Basque Center for Appied Mathematics (BCAM)

Introduction to Finite Element Methods

D. Pardo, D. Lasa

Basque Center for Applied Mathematics (BCAM)

TEAM MEMBERS:
D. Pardo (Research Professor)
A.-G. Saint-Guirons (Postdoctoral Fellow)
I. Martínez (PhD. Student)
D. Lasa (PhD. Student)

MAIN COLLABORATORS:
M. J. Nam, F. de la Hoz, M. Paszynski,
L.E. García-Castillo, I. Gómez,
C. Torres-Verdín, L. Demkowicz

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overview

1. Derivation of Variational Formulation in 1D:
   - Model problem: Helmholtz equation.
   - Derivation of the variational problem.
   - Final variational formulation.
   - Equivalence between strong and weak formulations.
2. Discretization and Discrete Finite Element Spaces:
   - Finite element spaces.
   - Linear algebraic problem.
   - Example: piecewise linear finite element space.
   - Additional remarks.

model problem: helmholtz equation

Strong formulation of Helmoltz equation:

$$\begin{cases} -\Delta u - k^2 u = f \quad \text{in} \quad \Omega \\ u\big|_{\Gamma_D} = u_0 \\ \dfrac{\partial u}{\partial n}\Big|_{\Gamma_N} = g \end{cases} \Rightarrow \begin{cases} -u'' - k^2 u = f \quad \text{in} \quad (0,1) \\ u(0) = u_0 \\ u'(1) = g \end{cases}$$

Objective: to derive the equivalent weak or variational formulation of the problem in order to solve it via Finite Element Methods (FEM).

derivation of the variational formulation (I)

To derive the variational formulation, we follow the next steps:

1. We multiply Helmholtz equation by an arbitrary test function v:

$$-u''v - k^2uv = fv$$

2. We integrate over the computational domain:

$$\int_0^1 -u''v - \int_0^1 k^2uv = \int_0^1 fv$$

3. We integrate by parts:

$$\int_0^1 u'v' - u'(1)v(1) + u'(0)v(0) - \int_0^1 k^2uv = \int_0^1 fv$$

4. We select test functions that vanish at the Dirichlet Boundary. In this case, we select $v(0) = 0$.

derivation of the variational formulation (II)

5. We apply the Neumann BC to obtain:

$$\int_0^1 u'v' - \int_0^1 k^2 uv = \int_0^1 fv + gv(1)$$ Variational form of the equation

For variational equation to make sense, we need all integrals to be finite. A sufficient condition is:

I. $u'v' \in L^1(0,1)$

II. $k^2 uv \in L^1(0,1)$

III. $fv \in L^1(0,1)$

At this point, we introduce Hölder's inequality, which will be helpful in order to establish the boundness of the mentioned integrals.

$$||f \cdot g||_{L^p} \leq ||f||_{L^q}||g||_{L^r} \quad \text{such that} \quad \frac{1}{q} + \frac{1}{r} = \frac{1}{p}$$

sufficient requirements for u, u', v, v'

Hölder's inequality assures us that if we select $q, r = 2$, then:

I. $u', v' \in L^2(0,1) \Rightarrow u'v' \in L^1(0,1)$

II. $k^2 \in L^\infty(0,1), \quad u, v \in L^2(0,1) \Rightarrow k^2 uv \in L^1(0,1)$

III. $f, v \in L^2(0,1) \Rightarrow fv \in L^1(0,1)$

In accordance to the above statements, we request:

- $v \in V = H_0^1(0,1) = \{v \in L^2(0,1), v' \in L^2(0,1), v(0) = 0\}$
- $u \in u_0 + V$

Thus, Sobolev spaces arise naturally as sufficient integrability conditions.

variational formulation of the problem

We are now in position to write the equivalent variational problem:

$$\begin{cases} u \in u_0 + V \\ \int_0^1 u'v' - \int_0^1 k^2 uv = \int_0^1 fv + gv(1) \quad \forall v \in V \end{cases}$$

Abstract Variational Formulation:

$$\begin{cases} u \in u_0 + V \\ b(u,v) = l(v) \quad \forall v \in V \end{cases}$$

Where b is a bilinear form and l is a linear form.

equivalence between strong and weak problem (I)

1. $u \in u_0 + V \Rightarrow u = u_0 + v, v \in V$. Since $v(0) = 0$, then $\boxed{u(0) = u_0}$.

