Numerical schemes for dispersive equations

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Outline

- Dispersive equations. Tools from Harmonic Analysis.
- Nonlinear Schrödinger equations. Stability of solutions.
- Approximation of NSE by complex Ginzburgh Landau
- Numerical schemes in finite differences with dispersive properties. Error estimates
- Splitting methods for NSE

Outline

1. Course I. Dispersive equations
   - Fourier Analysis Tools
   - Multipliers
   - Linear Schrödinger equation
   - Nonlinear Schrödinger equation

Dispersion relation

The dispersion relation $\omega = \omega(k)$ of a constant coefficient linear evolution equation determines how time oscillations $e^{i\omega t}$ are linked to spatial oscillations $e^{ik\cdot x}$ of wave number k. In other words, the dispersion relation is the function for which the plane waves $e^{ik\cdot x}e^{i\omega(k)t}$ solve the equation.

Note that for equations which are second-order in time rather than first-order, the dispersion relation is typically double-valued rather than single-valued.

The dispersion relation for the model constant-coefficient linear dispersive and wave equations are as follows:
Examples:
Phase rotation equation

$$iu_t + \omega_0 u = 0$$

the dispersion relation is constant: $\omega(k) = \omega_0$.
Transport equation

$$u_t + v \cdot u_x = 0$$

the dispersion relation is linear: $\omega(k) = -v \cdot k$.
Free Schrodinger equation

$$iu_t + \Delta u = 0$$

the dispersion relation is quadratic: $\omega(k) = -|k|^2$

For the Airy equation, $u_t + u_{xxx} = 0$ the dispersion relation is cubic: $\omega(k) = k^3$.
* For the free wave equation, $u_{tt} - \Delta u = 0$ the dispersion relation is $\omega(k) = \pm|k|$.
* For the Klein-Gordon equation, $u_{tt} - \Delta u + m^2 u = 0$the dispersion relation is $\omega(k) = \pm(m^2 + k^2)^{1/2}$.
* Non-time-reversible equations such as the heat equation do not have a dispersion relation, unless one permits $\omega(k)$ to be complex-valued.

Propagation

Let us assume that a PDE with constant coefficients admits a solution the monocromatic wave

$$u(x,t) = e^{i(\omega(\xi)t - \xi x)}$$

It propagates ritghward with t at the speed

phase speed $$c(\xi) = \frac{\omega(\xi)}{\xi}$$

The evolution of a wave packet containing several wave number is more complicated.

$$u(x,t) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i(\omega(\xi)t - \xi x)} d\xi = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{it(\omega(\xi) - \xi x/t)} d\xi$$

Choose $x/t = cont$. The main term in the asymptotic expansion of the integral comes from the points where

$$\frac{d}{d\xi}(\omega(\xi) - \frac{x}{t}) = 0,$$

group velocity: $$C(\xi) = \frac{d\omega}{d\xi} = \frac{x}{t}$$

The energy associated with wave number ξ moves asymptotically at the group speed.

The group velocity is the more important of the two velocities. Group velocity controls the motion of frequency envelopes and thus of energy and mass

An equation is dispersive if different frequencies propagate at different group velocities.
Thus, for instance, the phase rotation and transport equations are not dispersive, the Airy, Schrodinger, and Klein-Gordon equations are dispersive, and the wave equation is partialy dispersive (the group velocity depends on the direction of frequency but not on the magnitude).

If the group velocity is bounded we say that we have finite speed of propagation, otherwise we have infinite speed of propagation. Thus for instance, the phase rotation, transport, Klein-Gordon, and wave equations have finite speed of propagation, while the Schrodinger and Airy equation has infinite speed of propagation.
Intuitively, a dispersive equation should spread out the physical support of a solution over time. One way to capture this is via dispersive estimates, which in turn lead to Strichartz estimates; when there is infinite speed of propagation, dispersion can also be captured inside local smoothing estimates.

The Fourier Transform

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2i\pi x\cdot\xi} dx$$

Properties

1.$\|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1(\mathbb{R}^d)}$

2.$\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$

3. $(f * g)\hat{}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$

4. $\|f\|_{L^2(\mathbb{R}^d)} = \|\hat{f}\|_{L^2(\mathbb{R}^d)}$

Oscilatory integrals

Lemma

Let $f \in C_c^\infty([a,b])$ *and* $\phi'(x) \neq 0$ *for any* $x \in [a,b]$. *Then*

$$I(\lambda) = \int_a^b e^{i\lambda\phi(x)} f(x)dx = O(\lambda^{-k}), \qquad \text{as } \lambda \to \infty$$

for any $k \in \mathbb{Z}^+$.

