An Introduction to Matched Asymptotic Expansions
(A Draft)

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Preface

The Basque Center for Applied Mathematics (BCAM) is a newly founded institute which is one of the Basque Excellence Research Centers (BERC). BCAM hosts a series of training courses on advanced aspects of applied and computational mathematics that are delivered, in English, mainly to graduate students and postdoctoral researchers, who are in or from out of BCAM. The series starts in October 2009 and finishes in June 2010. Each of the 13 courses is taught in one week, and has duration of 10 hours.

This is the lecture notes of the course on Asymptotic Analysis that I gave at BCAM from Nov. 9 to Nov. 13, 2009, as the second of this series of BCAM courses. Here I have added some preliminary materials. In this course, we introduce some basic ideas in the theory of asymptotic analysis. Asymptotic analysis is an important branch of applied mathematics and has a broad range of contents. Thus due to the time limitation, I concentrate mainly on the method of matched asymptotic expansions. Firstly some simple examples, ranging from algebraic equations to partial differential equations, are discussed to give the reader a picture of the method of asymptotic expansions. Then we end with an application of this method to an optimal control problem, which is concerned with vanishing viscosity method and alternating descent method for optimal control of scalar conservation laws in presence of non-interacting shocks.

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Contents

Chapter 1

Introduction

In the real world, many problems (which arise in applied mathematics, physics, engineering sciences, $\cdots$, also pure mathematics like the theory of numbers) don't have a solution which can be written a simple, exact, explicit formula. Some of them have a complex formula, but we don't know too much about such a formula.

We now consider some examples. i) The Stirling formula:

$$n! \sim \sqrt{2n\pi}\mathrm{e}^{-n}n^n \left(1 + O\left(\frac{1}{n}\right)\right). \tag{1.0.1}$$

The Landau symbol the big Oh "O" and the Du Bois Reymond symbol "$\sim$" are used. Note that $n!$ grows very quickly as $n \to \infty$ and becomes so large that one can not have any idea about how big it is. But formula (1.0.1) gives us a good estimate of $n!$.

ii) From Algebra we know that in general there is no explicit solution to an algebraic equation with degree $n \geq 5$.

iii) Most of problems in the theory of nonlinear ordinary or partial differential equations don't have an exact solution.

And many others.

In practice, however, an approximation of a solution to such problems is usually enough. Thus the approaches to finding such an approximation is important. There are two main methods. One is numerical approximation, which is especially powerful after the invention of the computer and is now regarded as the third most import method for scientific research (just after the traditional two: theoretical and experimental methods). Another is analytical approximation with an error which is understandable and controllable, in particular, the error could be made smaller by some rational procedure. The term "analytical approximate solution" means that an analytic formula of an approximate solution is found and its difference with the exact solution.

Asymptotic analysis is powerful tool for finding analytical approximate solutions to complicated practical problems, which is an important branch of applied mathematics. In 1886 the establishment of rigorous foundation was done by Poincaré and Stieltjes. They published separately papers on asymptotic series. Later in 1905, Prandtl published a paper on the motion of a fluid or gas with small viscosity along a body. In the case of an airfoil moving through air, such problem is described by the Navier-Stokes equations with large Reynolds number. The method of singular perturbation was proposed.

Of course, the history of asymptotic analysis can be traced back to much earlier than 1886, even to the time when our ancestors studied the problem, as small as the measure of a rod, or as large as the study of the perturbed orbit of a planet. As we know, when we measure a rod, each measure gives a different value, so n-measures result in n-different values. Which one should choose to be the length of this rod? The best approximation to the real length of the rod is the mean value of these n-numbers, and each of the measures can be regarded as a perturbation of the mean value.

The Sun's gravitational attraction is the main force acting on each planet, but there are much weaker gravitational forces between the planets, which produce perturbations of their elliptical orbits; these make small changes in a planet's orbital elements with time. The planets which perturb the Earth's orbit most are Venus, Jupiter, and Saturn. These planets and the sun also perturb the Moon's orbit around the Earth-Moon system's center of mass. The use of mathematical series for the orbital elements as functions of time can accurately describe perturbations of the orbits of solar system bodies for limited time intervals. For longer intervals, the series must be recalculated.

Today, astronomers use high-speed computers to figure orbits in multiple body systems such as the solar system. The computers can be programmed to make allowances for the important perturbations on all the orbits of the member bodies. Such calculations have now been made for the Sun and the major planets over time intervals of up to several tens of millions of years.

As accurately as these calculations can be made, however, the behavior of celestial bodies over long periods of time cannot always be determined. For example, the perturbation method has so far been unable to determine the stability either of the orbits of individual bodies or of the solar system as a whole for the estimated age of the solar system. Studies of the evolution of the Earth-Moon system indicate that the Moon's orbit may become unstable, which will make it possible for the Moon to escape into an independent orbit around the Sun. Recent astronomers have also used the theory of chaos to explain irregular orbits.

The orbits of artificial satellites of the Earth or other bodies with atmospheres whose orbits come close to their surfaces are very complicated. The orbits of these satellites are influenced by atmospheric drag, which tends to bring the satellite down into the lower atmosphere, where it is either vaporized by atmospheric friction or falls to the planet's surface. In addition, the shape of Earth and many other bodies is not perfectly spherical. The bulge that forms at the equator, due to the planet's spinning motion, causes a stronger gravitational attraction. When the satellite passes by the equator, it may be slowed enough to pull it to earth.

The above argument gives us many problems with small perturbations, some of those perturbations can be omitted under suitable assumptions.

The main contents of asymptotic analysis are as follows: perturbation method, the method of multi-scale expansions, averaging method, WKBJ (Wentzel, Kramers, Brillouin and Jeffreys) approximation, the method of matched asymptotic expansions, asymptotic expansion of integrals, and so on. This course is mainly concerned with the method of matched asymptotic expansions. Firstly we study some simple examples arising in algebraic equation, ordinary differential equations, from which we will get key ideas of matched asymptotic expansion, though those examples are simple. Then we shall investigate matched asymptotic expansion for partial differential equations and finally take an optimal control problem as an application.

Let us now introduce some notations. $D \subset \mathbb{R}^d$ with $d \in \mathbb{N}$ denotes an open subset in $\mathbb{R}^d$. $f, g, h : D \to \mathbb{R}$ are real continuous functions. We denote a small quantity by ε. The Landau symbols the big Oh O and the little o will be used.

Definitions. *A sequence of gauge functions $\{\varphi_n(x)\}$ ($n = 0, 1, 2, \cdots$) is said to be form an asymptotic sequence as $x \to x_0$ if, for all n,*

$$\varphi_{n+1}(x) = o(\varphi_n(x)), \text{ as } x \to x_0.$$

If $\varphi_n(x)$ is an asymptotic sequence of gauge functions as $x \to x_0$, we say that

$$\sum_{n=1}^{\infty} a_n \varphi_n(x), \ a_n \text{ are constant function,}$$

is an asymptotic expansion (or asymptotic approximation) of a function $f(x)$ as $x \to x_0$, if for each N.

$$f(x) = \sum_{n=1}^{N} a_n \varphi_n(x) + o(\varphi_N(x)), \text{ as } x \to x_0.$$

Chapter 2

Algebraic equations

In this chapter we shall investigate some algebraic equations, which are very helpful for establishing the picture of asymptotic analysis in our mind, though those examples are quite simple. Let us consider algebraic equations with a small positive parameter which is denoted by ε in what follows.

2.1 Regular perturbation

Consider the following quadratic equation

$$x^2 - \varepsilon x - 1 = 0. \tag{2.1.1}$$

Suppose that $\varepsilon = 0$, equation (2.1.1) becomes

$$x^2 - 1 = 0. \tag{2.1.2}$$

It is easy to find the roots x^ε of (2.1.1), for any fixed ε, which read

$$x_1^\varepsilon = \frac{\varepsilon + \sqrt{\varepsilon^2 + 4}}{2}, \text{ and } x_2^\varepsilon = \frac{\varepsilon - \sqrt{\varepsilon^2 + 4}}{2}. \tag{2.1.3}$$

Correspondingly, the roots x^0 of (2.1.2) are $x^0_{1,2} = \pm 1$.

