Pricing and Hedging Derivative Securities

March 23, 2010

BCAM, March 15-19, 2010

1. No arbitrage hypothesis (complete vs. incomplete markets).

2. Risk neutral probabilities (pricing derivative securities).

3. Replicating portfolios (hedging derivative securities).

4. Stochastic differential equations, Black-Scholes-Merton (BSM)-PDE and relatives.

5. Stochastic optimal control, the dynamic programming principle and the Hamilton-Jacobi-Bellman (HJB) equation.

6. Jump processes.

The absence of arbitrage hypothesis

1. If two portfolios have the same payoff then their present value must be the same.

2. If the value of portfolio 1 is at least as big as the value of portfolio 2, then the present value of portfolio 1 is greater than or equal to the value of portfolio 2.

Example. Forward contract (Future contract). Model-less!!

We assume that our underlying pays no dividends and that it carries no costs (transactions, taxes, bid-ask spread, unlimited long-short positions, no restrictions on borrowing money).

We also assume that the time value of money is computed using a continuously compounded fixed interest rate r.

That is, a guaranteed amount of D dollars at time T is worth now De^{-rT} now.

Question: How much would you pay now to enter into a contract (no credit risk assumed) to get one unit of stock S_T at some time into the future T in exchange of K dollars?

T is called the maturity/expiration time of the contract and K is called the strike price.

Answer: Consider two portfolios:

Portfolio 1. Holds one forward contract as described above with payoff at T equal to $(S_T - K)$.

Portfolio 2. Borrow Ke^{-rT} dollars at the risk free rate r now.

It is clear that the time T payoffs of these two portfolios are equal to $(S_T - K)$, thus by no arbitrage, their present value should be the same.

Price of the forward contract $= S_0 - Ke^{-rT}$

In particular if we pick $K = S_0 e^{rT}$, the value of the forward now is zero. This K is called the forward price.

Forward (payoff always at maturity) vs Futures.

Differences: Marked to market and traded in the exchange vs OTC.

Observation: If risk free rates are deterministic, then the forward and the futures prices are the same.

Proof: By induction. Both prices are the same on settlement day and one day prior.

Need to show that if both prices are the same on day N say equal to P, then they are the same on day (N-1).

Say that the forward price on day (N-1) is G and the futures is F. Since risk free rates are deterministic, we know what that rate will be on day N for the time period from N to T, assume is r.

Strategy 1. Buy $e^{-r(T-N)}$ units of futures on day $(N-1)$, close the position on day N and invest the proceedings at the risk free rate r until the expiration date T. The time T payoff of this strategy is $e^{rT}e^{-rT}(P-F) = (P-F)$.

Strategy 2. Buy a forward contract on day $(N-1)$ and sell it on day N. This produces a gain/loss $(P-G)$ to be gotten on day T.

Thus, $F = G$, since if not, we could create an arbitrage since both strategies require no money to be invested on day $(N-1)$.

The put-call parity principle.

-European call option with strike price K and maturity time T has final time payoff $(S_T - K)_+$ and we denote it by $c(S_0, T, K)$.

-European put option with strike price K and maturity time T has final time payoff $(K - S_T)_+$ and we denote it by $p(S_0, T, K)$.

Put-call parity:

$$c(S_0, T, K) - p(S_0, T, K) = S_0 - Ke^{-rT}$$

"A long call plus a short put is equal to a long forward" where in the three we have the same strike price K and maturity time T.

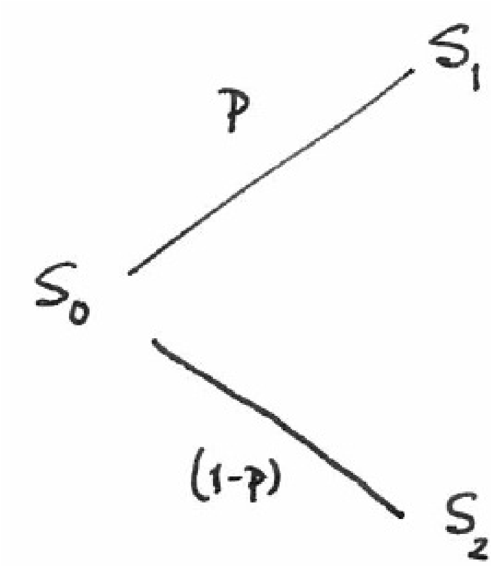
The proof is simply setting two different portfolios with the same final time T payoffs and applying the no arbitrage hypothesis.

The Binomial Model (one period)

-Consider an economy with just two securities: a stock (paying no dividend, initial unit price s_0 dollars) and a money market (carrying a risk free interest rate r).

-Just one maturity date T.

-Just two possible states for the stock price at time T: s_1 and s_2, with $s_1 < s_2$.

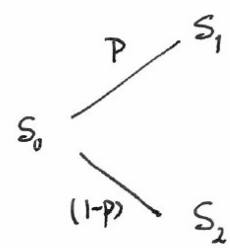


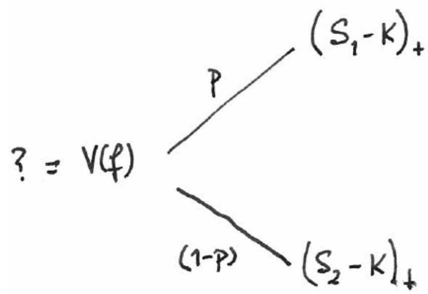
No Arbitrage in this economy means that

$$s_1 < s_0 \, e^{rT} < s_2.$$

Define a derivative security as follows. It pays $f_2 = (s_2 - K)_+$ at time T if the stock goes up to s_2 and it pays $f_1 = (s_1 - K)_+$ at time T if the stock goes down to s_1.

Question: What is the "fair price/no arbitrage" of this derivative at time 0?





Answer:

$$V(f) = e^{-rT}\ [(1-q)\ f_1 + q\ f_2]$$

where

$$q = \frac{s_0 e^{rT} - s_1}{s_2 - s_1}$$

In our example and if we assume that the risk free interest rate is flat $r = 0\%$, this price is \$2/3.

Option is overpriced

If someone is willing to pay \$1 for the option, we'll sell it to them and buy 2/3 units of the stock at the present \$2 price. This will costs us \$ 4/3. Thus we need to borrow \$1/3.

CASE 1. The stock moves up

At the end of the period the stock goes up to \$4 in which case we need to pay the buyer of the option \$2 and return \$1/3 for the money we borrowed. But we have \$ 2/3 units of thew stock at price \$4 which is worth \$ 8/3.

So our net worth is \$8/3 -\$2 -\$ 1/3 = \$1/3.

We started with an investment of \$0 dollars and we have made \$1/3.

CASE 2. The stock moves down

At the end of the period the stock goes down to \$1 in which case we do not need to pay the buyer anything for the option, we need to return \$1/3 for the money we borrowed. But we have \$ 2/3 units of thew stock at price \$1 which is worth \$ 2/3.

So our net worth is\$2/3 -\$ 1/3 = \$1/3.

Again, we started with an investment of \$0 dollars and we have made \$1/3.

We have created an arbitrage

Option is underpriced

Something similar can be done if someone thinks that the fair price of such an option is less than \$2/3, say \$1/3.

In that case we'll buy the option at that price and sell the portfolio of stock and money market. Namely, we'll short sell 2/3 units of the stock at the present \$2 price. This will generate us \$ 4/3. We'll pay the \$1/3 for the option and put the rest, \$1 in the money market account.

CASE 1. The stock moves up

At the end of the period the stock goes up to \$4 in which case we will get \$2 for the option, to close the short sell, we'll need to buy back 2/3 units of the stock at \$4, it will cost \$ 8/3 and we have \$1 in the bank.

So our net worth is \$2-\$8/3 -\$ 1 = \$1/3.

We started with an investment of \$0 dollars and we have made \$1/3.

CASE 2. The stock moves down

At the end of the period the stock goes down to \$1 in which case we do not get any money from the option, we need to return \$2/3 for close the short sell of the stock and we have \$1 in the bank.

So our net worth is\$-2/3 +\$ 1 = \$1/3.

Again, we started with an investment of \$0 dollars and we have made \$1/3.

We have created an arbitrage

The question is: Why?

The answer is: Because we can build a self financed replicating portfolio.

Set up a portfolio at time 0 consisting of ϕ units of the stock and ψ dollars in the money market account.

$$\phi s_0 + \psi.$$

So that its value at time T replicates the payoffs of the derivative security. That is

$$\phi s_1 + \psi \ e^{rT} = f_1$$

and

$$\phi s_0 + \psi \ e^{rT} = f_2$$

and solve for ϕ and ψ.

A little algebra give us that

$$\phi = \frac{f_2 - f_1}{s_2 - s_1}$$

and

$$\psi = e^{-rT} \ \frac{s_2 \ f_1 - f_2 \ s_1}{s_2 - s_1}.$$

Thus, by absence of arbitrage, the time 0 value of the portfolio should be the "fair price/no arbitrage" of the derivative security, i.e.

$$V(f) = \phi s_0 + \psi = \frac{f_2 - f_1}{s_2 - s_1} s_0 + e^{-rT} \ \frac{s_2 \ f_1 - f_2 \ s_1}{s_2 - s_1} =$$

$$= e^{-rT} \ [(1-q) \ f_1 + q \ f_2]$$

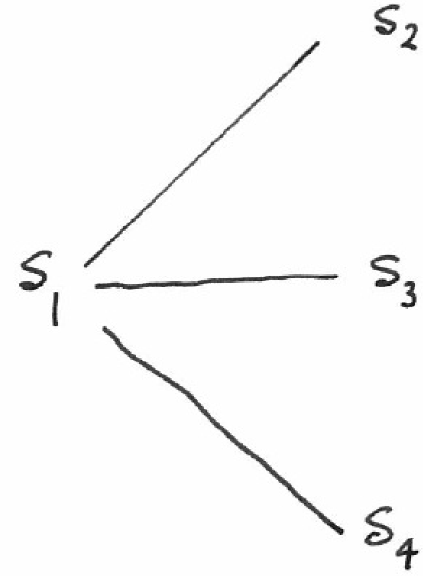
where

$$q = \frac{s_0 e^{rT} - s_1}{s_2 - s_1}.$$

q is the so called risk-neutral probability, and we can identify $V(f)$ as the risk neutral expectation of the discounted value of the payoff of the derivative security.

$$V(f) = E_{RN}[e^{-rT} \ f_T].$$

The Trinomial Model (one period)



Let, $s_2 < s_3 < s_4$ with the no arbitrage condition $s_2 < s_1 e^{rT}$ and $s_1 e^{rT} < s_4$.

If we minic what we did in the case of the binomial model for a contingent claim with payoff 3-vector (f_1, f_2, f_3), we obtain the following linear system:

$$\phi s_2 + \psi = f_1$$

$$\phi s_3 + \psi = f_2$$

$$\phi s_4 + \psi = f_3$$

but this is an overdetermined system of equations and thus not all the contingent claim 3-vectors (f_1, f_2, f_3) can be replicated, only a 2 dimensional subspace. This, as opposed to the binomial model, makes the market incomplete (i.e. no replicating portfolios or the existence of more than a unique risk neutral probability).

Thus, when one payoff vector is not replicatable, then the absence of arbitrage hypothesis can not uniquely specify the "fair price" at time zero of the contingent claim.

In this case, the most we can do, using no arbitrage is to find bounds for this price. That is:

$V(f) \leq$ value of any portfolio whose payoff dominates $f = (f_1, f_2, f_3)$

and

$V(f) \geq$ value of any portfolio whose payoff is dominated by $f = (f_1, f_2, f_3)$.

Therefore,

$$max_{\{\phi s_j + \psi \leq f_j, j=2,3,4\}}[\phi s_1 + e^{-rT}\psi] \leq V(f)$$

$$\leq min_{\{\phi s_j + \psi \leq f_j, j=2,3,4\}}[\phi s_1 + e^{-rT}\psi].$$

In this case, the actual price observed in the market must be determined using additional hypothesis besides the no arbitrage condition. For example, the "equilibrium" approach uses utility maximization.