Proof: Integration by parts.

Lemma

Let $k \geq 1$ integer and $|\phi^k(x)| \geq 1$ for any $x \in [a,b]$ with $\phi'(x)$ monotonic in the case $k = 1$. Then

$$| \int_a^b e^{i\lambda\phi(x)} dx| \leq c_k \lambda^{-1/k}$$

where the constant c_k is independent of a and b.

Proof on the blackboard + HW

Van der Corput Lemma

Lemma

Under the previous assumptions

$$I_\lambda = \left|\int_a^b e^{i\lambda\phi(x)} f(x)dx\right| \leq c_k \lambda^{-1/k}(\|f\|_{L^\infty} + \|f'\|_{L^1})$$

Proof: Define $G(x) = \int_a^x e^{i\lambda\phi(y)}dy$. The previous result implies $|G(x)| \leq c_k\lambda^{-1/k}$. Then

$$I_\lambda = \left|\int_a^b G' f dx\right| \leq |(Gf)|\Big|_a^b + \left|\int_a^b G f'\right| \leq c_k \lambda^{-1/k}(\|f\|_{L^\infty} + \|f'\|_{L^1}).$$

Examples: $\phi(x) = \sin(x), \phi(x) = x^2$, + HW

Definition

Given $1 \leq p < \infty$, we denote by $M_p(\mathbb{R}^d)$ the space of all bounded functions m on $\mathbb{R}^d$ such that the operator

$$T_m(f) = (\hat{f}m)^\vee$$

is bounded in $L^p(\mathbb{R}^d)$.

Properties

1. $\|m\|_{M_p} = \|T_m\|_{L^p - L^p}$
2. $M_2(\mathbb{R}^d) = L^\infty(\mathbb{R}^d)$
3. $\|m\|_{M_p} = \|m\|_{M_{p'}}$, for $1 < p < \infty$
4. M_p are nested: $1 < p < q < 2$ then $M_1 \subset M_p \subset M_q \subset M_2 = L^\infty$ and

$$\|m\|_\infty \leq \|m\|_{M_q} \leq \|m\|_{M_p} \leq \|m\|_{M_1}$$

A nontrivial example:

$m = \chi_{(a,b)}$ with $-\infty < a < b < \infty$ is belongs to $M_p(\mathbb{R})$ for any $1 < p < \infty$

Other properties:

1. if m is a L^p-multiplier then the translation, $m(\xi + a)$, the dilatation $m(\lambda\xi)$ and the orthogonal transformation, $m(A\xi)$, is also a L^p-multiplier

2. $m \in M_p(\mathbb{R})$ then $\tilde{m}(\xi_1, \xi_2) = m(\xi_1)$ is in $M_p(\mathbb{R}^2)$

3. The characteristic function of any convex polyhedron is a M_p-multiplier

4. In dimension $d > 1$ the characteristic function of a ball centered at origin is not a $L^p(\mathbb{R}^d)$ multiplier, $p \neq 2$.

More examples in Duoandikoetxea, Fourier Analysis, AMS.

Sufficient conditions to belong to M_p

Theorem (Marcinkiewicz)

Let m be a bounded function which has uniformly bounded variation on each dyadic interval in $\mathbb{R}$ Then m is a multiplier on $L^p(\mathbb{R})$.

$$\sup_j \Big(\int_{-2^{j+1}}^{-2^j} |m'(\xi)| d\xi + \int_{2^j}^{2^{j+1}} |m'(\xi)| d\xi \Big) \leq A < \infty, j \in \mathbb{Z}$$

Obs: In $\mathbb{R}^d$ there exists a similar condition but .... see Grafakos, Fourier Analysis, vol. 1
Examples: $m(\xi) = |\xi| 2^{-[\log_2 |\xi|]}$, $m(\xi) = |\xi|^s \chi_{(-1,1)}$, $s > 0$.

Theorem (Hörmander-Mihlin)

Let $m(\xi)$ be a bounded complex-valued function on $\mathbb{R}^d \setminus \{0\}$ that satisfies either
a) Mihlin's condition

$$|\partial_\xi^\alpha m(\xi)| \leq A|\xi|^{-|\alpha|}$$

for all multi-indices $|\alpha| \leq 1 + [d/2]$
or
b) Hörmander's condition

$$\sup_{R>0} R^{-d+2|\alpha|} \int_{R<|\xi|<2R} |\partial_\xi^\alpha m(\xi)|^2 d\xi \leq A^2 < \infty$$

for all multi-indices $|\alpha| \leq 1 + [d/2]$
Then for all $1 < p < \infty$, $m \in M_p(\mathbb{R}^d)$.