A natural question arises: Does x^ε converge to x^0? We prove easily that

$$x_1^\varepsilon \to x_1, \text{ and } x_2^\varepsilon \to x_2. \tag{2.1.4}$$

So (2.1.1) is called a *regular* perturbation of (2.1.2). The perturbation term is $-\varepsilon x$. A perturbation is called *singular* if it is not regular.

For most of practical problems, however, we don't have the explicit formulas like (2.1.1). In this case, how can we get the knowledge about the limits as the small parameter ε goes to zero?

The method of asymptotic expansions is a powerful tool for such an investigation. Note that x^ε depends on ε, to construct an asymptotic expansion, we define an ansatz as follows

$$x^\varepsilon = \varepsilon^{\alpha_0}(x_0 + \varepsilon^{\alpha_1}x_1 + \varepsilon^{\alpha_2}x_2 + \cdots), \tag{2.1.5}$$

here, α_i $(i = 0, 1, 2, \cdots)$ are constant to be determined, and we assume, without loss of generality, that x_0, x_1, differ from zero, and $0 < \alpha_1 < \alpha_2 < \cdots$.

We first determine α_0. There are three cases. i) $\alpha_0 > 0$, ii) $\alpha_0 < 0$ and iii) $\alpha_0 = 0$. We will show that only case iii) is possible to get an asymptotic expansion. Inserting ansatz (2.1.5) into equation (2.1.1). Balancing the resulting equation, we obtain

$$x^{2\alpha_0}x_0^2 + 2\varepsilon^{2\alpha_0+\alpha_1}x_0x_1 + \varepsilon^{2(\alpha_0+\alpha_1)}x_1^2 - x^{\alpha_0}x_0 - \varepsilon^{\alpha_0+\alpha_1}x_1 - 1 + \cdots) = 0 \tag{2.1.6}$$

Suppose now that case i) happens, i.e. $\alpha_0 > 0$, which implies α_0 is the smallest power. Thus from (2.1.6) it follows that the coefficient of ε^{α_0} should be zero, namely $x_0 = 0$, this is contradict to our assumption that $x_0 \neq 0$.

For case ii), namely $\alpha_0 < 0$, we have $2\alpha_0 < \alpha_0 < \alpha_0 + \alpha_1$, thus $2\alpha_0$ is the smallest power and the coefficient of $\varepsilon^{2\alpha_0}$ should be zero, so $x_0^2 = 0$, which violates our assumption too.

Therefore, we assert that only the case $\alpha_0 = 0$ is possible, and (2.1.5) becomes

$$x^\varepsilon = x_0 + \varepsilon^{\alpha_1}x_1 + \varepsilon^{\alpha_2}x_2 + \cdots , \tag{2.1.7}$$

moreover, (2.1.6) now is

$$x_0^2 + 2\varepsilon^{\alpha_1}x_0x_1 + \varepsilon^{2\alpha_1}x_1^2 - x_0 - \varepsilon^{\alpha_1}x_1 - 1 + \cdots) = 0. \tag{2.1.8}$$

Similar to the above procedure for deciding α_0, we can determine α_1, α_2 etc., which are $\alpha_1 = 1, \alpha_2 = 2, \cdots$. So ansatz (2.1.5) takes the following form

$$x^\varepsilon = x_0 + \varepsilon^1 x_1 + \varepsilon^2 x_2 + \cdots , \tag{2.1.9}$$

and the following expansions are obtained

$$\varepsilon^0 : \quad x_0^2 - 1 = 0, \tag{2.1.10}$$

$$\varepsilon^1 : \quad 2x_0x_1 - x_0 = 0, \tag{2.1.11}$$

$$\varepsilon^2 : \quad 2x_0x_2 + x_1^2 - x_1 = 0. \tag{2.1.12}$$

Solving (2.1.10) we have $x_0 = 1$ or $x_0 = -1$. We take the first case as an example and construct the asymptotic expansion. From (2.1.11) and (2.1.12) we get, respectively,

$$x_1 = \frac{1}{2}, \ x_2 = \frac{1}{8}.$$

Up to i-terms ($i = 1, 2, 3$), we expand x^ε as follows

$$X_1^\varepsilon = 1, \tag{2.1.13}$$
$$X_2^\varepsilon = 1 + \frac{\varepsilon}{2}, \tag{2.1.14}$$
$$X_3^\varepsilon = 1 + \frac{\varepsilon}{2} + \frac{\varepsilon^2}{8}. \tag{2.1.15}$$

The next question then arise: How precisely does X_i^ε ($i = 1, 2, 3$) satisfy the equation of x^ε?

Straightforward computations yield that

$$(X_1^\varepsilon)^2 - \varepsilon X_1^\varepsilon - 1 = O(\varepsilon), \tag{2.1.16}$$
$$(X_2^\varepsilon)^2 - \varepsilon X_2^\varepsilon - 1 = O(\varepsilon^2), \tag{2.1.17}$$
$$(X_3^\varepsilon)^2 - \varepsilon X_3^\varepsilon - 1 = O(\varepsilon^4). \tag{2.1.18}$$

From which it is easy to see that X_i^ε satisfies very well the equation when ε is small, and the error becomes smaller as i is larger, which means that we take more terms.

2.2 Iterative method

In this section we are going to make use the so called iterative method to construct asymptotic expansions again for equation (2.1.1). We rewrite (2.1.1) as follows

$$x = \pm\sqrt{1 + \varepsilon x},$$

where $x = x^\varepsilon$. This formula suggests us an iterative procedure,

$$x_{n+1} = \sqrt{1 + \varepsilon x_n} \tag{2.2.1}$$

for any $n \in \mathbb{N}$. Here we only take the positive root as an example. Let x_0 be a fixed real number. One then obtains from (2.2.1) that

$$x_1 = 1 + \frac{\varepsilon}{2} x_0 + \cdots, \tag{2.2.2}$$

so we find the first term of an asymptotic expansion, however the second term in (2.2.2) still depends on x_0. To get the second term of an asymptotic expansion, we iterate once again and arrive at

$$x_2 = 1 + \frac{\varepsilon}{2} x_1 + \cdots = 1 + \frac{\varepsilon}{2} + \cdots, \tag{2.2.3}$$

this gives us the desired result. After iterating twice, we then construct an asymptotic expansion:

$$x^\varepsilon = 1 + \frac{\varepsilon}{2} + \cdots . \tag{2.2.4}$$

The shortage of this method for constructing an asymptotic expansion is that we don't have an explicit formula, like (2.2.1), which guarantees the iteration converges.

2.3 Singular perturbation

We now investigate the following equation which will give us very different results.

$$\varepsilon x^2 - x - 1 = 0. \tag{2.3.1}$$

Suppose that $\varepsilon = 0$, equation (2.3.1) becomes

$$-x - 1 = 0. \tag{2.3.2}$$

Therefore we see that one root of (2.3.1) disappears as ε becomes 0. This is very different from (2.1.1). It is not difficult to get the roots of (2.3.1) which read

$$x^\varepsilon = \frac{1 \pm \sqrt{1+4\varepsilon}}{2\varepsilon}.$$

There hold, as $\varepsilon \to 0$, that

$$x_-^\varepsilon = \frac{1 - \sqrt{1+4\varepsilon}}{2\varepsilon} \to -1,$$

and by a Taylor expansion,

$$\begin{aligned} x_+^\varepsilon &= \frac{1 + \sqrt{1+4\varepsilon}}{2\varepsilon} \\ &= \frac{1}{2\varepsilon}\left(1 + 1 + 2\varepsilon - 2\varepsilon^2 + \cdots\right) \\ &= \frac{1}{\varepsilon} + 1 - 2\varepsilon + \cdots . \end{aligned} \tag{2.3.3}$$

Therefore we see that one root i.e. x_-^ε converges to that of the limit equation (2.3.2), another blows up at the rate $\frac{1}{\varepsilon}$. Thus we can not expect that an asymptotic expansion like we did for a regular problem is valid too in this case. How to find a suitable scale for such a problem? We shall make use of the rescaling technique.