The above inequalities are not in a similar form to the "risk neutral" probability of the binomial model, but we can arrive to that form by using the duality theory of linear programming. Namely, let us focus on the upper bound for $V(f)$.

$$min_{\{\phi s_j+\psi\leq f_j, j=2,3,4\}}[\phi s_1 + e^{-rT}\psi] = min_{\phi,\psi} max_{\pi_j\geq 0}[\phi s_1 + e^{-rT}\psi +$$

$$+\sum_{j=2}^{4}\pi_j(f_j - \phi s_j - \psi)]$$

$$= max_{\pi_j\geq 0} min_{\phi,\psi}[\phi s_1 + e^{-rT}\psi + \sum_{j=2}^{4}\pi_j(f_j - \phi s_j - \psi)]$$

$$= max_{\pi_j\geq 0} min_{\phi,\psi}[\phi(s_1 - \sum_{j=2}^{4}\pi_j s_j) + \psi(e^{-rT} - \sum_{j=2}^{4}\pi_j) + \sum_{j=2}^{4}\pi_j f_j]$$

$$max_{\{\sum_{j=2}^{4}\pi_j s_j = s_1,\ \sum_{j=2}^{4}\pi_j = e^{-rT},\ \pi_j' s\geq 0\}} \sum_{j=2}^{4}\pi_j f_j.$$

The first line holds because, $max_{\pi_j \geq 0} \pi_j(f_j - \phi s_j - \psi) = 0$ if $\phi s_j + \psi \leq f_j$ or is equal to ∞ otherwise.

The second line holds by the duality theorem of linear programming, and the third is obtained rearranging the second, and the fourth by an argument similar to the first. Hence we have shown

$$min_{\{\phi s_j + \psi \leq f_j, j=2,3,4\}}[\phi s_1 + e^{-rT}\psi] =$$

$$= max_{\{\sum_{j=2}^{4} \pi_j s_j = s_1,\ \sum_{j=2}^{4} \pi_j = e^{-rT},\ \pi_j' s \geq 0\}} \sum_{j=2}^{4} \pi_j f_j.$$

Letting $\tilde{\pi}_j = \pi_j e^{rT},\ j = 2, 3, 4$ we finally obtain

$$min_{\{\sum_{j=2}^{4} \tilde{\pi}_j s_j = e^{rT} s_1,\ \sum_{j=2}^{4} \tilde{\pi}_j = 1,\ \tilde{\pi}_j' s \geq 0\}} e^{-rT} \sum_{j=2}^{4} \tilde{\pi}_j f_j \leq V(f)$$

$$\leq max_{\{\sum_{j=2}^{4} \tilde{\pi}_j s_j = e^{rT} s_1,\ \sum_{j=2}^{4} \tilde{\pi}_j = 1,\ \tilde{\pi}_j' s \geq 0\}} e^{-rT} \sum_{j=2}^{4} \tilde{\pi}_j f_j.$$

Which illustrates the incompleteness of this trinomial model. In general the number of random factors in the models is greater than the number of underlyings used in the replicating portfolios, the market will be incomplete.

Arbitrage Theorems-The general one period model

*N securities, $i = 1, \ldots, N$.

*M final states, $\alpha = 1, \ldots, M$.

*Fixed initial values, i.e. one unit of security i is worth now p_i dollars.

*State dependant final values for each security $D = D_{i\alpha}$'s matrix (N x M).

-Binomial model: The matrix D is (2 x 2).

-Trinomial model: The matrix D is (2 x 3).

Principle of no arbitrage

Let's consider a portfolio of those N securities composed by θ_i units of each security for $i = 1, \ldots, N$,

a) $\sum_{i=1}^{N} \theta_i D_{i\alpha} \geq 0$ for all $\alpha = 1, \ldots, M$ implies that $\sum_{i=1}^{N} \theta_i p_i \geq 0$.

b) If we have both $\sum_{i=1}^{N} \theta_i D_{i\alpha} \geq 0$ for all $\alpha = 1, \ldots, M$ and $\sum_{i=1}^{N} \theta_i p_i = 0$ then we must have $\sum_{i=1}^{N} \theta_i D_{i\alpha} = 0$ for every $\alpha = 1, \ldots, M$.

Theorem 1 *The economy satisfies condition a) above if and only if there exists a risk neutral probability $\{\pi_\alpha\}$ such that*

$$\sum_{\alpha=1}^{M} D_{i\alpha} \pi_\alpha = p_i,$$

for each $i = 1, \ldots, N$.

If the economy satisfies conditions a) and b) then the π_α's are all strictly positive.

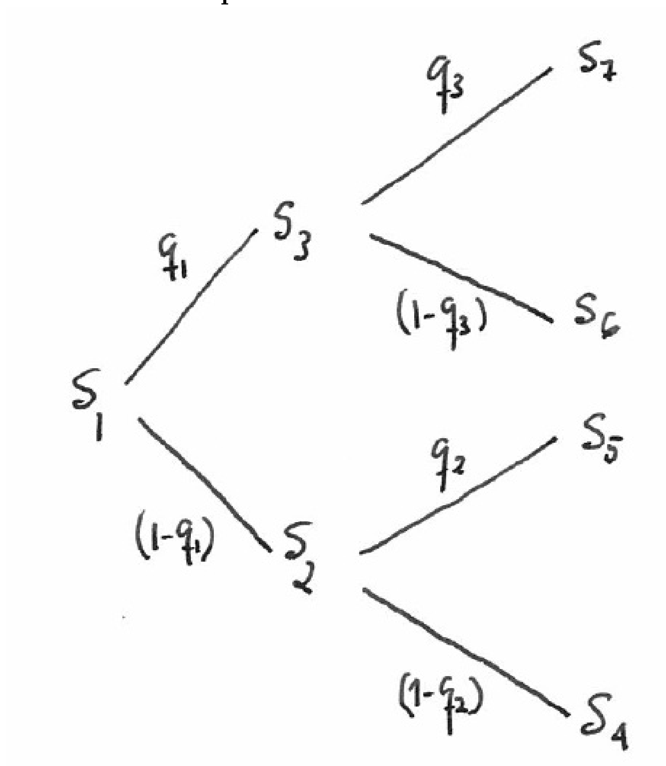
Proof 1 *The "if" direction of the proof is trivial (sketch it!!), as for the other direction, it follows from Farkas' Lemma.*

Let's go back for a moment to the binomial and trinomial models and find their corresponding D matrices.

$$\begin{pmatrix} 1 & 1 \\ S_2 & S_3 \end{pmatrix} = D(\text{binomial}) \qquad ; \quad D(\text{trinomial}) = \begin{pmatrix} 1 & 1 & 1 \\ S_2 & S_3 & S_4 \end{pmatrix}.$$

Theorem 2 *In an arbitrage free economy with N securities, each derivative security can be replicated if and only if there exists a unique risk neutral probability.*

Multiperiod binomial models



$$f_3 = e^{-r\ \delta t}\ [q_3\ f_7 + (1-q_3)\ f_6]$$

$$f_2 = e^{-r\ \delta t}\ [q_2\ f_5 + (1-q_2)\ f_4]$$

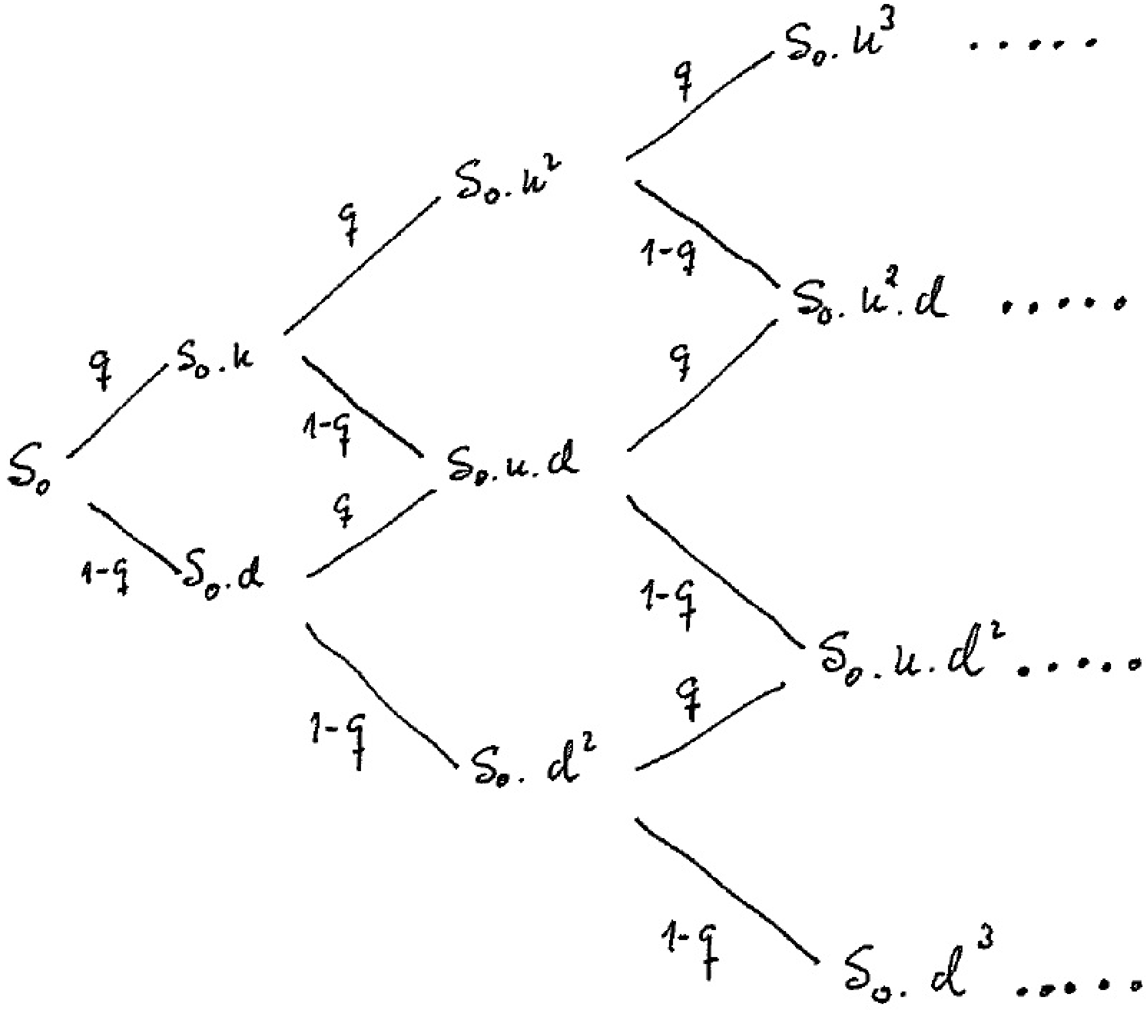
and

$$f_1 = e^{-r\ \delta t}\ [q_1\ f_3 + (1-q_1)\ f_2] =$$

$$= e^{-2r\ \delta t}\ [q_1\ q_3\ f_7 + q_1\ (1-q_3)\ f_6 + (1-q_1)\ q_2\ f_5 + (1-q_1)(1-q_2)\ f_4].$$

For N periods

$$V(f) = e^{-N\ r\delta t} \sum_{final\ states} [probability\ of\ the\ path]\,[payoff\ of\ the\ state].$$



$$V(f) = e^{-N\ r\delta t} \sum_{j=0}^{N} [\ \binom{N}{j}\ q^j\ (1-q)^{N-j}\ f(s_0 u^j d^{N-j})]$$

$$= e^{-N\ r\delta t}\ E_{RN}[f(S_T)].$$

$$q = \frac{e^{r\delta t} - d}{u - d}.$$

Continuous Time Models. Lognormal stock price dynamics

Break up the interval $[0, T]$ into N intervals of length δt and assume that for each subinterval $[j\delta t, (j+1)\delta t]$, the stock price satisfies that

$$s((j+1)\delta t) = e^{r_j \delta t} s(j\delta t),$$

where the $r_j \delta t$ are independent random variables, identically distributed and gaussian with mean $\mu\delta t$ and variance $\sigma\delta t$ with μ and σ constants.