2. Integrating the variational equation by parts:

$$-\int_0^1 u''v + u'v\Big|_0^1 - \int_0^1 k^2 uv = \int_0^1 fv + gv(1)$$

We consider the subspace for all $v \in V$ such that $v(1) = 0$.

$$-\int_0^1 v(-u'' - k^2 - f) = 0 \quad \forall v \in V : v(1) = 0$$

Now, we invoke Fourier's lemma:

Let f be a continous function defined on $(0, l)$ such that:

$$\int_0^l f(x)v(x)dx = 0$$

for every continous test function v that vanishes at the endpoints, $v(0) = v(l) = 0$. Then f must identically vanish, $f = 0$.

equivalence between strong and weak problem (II)

2. Thus,

$$\boxed{-u'' - k^2 u - f = 0 \quad \text{in} \quad (0,1)}$$

3. Since

$$-\int_0^1 u''v + u'v\Big|_0^1 - \int_0^1 k^2 uv = \int_0^1 fv + gv(1)$$

and

$$-u'' - k^2 - f = 0$$

then

$u'(1)v(1) = gv(1) \Rightarrow$ selecting $v(1) \neq 0$ we obtain $\boxed{u'(1) = g}$

At this point, we have totally recovered the problem in its original strong formulation.

Conclusion: both formulations are equivalent.

discretization of the variational problem (I)

Let be the abstract variational formulation of the Helmholtz unidimensional equation:

$$\begin{cases} u \in u_0 + V \\ b(u,v) = l(v) \quad \forall v \in V \end{cases}$$

V is an infinite dimensional space. Our objective is to aproximate this space with a finite dimensional subspace $V_{hp} \subset V$.

- We denote the elements of V_{hp} by $v_{hp} \in V_{hp}$
- We write our aproximate solution as $u_{hp} \in u_0 + V_{hp}$. For simplicity we shall assume tha $u_0 = 0$.

Then,

$$\begin{cases} u \in V \\ b(u_{hp}, v_{hp}) = l(v_{hp}) \quad \forall v_{hp} \in V_{hp} \end{cases}$$

discretization of the variational problem (II)

We have to solve the following aproximated problem:

$$\begin{cases} u \in u_0 + V \\ b(u_{hp}, v_{hp}) = l(v_{hp}) \quad \forall v_{hp} \in V_{hp} \end{cases}$$

For simplicity, we consider $u_0 = 0$. Then,

$$\begin{cases} u \in V \\ b(u_{hp}, v_{hp}) = l(v_{hp}) \quad \forall v_{hp} \in V_{hp} \end{cases}$$

discretization of the variational problem (III)

Let $\{e^i\}_{j=1}^N$ a basis of the N dimensional V_{hp} subspace.
We have:

$$u_{hp}(x) = \sum_1^N u_j e_j(x)$$

With the above definition, we have to solve this problem:

$$(1) = \begin{cases} \text{Find} \quad u_{hp} \quad \text{such that} \\ b(u_{hp}, v_{hp}) = l(v_{hp}) \quad \forall v_{hp} \in V_{hp} \end{cases}$$

The above problem is equivalent to

$$(2) = \begin{cases} \text{Find} \quad u_j \quad \text{such that} \\ \sum_j b(e_j(x), e_i) u_j = l(e_i) \quad i = 1, \ldots, N \end{cases}$$

- (1) $\Rightarrow$ (2): Let $u_{hp} = \sum_j u_j e_j$ and $e_i \in V_{hp}$. Then,

$$\sum_j b(e_j, e_i) u_j = l(e_i) \quad i = 1, \ldots, N$$

discretization of the variational problem (IV)

- **(2) $\Rightarrow$ (1):**

  1. Let $v_{hp} \in V_{hp}$. Then $v_{hp} \sum_i v_i e_i$. Thus,

$$
\begin{aligned}
b(u_{hp}, v_{hp}) &= b(u_{hp}, \sum_i v_i e_i) \\
&= \sum_i \sum_j v_i u_j b(e_j, e_i) \\
(2) \Rightarrow \quad &= \sum_i v_i l(e_i) \\
&= l\Big(\sum_i v_i e_i\Big) \\
&= l(v_{hp})
\end{aligned}
$$