2.3.1 Rescaling

Suppose that we don't know *a priori* the correct scale for constructing an asymptotic expansion, then the rescaling technique helps us to find it. This subsection is concerned with it, and we take (2.3.1) as an example.

Let δ be a real function in ε, and let

$$x = \delta X,$$

where $\delta = \delta(\varepsilon)$ and $X = O(1)$. A rescaling technique is to determine the function δ, consequently, a new variable X is found.

Rewriting (2.3.1) in X, we have

$$\varepsilon\delta^2 X^2 - \delta X - 1 = 0. \tag{2.3.4}$$

By comparing the coefficients of (2.3.4), namely,

$$\varepsilon\delta^2, \ \delta, \ 1,$$

we divide the rescaling argument into five cases.

Case i) $\delta << 1$. Then (2.3.4) can be written as

$$1 = \underbrace{\varepsilon\delta^2 X^2}_{o(1)} - \underbrace{\delta X}_{o(1)} = o(1), \tag{2.3.5}$$

which can not be true since the left hand side of (2.3.5) while the right hand side is a very small quantity.

Case ii) $\delta = 1$, which means there is no any change to (2.3.1). (2.3.4) becomes

$$\underbrace{\varepsilon X^2}_{o(1)} - X - 1 = 0, \tag{2.3.6}$$

thus it is impossible that $X = 1$ and we can construct a regular asymptotic expansion but cannot recover the lost root.

Case iii) $1 << \delta << \frac{1}{\varepsilon}$ which implies that $\delta\varepsilon << 1$. Dividing equation (2.3.4) by δ we obtain

$$\underbrace{\varepsilon\delta X^2}_{o(1)} - X - \underbrace{\frac{1}{\delta}}_{o(1)} = 0, \tag{2.3.7}$$

and $X = o(1)$. This is impossible.

Case iv) $\delta = \frac{1}{\varepsilon}$, namely $\delta\varepsilon = 1$, also $\delta >> 1$ since we assume that $\varepsilon << 1$. Consequently, we infer from (2.3.4) that

$$X^2 - X - \underbrace{\frac{1}{\delta}}_{o(1)} = 0, \tag{2.3.8}$$

Thus $X \sim 0$ or 1. This gives us the correct scale.

Case v) $\delta >> \frac{1}{\varepsilon}$, thus $\delta\varepsilon >> 1$. Multiplying (2.3.4) by $\varepsilon^{-1}\delta^{-2}$ yields

$$X^2 - \underbrace{(\varepsilon\delta)^{-1}X}_{o(1)} - \underbrace{\frac{1}{\varepsilon\delta^2}}_{o(1)} = 0, \tag{2.3.9}$$

and $X = o(1)$. So this is not a suitable scale either.

In conclusion, the suitable scale is $\delta = \frac{1}{\varepsilon}$, thus

$$x = \frac{X}{\varepsilon}.$$

(2.3.4) is turn out to be

$$X^2 - X - \varepsilon = 0. \tag{2.3.10}$$

A singular problem is reduced to a regular one.

We now turn back to equation (2.3.1). Rescaling suggests us to use the following ansatz:

$$x^\varepsilon = \varepsilon^{-1}x_{-1} + x_0 + \varepsilon x_1 + \cdots . \tag{2.3.11}$$

Inserting into (2.3.1) and comparing the coefficients of ε^1 $(i = -1, 0, 1, \cdots)$ on both sides of (2.3.1), we obtain

$$\varepsilon^{-1}: \quad x_{-1}^2 - x_{-1} = 0, \tag{2.3.12}$$

$$\varepsilon^0: \quad 2x_0x_{-1} - x_0 - 1 = 0, \tag{2.3.13}$$

$$\varepsilon^1: \quad x_0^2 + 2x_{-1}x_1 - x_1 = 0. \tag{2.3.14}$$

The roots of (2.3.12) are $x_{-1} = 1$ and $x_{-1} = 0$. The second root does not yield a singular asymptotic expansion, thus it can be excluded easily. We consider now $x_{-1} = 1$. From (2.3.13) and (2.3.14) one solves

$$x_0 = 1, \ x_1 = -1.$$

Therefore, we construct approximations X_i^ε $i = 0, 1, 2$, of the root x_+^ε by

$$X_2^\varepsilon = \frac{1}{\varepsilon} + 1 - \varepsilon, \tag{2.3.15}$$

$$X_1^\varepsilon = \frac{1}{\varepsilon} + 1, \tag{2.3.16}$$

and

$$X_0^\varepsilon = \frac{1}{\varepsilon}. \tag{2.3.17}$$

How precisely do they satisfy equation (2.3.1)? Computations yield

$$\varepsilon(X_0^\varepsilon)^2 - X_0^\varepsilon - 1 = -1,$$

correspondingly, $x_+^\varepsilon - X_0^\varepsilon = 1 - 2\varepsilon + o(\varepsilon) \not\to 0$. This means that an expansion with one term is not a good approximation.

$$\varepsilon(X_1^\varepsilon)^2 - X_1^\varepsilon - 1 = \varepsilon,$$

and

$$\varepsilon(X_2^\varepsilon)^2 - X_2^\varepsilon - 1 = O(\varepsilon^2),$$

meanwhile, $x_+^\varepsilon - X_1^\varepsilon = O(\varepsilon)$, and $x_+^\varepsilon - X_2^\varepsilon = O(\varepsilon^2)$.

Thus X_1^ε, X_2^ε are a good approximation to x_+^ε. Moreover, we can conclude that the more terms we take, the more precise the approximation is. We also figure out the profile approximately of x_+^ε, in other word, we know now how the root disappears by blowing up as ε goes to zero.

2.4 Non-ingeral powers

Any of the asymptotic expansions in Sections 1.1 and 1.2 are a series with integral powers. However this is in general not true. Here we give an example. Consider

$$(1 - \varepsilon)x^2 - 2x + 1 = 0. \tag{2.4.1}$$

Define, as in previous sections, an ansatz as follows

$$x^\varepsilon = x_0 + \varepsilon x_1 + \varepsilon x_2 + \cdots, \tag{2.4.2}$$

inserting (2.4.2) into (2.4.1) and balancing both sides yield

$$\varepsilon^0: \quad x_0^2 - 2x_0 + 1 = 0, \tag{2.4.3}$$
$$\varepsilon^0: \quad 2x_0x_1 - 2x_1 - x_0^2 = 0, \tag{2.4.4}$$
$$\varepsilon^1: \quad 2x_0x_2 - 2x_2 + x_1^2 - 2x_0x_1 = 0. \tag{2.4.5}$$

From (2.4.3) one gets $x_0 = 1$, whence (2.4.4) implies $2x_1 - 2x_1 - 1^2 = 0$, that is $1 = 0$, a contradiction. So (2.4.2) is not well-defined.

Now we define an ansatz as

$$x^\varepsilon = x_0 + \varepsilon^\alpha x_1 + \varepsilon^\beta x_2 + \cdots , \tag{2.4.6}$$

where $0 < \alpha < \beta < \cdots$ are constants to be determined. inserting this ansatz into (2.4.1) and balancing both sides, we obtain there must hold that

$$\alpha = \frac{1}{2}, \ \beta = 1, \cdots .$$

and the correct ansatz is

$$x^\varepsilon = x_0 + \varepsilon^{\frac{1}{2}} x_1 + \varepsilon x_2 + \varepsilon^{\frac{3}{2}} x_3 + \cdots . \tag{2.4.7}$$

The remaining part of construction is similar to previous ones. We have $x^\varepsilon = 1 \pm \varepsilon^{\frac{1}{2}} + \varepsilon \pm \varepsilon^{\frac{3}{2}} + \cdots$.