1) For any time interval (t_1, t_2), $\ln s(t_2) - \ln s(t_1)$ is a gaussian random variable with mean $\mu(t_2 - t_1)$ and variance $\sigma(t_2 - t_1)$.

2) These gaussian random variables are independent when associated with different time intervals.

3) $\ln s(t)$ executes a Brownian motion with drift.

Lognormal model as the limit of the multiperiod binomial tree model

Let $u = e^{u\delta t + \sigma\sqrt{\delta t}}$ and $d = e^{u\delta t - \sigma\sqrt{\delta t}}$ in our multiperiod binomial tree model.

$$s(t) = s(N\delta t) = s(0)e^{[N\mu\delta t + j\sigma\sqrt{\delta t} - (N-j)\sigma\sqrt{\delta t}]} =$$

$$= s(0)e^{[N\mu\delta t + (2j-N)\sigma\sqrt{\delta t}]}.$$

Let X_N be the random variable "number of times you get heads when you flip a coin N times".

X_N is the sum of N independent random variables, each one takes value 1 if you get Heads and 0 if Tails.

$$s(T) = s(0)e^{[N\mu\delta t + (2X_N - N)\sigma\sqrt{\delta t}]} =$$

$$= s(0)e^{[\mu T + \frac{(2X_N - N)}{\sqrt{N}}\sigma\sqrt{T}]}.$$

Consider the risk neutral probability given by

$$q = \frac{e^{r\delta t} - d}{u - d} =$$

$$= \frac{1}{2}(1 - \sqrt{\delta t}\frac{\mu - r + \frac{1}{2}\sigma^2}{\sigma}) + O(\delta t).$$

X_N is the sum of N independent random variables with mean q and variance $q(1-q)$, by the CLT the limiting distribution as $\delta t \to 0$ of $\frac{(2X_N - N)}{\sqrt{N}}$ is gaussian with mean $\sqrt{T}(\frac{r-\mu-\frac{1}{2}\sigma^2}{\sigma})$ and variance 1.

$$s(T) = s(0)e^{[\mu T + (\sqrt{T}(\frac{r-\mu-\frac{1}{2}\sigma^2}{\sigma}) + Z)\sigma\sqrt{T}]} =$$

$$= s(0)e^{[(r-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z]},$$

where Z is a standard normal random variable $N(0,1)$.

Using this we can price a derivative security with final payoff function $f(S_T)$ as

$$V(f) = e^{-r\ T}\ E_{RN}[f(S_T)] = e^{-r\ T}\ E[f(s(0)e^{[(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}Z]})] =$$

$$= e^{-r\ T}\ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s_0\ e^{[(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}x]})\ e^{-\frac{x^2}{2}}\ dx.$$

These lead to the famous Black-Scholes formulas for the values of the european calls and puts with strike price K and maturity time T when we choose the final time payoff function f to be $(x-K)_+$ (call) and $(K-x)_+$ (put).

Namely, after integration

$$c(s_0, T, K) = s_0 \; N(d_1) - K \; e^{-r \; T} \; N(d_2),$$

where

$$d_1 = \frac{\log(\frac{s_0}{K}) + (r + \frac{1}{2}\sigma^2)\, T}{\sigma \; \sqrt{T}},$$

$$d_2 = d_1 - \sigma\sqrt{T},$$

and

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \; e^{-\frac{s^2}{2}} \; ds,$$

is the cumulative normal distribution function corresponding to the standard $N(0,1)$ normal random variable.

We have just obtained, by probability means, the Black-Scholes model in continuous time for the risk neutral dynamics of the underlying stock.

We would like to find the stochastic differential equation that leads to this model, since it will be a more workable tool.

For this we need to introduce Brownian motion.

Brownian Motion

1. $\{W_t\}_{t\geq 0}$ the brownian stochastic process.

2. $W_0 = 0$, $W_t - W_s$ is gaussian with mean 0 and variance (t-s).

3. $W_{t_2} - W_{t_1}$ is independent of $W_{t_4} - W_{t_3}$ as long as the time intervals do not overlap.

4. A filtration$\{F_t\}_{t\geq 0}$ of σ-algebras.

Ito's Calculus

Is based on the fact that the Quadratic Variation of Brownian Motion is dt. Namely,

$$(dW_t)^2 = dt$$

$$dS_t = \alpha_t \ S_t \ dt + \sigma_t \ S_t \ dW_t$$

Applying Ito's formula (the stochastic calculus version of the Chain rule), we obtain

$$df(t, S_t) = \frac{\partial f}{\partial t} \ dt + \frac{\partial f}{\partial S} \ dS_t + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} (dS_t)^2$$

$$(dS_t)^2 = (\sigma_t)^2 \ (S_t)^2 \ dt.$$

Using Ito's calculus, we can show that

$$s(T) = s(0)e^{[(r-\frac{1}{2}\sigma^2)T+\sigma W_T]}$$

is a solution of the following SDE

$$dS_t = r\ S_t\ dt + \sigma\ S_t\ dW_t,$$

the risk neutral dynamics for the stock price stochastic process of the log-normal model.

Girsanov's theorem

The existence of the risk neutral probabilities is guaranteed in this model by Girsanov's theorem.

$$dS_t = \alpha\ S_t\ dt + \sigma\ S_t\ dW_t =$$

$$= r\ S_t\ dt + \sigma\ S_t\ ((\frac{\alpha - r}{\sigma})\ dt + dW_t) =$$

$$= r\ S_t\ dt + \sigma\ S_t(\theta\ dt + dW_t).$$

Let $Z_t = e^{(-\theta W_t - \frac{1}{2}\theta^2 t)}$, it is immediate to show that $E[Z_t] = 1$ for all t and that Z_t is a martingale with respect to the Brownian filtration, i.e. $E[Z_t/F_s] = Z_s$ for $s < t$.

If we define $E_{RN}[X] = E[Z_t X]$, for X, F_t measurable, we can consider Z_t being the Radon-Nykodim derivative process of the RN-probability with respect to the real world probability.

It is clear that:

If $s < t$ and the random variable X is F_t measurable, then

$$E_{RN}[X/F_s] = \frac{1}{Z_s} E[Z_t X/F_s].$$

It also holds that $(\theta t + W_t)$ wehere W_t is Brownian morion under the real world probability is just a Brownian motion under the RN-probability.

Therefore, the risk neutral probability is the one that removes the drift θ from the SDE above, and according to Girsanov's theorem, that is achieved by E_{RN}.

Replicating portfolios in continuous time models (self financed)

$$dS_t = \alpha \ S_t \ dt + \sigma \ S_t \ dW_t$$

$$X_t = \Delta_t S_t + (X_t - \Delta_t S_t)$$

That is, the replicating portfolio is composed of Δ units of the stock at time t and the rest $(X_t - \Delta_t S_t)$ in the money market. Then,

$$dX_t = \Delta_t dS_t + r \ (X_t - \Delta_t S_t) \ dt$$

We want our replicating portfolio to be self-financed, and we are going to explore what the continuous time "self-financed" condition is.

Suppose that $X_t = \Delta_t\ S_t + \Gamma_t\ M_t$, where Γ_t is the number of units in the money market at time t.

Consider an infinitesimal interval of time from t to $t + \delta t$, we assume that during that interval, we hold the number of units of the stock and on the money market constant, thus its values at time $t + \delta t$ is equal to

$$X_{t+\delta t} = \Delta_t\ S_{t+\delta t} + \Gamma_t\ M_{t+\delta t}$$

At time $t + \delta t$ we rebalance the portfolio in a self financed way, namely

$$X_{t+\delta t} = \Delta_{t+\delta t}\ S_{t+\delta t} + \Gamma_{t+\delta t}\ M_{t+\delta t}$$

remember that $M_{t+\delta t} = M_t\ e^{r\delta t}$, hence $M_{t+\delta t} - M_t = r\ \delta t$. Subtracting the above two expressions for $X_{t+\delta t}$ we obtain

$$0 = \Delta_{t+\delta t}\ S_{t+\delta t} + \Gamma_{t+\delta t}\ M_{t+\delta t} - \Delta_t\ S_{t+\delta t} + \Gamma_t\ M_{t+\delta t} =$$

$$(\Delta_{t+\delta t} - \Delta_t)\ S_{t+\delta t} + (\Gamma_{t+\delta t} - \Gamma_t)\ M_{t+\delta t},$$

adding and subtracting $(\Delta_{t+\delta t} - \Delta_t)\ S_t$ and $(\Gamma_{t+\delta t} - \Gamma_t)\ M_t$

$$0 = (\Delta_{t+\delta t} - \Delta_t)\ (S_{t+\delta t} - S_t) + (\Gamma_{t+\delta t} - \Gamma_t)\ (M_{t+\delta t} - M_t) +$$

$$(\Delta_{t+\delta t} - \Delta_t)\ S_t + (\Gamma_{t+\delta t} - \Gamma_t)\ M_t.$$

Which suggests that the self finance condition in continuous time should be

$$d\Delta_t\ dS_t + d\Delta_t\ S_t + d\Gamma_t\ dM_t + d\Gamma_t\ M_t = 0.$$

This can be immediately proved by differentiating $X_t = \Delta_t\ S_t + \Gamma_t\ M_t$ the fact that $dX_t = \Delta_t\ dS_t + r\ (X_t - \Delta_t\ S_t)\ dt$, and that $r\ (X_t - \Delta_t\ S_t)\ dt = r\ \Gamma_t\ M_t\, dt = \Gamma_t\ dM_t$.

After cancellation, the self finance condition in continuous time appears.

In real life, the continuous time self financed rebalancing of the replicating portfolio is not feasible, yet it can be shown that if we rebalance the portfolio every δt interval of time throughout the life of the derivative, the expected total cost of refinancing the portfolio to be replicating goes to zero as δt goes to zero.

Sketch the proof

The Black-Scholes-Merton PDE

Let $dS_t = \alpha\ S_t\ dt + \sigma\ S_t\ dW_t$ be the real world dynamics of the underlying stock,

Applying to's formula to the option value process $V_t = V(t, S_t)$, we obtain

$$dV_t = \frac{\partial V}{\partial t}\ dt + \frac{\partial V}{\partial S}\ dS_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\ (dS_t)^2$$

$$= [\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}\ \alpha\ S_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\ \sigma^2\ (S_t)^2]\ dt + \frac{\partial V}{\partial S}\ \sigma\ S_t\ dW_t$$

Since we want our portfolio X_t to replicate the value of the option V_t, this will hold if $d(e^{-rt}V_t) = d(e^{-rt}X_t)$ since $V_0 = X_0$.

$$d(e^{-rt}X_t) = e^{-rt}[\Delta_t dS_t + r\ (X_t - \Delta_t S_t)\ dt] - re^{-rt}X_t\ dt =$$

$$= e^{-rt}[\Delta_t \alpha S_t dt + \Delta_t \sigma S_t dW_t + r\ (X_t - \Delta_t S_t)\ dt - r\ X_t\ dt] =$$

$$= e^{-rt}[\alpha\ \Delta_t\ S_t - r\Delta_t\ S_t]\ dt + e^{-rt}\sigma\Delta_t\ S_t\ dW_t.$$

In a similar fashion we can compute

$$d(e^{-rt}V_t) = e^{-rt}[[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}\ \alpha\ S_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\ \sigma^2\ (S_t)^2]\ dt$$

$$+\frac{\partial V}{\partial S}\ \sigma\ S_t\ dW_t] - r\ e^{-rt}\ V_t\ dt.$$

We first observe that in order to cancel the dW_t terms, $\Delta_t = \frac{\partial V}{\partial S}$, and this leads to the famous Black-Scholes-Merton PDE

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}\ r\ S_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\ \sigma^2\ (S_t)^2 - rV = 0$$

with final time condition $V(T, S) = f(S)$.