Examples:
1. $m(\xi) = |\xi|^{i\tau}$ with $\tau \in \mathbb{R}$.
2. If m is real valued and satisifes Mihlin's condition then $e^{im(\xi)}$ is in M_p, $1 < p < \infty$

Exercise: Consider $L_1 = \partial_1 - \partial_2^2 + \partial_3^4$, $L_2 = \partial_1 + \partial_2^2 + \partial_3^2$. Prove that for every $1 < p < \infty$

$$\|\partial_2 \partial_3^2 f\|_{L^p(\mathbb{R}^3)} \leq C_p \|L_1 f\|_{L^p(\mathbb{R}^3)}$$

$$\|\partial_1 f\|_{L^p(\mathbb{R}^3)} \leq C_p \|L_2 f\|_{L^p(\mathbb{R}^3)}$$

More references: Grafakos, vol 1, Duoandikoetxea

Linear equation

$$\begin{cases} iu_t + \Delta u = F(t,x) & t \in \mathbb{R}, x \in \mathbb{R}^d, \\ u(x,0) = u_0(x) & x \in \mathbb{R}^d. \end{cases}$$

Basic properties 1.

$$e^{it\Delta} u_0 = \frac{1}{(4\pi i t)^{d/2}} e^{-|x|^2/4it} * \varphi = (e^{-4\pi^2 it|\xi|^2} \hat{u_0})^{\vee}.$$

2. $\|e^{it\Delta} u_0\|_{L^2} = \|u_0\|_{L^2}$
3. $\|e^{it\Delta} u_0\|_{L^\infty(\mathbb{R}^d)} \lesssim |t|^{-d/2} \|u_0\|_{L^1(\mathbb{R}^d)}$

The decay is "optimal". Choose as initial data $u_0 = \exp(-\pi|x|^2)$, then

$$u(t,x) = (e^{-(1+4\pi it)\pi|\xi|^2})^{\vee} = \frac{1}{(1+4\pi it)^{d/2}} \exp(-\frac{\pi|x|^2}{1+4\pi it})$$
$$= (1+4\pi it)^{-d/2} \exp(-\frac{\pi|x|^2}{1+16\pi^2t^2}) \exp(\frac{4\pi^2 it|x|^2}{1+16\pi^2t^2})$$

We easily obtain

$$ct^{-d/2}\chi_{\{|x|<t\}}(x) \leq |u(t,x)| \leq Ct^{-d/2}$$

A first estimate

Theorem

If $t \neq 0$ *and* $p' \in [1, 2]$ *then* $\exp(it\Delta) : L^{p'}(\mathbb{R}) \to L^p(\mathbb{R}^d)$ *is continuous and*

$$\|\exp(it\Delta)f\|_p \leq c|t|^{-d/2(1/p'-1/p)}\|f\|_{p'}$$

Proof: $\exp(it\Delta) : L^2 \to L^2$ with norm one and $\exp(it\Delta) : L^1 \to L^\infty$ with norm less than $|t|^{-d/2}$.

HW: Let p and q such that $\exp(i\Delta)$ is continuous from $L^q(\mathbb{R})$ to $L^p(\mathbb{R})$. Prove that $q = p'$ and $p \geq 2$. Hint: Scaling, consider $\varphi_\lambda(x) = \varphi(\lambda x)$ and make λ small and large.

TT^* argument

Theorem

Let H be a Hilbert space, B and its dual B', Banach spaces and a linear operator T. The following are equivalent
i) T is bounded from H to B:

$$\|Tf\|_B \leq C\|f\|_H$$

ii) T^* is bounded from B' to H:

$$\|T^*F\|_H \leq C\|F\|_{B'}$$

iii) TT^* is bounded from B' to B:

$$\|TT^*F\|_B \leq C^2\|F\|_{B'}$$

A particular case, let us choose $H = L^2(\mathbb{R}^d)$, $B = L^q(\mathbb{R}, L^r(\mathbb{R}^d))$, $B' = L^{q'}(\mathbb{R}, L^{r'}(\mathbb{R}^d))$ and $T = \exp(it\Delta)$ which maps $\varphi \in L^2(\mathbb{R}^d)$ to $\exp(it\Delta) \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^d))$.
The formal adjoint of T is