Chapter 3

Ordinary differential equations

We start this chapter with some definitions. Consider

$$L_0[u] + \varepsilon L_1[u] = f_0 + \varepsilon f_1, \text{ in } D. \tag{3.0.1}$$

and the associated equation corresponding to the case that $\varepsilon = 0$

$$L_0[u] = f_0. \text{ in } D. \tag{3.0.2}$$

Here, $L_0,\ L_1$ are known operators, either ordinary or partial; $f_0,\ f_1$ are given functions.

The terms $\varepsilon L_1[u]$ and εf_1 are called *perturbations*.

E_ε (E_0 respectively) denotes the problem consisting equation (3.0.1) ((3.0.2) respectively) and suitable boundary/initial conditions. The solution to problem E_ε (or E_0) is denoted by u^ε (or u_0).

Definition. *Problem E_ε is regular if*

$$\|u^\varepsilon - u^0\|_D \to 0,$$

as $\varepsilon \to 0$. Otherwise, Problem E_ε is referred to a singular one.

Here $\|\cdot\|_D$ is a suitable norm over domain D. Note that a problem is regular or singular depends on the choice of the norm, which can be clarified by the following problem.

Example. Let $D = (0, 1)$. A real function $\varphi : D \to \mathbb{R}$ is a solution to

$$\varepsilon \frac{d^2\varphi}{dx^2} + \frac{d\varphi}{dx} = 0, \text{ in } D, \tag{3.0.3}$$

$$\varphi|_{x=0} = 0,\ \varphi|_{x=1} = 1. \tag{3.0.4}$$

The solution φ is

$$\varphi = \varphi(x;\varepsilon) = \frac{1 - \mathrm{e}^{-\frac{x}{\varepsilon}}}{1 - \mathrm{e}^{-\frac{1}{\varepsilon}}},$$

which is monotone increasing.

Now we define two norms. i) $\|\varphi\|_D = \max_{\bar{D}} |\varphi|$, then problem (3.0.3) – (3.0.4) is singular since $\|\varphi - \varphi^0\|_D = 1$ where $\varphi_0 = 0$ or $\varphi_0 = 1$.

ii) Define

$$\|\varphi\|_D = \left(\int_D |\varphi|^2 \right)^{\frac{1}{2}},$$

choose $\varphi^0 = 1$ which satisfies $\frac{d\varphi}{dx} = 0$, then we can prove easily that $\|\varphi - \varphi^0\|_D \to 0$ as $\varepsilon \to 0$, whence problem (3.0.3) – (3.0.4) is regular.

In what follows, we restrict ourself to consider only the *maximum* norm, and in the remaining part of this chapter the domain D is defined by $D = (0, 1)$.

3.1 First order ODEs

3.1.1 Regular

In this subsection we first consider a regular problem of ordinary differential equations of first order. Consider

$$\frac{du}{dx} + u = \varepsilon x, \text{ in } D. \tag{3.1.1}$$

$$u(0) = 1, \tag{3.1.2}$$

and its associated problem

$$\frac{du}{dx} + u = 0, \text{ in } D. \tag{3.1.3}$$

$$u(0) = 1, \tag{3.1.4}$$

We can solve easily these problems whose solution reads

$$u^\varepsilon(x) = (1 + \varepsilon)\mathrm{e}^{-x} + \varepsilon(x - 1), \; u^0(x) = \mathrm{e}^{-x}. \tag{3.1.5}$$

Calculating the difference of these two solutions yields

$$\|u^\varepsilon - u^0\|_D = \varepsilon \max_{\bar{D}} |\mathrm{e}^{-x} + x - 1| \to 0$$

as $\varepsilon \to 0$. Therefore, problem (3.1.1) – (3.1.2) is regular, and the term εx is a regular perturbation.

But, in general, one can not expect that there are the explicit simple formulas, like (3.1.5), of exact solutions. Thus we next deal with this problem in a general way, and employ the method of asymptotic expansions. To this end, we define an ansatz

$$u^\varepsilon(x) = u_0(x) + \varepsilon u_1(x) + \varepsilon^2 u_2(x) + \cdots, \tag{3.1.6}$$

insert it into equation (3.1.1) to get

$$\begin{aligned}
\varepsilon^0: &\quad u_0' + u_0 = 0,\ u_0(0) = 1, &(3.1.7)\\
\varepsilon^0: &\quad u_1' + u_1 = x,\ u_1(0) = 0, &(3.1.8)\\
\varepsilon^1: &\quad u_2' + u_2 = 0,\ u_2(0) = 0. &(3.1.9)
\end{aligned}$$

The condition $u_0(0) = 1$ follows from (3.1.2) and ansatz (3.1.6). In fact, there holds

$$1 = u^\varepsilon(0) = u_0(0) + \varepsilon u_1(0) + \varepsilon^2 u_2(0) + \cdots \to u_0(0), \tag{3.1.10}$$

thus, $u_0(0) = 1$. With this in hand, we use again (3.1.10) to derive the condition $u_1(0) = 0$. We obtain

$$\begin{aligned}
0 &= \varepsilon u_1(0) + \varepsilon^2 u_2(0) + \cdots, \text{ whence}\\
0 &= u_1(0) + \varepsilon u_2(0) + \cdots &(3.1.11)
\end{aligned}$$

Letting $\varepsilon \to 0$ we get $u_1(0) = 0$. In a similar manner, we derive the condition $u_2(0) = 0$.

Solving problems (3.1.7), (3.1.8) and (3.1.9) we have

$$u_0(x) = \mathrm{e}^{-x},\ u_1(x) = x - 1 + \mathrm{e}^{-x},\ u_2(x) = 0. \tag{3.1.12}$$

Thus the approximations can be constructed:

$$\begin{aligned}
U_0^\varepsilon(x) &= \mathrm{e}^{-x}, &(3.1.13)\\
U_1^\varepsilon(x) &= U_2^\varepsilon(x) = (1+\varepsilon)\mathrm{e}^{-x} + \varepsilon(x-1). &(3.1.14)
\end{aligned}$$

Simple calculation shows that $U_0^\varepsilon(x)$ satisfies (3.1.1) with a small error

$$(U_1^\varepsilon(x))' + U_1^\varepsilon(x) - \varepsilon x = O(\varepsilon),$$

and condition (3.1.2) is satisfied exactly. Note that $U_1^\varepsilon(x) = U_2^\varepsilon(x)$ are equal to the exact solution (3.1.5), so they solve problem (3.1.1) – (3.1.2), and are a very good "approximation".

3.1.2 Singular

Now we are going to study singular perturbation and boundary layers. The perturbed problem is

$$\varepsilon\frac{du}{dx} + u = x, \text{ in } D. \tag{3.1.15}$$
$$u(0) = 1, \tag{3.1.16}$$

and the associated one is

$$u = x, \text{ in } D. \tag{3.1.17}$$
$$u(0) = 1, \tag{3.1.18}$$

The exact solutions of problem (3.1.15) – (3.1.16) is

$$u^\varepsilon(x) = (1+\varepsilon)\mathrm{e}^{-x} + x - \varepsilon, \tag{3.1.19}$$

Let $u^0(x) = x$. Computation yields

$$\|u^\varepsilon - u^0\|_D = \max_{\bar{D}} |(1+\varepsilon)\mathrm{e}^{-x} - \varepsilon| = 1,$$

for any positive ε. Therefore, by definition, problem (3.1.15) – (3.1.16) is singular, and the term $\varepsilon\frac{du}{dx}$ is a singular perturbation.