The Feynman-Kac Theory

Under the risk neutral dynamics

$$dS_t = r\ S_t\ dt + \sigma\ S_t\ dW_t$$

$$X_t = \Delta_t S_t + (X_t - \Delta_t S_t)$$

$$dX_t = \Delta_t dS_t + r\ (X_t - \Delta_t S_t)\ dt =$$

$$= r\ X_t\ dt + \sigma\ \Delta_t S_t\ dW_t.$$

We have shown that under the risk neutral dynamics, the process $e^{-rt}X_t$ is a martingale, that is, for $s < t$

$$e^{-rs}X_s = E_{RN}[e^{-rt}X_t/F_s]$$

Since X_t replicates V_t, the same is true for $e^{-rt}V_t$, thus

$$e^{-rs}V_s = E_{RN}[e^{-rt}V_t/F_s]$$

but according to Feynmac-Kac's theory, this is only true if the dt part of the SDE for $d(e^{-rt}V_t)$ is equal to 0, which leads to the BMS partial differential equation.

This arguments works both ways.

Consider the following PDE with boundary condition;

$$-\frac{1}{2}\Delta u + c\,u = f, \quad in\ U$$

$$u = 0 \quad on\ \partial U.$$

where $c,\ f$ are smooth functions with $c \geq 0$ in U.

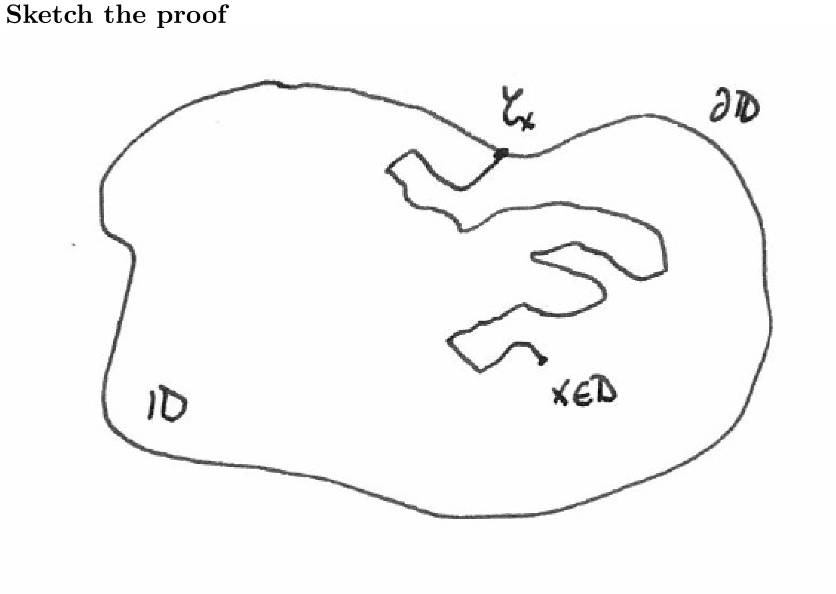
Feynman-Kac formula

For each $x \in U$,

$$u(x) = E(\int_0^{\tau_x} f(X_t)\ e^{-\int_0^t c(X_s)\ ds}\ dt)$$

where, $X_t = x + W_t$ is a brownian motion starting at x, and τ_x denotes the first hitting time (stopping time) of ∂U.

Sketch the proof



The Martingale Representation Theorem and

self-financed replicating portfolios

$$e^{-rt}X_t = E_{RN}[e^{-rT}X_T/F_t]$$

$$e^{-rt}V_t = E_{RN}[e^{-rT}X_T/F_t]$$

Thus $e^{-rt}\ V_t$ is a martingale under RN. And thus so is $e^{-rt}\ S_t$. The MRT says that there exists an adapted process Δ_t such that

$$d(\frac{V_t}{e^{rt}}) = \Delta_t\ d(\frac{S_t}{e^{rt}})$$

Then it can be shown that the replicating portfolio

$$V_t = \Delta_t\ S_t + (V_t - \Delta_t\ S_t)$$

is self financed, that is $dV_t = \Delta_t\ dS_t + r\ (V_t - \Delta_t\ S_t)\ dt$.

Sketch the proof

SDE's lead to PDE's

A SDE with its initial condition determines a diffusion process. This process defines a deterministic function of space and time in two different ways:

1) Considering the expected value of some payoff, as a function of the initial time and position.

2) Considering the probability (transition) of being in a certain state at a given time, knowing the initial time and position.

The study of 1) leads backward Kolmogorov PDE and the Feynmac-Kac formula.

The study of the second (which is the dual of the first as we'll see later) leads to the forward Kolmogorov PDE or what some people call the Fokker-Plank equation.

The backward Kolmogorov equation (R.V. Kohn)

Let us consider the following diffusion

$$dy_t = f(t, y_t)\ dt + g(t, y_t)\ dW_t,$$

Let,

$$u(t, x) = E_{y_t=x}[\Phi(y(T))].$$

Then, $u(t, x)$ solves the following backward Kolmogorov equation (BKE)

$$u_t + f(t, x)\ u_x + \frac{1}{2}\ g^2(t, x)\ u_{xx} = 0$$

with final time condition

$$u(x, T) = \Phi(x).$$

The version of the BKE for vector-valued diffusions

$$dy_i(t) = f_i(t, y(t))\ dt + \sum_j g_{ij}(t, y(t))\ dW_j(t),$$

where $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Where, $(W_1, \ldots, W_m)$ is a m-dimensional Brownian motion. Then,

$$u(t, x) = E_{y_t = x}[\Phi(y(T))]$$

solves the following BKE

$$u_t + L\ u = 0,$$

with final time condition

$$u(x, T) = \Phi(x),$$

where the differential operator L is given by

$$L\ u = \sum_i f_i\ \frac{\partial u}{\partial x_i}\ +\ \frac{1}{2}\ \sum_{i,j,k} g_{ik}\ g_{jk}\ \frac{\partial^2 u}{\partial x_i\ \partial x_j}.$$

Feynman-Kac revisited

$$u(t,x) = E_{y(t)=x}[e^{-\int_t^T b(s,y(s))\ ds}\ \Phi(y(T))]$$

is a solution of the following PDE

$$u_t + f(t,x)\ u_x + \frac{1}{2}\ g^2(t,x)\ u_{xx} - b(t,x)\ u = 0,$$

with final time condition

$$u(T,x) = \Phi(x).$$

This leads naturally under the lognormal dynamics of the underlying to the BSM equation.

Sketch the proof

Running payoff

Suppose we are interested in computing

$$u(t,x) = E_{y(t)=x}[\int_t^T \Psi(s,y(s))\ ds].$$

Then $u(t,x)$ solves the following PDE

$$u_t + Lu + \Psi(t,x) = 0,$$

with the final time condition

$$u(T,x) = 0.$$

Running and final time payoffs

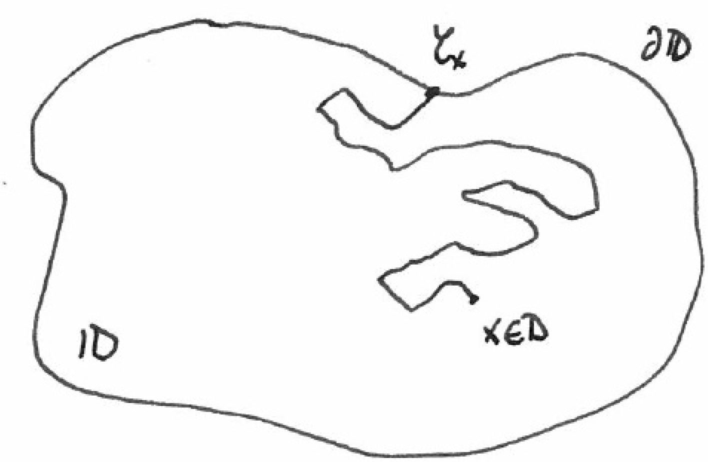
Let $y(t) = (y_1(t), \ldots, y_n(t))$ be a vector valued diffusion:

$$dy_i(t) = f_i(t, y(t))\ dt + \sum_j g_{ij}(t, y(t))\ dW_j(t),$$

where $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Where, $(W_1, \ldots, W_m)$ is a m-dimensional Brownian motion.

Let D be a region in R^n and pick an $x \in D$. Define the following stopping time τ_x:

It is the first time the diffusion $y(t)$ exits from D, if prior to T; otherwise $\tau_x = T$.



Then the function

$$u(t,x) = E_{y(t)=x}[\int_t^{\tau_x} \Psi(s, y(s))\ ds\ + \Phi(\tau_x, y(\tau_x))]$$

solves the following PDE

$$u_t + L\, u + \Psi(t,x) = 0$$

for $x \in D$. With boundary condition

$$u(t,x) = \Phi(t,x)$$

for $x \in \partial D$, and final time condition

$$u(t,x) = \Phi(T,x)$$

for all $x \in D$.
where the differential operator L is given by

$$L\ u = \sum_i f_i\ \frac{\partial u}{\partial x_i}\ +\ \frac{1}{2}\ \sum_{i,j,k} g_{ik}\ g_{jk}\ \frac{\partial^2 u}{\partial x_i\ \partial x_j}.$$

Sketch the proof

The Kolmogorov forward/Fokker-Planck equation

As above, let $y(t) = (y_1(t), \ldots, y_n(t))$ be a vector valued diffusion:

$$dy_i(t) = f_i(t, y(t))\ dt + \sum_j g_{ij}(t, y(t))\ dW_j(t),$$

where $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Where, $(W_1, \ldots, W_m)$ is a m-dimensional Brownian motion.

The solution $y(t)$ is a Markov process, and as such, it has a well defined transition probability $p(z, s; x, t)$ which denotes the probability of the process being at z at time s, given that started at x at a previous time $t < s$. This transition probability density functions satisfy the Chapman-Kolmogorov equation,

$$p(z, s; x, t) = \int_{R^n} p(z, s; z_1, s_1)\ p(z_1, s_1; x, t)\ dz_1,$$

for any intermediate time $t < s_1 < s$. If the initial time condition is a random variable y_0 with density $\rho_0(x)$, then the density function of the y process at time $s > t$ is given by

$$\rho(z, s) = \int_{R^n} p(z, s; x, t)\ \rho_0(x)\ dx.$$

Theorem 3 *$p(z,s;x,t)$ satisfies the following PDE is (z,s)*

$$-p_s - \sum_i \frac{\partial}{\partial z_i}(f_i(z,s)\ p) + \frac{1}{2} \sum_{i,j,k} \frac{\partial^2}{\partial z_i \partial z_j}(g_{ik}(z,s)\ g_{jk}(z,s)\ p) = 0,$$

for $s > t$, with initial condition $p = \delta_x(z)$ at $s = t$, where δ_x is the Dirac delta function at x.

We can rewrite this FKE/F-PE as

$$-p_s + L^* p = 0$$

where

$$L^* p = \sum_i \frac{\partial}{\partial z_i}(f_i(z,s)\ p) + \sum_{i,j} \frac{\partial^2}{\partial z_i \partial z_j}(a_{ij}(z,s)\ p),$$

where $a_{ij} = \frac{1}{2}(g\ g^T)_{ij}$.

It immediately follows that the densities defined above $\rho(z, s)$ for $s > t$ also solve the FKE with initial condition $\rho(z, s) \to \rho_0(z)$ as $s \to t$.

Before we outline the proof of the above theorem, let's examine some consequences.

1. $\int \rho(z, s)\ dz = 1$. This follows from the fact that

$$\frac{d}{ds} \int \rho(z, s)\ dz = 0,$$

and that this holds for the initial time 0 distribution.