Thus, the two approaches are equivalent and we have to solve

$$
\boxed{\sum_j b(e_j, e_i) u_j = l(e_i) \quad i = 1, \ldots, N}
$$

establishment of the algebraic linear problem

Defining:

$$b_{ij} = b(e_j, e_i) \quad i,j = 1,\dots,N, \quad u = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}, \quad l = \begin{pmatrix} l(e_1) \\ \vdots \\ l(e_N) \end{pmatrix}$$

The problem reduces to:

$$\boxed{Bd = l}$$

Where B is the so called *stiffness matrix*, l is *the load vector* are data that we have to compute and u is the unkown solution of the problem.

example: piecewise linear finite element space I

In the case of one dimensional Helmholtz equation we proceed partitioning the interval $(0,1)$ into N subintervals:

$$0 = x_0 < x_1 < \cdots < x_K < x_{K+1} < \cdots < x_N = 1$$

Each of the subintervals (x_K, x_{K+1}), will be called *finite element*, and it will have two parameters associated with it: element length $h_K = x_{k+1} - x_K$, and element local polynomial order of approximation p_K ($p_K = 1$ if we restricted ourselfs to the piecewise linear space).

The basis functions are defined as follows:

$$e_K(x) = \begin{cases} \frac{x - x_{K-1}}{x_K - x_{K-1}} & \text{if} \quad x_{K-1} < x < x_K \\ \frac{x_{K+1} - x}{x_{K+1} - x_{K-1}} & \text{if} \quad x_K < x < x_{K+1} \\ 0 & \text{otherwise} \end{cases}$$

example: piecewise linear finite element space II

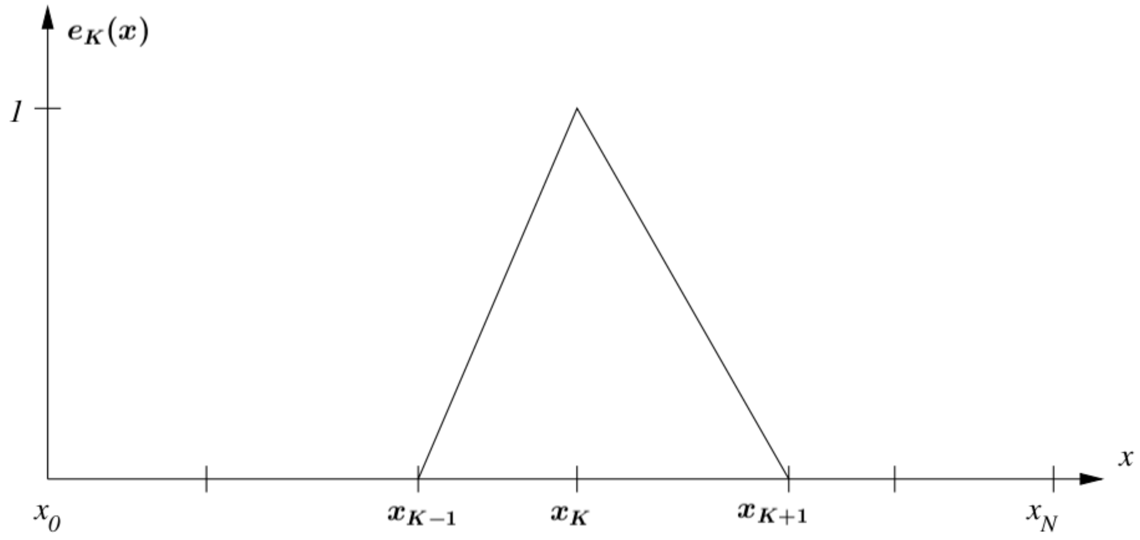

Figure 1: One-dimensional finite element space element mesh and piecewise linear hat function.

example: piecewise linear finite element space III

The basis functions e_K are zero outside the neighborhood of node K. As a result, most of the entries of B are zero since $b(u,v)$ contains integrals involving products of uv and $u'v'$.
Thus, B is *sparse*

$$B = \begin{pmatrix} B_{11} & B_{12} & 0 & & \cdots & 0 \\ B_{21} & B_{22} & B_{23} & \ddots & & \vdots \\ 0 & B_{32} & B_{33} & B_{34} & \ddots & \\ 0 & & \ddots & & & 0 \\ \vdots & & & B_{n-1,n-1} & B_{n-1,n-1} & B_{n-1,n} \\ 0 & & & & B_{n,n-1} & B_{nn} \end{pmatrix}$$