We next want to employ the method of asymptotic expansions to study this singular problem, for such a problem at least one boundary layer arises and a matched asymptotic expansion is suitable for it. We will construct outer and inner expansions which are valid in the so-called outer and inner regions, respectively. Then we derive matching conditions which are enable us to establish an asymptotic expansion which is valid uniformly in the whole domain. Thus we start with the construction of outer expansions.

3.1.2.1 Outer expansions

The ansatz for deriving outer expansion is just of the form of a regular expansion:

$$u^\varepsilon(x) = u_0(x) + \varepsilon u_1(x) + \varepsilon^2 u_2(x) + \cdots, \tag{3.1.20}$$

Similar to the approach for asymptotic expansion of the regular problem (3.1.1) – (3.1.2), we obtain

$$\varepsilon^0: \quad u_0(x) = x, \tag{3.1.21}$$
$$\varepsilon^1: \quad u_1(x) + u_0'(x) = 0, \tag{3.1.22}$$
$$\varepsilon^2: \quad u_2(x) + u_1'(x) = 0. \tag{3.1.23}$$

Solving the above problems yields

$$u_0(x) = x, \tag{3.1.24}$$
$$u_1(x) = -1, \tag{3.1.25}$$
$$u_2(x) = 0. \tag{3.1.26}$$

Then we get approximations:

$$O_2^\varepsilon(x) = x - \varepsilon. \tag{3.1.27}$$

Moreover, from (3.1.24) we obtain that $u_0(0) = 0$ which differs from the given condition $u_0(0) = 1$, thus there is a *boundary layer* appearing at $x = 0$.

3.1.2.2 Inner expansions

To construct inner expansions, we introduce a new variable, the so-called fast variable:

$$\xi = \frac{x}{\varepsilon},$$

On one hand, for a beginner to the theory of asymptotic analysis, it may be not easy to understand why we define ξ in this form, the power of ε is 1? To convince oneself, one may assume a more general form as $\xi = \frac{x}{\varepsilon^\alpha}$ with $\alpha \in \mathbb{R}$. Then repeating the procedure we will carry out later in next subsection, we prove that α must be equal to 1 in order to get an asymptotic expansion. On the other hand we already assume that a *boundary layer* (Definition!) occurs at $x = 0$. If the assumption is incorrect, the procedure will break down when you try to match the inner and outer expansions in the intermediate region. At this point one may assume that there exists a boundary layer near a point $x = x_0$. The following analysis is the same, except that the scale transformation in the boundary layer is $\xi = \frac{x - x_0}{\varepsilon^\delta}$. We shall carry out the analysis for determining δ in the next section.

An inner expansion is in terms of ξ, we assume that

$$u^\varepsilon(x) = U_0(\xi) + \varepsilon U_1(\xi) + \varepsilon^2 U_2(\xi) + \cdots. \tag{3.1.28}$$

It is easy to compute that for $i = 0, 1, 2, \cdots$,

$$\frac{dU_i(\xi)}{dx} = \frac{1}{\varepsilon}\frac{dU_i(\xi)}{d\xi}.$$

Invoking equation (3.1.1) we arrive at

$$\varepsilon^{-1}: \quad U_0' + U_0 = 0, \tag{3.1.29}$$
$$\varepsilon^0: \quad U_1' + U_1 = \xi, \tag{3.1.30}$$
$$\varepsilon^1: \quad U_2' + U_2 = 0. \tag{3.1.31}$$

From which we have

$$U_0(\xi) = C_0 e^{-\xi},\ U_1(\xi) = C_1 e^{-\xi} + \xi - 1,\ U_2(\xi) = C_2 e^{-\xi}. \tag{3.1.32}$$

Next step is to determine the constants C_i with $i = 0, 1, 2$. To this end, we use the condition at $x = 0$ which implies that $\xi = 0$ too to conclude that $U_0(0) = 1$, thus $C_0 = 1$. Similarly we have $C_1 = 1$, $C_2 = 0$. Therefore, an inner expansion can be obtained

$$I_2^\varepsilon(\xi) = (1 + \varepsilon)e^{-\xi} + \varepsilon(\xi - 1). \tag{3.1.33}$$

3.1.2.3 Matched asymptotic expansions

There are two main approaches to combine together the inner and outer expansions. The first one is to take the sum of the inner expansion (3.1.33) and the outer expansion (3.1.27), then subtract their common part which is valid in the intermediate region. To get a matched asymptotic expansion, it remains to find the common part. We start with

$$U_0(\xi) + \varepsilon U_1(\xi) + \varepsilon^2 U_2(\xi) = u_0(x) + \varepsilon u_1(x) + \varepsilon^2 u_2(x) + O(\varepsilon^3).$$

Following Fife [14], we rewrite $x = \varepsilon\xi$ and expand the right hand side in terms of ξ. There hold

$$\begin{aligned} U_0(\xi) + \varepsilon U_1(\xi) + \varepsilon^2 U_2(\xi) &= u_0(\varepsilon\xi) + \varepsilon u_1(\varepsilon\xi) + \varepsilon^2 u_2(\varepsilon\xi) + O(\varepsilon^3) \\ &= u_0(0) + u_0'(0)\varepsilon\xi + \frac{1}{2}u_0''(0)(\varepsilon\xi)^2 \\ &\quad + \varepsilon(u_1(0) + u_1'(0)\varepsilon\xi) + u_2(0) + O(\varepsilon^3) \\ &= u_0(0) + \varepsilon(u_0'(0)\xi + u_1(0)) \\ &\quad + \varepsilon^2\left(\frac{1}{2}u_0''(0)\xi^2 + u_1'(0)\xi) + u_2(0)\right) + O(\varepsilon^3). \end{aligned} \tag{3.1.34}$$

Therefore we obtain the following matching conditions

$$\begin{aligned} U_0(\xi) &= u_0(0) = 0, && (3.1.35) \\ U_1(\xi) &\sim u_0'(0)\xi + u_1(0) = \xi - 1, && (3.1.36) \\ U_2(\xi) &\sim \frac{1}{2}u_0''(0)\xi^2 + u_1'(0)\xi + u_2(0) = 0, && (3.1.37) \end{aligned}$$

for $\xi \to \infty$.

The common part is $U_0(\xi) + \varepsilon U_1(\xi) + \varepsilon^2 U_2(\xi) = \varepsilon(\xi - 1)$. The matched asymptotic expansion then is

$$\begin{aligned} U_2^\varepsilon(x) &= I_2^\varepsilon(\xi) + O_2^\varepsilon(x) - \text{Common part} \\ &= (1 + \varepsilon)e^{-\xi} + \varepsilon(\xi - 1) + (x - \varepsilon) - \varepsilon(\xi - 1) \\ &= (1 + \varepsilon)e^{-\xi} + x - \varepsilon. \end{aligned} \tag{3.1.38}$$

So $U_2^\varepsilon(x)$ is just the exact solution to problem (3.1.15) – (3.1.16).

The second method for constructing a matched asymptotic expansion from inner and outer expansions is to make use of a suitable cut-off function to form a linear combination of inner and outer expansions.

We define a function $\chi = \chi(\xi) : \mathbb{R} \to \mathbb{R}^+$ which is smooth such that

$$\chi(\xi) = \begin{cases} 1 & \text{if } |\xi| \le 1, \\ 0 & \text{if } |\xi| \ge 2, \end{cases} \tag{3.1.39}$$

and $0 \le \chi(\xi) \le 1$ if $\xi \in [1, 2]$. And let

$$\chi_\varepsilon(x) = \chi(\varepsilon^{-\gamma} x), \tag{3.1.40}$$

which is easily seen that

$$\text{supp}(\chi_\varepsilon) \subset [0, 2\varepsilon^\gamma], \quad \text{supp}(\chi'_\varepsilon),\ \text{supp}(\chi''_\varepsilon) \subset [\varepsilon^\gamma, 2\varepsilon^\gamma].$$

Here $\gamma \in (0, 1)$ is a fixed number.