2. If the original diffusion has no drift (i.e. the f's are zero), then the expected position of the y process is time independent, i.e. $\frac{d}{ds} E[y(s)] = 0$. This follows from the FKE

$$\frac{d}{ds} E[y(s)] = \frac{d}{ds} \int z\ \rho(z, s)\ dz =$$

$$= \int z\ \rho_s(z, s)\ dz =$$

$$= \int z\ L^* \rho(z, s)\ dz = 0$$

when the drift f is equal to zero. Since,

$$\int z\ L^* \rho(z, s)\ dz = \frac{1}{2} \int z\ (g^2\ \rho)_{zz}\ dz =$$

$$= -\frac{1}{2} \int (g^2\ \rho)_z\ dz = 0.$$

We pass to sketch the proof of the FKE.

We already know that

$$u(x,t) = E_{y(t)=x}[\Phi(y(T))] = \int \Phi(z)\ p(z,T;x,t)\ dz$$

is a solution of the BKE equation $(u_t + Lu) = 0$ with the final time condition $u(x,T) = \Phi(x)$.

Let us consider the standard inner product in $C_0^\infty(R^n)$, i.e.

$$(v, w) = \int_{R^n} v(x)\ w(x)\ dx.$$

If we have an operator L defined on $C_0^\infty(R^n)$, its dual L^* is defined to be the operator such that

$$(L\, v, w) = (v, L^*\, w)$$

for all v, w.

We have the following

$$\int_R (f\ v_x + \frac{1}{2} g^2\ v_{xx})\ w\ dx = \int_R v\ (-(f\ w)_x + \frac{1}{2}(g^2\ w)_{xx})\ dx,$$

which follows by integration by parts. This is the scalar case, the vector valued one will follow a similar argument.

From the BKE, we have that

$$E_{y(t)=x}[\phi(T, y(T))] - \phi(t, x) = E_{y(t)=x}[\int_t^T (\phi_s + L\,\phi)\ (s, y(s))\ ds],$$

for any function $\phi(s, y)$. Now we are going to express the two expectations on the above equation using the transition probability density functions $p(z, s; x, t)$. Thus,

$$\int_{R^n} \phi(T, z)\, p(z, T; x, t)\, dz - \phi(t, x) = \int_t^T \int_{R^n} (\phi_s + L\,\phi)\,(s, z)\, p(z, s; x, t)\, dz\, ds.$$

Integrating by parts on time and using the dual of the operator L we have that the right hand side of the above equation is equal to

$$\int_t^T \int_{R^n} (-\phi\ p_s + \phi\ L^* p)\ (s, z)\ dz\ ds + \int_{R^n} \{\phi(s, z)\ p(z, s; x, t)\ dz\}_{s=t}^{s=T}.$$

The second term in the above sum and the left hand side of the equation only involve the initial and final time t and T, thus we get that

$$\int_t^T \int_{R^n} (-\phi\ p_s + \phi\ L^* p)\ (s,z)\ dz\ ds = 0,$$

for any function $\phi(t,x)$. Hence,

$$(-p_s + \ L^* p)\ (s,z) = 0.$$

Therefore,

$$\int_{R^n} \phi(T,z)\ p(z,T;x,t)\ dz - \phi(t,x) = \int_{R^n} \{\phi(s,z)\ p(z,s;x,t)\ dz\}_{s=t}^{s=T} =$$

$$= \int_{R^n} \phi(T,z)\ p(z,T;x,t)\ dz - \int_{R^n} \phi(t,z)\ p(z,t;x,t)\ dz$$

after simplifying, we get that

$$\phi(t,x) = \int_{R^n} \phi(t,z)\ p(z,t;x,t)\ dz,$$

which is what we mean by the initial condition on the transition probability density function $p(z,s;x,t) = \delta_x$ at $s = t$, and we are done.

Stochastic optimal control, dynamic programming

and the HJB equation

We'll start with the deterministic case. We have a system whose state at time t is described by a vector $y = y(s) \in R^n$. The system evolves in time and we have the influence to affect its evolution through a vector valued control $\alpha(s) \in R^n$.

$$\dot{y}(s) = f(y(s), \alpha(s)),$$

with initial condition $y(0) = x$. Our goal is to choose a control $\alpha(s)$ for $0 < s < T$ so as to maximize some utility or minimize some cost, e.g.

$$max_{\alpha}[\int_0^T h(y(s), \alpha(s))\, dz + g(y(T))].$$

Example. Consider an individual whose wealth today is x, and who will live exactly T years.

$$\dot{y} = r\ y - \alpha,$$

with initial condition $y(0) = x$.The control $\alpha(s) \geq 0$, and $y(s) \geq 0$. The goal is

$$max_{\alpha}[\int_0^T e^{-\rho\, s}\ h(\alpha(s))\ ds].$$

A typical choice for the utility of consumption function is the monotone increasing, concave function $h(\alpha) = \alpha^{\gamma}$ where $0 < \gamma < 1$. For simplicity we have assigned no utility to final time wealth (at T), so that the final time boundary condition is $y(T) = 0$.

The $y \geq 0$ constrain can be avoided by defining a time τ to be the first time y reaches 0 if before T or equal to T if that happens after time T, i.e. $y \geq 0$ for all $s < T$.

To make the problem generic, rather than starting at time 0 we assume we'll start at time t. Then, the value function is defined to be

$$u(t, x) = max_{\alpha}[\{\int_t^T h(y(s), \alpha(s))\, dz\} + g(y(T))].$$

In the case of our example,

$$u(t, x) = max_{\alpha}[\int_t^T e^{-\rho\, s}\ h(\alpha(s))\ ds],$$

this if we discount to time 0, in general the utility of consumption discounted to time t is

$$u(t, x) = max_{\alpha}[\int_t^T e^{-\rho\, (s-t)}\ h(\alpha(s))\ ds].$$

In general y will depend on x, t and α, so we'll denote $y = y_{x,t,\alpha}$, and thus

$$u(t,x) = max_\alpha[\{\int_t^T h(y_{x,t,\alpha}(s), \alpha(s))\, dz\} + g(y_{x,t,\alpha}(T))].$$

And in particular for our utility of consumption problem we have

$$u(t,x) = max_\alpha[\int_t^T e^{-\rho\,(s-t)}\ h(y_{x,t,\alpha}, \alpha(s))\ ds].$$

Dynamic programming

1) At time T, $u(T,x) = g(x)$. In this case, the dynamics of y, the control and the running utility are irrelevant.

2) Let $t = T - \Delta t$, and approximate the dynamics of y by

$$y(s + \Delta t) = y(s) + f(y(s), \alpha(s))\ \Delta t.$$

We are assuming that the controls are piecewise constant on the t-mesh, then

$$u(T - \Delta t, x) = max_\alpha\{h(x,\alpha)\,\Delta t + g(x + f(x,\alpha)\,\Delta t)\}.$$

3) At time $t = T - 2\Delta t$, we have that letting $y(T - 2\,\Delta t) = x$

$$u(T - 2\Delta t, x) = max_\alpha\{h(x,\alpha)\,\Delta t + u(x + f(x,\alpha)\,\Delta t, T - \Delta t)\},$$

and continue backward until time t. This pretty much describes the numerical scheme used to solve the PDE associated to the value function $u(t,x)$, i.e. the HJB equation.

Let us derive the HJB equation, basically all we need to do is to let $\Delta t \to 0$ in our above argument.

It follows from the dynamic programming principle,

$$u(t,x) = max_{\alpha}\{\int_t^{t'} h(y_{x,t,\alpha}(s), \alpha(s))\, ds + u(y_{x,t,\alpha}(t'), t')\},$$

whenever $t < t' < T$. let $t' = t + \Delta t$, then under suitable conditions on u and the control, we have that

$$u(t,x) \geq h(x,a)\,\Delta t + u(x + f(x,a)\,\Delta t, t + \Delta t) + errors$$

with equality achieved when the a is chosen optimally. Using Taylor's for u we have

$$u(t,x) \geq h(x,a)\,\Delta t + u(x,t) + (\nabla u\ f(x,a) + u_t)\,\Delta t + errors,$$

with equality again when a is chosen optimally. Taking the limit as $\Delta t \to 0$ this give

$$0 = u_t + max_{\alpha}\{\nabla u\ f(x,\alpha) + h(x,\alpha)\},$$

where the Hamiltonian $H(p,x) = max_{\alpha}\{p\ f(x,\alpha) + h(x,\alpha)\}$. The final time condition is obvious if $t = T$, then $u(T,x) = g(x)$.

The verification argument

We need to show that the solution of the HJB equation $w(t,x)$ is the value function $u(t,x)$.

Obviously the solution of the HJB equation $w(t,x)$ once known defines as a feedback law an admissible feedback control that leads to w, and since the value function is the maximum over all possible controls, then clearly

$$w(t,x) \leq u(t,x).$$

Let's work on the other inequality. To show this, let's take any admissible control $\alpha(s)$ and its associated dynamics $y_{x,t,\alpha}(s)$ that start at x at time t. By the chain rule, we have

$$\frac{d}{ds}w(s, y(s)) = w_s(s, y(s)) + \nabla w(s, y(s))\ \dot{y}(s)$$

$$= w_s(s, y(s)) + \nabla w(s, y(s))\ f(y(s), \alpha(s))$$

$$\leq w_s + H(\nabla w, y) - h(y(s), \alpha(s))$$

$$= -h(y(s), \alpha(s)),$$

now integrating between t and T:

$$w(T, y(T)) - w(t, x) \leq -\int_t^T h(y(s), \alpha(s))\ ds.$$

Since $w(T, y(T)) = g(y(T))$ we have that

$$w(t, x) \geq \int_t^T h(y(s), \alpha(s))\ ds + g(y(T)),$$

this argument applies to any control $\alpha(s)$, thus maximizing the right hand side over all the admissible controls $\alpha(s)$, we finally have

$$u(t, x) \leq w(t, x).$$

Stochastic optimal control

A simple adjustment using Ito Calculus of the above deterministic optimal control problem will lead us to the stochastic optimal control Hamilton-Jacobi-Bellman PDE for a controlled diffusion $y_t = y(t)$. Namely, if we have a vector valued difussion

$$dy_i(t) = f_i(t, y_t, \alpha_t)\, dt + \sum_{k=1}^{m} g_{ik}(t, y_t, \alpha_t)\ dW_k(t),$$

for $i = 1, \ldots, n$ with the f's in L_1 and the g's in L^2, then if $y(t) = x \in R^n$, if w elet the value function

$$u(t, x) = max_\alpha[\{\int_t^T h(y(s), \alpha(s))\ ds + g(y(T))\}]$$

then $u(t,x)$ solves the following HJB equation

$$0 = u_t + \max_{\alpha}[\nabla_x u(t,x)\ f(t,x,\alpha) + h(x,\alpha)+$$

$$+\frac{1}{2}\ \sum_{ij}[\sum_{k} g_{ik}(t,x,\alpha)\ g_{jk}(t,x,\alpha)]\ \frac{\partial^2 u}{\partial x_i\ \partial x_j}],$$

with final time condition $u(T,x) = g(x)$.

Merton's example: Optimal portfolio selection and consumption

Let us consider a risk-free asset (with fixed constant return r) and a risky one (we choose just one risky asset for the sake of simplicity, the multiple risky asset case can be treated similarly); whose SDE's are

$$dy_1(t) = r\ y_1(t)\,dt$$

and

$$dy_2(t) = \mu\, y_2(t)\,dt + \sigma\, y_2(t)\,dW(t),$$

where as usual $W(t)$ is a one-dimensional Brownian motion, (the multidimensional case will work in a similar fashion). Let $\alpha_1(t)$ the fraction of the total wealth invested in the risky asset at time t, and $\alpha_2(t)$ the rate of wealth consumption at time t.