mesh refinements: h-adaptivity

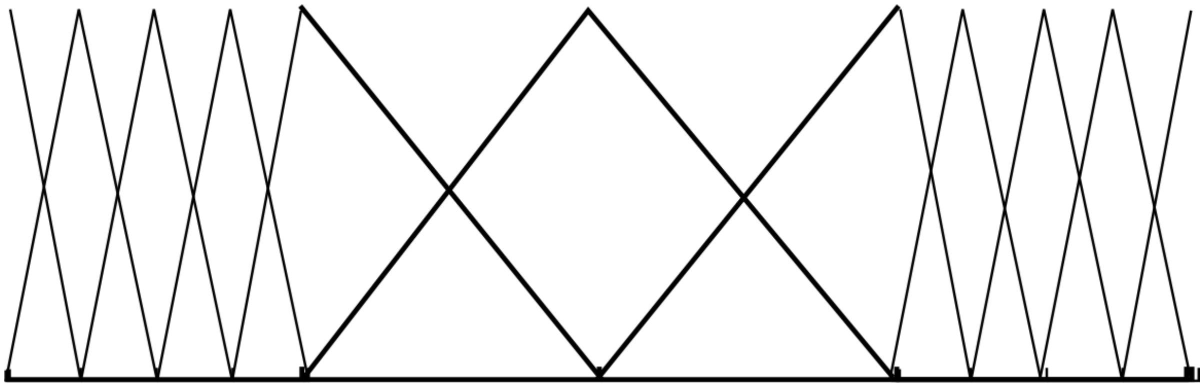

Figure 2: One-dimensional finite element space element mesh and piecewise linear hat function.

1. Convergence limited by the polynomial degree, and large material contrasts.
2. Optimal h-grids do NOT converge exponentially in real applications.
3. They may “lock” (100$\%$ error).

mesh refinements: p-adaptivity

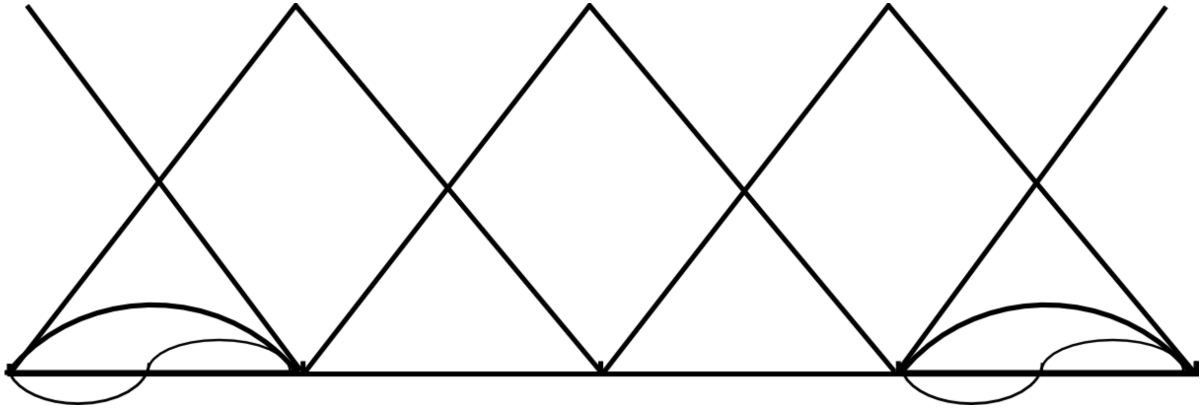

Figure 3: Polynomial order of the basis functions is increased in order to achieve more accurate results.

1. **Exponential convergence feasible for analytical (“nice”) solutions.**
2. **Optimal *p*-grids do NOT converge exponentially in real applications.**
3. **If initial *h*-grid is not adequate, the *p*-method will fail miserably.**

mesh refinements: k-adaptivity

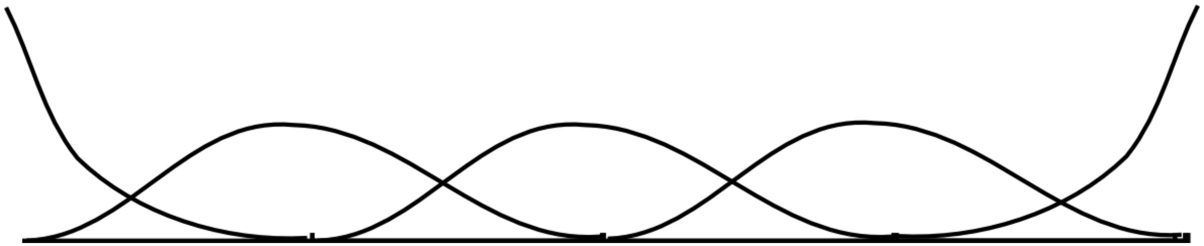

Figure 4: This refinement technique makes use of smooth $\boldsymbol{C^k}$ functions.