Now we are able to define an approximation by

$$U_2^\varepsilon(x) = (1 - \chi_\varepsilon(x)) O_2^\varepsilon(\xi) + \chi_\varepsilon(x) I_2^\varepsilon(x). \tag{3.1.41}$$

By this method, we don't need to find the common part and the argument is simpler, however, the price we should pay is that $U_2^\varepsilon(x)$ does not satisfy equation (3.1.15) precisely any more. Instead, an error occurs:

$$\begin{aligned} \varepsilon \frac{dU_2^\varepsilon(x)}{dx} + U_2^\varepsilon(x) - x &= \chi'_\varepsilon(x))(I_2^\varepsilon(\xi) - O_2^\varepsilon(x))\varepsilon^{1-\gamma} \\ &= O(\varepsilon^{1-\gamma}). \end{aligned} \tag{3.1.42}$$

3.2 Second order ODEs and boundary layers

The previous examples have given us some ideas about the method of asymptotic expansions. However, they can not outline all the features of this method since they are really too simple. In this section, we are going to study a more complex problem which possesses all aspects of asymptotic expansions. Consider the problem for second order ordinary differential equation:

$$\varepsilon \frac{d^2 u}{dx^2} + (1 + \varepsilon) \frac{du}{dx} + u = 0, \text{ in } D. \tag{3.2.1}$$

$$u(0) = 0,\ u(1) = 1. \tag{3.2.2}$$

First of all, we explain that problem (3.2.1) – (3.2.2) is not regular. We set

$$u^{\varepsilon}(x) = u_0(x) + \varepsilon u_1(x) + \varepsilon^2 u_2(x) + \cdots . \tag{3.2.3}$$

Inserting into equation (3.2.1) and comparing the coefficients of ε^i in both sides, one has

$$\frac{du_0}{dx} + u_0 = 0, \tag{3.2.4}$$

$$u_0(0) = 0, \ u_0(1) = 1. \tag{3.2.5}$$

Then we further get

$$\frac{du_1}{dx} + u_1 + \frac{d^2u_0}{dx^2} + \frac{du_0}{dx} = 0, \tag{3.2.6}$$

$$u_1(0) = 0. \tag{3.2.7}$$

$$\frac{du_2}{dx} + u_2 + \frac{d^2u_1}{dx^2} + \frac{du_1}{dx} = 0, \tag{3.2.8}$$

$$u_2(0) = 0. \tag{3.2.9}$$

Since equation (3.2.4) is of first order, only one of conditions (3.2.5) can be satisfied. The solution to (3.2.4) is

$$u_0(x) = C_0 \mathrm{e}^{-x}.$$

We shall see that even we require $u_0(x)$ meets only one condition in (3.2.5), there still is no asymptotic expansion of the form (3.2.3). There are two cases.

Case i) Suppose that the condition at $x = 0$ is satisfied (we don't care at this moment another condition), then $C_0 = 0$, hence $u_0(x) = 0$. Solving problems (3.2.6) – (3.2.7) and (3.2.8) – (3.2.9) one has

$$u_1(x) = u_2(x) = 0.$$

Thus no asymptotic expansion can be found.

Case ii) Assume that the condition at $x = 1$ is satisfied, then $C_0 = e$, and $u_0(x) = \mathrm{e}^{1-x}$. Consequently, equations (3.2.6) and (3.2.8) become

$$\frac{du_1}{dx} + u_1 = 0, \ \frac{du_2}{dx} + u_2 = 0. \tag{3.2.10}$$

Hence,

$$u_1(x) = C_1 \mathrm{e}^{-x}, \ u_2(x) = C_2 \mathrm{e}^{-x}.$$

But from condition $u_1(1) = u_2(1) = 0$ which can be derived from ansatz (3.2.3), it follows that $C_1 = C_2 = 0$, whence

$$u_1(x) = u_2(x) = 0,$$

and the "possible" asymptotic expansion is $U_i^\varepsilon(x) = \mathrm{e}^{1-x}$ for any $i = 0, 1, 2, \cdots$. Note that $|U_i^\varepsilon(0) - u^\varepsilon(0)| = e \not\to 0$.

Therefore, (3.2.1) – (3.2.2) is singular.

3.2.1 Outer expansions

This subsection is concerned with outer expansions. We begin with the definition of an ansatz

$$u^\varepsilon(x) = u_0(\xi) + \varepsilon u_1(\xi) + \varepsilon^2 u_2(\xi) + \cdots . \tag{3.2.11}$$

For simplicity of notations, we will denote the derivative of a one-variable function by $'$, namely, $f'(x) = \frac{df}{dx}$, $f'(\xi) = \frac{df}{d\xi}$, etc. Inserting (3.2.11) into equation (3.2.1) and equating the coefficients of ε^i of both sides yield

$$\varepsilon^0 : \quad u_0' + u_0 = 0, \ u_0(1) = 1, \tag{3.2.12}$$
$$\varepsilon^1 : \quad u_1' + u_1 + u_0'' + u_0' = 0, \ u_1(1) = 0, \tag{3.2.13}$$
$$\varepsilon^2 : \quad u_2' + u_2 + u_1'' + u_1' = 0, \ u_2(1) = 0. \tag{3.2.14}$$

The solutions to (3.2.12), (3.2.13) and (3.2.14) are respectively,

$$u_0(x) = \mathrm{e}^{1-x}, \ u_1(x) = u_2(x) = 0.$$

Thus outer approximations (up to $i+1$-terms) can be constructed as follows

$$O_i^\varepsilon(x) = \mathrm{e}^{1-x}, \tag{3.2.15}$$

here $i = 0, 1, 2$.

3.2.2 Inner expansions

The construction of an inner expansion is more complicated than that for an outer expansion. Firstly a correct scale should be decided, by using the rescaling technique.

3.2.2.1 Rescaling

Introduce a new variable

$$\xi = \frac{x}{\delta}, \tag{3.2.16}$$

where $\delta = \delta(\varepsilon)$. In what follows, we shall prove that in order to get an inner expansion which matches well the outer expansion, δ should be very small, so ξ is called *fast* variable. The first goal of this subsection is to find a correct formula of δ. Rewriting equation (3.1.1) in terms of ξ gives

$$\frac{\varepsilon}{\delta^2}\frac{d^2U}{d\xi^2} + \frac{1+\varepsilon}{\delta}\frac{dU}{d\xi} + U = 0. \tag{3.2.17}$$

To investigate the relation of the coefficients of (3.2.17), i.e.

$$\frac{\varepsilon}{\delta^2}, \quad \frac{1+\varepsilon}{\delta}, \quad 1,$$

there are five cases which should be taken into account. Note that

$$\frac{1+\varepsilon}{\delta} \sim \frac{1}{\delta}$$

since $\varepsilon << 1$.

Case i) $\delta >> 1$. Recalling that $\varepsilon << 1$, one has

$$\frac{\varepsilon}{\delta^2} << \frac{1}{\delta^2} << \frac{\varepsilon}{\delta} << 1.$$

Thus equation (3.2.17) becomes

$$\underbrace{\frac{\varepsilon}{\delta^2}\frac{d^2U}{d\xi^2}}_{o(1)} + \underbrace{\frac{1+\varepsilon}{\delta}}_{o(1)}\frac{dU}{d\xi} + \underbrace{U}_{O(1)} = 0, \tag{3.2.18}$$

so $U = o(1)$. This large δ is not a correct scale.

Case ii) $\delta \sim 1$. This implies $\xi \sim x$, and (3.2.16) changes nothing. In the present case, only a regular expansion can be expected, so that is not what we want.

Case iii) $\delta << 1$ and $\frac{\varepsilon}{\delta^2} >> \frac{1}{\delta}$. From which it follows that $\varepsilon >> \delta$. Dividing equation (3.2.17) by $\frac{\varepsilon}{\delta^2}$ yields

$$\frac{d^2U}{d\xi^2} + \underbrace{\frac{1+\varepsilon}{\varepsilon}\delta\frac{dU}{d\xi}}_{o(1)} + \underbrace{\frac{\delta^2}{\varepsilon}U}_{o(1)} = 0, \tag{3.2.19}$$

from which it follows that

$$\frac{d^2U}{d\xi^2} = o(1).$$

Thus this scale would not lead to an inner expansion either.