If we denote by $y(t)$ the investor's wealth at time t we have that

$$y(t) = (1 - \alpha_1(t))\, y(t) + \alpha_1(t)\, y(t) - \alpha_2(t)\, dt$$

$$= (1 - \alpha_1(t))\, \frac{y(t)}{y_1(t)}\, y_1(t) + \alpha_1(t)\, \frac{y(t)}{y_2(t)}\, y_2(t) - \alpha_2(t)\, dt,$$

taking differentials we get that

$$dy(t) = (1 - \alpha_1(t))\, r\, y(t)\, dt + \alpha_1(t)\, y(t)\, (\mu\, dt + \sigma\, dW(t)) - \alpha_2(t)\, dt.$$

With the restrictions $0 \leq \alpha_1(t) \leq 1$ and $\alpha_2(t) \geq 0$ as long as $y(t) \geq 0$.

We define the time τ as in the deterministic case; the minimum $\{T, \inf\{s \geq t \; s.t. \; y(s) = 0\}\}$. It is clear that this is a stopping time with respect to the brownian filtration. We seek to maximize

$$u(t,x) = max_{\alpha_1,\alpha_2} E_{y(t)=x} \; [\int_t^T e^{-\rho\, s} \; h(\alpha(s)) \; ds].$$

A typical choice for the utility of consumption function is the monotone increasing, concave function $h(\alpha) = \alpha^{\gamma}$ where $0 < \gamma < 1$. Arguing as in the deterministic case we get

$$u(t,x) == \max_{\alpha_1,\alpha_2} \{e^{\rho t} \, h(\alpha_2) \, \Delta t + E_{y(t)=x} \; [u(t+\Delta t, y(t+\Delta t))]\}.$$

Applying Ito Calculus and taking the limit as $\Delta t \to 0$ we obtain the HJB equation for this problem, namely

$$u_t + \max_{\alpha_1,\alpha_2} \{e^{\rho t} \, h(\alpha_2) \; + \; [(1-\alpha_1) \, x \, r + \alpha_1 \, x \, \mu - \alpha_2] \, u_x + \frac{1}{2} \, x^2 \, \alpha_1^2 \, \sigma^2 \, u_{xx}\} = 0.$$

with the final time condition $u(T,x) = 0$.

Let's try to solve this HJB equation. First, let's make some observations. The value function $u(t,x)$ is increasing on x, thus $u_x > 0$, and $u_{xx} < 0$ which reflects the concavity of the utility function h. Thus the optimal control α_1 is given by

$$\alpha_1^* = -\frac{(\mu - r)\, u_x}{\sigma^2\, x\, u_{xx}}$$

which is positive then (remember we are assuming that $\mu > r$). We'll later show that is also strictly less than 1 as required.

The optimal $\alpha_2 = \alpha_2^*$ satisfies

$$h'(\alpha_2^*) = e^{\rho\, t}\, u_x,$$

and we can be sure that this α_2^* is positive as required by assuming that $h'(0) = \infty$.

The case $h(x) = x^\gamma$ with $0 < \gamma < 1$

It is not difficult to show that the solution must have the form $u(t,x) = g(t)\,x^\gamma$, and the associated optimal controls are

$$\alpha_1^* = \frac{(\mu - r)}{\sigma^2\,(1-\gamma)}$$

and

$$\alpha_2^* = [e^{\rho\,t}\,g(t)]^{\frac{1}{(\gamma-1)}}\,x.$$

We are going to assume that $\mu - r < \sigma^2\,(1-\gamma)$ so that the control $\alpha_1^* < 1$. Plugging these values into the HJB equation we obtain that

$$\frac{dg}{dt} + \nu\,\gamma\,g(t) + (1-\gamma)\,g(t)\,(e^{\rho\,t}\,g(t))^{\frac{1}{(\gamma-1)}} = 0$$

with $\nu = r + \frac{(\mu-r)}{2\,\sigma^2\,(1-\gamma)}$. And we must solve this ODE with the final time condition $g(T) = 0$.

If we apply the transformation $G(t) = e^{\rho\, t}\, g(t)$, the ODE becomes

$$G_t + (\nu\,\gamma - \rho)\, G + (1-\gamma)\, G^{\frac{\gamma}{(\gamma-1)}} = 0.$$

Next, we multiply this ODE by $\frac{1}{(\gamma-1)}\, G^{\frac{\gamma}{(1-\gamma)}}$ to see that $H(t) = G^{\frac{1}{(1-\gamma)}}$ satisfies the linear ODE

$$H_t - \eta\, H + 1 = 0$$

with $\eta = \frac{\rho-\nu\,\gamma}{1-\gamma}$. The solution of this linear ODE satisfying $H(T) = 0$ is given by

$$H(t) = \eta^{-1}\ (1 - e^{-\eta\,(T-t)}).$$

Going back in the transformations we obtain that

$$g(t) = e^{-\rho\, t} [\frac{1-\gamma}{\rho - \nu\, \gamma}\, (1 - e^{-\frac{(\rho-\nu\, \gamma)\, (T-t)}{1-\gamma}})]^{1-\gamma}.$$

A verification type of argument shows that $u(t,x)$ is the value function. Namely,

$$u(t,x) = x^{\gamma}\ e^{-\rho\, t} [\frac{1-\gamma}{\rho - \nu\, \gamma}\, (1 - e^{-\frac{(\rho-\nu\, \gamma)\, (T-t)}{1-\gamma}})]^{1-\gamma},$$

with the optimal controls being

$$\alpha_1^* = \frac{(\mu - r)}{\sigma^2\, (1-\gamma)}$$

which is constant in time, and

$$\alpha_2^*(t) = [e^{\rho\, t}\, g(t)]^{\frac{1}{(\gamma-1)}}\, y(t) =$$

$$= y(t)\ [\frac{1-\gamma}{\rho - \nu\, \gamma}\, (1 - e^{-\frac{(\rho-\nu\, \gamma)}{1-\gamma} (T-t)})]^{-1},$$

with $\nu = r + \frac{(\mu - r)}{2\, \sigma^2\, (1-\gamma)}$.

Jump-diffusion processes-S. Shreve-Continuous time finance

The Poisson process

Theorem 4 *let $N(t)$ be a Poisson process with intensity $\lambda > 0$, and let $0 = t_0 < t_1 < \ldots, < t_n$ be given. Then the increments*

$$N(t_1) - N(t_0),\;\; N(t_2) - N(t_1),\;\; N(t_3) - N(t_2),\;\; \ldots, N(t_n) - N(t_{n-1})$$

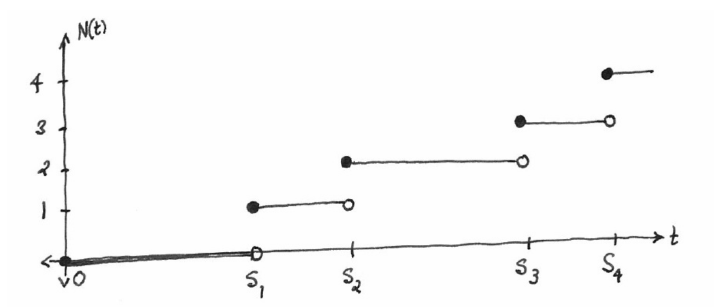
are stationary and independent, and

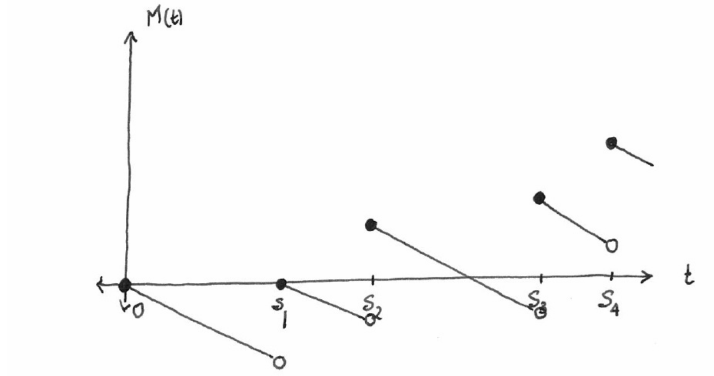
$$P\{N(t_{j+1}) - N(t_j) = k\} = \frac{\lambda^k\,(t_{j+1} - t_j)^k}{k!}\,e^{-\lambda\,(t_{j+1}-t_j)},$$

for $k = 0, 1, \ldots$.

The process $M(t) = N(t) - \lambda\,t$ is a martingale and is called the compensated Poisson process.

Sample paths for a Poisson process and a compensated Poisson process





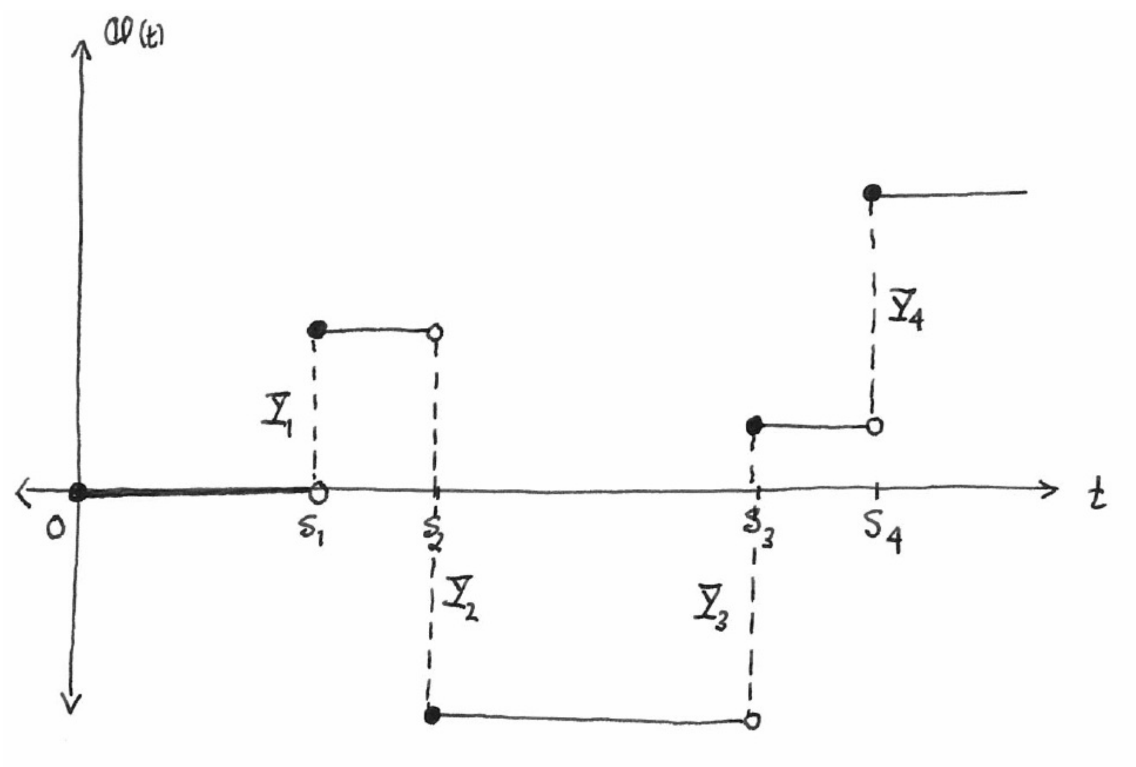
The compound Poisson process

Let $\{Y_1, Y_2, \ldots\}$ be a sequence of i.i.d. with common mean $E[Y_i] = \beta$. We also assume that the Y_i's are independent of some Poisson process $N(t)$. We define the compound Poisson process to be

$$Q(t) = \sum_{i=1}^{N(t)} Y_i,$$

for any $t \geq 0$.

Sample path for a compound Poisson process



Theorem 5 *The compensated Poisson process $Q(t) - \beta\,\lambda\,t$ is a martingale. Let $0 = t_0 < t_1 < \ldots, < t_n$ be given. The increments,*

$$Q(t_1) - Q(t_0),\;\; Q(t_2) - Q(t_1),\;\; Q(t_3) - Q(t_2),\;\; \ldots, Q(t_n) - Q(t_{n-1})$$

are stationary and independent. In particular, the distribution of $Q(t_{j+1}) - Q(t_j)$ is the same as the distribution of $Q(t_{j+1} - t_j)$.