$$T^*F = \int_{-\infty}^{\infty} \exp(-is\Delta)F(s)ds$$

and

$$TT^*F = \int_{-\infty}^{\infty} \exp(i(t-s)\Delta)F(s)ds$$

Show on Blackboard

Strichartz Estimates

1. $\| \exp(it\Delta)\varphi\|_{L^q(\mathbb{R},L^r(\mathbb{R}^d))} \leq C\|\varphi\|_{L^2(\mathbb{R}^d)}$
2. $\| \int_{-\infty}^{\infty} \exp(-is\Delta)F(s)ds\|_{L^2(\mathbb{R}^d)} \leq \|F\|_{L^{q'}(\mathbb{R},L^{r'}(\mathbb{R}^d))}$
3. $\| \int_{-\infty}^{\infty} \exp(i(t-s)\Delta)F(s)ds\|_{L^q(\mathbb{R},L^r(\mathbb{R}^d))} \leq C\|F\|_{L^{q'}(\mathbb{R},L^{r'}(\mathbb{R}^d))}$

Retarded estimate

4. $\| \int_0^t \exp(i(t-s)\Delta)F(s)ds\|_{L^q(\mathbb{R},L^r(\mathbb{R}^d))} \leq C\|F\|_{L^{q'}(\mathbb{R},L^{r'}(\mathbb{R}^d))}$

for admissible pairs $2 \leq r < \frac{2d}{d-2}$ if $d \geq 3$, $2 \leq r < \infty$ if $d = 2$ and $2 \leq r \leq \infty$ if $d = 1$: $1/q = d/2(1/2 - 1/r)$

Proof of estimate 3.

$$
\begin{aligned}
\| \int_{-\infty}^{\infty} \exp(i(t-s)\Delta)F(s)ds\|_{L^r(\mathbb{R}^d)} \\
&\leq \int_{-\infty}^{\infty} \| \exp(i(t-s)\Delta)F(s)\|_{L^r(\mathbb{R}^d)} ds \\
&\leq C \int_{-\infty}^{\infty} \frac{\|F(s,\cdot)\|_{L^{r'}(\mathbb{R}^d)}}{|t-s|^{d/2(1/r'-1/r)}} ds.
\end{aligned}
$$

Hardy-Littlewod-Sobolev inequality

Theorem

Let $0 < \alpha < d$, $1 < p < q < \infty$ *with* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$. *The integral (Riesz-potential of order* α*)*

$$I_\alpha f = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}} dy$$

satisfies

$$\|I_\alpha f\|_{L^q(\mathbb{R}^d)} \leq c_{p,d,\alpha} \|f\|_{L^p(\mathbb{R}^d)}$$

$$
\begin{aligned}
\| \int_{-\infty}^{\infty} \exp(i(t-s)\Delta)F(s)ds\|_{L_t^q L_x^r} &\lesssim \| \int_{-\infty}^{\infty} \frac{\|F(s,\cdot)\|_{L^{r'}(\mathbb{R}^d)}}{|t-s|^{d/2(1/r'-1/r)}} ds\|_{L_t^q} \\
&= \|I_\alpha(\|F\|_{L_x^{r'}})\|_{L_t^q} \leq \Big\| \|F\|_{L_x^{r'}} \Big\|_{L_t^{q'}} = \|F\|_{L_t^{q'} L_t^{q'}}
\end{aligned}
$$

HW: Show that if estimate 1. holds then the pair (q, r) should be admissible

Retarded estimate, Christ and Kiselev's Lemma

Lemma

Consider the operator $W : L^{p_1}(\mathbb{R}, B_1) \to L^{p_2}(\mathbb{R}, B_2)$

$$(Wf)(t) = \int_{\mathbb{R}} K(t,s) f(s) ds$$

where $K(t,s) \in \mathcal{B}(B_1, B_2)$ *and* $K(t,s)$ *locally integrale. If* $p_1 < p_2$ *then the operator*

$$\tilde{W} f(t) = \int_{s<t} K(t,s) f(s) ds$$

is bounded from $L^{p_1}(\mathbb{R}, B_1)$ *to* $L^{p_2}(\mathbb{R}, B_2)$ *and*

$$\|\tilde{W}\|_{L^{p_1}(\mathbb{R},B_1)\to L^{p_2}(\mathbb{R},B_2)} \leq c(p_1,p_2) \|W\|_{L^{p_1}(\mathbb{R},B_1)\to L^{p_2}(\mathbb{R},B_2)}$$