Case iv) $\delta << 1$ and $\frac{\varepsilon}{\delta^2} \sim \frac{1}{\delta}$. We have $\varepsilon \sim \delta$. Multiplying equation (3.2.17) by δ to get

$$\underbrace{\frac{\varepsilon}{\delta}}_{\sim 1} \frac{d^2U}{d\xi^2} + \frac{dU}{d\xi} + \underbrace{\varepsilon \frac{dU}{d\xi} + \delta U}_{o(1)} = 0, \tag{3.2.20}$$

this will lead to a correct scale. We just choose the simple relation $\delta = \varepsilon$, and (3.2.16) turns out to be $\xi = \frac{x}{\varepsilon}$.

Case v) $\delta << 1$ and $\frac{\varepsilon}{\delta^2} << \frac{1}{\delta}$, which implies $\varepsilon << \delta$. Multiplying equation (3.2.17) by δ we obtain

$$\underbrace{\frac{\varepsilon}{\delta}}_{o(1)} \frac{d^2U}{d\xi^2} + \underbrace{(1+\varepsilon)}_{\sim 1} \frac{dU}{d\xi} + \underbrace{\delta U}_{o(1)} = 0, \tag{3.2.21}$$

which implies that $\frac{dU}{d\xi} = o(1)$. This case is not what we want.

Now we turn back to construction of inner expansions. From rescaling we can define

$$\xi = \frac{x}{\varepsilon}, \tag{3.2.22}$$

and an ansatz as follows

$$u^\varepsilon(x) = U_0(\xi) + \varepsilon U_1(\xi) + \varepsilon^2 U_2(\xi) + \cdots. \tag{3.2.23}$$

It is easy to compute that for $i = 0, 1, 2, \cdots$,

$$\frac{dU_i(\xi)}{dx} = \frac{1}{\varepsilon}\frac{dU_i(\xi)}{d\xi}, \quad \frac{d^2U_i(\xi)}{dx^2} = \frac{1}{\varepsilon^2}\frac{d^2U_i(\xi)}{d\xi^2}.$$

Then invoking equation (3.1.1) we arrive at

$$\varepsilon^{-1}: \quad U_0'' + U_0' = 0, \tag{3.2.24}$$
$$\varepsilon^{0}: \quad U_1'' + U_1' + U_0' + U_0 = 0, \tag{3.2.25}$$
$$\varepsilon^{1}: \quad U_2'' + U_2' + U_1' + U_1 = 0. \tag{3.2.26}$$

From which we obtain the general solutions

$$U_0(\xi) = C_{01}\mathrm{e}^{-\xi} + C_{02}, \tag{3.2.27}$$
$$U_1(\xi) = C_{11}\mathrm{e}^{-\xi} + C_{12} - C_{02}\xi, \tag{3.2.28}$$
$$U_2(\xi) = C_{21}\mathrm{e}^{-\xi} + C_{22} + \frac{C_{02}}{2}\xi^2 - C_{12}\xi. \tag{3.2.29}$$

Here C_{ij} with $i = 0, 1, 2; j = 1, 2$ are constants. Next step is to determine these constants. To this end, we use the condition at $x = 0$ which implies that $\xi = 0$ too, to conclude that $U_0(0) = 0$, $U_1(0) = 0$ and $U_2(0) = 0$, thus

$$C_{i1} = -C_{i2} =: A_i, \tag{3.2.30}$$

for $i = 0, 1, 2$. Hence, (3.2.27) – (3.2.29) are reduced to

$$U_0(\xi) = A_0(\mathrm{e}^{-\xi} - 1), \tag{3.2.31}$$
$$U_1(\xi) = A_1(\mathrm{e}^{-\xi} - 1) + A_0\xi, \tag{3.2.32}$$
$$U_2(\xi) = A_2(\mathrm{e}^{-\xi} - 1) - \frac{A_0}{2}\xi^2 + A_1\xi. \tag{3.2.33}$$

Therefore we still need to find constants A_i. For this purpose we need matching conditions. An inner region is near the boundary layer and is usually very thin, is of $O(\varepsilon)$ in the present problem, while an outer region is far from the boundary layer. Thus there is an intermediate (or, matching, overlapping) region between them, the scale of the distance of this region to the boundary layer is of $O(\varepsilon^\alpha)$ where $\alpha \in (0, 1)$. Inner and outer expansions are valid (by this word we mean that an expansion satisfies well the associated equation) respectively, over inner and outer regions. Roughly speaking, matching conditions are the conditions which are given over the intermediate region so that the outer and inner expansions coincide there. The task of the next subsection is to find such conditions.

3.2.3 Matching conditions

Now we expect reasonably that the inner expansions coincide with the outer ones in the intermediate region, and write

$$U_0(\xi) + \varepsilon U_1(\xi) + \varepsilon^2 U_2(\xi) = u_0(x) + \varepsilon u_1(x) + \varepsilon^2 u_2(x) + O(\varepsilon^3).$$

To derive the matching conditions, we shall employ two main methods.

3.2.3.1 Matching by expansions

(Relation with the intermediate variable method?) Following Fife [14], we rewrite $x = \varepsilon\xi$ and expand the right hand side in terms of ξ. We then obtain the matching conditions

$$\begin{aligned} U_0(\xi) &\sim u_0(0) = e, & (3.2.34)\\ U_1(\xi) &\sim u_0'(0)\xi + u_1(0) = -e\xi, & (3.2.35)\\ U_2(\xi) &\sim \frac{1}{2}u_0''(0)\xi^2 + u_1'(0)\xi + u_2(0) = \frac{e}{2}\xi^2, & (3.2.36)\end{aligned}$$

for $\xi \to \infty$. From (3.2.31) it follows that

$$U_0(\xi) \to A_0,$$

as $\xi \to \infty$. Combination with (3.2.34) yields

$$A_0 = -e. \qquad (3.2.37)$$

Hence, (3.2.31) – (3.2.33) are now

$$\begin{aligned} U_0(\xi) &= -e(\mathrm{e}^{-\xi} - 1), & (3.2.38)\\ U_1(\xi) &= A_1(\mathrm{e}^{-\xi} - 1) - e\xi, & (3.2.39)\\ U_2(\xi) &= A_2(\mathrm{e}^{-\xi} - 1) + \frac{e}{2}\xi^2 + A_1\xi. & (3.2.40)\end{aligned}$$

So the leading term of inner expansion is obtained. Comparing (3.2.35) with (3.2.39) for large ξ we have

$$A_1 = 0. \qquad (3.2.41)$$

In a similar manner, from (3.2.36) with (3.2.40) one gets

$$A_2 = 0. \qquad (3.2.42)$$

Therefore, the first three term of the inner expansion are determined, which read

$$\begin{aligned} U_0(\xi) &= e(1 - e^{-\xi}), & (3.2.43)\\ U_1(\xi) &= -e\xi, & (3.2.44)\\ U_2(\xi) &= \frac{e}{2}\xi^2. & (3.2.45)\end{aligned}$$

Using these functions, we define approximations up to $i + 1$-terms ($i = 0, 1, 2$) as follows

$$\begin{aligned} I_0^\varepsilon(\xi) &= e(1 - e^{-\xi}), & (3.2.46)\\ I_1^\varepsilon(\xi) &= e(1 - e^{-\xi}) - \varepsilon e\xi, & (3.2.47)\\ I_2^\varepsilon(\xi) &= e(1 - e^{-\xi}) - \varepsilon e\xi + \varepsilon^2\frac{e}{2}\xi^2. & (3.2.48)\end{aligned}$$

3.2.3.2 Van Dyke's rule for matching

Matching with an intermediate variable can be tiresome. The following Van Dyke's rule [33] for matching usually works and is more convenient.