Let us assume that the compound Poisson process can have only finitely many different jump sizes $y-1, y_2, \dots, y_M$ with their corresponding probabilities $p(y_1), p(y_2), \dots, p(y_M)$ that add up to 1. Let $N(t)$ be the defining Poisson process. That is,

$$Q(t) = \sum_{i=1}^{N(t)} Y_i,$$

where the Y_i's are i.i.d. Let, for $m = 1, 2, \dots, M$ $N_m(t)$ denote the number of jumps in Q of size y_m up to time t. Then,

$$N(t) = \sum_{i=1}^{M} N_m(t), \quad and \quad Q(t) = \sum_{i=1}^{M} y_m \, N_m(t).$$

The processes $N_1, N_2, \dots, N_M$ are independent Poisson processes, and each N_m has intensity $\lambda \, p(y_m)$.

Jump processes and their integrals

We wish to define

$$\int_0^t \Phi(s)\, dX(s),$$

where the process $X(s)$ has jumps. Our process $X(t) = X(0)+R(t)+I(t)+J(t)$, that is, is composed of one Riemann integral dt, an Ito integral $dW(t)$ and the pure jump integral $J(t)$. More precisely,

$$X(t) = X^c(t) + J(t) = X(0) + \int_0^t \Theta(s)\, ds + \int_0^t \Gamma(s)\, dW(s) + J(t),$$

where $\Theta \in L^1$, $\Gamma \in L^2$ and $J(t)$ is and adapted, right-continuous pure jump process with $J(0) = 0$. The left continuous version of this pure jump process will be $J(t-)$. We also assume that we have only finitely jumps on each finite time interval and is constant between jumps, that's why we call it "pure". Then, we define the integral

$$\int_0^t \Phi(s)\, dX(s) = \int_0^t \Phi(s)\,\Gamma(s)\, dW(s) + \int_0^t \Phi(s)\,\Theta(s)\, ds + \sum_{0<s\leq t} \Phi(s)\,\Delta J(s).$$

In the context of processes that can jump, we would also like to have that the stochastic integral with respect to a martingale is still a martingale, as is the case of the Ito stochastic integral when the integrand is and adapted L^2 process, namely we would like that if $x(t)$ is a jump-difussion process which is a martingale then

$$\int_0^t \Phi(s)\, dX(s)$$

is also a martingale. This is the case if we require that the integrand process $\Phi(t)$ is left-continuous and adapted to the corresponding filtration. More precisely, if $\Phi(t)$ is left continuous and adapted and

$$E[\int_0^t \Gamma^2(s)\, \Phi^2(s)\, ds] < \infty,$$

for all $t \geq 0$. Then the stochastic integral $\int_0^t \Phi(s)\, dX(s)$ is also a martingale.

Quadratic variation and stochastic calculus for jump processes

Let $X_1(t)$ and $X_2(t)$ be two different jump-difussion processes with SDE's

$$dX_1(t) = \Theta_1(t)\,dt + \Gamma_1(t)\,dW(t) + dJ_1(t),$$

and

$$dX_2(t) = \Theta_2(t)\,dt + \Gamma_2(t)\,dW(t) + dJ_2(t),$$

where both J_1 and J_2 are right continuous pure jump processes. Then the quadratic variation of X_1 and X_2 is given by

$$[X_1, X_2](T) = [X_1^c, X_2^c](T) + [J_1, J_2](T) = \int_0^T \Gamma_1(s)\,\Gamma_2(s)\,ds + \sum_{0<s\leq T} \Delta J_1(s)\,\Delta J_2(s).$$

Or in differential notation,

$$dX_1(t)\,dX_2(t) = \Gamma_1(s)\,\Gamma_2(s)\,ds + dJ_1(s)\,dJ_2(s).$$

In particular, $[W, M](t) = 0$ for all $t \geq 0$.

Ito's formula for jump-diffusion processes

Let $X(t)$ be a jump diffusion process,

$$X(t) = X^c(t) + J(t) = X(0) + \int_0^t \Theta(s)\, ds + \int_0^t \Gamma(s)\, dW(s) + J(t),$$

in differential form,

$$dX(t) = dX^c(t) + dJ(t) = X(0) + \Theta(t)\, dt + \Gamma(t)\, dW(t) + dJ(t),$$

then

$$f(X(t)) = f(X(0)) + \int_0^t f'(X(s))\, dX^c(s) + \frac{1}{2} \int_0^t f''(X(s))\, dX^c(s)\, dX^c(s)$$

$$+ \sum_{0<s\leq t} [f(X(s)) - f(X(s-))].$$

Ito's formula for multiple jump-diffusion processes is a simple generalization of the above, namely, let $X_1(t)$ and $X_2(t)$ be two jump processes, then

$$f(t, X_1(t), X_2(t)) = f(0, X_1(0), X_2(0)) + \int_0^t f_t(s, X_1(s), X_2(s))\, ds$$

$$+ \int_0^t f_{x_1}(s, X_1(s), X_2(s))\, dX_1^c(s) + \int_0^t f_{x_2}(s, X_1(s), X_2(s))\, dX_2^c(s)$$

$$+\frac{1}{2}\int_0^t f_{x_1,x_1}(s, X_1(s), X_2(s))\, dX_1^c(s))\, dX_1^c(s))$$

$$+\frac{1}{2}\int_0^t f_{x_2,x_2}(s, X_1(s), X_2(s))\, dX_2^c(s))\, dX_2^c(s))$$

$$+ \int_0^t f_{x_1,x_2}(s, X_1(s), X_2(s))\, dX_1^c(s))\, dX_2^c(s))$$

$$+ \sum_{0<s\leq t} [f(s, X_1(s), X_2(s)) - f(s, X_1(s-), X_2(s-))].$$

Ito's product rule for jump processes

$$X_1(t)\,X_2(t) = X_1(0)\,X_2(0) + \int_0^t X_2(s-)\,dX_1(s) + \int_0^t X_1(s-)\,dX_2(s)$$

$$+[X_1, X_2](t).$$

The Doleans-Dade exponentials

Let $X(t)$ be a jump-diffusion process and define the following SDE

$$dZ^X(t) = Z^X(t-)\, dX(t),$$

the the solution is given by

$$Z^X(t) = exp\{X^c(t) - \frac{1}{2}[X^c, X^c](t)\} \prod_{0<s\leq t}(1 + \Delta X(s)).$$

With $Z^X(0) = 1$

Changing the measures

Poisson process

Let $N(t)$ be a Poisson process with intensity λ, then if we change the probability using the Radon Nikodym process

$$Z(t) = e^{(\lambda - \tilde{\lambda})} \, (\frac{\tilde{\lambda}}{\lambda})^{N(t)},$$

which satisfies the SDE

$$dZ(t) = \frac{\tilde{\lambda} - \lambda}{\lambda} \, Z(t-) \, dM(t),$$

where $M(t)$ is the compensated Poisson process which is a martingale. It follows that $Z(t)$ is also a martingale, and if we define a new probability $\tilde{P}$ by

$$\frac{d\tilde{P}}{dP}|_t = Z(t),$$

then under this new probability $\tilde{P}$, the Poisson process $N(t)$ is also a Poisson process with intensity $\tilde{\lambda}$.

The compound Poisson process

Let $Q(t) = \sum_{i=1}^{N(t)} Y_i$ be a compound Poisson process with intensity λ and such that $E[Y_i] = \beta$. Let us assume that the iid Y_i's have a continuous density function $f(y)$. Then the Radon-Nikodym derivative process defined by

$$Z(t) = e^{(\lambda - \tilde{\lambda})\, t} \prod_{i-1}^{N(t)} \frac{\tilde{\lambda}\, \tilde{f}(Y_i)}{\lambda\, f(Y_i)}.$$

where $\tilde{f}(y)$ is also a continuous density function for the Y_i's; is a martingale since it is the solution of the following SDE

$$dZ(t) = Z(t-)\, dH(t) - Z(t-)\, dM(t),$$

where both $H(t)$ is a compensates compound Poisson process and $M(t)$ is a compensated Poisson process. Hence, being both martingales, the process $Z(t)$ is also a martingale.

If we define a new probability $\tilde{P}$ by

$$\frac{d\tilde{P}}{dP}|_t = Z(t),$$

then under this new probability $\tilde{P}$, the compound Poisson process $Q(t)$ is still a compound Poisson process with intensity $\tilde{\lambda}$ and new density function for the iid's Y_i's equal to $\tilde{f}(y)$.

Changing the measure simultaneously for both

the compound Poisson process and the Brownian motion

Let $\Theta(t)$ be an L^2 adapted process and let λ and $\tilde{\lambda}$ be positive constants. We define

$$Z_\theta(t) = exp\{-\int_0^t \Theta(s)\, dW(s) - \frac{1}{2}\int_0^t \Theta^2(s)\, ds\},$$

$$Z_2(t) = e^{(\lambda-\tilde{\lambda})\, t} \prod_{i=1}^{N(t)} (\frac{\tilde{\lambda}\tilde{f}(Y_i)}{\lambda\, f(Y_i)})^{N(t)},$$

$$Z(t) = Z_\theta(t)\, Z_2(t),$$

then $Z(t)$ is a martingale, this follows from the fact that

$$dZ(t) = Z_\theta(t-)\, dZ_2(t) + Z_2(t-)\, dZ_\theta(t),$$

and since both processes $Z_\theta(t)$ and $Z_2(t)$ are martingales, it follows that $Z(t)$ is also a martingale.

If we define a new probability $\tilde{P}$ by means of the Radon-Nikodym derivative process $Z(t)$ as

$$\frac{d\tilde{P}}{dP}|_t = Z(t),$$

then under this new probability $\tilde{P}$, the compound Poisson process $Q(t)$ is still a compound Poisson process with intensity $\tilde{\lambda}$ and new density function for the iid's Y_i's equal to $\tilde{f}(y)$; and

$$\tilde{W}(t) = W(t) + \int_0^t \Theta(s)\, ds$$

is a brownian motion independent of $Q(t)$.

Asset driven by a Brownian motion and

a compound Poisson process

Our goal is to find the risk neutral dynamics (SDE jump diffusion) for the underlying asset. Once this is achieved, we will be able to price and hedge derivative securities on this underlying.

We will assume that the random variable for the size of the jumps is discrete with possible jumps given by $-1 < y_1 < y_2 < \ldots, y_M$ nonzero numbers. According to our previous results, the underlying Poisson process and compound Poisson process can be decomposed into M independent Poisson processes as

$$N(t) = \sum_{m=1}^{M} N_m(t), \quad Q(t) = \sum_{m=1}^{M} y_m \, N_m(t).$$

The M Poisson processes with intensity $\lambda = \sum_{m=1}^{M} \lambda_m$.

If we denote by Y_i the size of the j-th jump of Q, then Q can be written as

$$Q(t) = \sum_{j=1}^{N(t)} Y_j.$$

Let $p(y_m) = \frac{\lambda_m}{\lambda}$, where $p(y_m) = P\{Y_j = y_m\}, m = 1, 2, \ldots, M$ We also have that $\beta = E\,[Y_j] = \sum_{m=1}^{M} y_m\, p(y_m) = \frac{1}{\lambda} \sum_{m=1}^{M} \lambda_m\, y_m$.