1. **Makes use of C^k functions that generally offer faster convergence for smooth solutions.**
2. **Support of basis functions becomes larger as we increase k.**
3. **The method is still under intensive research.**

mesh refinements: r-adaptivity

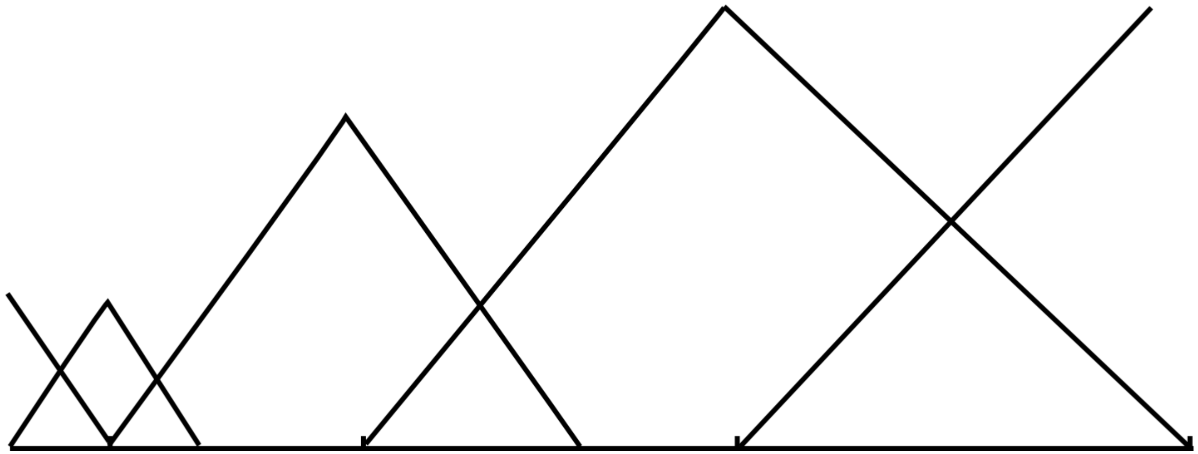

Figure 5: Node relocation strategies.

1. **Makes use of remeshing strategies, that is, mesh-node relocation, which enables us to calculate the new ‘optimal’ node position from the estimated error.**
2. **The method is difficult to implement.**

some remarks about the method I

- The method described above is called (Bubnov-) Galerkin method because we use the same space to approximate the solution and the test function. Otherwise, if we use different spaces the method is called Petrov-Galerkin.
- The use of piecewise polynomial basis functions simplifies the evaluation of integrals. However, it is possible to consider non-polynomial basis functions.
- Selecting a basis with small support leads to a sparse linear algebraic system.

some remarks about the method II

- Let X be a Hilbert space with norm $||\cdot||$, $V \subset X$ and let $b(u,v)$ denote a bilinear form defined on $X \times X$, which is continous,i.e., there exists $M > 0$ such that

$$|b(u,v)| \leq M||u||||v||, \quad \forall u,v \in X$$

and it is V-coercive, i.e., there exists $\alpha > 0$ such that

$$\alpha||v||^2 \leq b(v,v), \quad \forall v \in V$$

Let $l(v)$ be an arbitrary linear and continous functional defined on V, i.e.,

$$|l(v)| \leq C||v||, \quad \forall v \in V$$

some remarks about the method III

- Consider the problem of finding an element u in V such that

$$b(u,v) = l(v), \forall v \in V$$

Consider the same problem on a finite-dimensional subspace V_{hp} of V, so, u_{hp} in V_{hp} satisfies

$$b(u_{hp}, v_{hp}) = l(v_{hp}), \forall v \in V$$

By the Lax-Milgram theorem, each of these problems has exactly one solution. Céa's lemma states that

$$||u - u_{hp}|| \leq \frac{M}{\alpha} \inf_{v \in V_{hp}} ||u - v||$$

That is to say, the subspace solution u_{hp} is "the best" approximation of u in V_{hp} up to the constant M/α.