frac{A^k}{k!}\right] \left[\sum_{j=0}^{\infty} \frac{B^j}{j!}\right] \qquad (5) \\
&= \left[I + A + \frac{A^2}{2} + \cdots\right] \cdot \left[I + B + \frac{B^2}{2} + \cdots\right] \\
&= I + [A + B] + AB + \frac{A^2}{2} + \frac{B^2}{2} + \cdots
\end{aligned}
$$

There is a gap or error on the quadratic terms: $\left[A^2 + B^2 + AB + BA\right] \Big/ 2$ and $\left[A^2 + B^2 + 2AB\right] \Big/ 2$. The gap is

$$[BA - AB] \Big/ 2.$$

The difference only vanishes when the commutator

$$[A, B] = BA - AB$$

vanishes.

But, according to Lie's Theorem, even when the commutation does not occur:

$$e^{A+B} \sim \left(e^{\frac{A}{n}} e^{\frac{B}{n}}\right)^n,$$

and, for n fixed,

$$\left(e^{\frac{A}{n}} e^{\frac{B}{n}}\right)^n = e^{\frac{A}{n}} e^{\frac{B}{n}} \cdots e^{\frac{A}{n}} e^{\frac{B}{n}}.$$

It is the iterated product n times, of $e^{A/n} e^{B/n}$.

We have

$$\left[e^{A/n}e^{B/n}\right]x_0 = e^{A/n}\left[e^{B/n}x_0\right].$$

But $e^{B/n}x_0 = y_0$ is the value at time $t = 1/n$ of the solution of

$$\dot{y} = By;\ y(0) = x_0,$$

while $e^{A/n}y_0 = z_0$ is the value at time $t = 1/n$ of the solution of

$$\dot{z} = Az,\ z(0) = y_0.$$

Then, in view of,

$$x(t) = e^{(A+B)t}x_0 = \lim_{n\to\infty}\left(e^{\frac{At}{n}}e^{\frac{Bt}{n}}\right)^n x_0, \tag{6}$$

we approximate $x(t)$ by

$$x_n(t) = \left(e^{\frac{At}{n}}e^{\frac{Bt}{n}}\right)^n x_0$$

which represents the iteration (n times) of a procedure in which we solve in every subinterval the two isolated equations

$$x' = Ax \tag{7}$$

and

$$x' = Bx \tag{8}$$

SOLVING EACH SYSTEM SEPARATELY AND ITERATING WE EVENTUALLY GET THE SOLUTION OF THE GLOBAL SYSTEM.

PROOF=EXERCISE.

This idea can be applied in great generality:

- In the context of the Theory of Semigroups when A and B are generators os semigroups of contractions in Banach spaces. Thus, with many applications to PDE's.

- In the nonlinear context.

  In this way, for instance, the viscous Burgers equation

  $$u_t - \nu u_{xx} + u u_x = 0,$$

  can be viewd as the superposition of the linear heat equation

  $$u_t - \nu u_{xx} = 0,$$

and the hyperbolic equation

$$u_t + uu_x = 0.$$

Viscous Burgers eqn. $=$ Heat eqn. $+$ inviscid Burgers eqn..

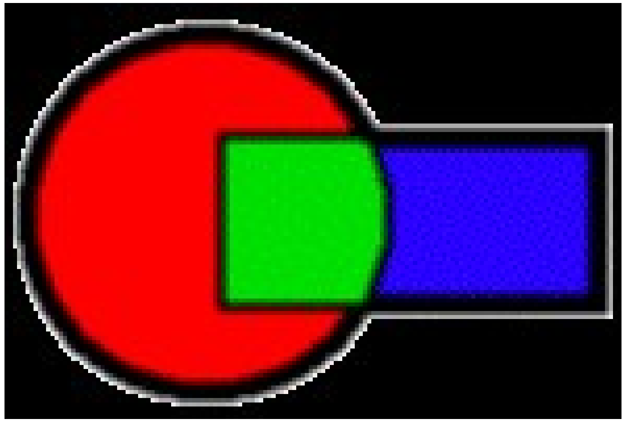
From a purely analytical point of view this does not seem to be a very good idea since solving hyperbolic equations is more difficult than solving parabolic ones. But it may be useful in numerics where we can exploit the properties of each specific system: characteristic methods for the hyperbolic component $+$ finite elements for the heat component, for instance.

- J. M. Burgers, Application of a model system to illustrate some points of the statistical theory of free turbulence, Proc. Konink. Nederl. Akad. Wetensch. 43, 212 (1940).

- E. Hopf, The partial dfferential equation $u_t + uu_x = u_{xx}$, Comm. Pure Appl. Math. 3, 201–230 (1950).

- J. D. Cole, On a quasi-linear parabolic equation occurring in aerodynamics, Quart. Appl. Math. 9, 225 – 236 (1951).

• Domain decomposition = "Divide and conquer"



Karl Hermann Amandus Schwarz (25 January 1843 – 30 November 1921). The Schwarz alternating method is an iterative method to find the solution of a PDE on a complex domain, iteration the solution in each subdomain.

THE DRAWBACK:

These ideas (Lie's iteration, domain decomposition, etc.) hardly allow obtaining sharp qualitative properties for evolution problems. This is particularly true from a dynamical and control theoretical viewpoint, and even more when the dynamical systems involved are of hyperbolic nature with weak damping, finite speed of propagation, etc.