We know that

$$Q(t) - \beta\,\lambda\,t$$

is a martingale. We'll model the dynamics of the underlying asset by the following SDE

$$dS(t) = \alpha\, S(t)\, dt + \sigma\, S(t)\, dw(t) + S(t-)\, d(Q(t) - \beta\,\lambda\, t)$$

$$(\alpha - \beta\,\lambda)\, S(t)\, dt + \sigma\, S(t)\, dw(t) + S(t-)\, d\,Q(t).$$

The solution to the above jump-diffusion SDE is given by

$$S(t) = S(0)\ exp\{(\alpha - \beta\,\lambda - \frac{1}{2}\,\sigma^2)\,t + \sigma\,W(t)\}\ \prod_{i=1}^{N(t)} (Y_i + 1).$$

But this are not the risk neutral dynamics. We willl undertake the effort of constructing the risk neutral measure using previously stated change of measure results. Namely, let $\theta(t)$ and $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_M$ be positive constants. We define

$$Z_\theta(t) = exp\{-\theta\, W(t) - \frac{1}{2}\,\theta\, t\},$$

$$Z_m(t) = e^{(\lambda_m - \tilde{\lambda}_m)\, t}\, (\frac{\tilde{\lambda}_m}{\lambda_m})^{N_M(t)}, \quad m = 1, \ldots, M$$

$$Z(t) = Z_\theta(t) \prod_{m=1}^{M} Z_m(t),$$

and since both processes $Z_\theta(t)$ and $\prod_{m=1}^{M} Z_m(t)$ are martingales, it follows that $Z(t)$ is also a martingale. Therefore under the probability

$$\tilde{P}(A) = \int_A Z(T)\, dP,$$

for any $F(T)$ measurable set A, we have that:

(i) $\tilde{W}(t) = (W(t) + \theta\, t)$ is a brownian motion,

(ii) Each $N_m(t)$ is a Poisson process with intensity $\tilde{\lambda}_m$, and

(iii) $\tilde{W}$ and $N_1, \ldots, N_M$ are independent of each other.

Let us define now, $\tilde{\lambda} = \sum_{m=1}^{M} \tilde{\lambda}_m$, and $\tilde{p}(y_m) = \frac{\tilde{\lambda}_m}{\tilde{\lambda}}$. Then under P the process $N(t) = \sum_{m=1}^{M} N_m(t)$ is a Poisson process with intensity $\tilde{\lambda}$, and the jump size random variables $Y_1, Y_2, \ldots$ are independent and identically distributed with $\tilde{P}\{Y_i = y_m\} = \tilde{p}(y_m)$, and $Q(t) - \tilde{\beta}\tilde{\lambda}\, t$ is a martingale compensated compund Poisson process, where

$$\tilde{\beta} = \tilde{E}[Y_i] = \sum_{m=1}^{M} y_m\, \tilde{p}(y_m) = \frac{1}{\tilde{\lambda}} \sum_{m=1}^{M} \tilde{\lambda}_m\, y_m.$$

We claim that under $\tilde{P}$ the dynamins of the underlying asset $S(t)$ are risk neutral, i.e. the mean return is equal to r.

$$dS(t) = (\alpha - \beta\,\lambda)\, S(t)\, dt + \sigma\, S(t)\, dW(t) + S(t-)\, d\, Q(t)$$

$$= r\, S(t)\, dt + \sigma\, S(t)\, d\tilde{W}(t) + S(t-)\, d\, (Q(t) - \tilde{\beta}\tilde{\lambda}\, t).$$

This is equivalent to the equation

$$\alpha - \beta\lambda = r + \sigma\theta - \tilde{\beta}\tilde{\lambda},$$

which is equivalent to

$$\alpha - r = \sigma\theta + \sum_{m=1}^{M} (\lambda_m - \tilde{\lambda}_m)\, y_m.$$

which is called the market price of risk equation for this model. So, we have one equation on $(M+1)$ unknowns $\theta, \tilde{\lambda}_1, \ldots, \tilde{\lambda}_M$, an undetermined system, therefore we have numerous risk neutral probabilities and the market is incomplete.

In any case, let's choose one solution to the market price of risk equation, then

$$dS(t) = r\,S(t)\,dt + \sigma\,S(t)\,d\tilde{W}(t) + S(t-)\,d\,(Q(t) - \tilde{\beta}\tilde{\lambda}\,t)$$

$$= (r - \ \beta\tilde{\lambda})\,S(t)\,dt + \sigma\,S(t)\,d\tilde{W}(t) + S(t-)\,d\,Q(t),$$

whose solution we already know it is

$$S(t) = S(0)\ exp\{(r - \tilde{\beta}\,\tilde{\lambda} - \frac{1}{2}\,\sigma^2)\,t + \sigma\,\tilde{W}(t)\}\ \prod_{i=1}^{N(t)}(Y_i + 1).$$

Pricing and hedging an european call option

on a jump-diffusion model

Since we know that $\tilde{\beta}\,\tilde{\lambda} = \sum_{m=1}^{M} \tilde{\lambda}_m\, y_m$, and this appears explicitely in our model, as opposed to θ; thus we can choose the $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_M$'s to calibrate the model to market data, and once the $\tilde{\lambda}$'s are fixed, use the market price of risk equation to find the θ and come up with a unique risk neutral probability.

Let's introduce some notation,

$$k(x, \tau) = x\ N(d_+(\tau, x)) - K\ e^{-r\ T}\ N(d_-(\tau, x)),$$

where

$$d_{+-} = \frac{\log(\frac{x}{K}) + (r + -\frac{1}{2}\sigma^2)\,\tau}{\sigma\ \sqrt{\tau}},$$

and

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{s^2}{2}}\ ds,$$

is the cumulative normal distribution function corresponding to the standard $N(0,1)$ normal random variable.

Theorem 6 *For $0 \le t < T$, the risk neutral price of the call,*

$$V(t) = \tilde{E}[e^{-r(T-t)}\,(S(T)-K)_+|F(t)],$$

is given by $V(t) = c(t, S(t))$, where

$$c(t,x) = \sum_{j=0}^{\infty} e^{-\tilde{\lambda}(T-t)}\,\frac{\tilde{\lambda}^j\,(T-t)^j}{j!}\,\tilde{E}[k(T-t, x\,e^{-\tilde{\beta}\tilde{\lambda}(T-t)}\,\prod_{i=1}^{j}(Y_i+1)].$$

The PDE for the jump difussion process

Theorem 7 *$c(t,x)$ satisfies the following integro-differential PDE,*

$$-r\,c(t,x) + c_t(t,x) + (r - \tilde{\beta}\tilde{\lambda})\,x\,c_x(t,x) + \frac{1}{2}\,\sigma^2\,x^2\,c_{xx}(t,x) +$$

$$+\tilde{\lambda}[\sum_{m=1}^{M} \tilde{p}(y_m)\,c(t,(y_m+1)\,x) - c(t,x)] = 0,$$

for $0 \leq t < T$, and $x \geq 0$, and the terminal condition $c(T,x) = (x-K)_+$, for $x \geq 0$.

This theorem holds in the discrete jump size case, in the continous case, the integro-differential PDE becomes

$$-r\,c(t,x) + c_t(t,x) + (r - \tilde{\beta}\tilde{\lambda})\,x\,c_x(t,x) + \frac{1}{2}\,\sigma^2\,x^2\,c_{xx}(t,x) +$$

$$+\tilde{\lambda}\,E[\,c(t,(Y+1)\,x) - c(t,x)] = 0,$$

where the expectation in the last term of the above equation is with respect to the distribution of the iid's Y_i's. More precisely if $\tilde{f}(y)$ is the common density function of those random variables,

$$E[\,c(t,(Y+1)\,x) - c(t,x)] = \int_{-1}^{\infty} c(t,(y+1)x)\,\tilde{f}(y)\,dy - c(t,x).$$

Where the market price of risk equation in the continuous case is

$$\alpha - r = \sigma\,\theta + \beta\,\lambda - \tilde{\beta}\,\tilde{\lambda},$$

where the unknowns are θ, $\tilde{\lambda}$ and $\tilde{\beta}$.

The hedging

Let us consider the following replicating portfolo

$$X(t) = \Gamma(t)\, S(t) + (X(t) - \Gamma(t)\, S(t)),$$

made of $\Gamma(t-)$ units of the underlying asset and the rest invested in the risk free money market. Then,

$$dX(t) = \Gamma(t-)\, dS(t) + r\,(X(t) - \Gamma(t)\, S(t))\, dt,$$

if we want the portfolio to replicate the value of the option $c(t, S(t))$, we need to have that

$$d(e^{rt}\, c(t, S(t))) = d(e^{rt}\, X(t)).$$

The left hand side of the above equality is equal (Ito Calculus and above theorem) to

$$d(e^{rt}\,c(t,S(t))) = e^{-rt}\,\sigma\,S(t)\,c_x(t,S(t))\,d\tilde{W}(t)+$$

$$+e^{-rt}\,[c(t,(Y+1)\,S(t-)) - c(t,S(t-))]\,\,(dN(t) - \tilde{\lambda}\,dt).$$

As for the right hand side, it is equal to

$$d(e^{rt}\,X(t)) = e^{-rt}\,[\Gamma(t-)\,\sigma\,S(t)\,d\tilde{W}(t) + \Gamma(t-)\,S(t-)\,d(Q(t) - \tilde{\beta}\,\tilde{\lambda}\,t)],$$

where $dQ(t) = Y\,dN(t)$, and it is not difficult to see that $\tilde{\beta}$ being equal to the $\tilde{E}[Y_i]$'s, in our argument for the hedging, we can replace $\tilde{\beta}$ with $Y = Y_i$'s. Thus,

$$d(e^{rt}\,X(t)) = e^{-rt}\,[\Gamma(t-)\,\sigma\,S(t)\,d\tilde{W}(t) + \Gamma(t-)\,S(t-)\,Y\,d(N(t) - \tilde{\lambda}\,t)].$$

If we take $\Gamma(t) = c_x(t, S(t))$ the $d\tilde{W}(t)$ terms of both $d(e^{rt}\, c(t, S(t)))$ and $d(e^{rt}\, X(t))$ cancel and we have that

$$d(e^{rt}\, c(t, S(t))) - d(e^{rt}\, X(t)) =$$

$$= e^{-rt}\, [c(t, (Y+1)\, S(t-)) - c(t, S(t-))]\ (dN(t) - \tilde{\lambda}\, dt) -$$

$$-c_x(t, S(t-))\, S(t-)\, Y\, d(N(t) - \tilde{\lambda}\, t) =$$

$$= e^{-rt}\, [c(t, (Y+1)\, S(t-)) - c(t, S(t-)) - c_x(t, S(t-))\, S(t-)\, Y]\, (dN(t) - \tilde{\lambda}\, dt),$$

and since the function $c(t, x)$ is strictly convex in the x variable, we have that

$$c(t, (Y+1)\, S(t-)) - c(t, S(t-)) - c_X(t, S(t-))\, S(t-)\, Y > 0$$

since $Y > -1$. This shows that when a jump occurs, i.e. $dt = 0$ and $dN(t) = 1$, $d(e^{rt}\, c(t, S(t))) - d(e^{rt}\, X(t)) > 0$ and when no jump occurs, i.e. $-\tilde{\lambda} dt < 0$ and $dN(t) = 0$, then

$$c(t, (Y+1)\, S(t-)) - c(t, S(t-)) - c_x(t, S(t-))\, S(t-)\, Y\, d(N(t) - \tilde{\lambda}\, t) < 0$$

thus, $d(e^{rt}\, c(t, S(t))) - d(e^{rt}\, X(t)) < 0$ that is, between jumps the hedging portfolio outperforms the option, and at jump times it is the other way around in a way that throughout the life of the option they offset each other since $(dN(t) - \tilde{\lambda}\, dt)$ is a martingale.

References

[1] R. Merton. *Continuous Time Finance*, Blackwell (1992)

[2] B.K. Oksendal. *Stochastic Differential Equations: an Introduction with applications*, Springer (1998)

[3] R.V. Kohn. *Course Notes from NYU on Derivative Securities and PDE in Finance*, (2004-2007)

[4] S. Shreve. *Stochastic Calculus for Finance II: Continuous Time Models*, Springer (2004)

[5] M. Baxter and A. Rennie. *Financial Calculus: an Introduction to derivative pricing*, Cambridge University Press (1996)