Application to the retarded estimate

Choose

$$K(t,s) = 1_{[0,T](t)}1_{[0,T]}(s)e^{i(t-s)\Delta}$$

$$B_1 = L^{r'}(\mathbb{R}^d), B_2 = L^r(\mathbb{R}^d)$$

$$p_1 = q' < p_2 = q$$

and

$$Wf = TT^*f = \int_0^T e^{i(t-s)\Delta} f(s)ds.$$

Details on blackboard.

Smoothing effect

Theorem

if $d = 1$ *then*

$$\sup_{x\in\mathbb{R}} \int_{-\infty}^{\infty} |D_x^{1/2} e^{it\Delta} f(x)|^2 dt \leq C\|f\|_{L^2(\mathbb{R})^2}$$

Obs: For initial data $f \in L^2(\mathbb{R})$ the solution satisfy

$$\exp(it\Delta) \in L_t^2(\mathbb{R}, H_{loc}^{1/2}(\mathbb{R}))$$

Obs: A similar result holds in any dimension d
Proof: Blackboard

Nonlinear problems

$$\begin{cases} iu_t + \partial_x^2 u = |u|^p u, \; x \in \mathbb{R}, \; t \neq 0, \\ u(0,x) = \varphi(x), \; x \in \mathbb{R}, \end{cases} \tag{1}$$

Conservation of the $L^2(\mathbb{R})$-norm:

$$\frac{d}{dt} \int_{\mathbb{R}} |u(t,x)|^2 dx = 2\Re\Big(\int_{\mathbb{R}} u(t) \overline{u}_t(t) \Big) = 0.$$

Theorem

Let $f(u) = |u|^p u$ *with* $p \in (0,4)$. *Then*
i) (Global existence and uniqueness) For any $\varphi \in L^2(\mathbb{R})$, *there exists a unique global solution* u *of* (1) *in the class*

$$u \in C(\mathbb{R}, L^2(\mathbb{R})) \cap L^q_{loc}(\mathbb{R}, L^r(\mathbb{R}))$$

for all $1/2$*-admissible pairs* (q,r) *such that*

$$\|u(t)\|_{L^2(\mathbb{R})} = \|\varphi\|_{L^2(\mathbb{R})}, \, \forall\, t \in \mathbb{R}.$$

ii) (Stability) Let φ *and* ψ *be two* $L^2(\mathbb{R})$ *functions, and* u *and* v *the corresponding solutions of the NSE. Then for any* $T>0$ *there exists a positive constant* $C(T, \|\varphi\|_{L^2(\mathbb{R})}, \|\psi\|_{L^2(\mathbb{R})})$ *such that the following holds*

$$\|u - v\|_{L^\infty(0,T;\, L^2(\mathbb{R}))} \leq C(T, \|\varphi\|_{L^2(\mathbb{R})}, \|\psi\|_{L^2(\mathbb{R})}) \|\varphi - \psi\|_{L^2(\mathbb{R})} \quad (2)$$

An estimate:

Lemma

Let $\varphi \in L^2(\mathbb{R})$ *and* u *be the solution of the NSE with initial data* φ *and nonlinearity* $f(u) = |u|^p u$, $p \in (0,4)$, *as in Theorem 12. There exists* $c(p) > 0$ *and* $T_0 = c(p)\|\varphi\|_{L^2(\mathbb{R})}^{-4p/(4-p)}$ *such that for any* $1/2$*-admissible pairs* (q,r), *there exists a positive constant* $C(p,q)$ *such that*

$$\|u\|_{L^q(I;\, L^r(\mathbb{R}))} \leq C(p,q)\|\varphi\|_{L^2(\mathbb{R})} \tag{3}$$

holds for all intervals I *with* $|I| \leq T_0$.

A short proof

Regularity

Moreover if $\varphi \in H^s(\mathbb{R})$, $s \in (0, 1/2)$ then

$$u \in C(\mathbb{R}, H^s(\mathbb{R})) \cap L^q_{loc}(\mathbb{R}, B^s_{r,2}(\mathbb{R}))$$

for every admissible pairs (q, r).
Also if $\varphi \in H^1(\mathbb{R})$ then $u \in C(\mathbb{R}, H^1(\mathbb{R}))$