For a function f, we have corresponding inner and outer expansions which are denoted respectively, $f = \sum_n \varepsilon^n f_n(x)$ and $f = \sum_n \varepsilon^n g_n(\xi)$. We define

Definition. *Let P, Q be non-negative integers.*

$$\begin{aligned} E_P f &= \text{outer limit}(x \text{ fixed } \varepsilon \downarrow 0) \text{ retaining } P+1 \text{ terms of an outer expansion} \\ &= \sum_{n=0}^{P} \varepsilon^n f_n(x), \end{aligned} \tag{3.2.49}$$

and

$$\begin{aligned} H_Q f &= \text{inner limit}(\xi \text{ fixed } \varepsilon \downarrow 0) \text{ retaining } Q+1 \text{ terms of an inner expansion} \\ &= \sum_{n=0}^{Q} \varepsilon^n g_n(\xi), \end{aligned} \tag{3.2.50}$$

Then the Van Dyke matching rule can be stated as

$$E_P H_Q f = H_Q E_P f.$$

Example. Let $P = Q = 0$. For our problem in this section, we define $f = u^\varepsilon$, and $H_0 g := A_0(e^{-\xi} - 1)$, $E_0 f := e^{1-x}$. Then

$$\begin{aligned} E_0 H_0 g &= E_0\{A_0(e^{-\xi} - 1)\} \\ &= E_0\{A_0(e^{-x/\varepsilon} - 1)\} \\ &= -A_0. \end{aligned} \tag{3.2.51}$$

and

$$\begin{aligned} H_0 E_0 f &= H_0\{e^{1-x}\} \\ &= H_0\{e^{1-\varepsilon\xi}\} \\ &= e. \end{aligned} \tag{3.2.52}$$

By the Van Dyke rule, (3.2.51) must coincide with (3.2.52), and we obtain

$$A_0 = -e,$$

which is (3.2.37). We can also derive the matching conditions of higher order.

3.2.4 Matched asymptotic expansions

In this subsection we shall make use of the inner and outer expansions to construct approximations. Also we do it in two ways.

i) The first method: Adding inner and outer expansions, then subtracting the common part, we obtain

$$\begin{aligned} U_0^\varepsilon(x) &= e^{1-x} + e(1-e^{-\xi}) - e = e(e^{-x} - e^{-\frac{x}{\varepsilon}}) \\ U_1^\varepsilon(x) &= e(e^{-x} - e^{-\frac{x}{\varepsilon}}) - ex - (-e\varepsilon\xi) = e(e^{-x} - e^{-\frac{x}{\varepsilon}}) \\ U_2^\varepsilon(x) &= e(e^{-x} - e^{-\frac{x}{\varepsilon}}) + \frac{1}{2}ex^2 - \frac{1}{2}e(\varepsilon\xi)^2 = e(e^{-x} - e^{-\frac{x}{\varepsilon}}). \quad (3.2.53) \end{aligned}$$

From which one asserts that

$$U_0^\varepsilon(x) = U_1^\varepsilon(x) = U_2^\varepsilon(x).$$

The more terms we take, but the accuracy does not increase! This is different from what we had for algebraic equations.

ii) The second method: Employing the cut-off function defined in previous subsection. Then we get

$$U_i^\varepsilon(x) = (1 - \chi_\varepsilon(x))O_i^\varepsilon(x) + \chi_\varepsilon(x)I_i^\varepsilon(\xi). \quad (3.2.54)$$

Here, $i = 0, 1, 2$.

3.3 Examples

The following examples will help us to understand the method of asymptotic expansions.

Example 1. In a given singular perturbation problem, more than one boundary layer can occur. This is exemplified by

$$\begin{aligned} \varepsilon\frac{d^2u}{dx^2} - u &= A, \text{ in } D. \quad (3.3.1) \\ u(0) = \alpha, \ u(1) &= \beta. \quad (3.3.2) \end{aligned}$$

Here $A \neq 0, \beta \neq 0$.

Example 2. A problem can be singular although ε does not multiply the highest order derivative of the equation. A simple example is the following:

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} - \varepsilon\frac{\partial u}{\partial y} &= 0, \text{ in } D = \{(x,y) \mid 0 < x < 1, \ 0 < y < y_0\}. \quad (3.3.3) \\ u(x,0;\varepsilon) &= f(x), \text{ for } 0 \leq x \leq 1, \quad (3.3.4) \\ u(0,y;\varepsilon) &= g_1(y), \ u(1,y;\varepsilon) = g_2(y), \text{ for } 0 \leq y \leq y_0. \quad (3.3.5) \end{aligned}$$

Here we take $y_0 > 0$, and choose u_0 satisfying $\frac{\partial^2 u_0}{\partial x^2} = 0$ as follows

$$u_0(x, y; \varepsilon) = g_1(y) + (g_2(y) - g_1(y))x.$$

However, in general, $u_0(x, 0; \varepsilon) \neq f(x)$, so that u_0 is not an approximation of u in $\bar{D}$.

This can be easily understood by noting that (3.3.3) is parabolic while it becomes elliptic if $\varepsilon = 0$.

Example 3. In certain perturbation problems, there exists a uniquely defined function u_0 in $\bar{D}$ satisfying the limit equation $L_0 u_0 = f_0$ and *all* the boundary conditions imposed on u, and yet u_0 is not an approximation of u:

$$(x + \varepsilon)^2 \frac{du}{dx} + \varepsilon = 0, \text{ for } 0 < x < A, \tag{3.3.6}$$

$$u(0; \varepsilon) = 1. \tag{3.3.7}$$

We have the exact solution to problem (3.3.6) – (3.3.7):

$$u = u(x; \varepsilon) = \frac{\varepsilon}{x + \varepsilon},$$

and

$$L_0 = x^2 \frac{d}{dx}.$$

The function $u_0 = 1$ satisfies $L_0 u_0 = 0$ and also the boundary conditions. But

$$\lim_{\varepsilon \to 0} u(x; \varepsilon) = \begin{cases} 0 & \text{if} \quad x \neq 0, \\ 1 & \text{if} \quad x = 0, \end{cases} \tag{3.3.8}$$

and

$$\max_{\bar{D}} |u - u_0| = \frac{A}{A + \varepsilon} \not\to 0$$

as $\varepsilon \to 0$.

Example 4. Some operators L_ε cannot be decomposed into an "unperturbed ε-independent part" and "a perturbation".

$$\frac{du}{dx} - \varepsilon \exp(-(u - 1)/\varepsilon) = 0, \text{ for } D = \{0 < x < A\}, \ A > 0, \tag{3.3.9}$$

$$u(0; \varepsilon) = 1 - \alpha, \ \alpha > 0. \tag{3.3.10}$$

Note that

$$\lim_{\varepsilon \to 0} \left\{ \max_{\bar{D}} \left(\varepsilon \exp(-(g - 1)/\varepsilon) \right) \right\} = 0$$

if and only if $g > 1$ for $x \in \bar{D}$. Thus we do not have the decomposition with $L_0 = \frac{d}{dx}$. Moreover, from the exact

$$u(x; \varepsilon) = 1 + \varepsilon \log(x + \exp(-\alpha/\varepsilon)),$$

we assert easily that none of the "usually" successful methods produces an approximation of u in $\bar{D}$.

Chapter 4

Partial differential equations

4.1 Regular problem

Let Ω be an open bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, here $n \in \mathbb{N}$. Consider

$$\Delta u + \varepsilon u = f_0 + \varepsilon f_1, \tag{4.1.1}$$

$$u|_{\partial\Omega} = 0. \tag{4.1.2}$$

We shall prove this is a regular problem.

4.2 Conservation laws and vanishing viscosity method

In this section we will study the inviscid limit of scalar conservation laws with viscosity.

$$u_t + (F(u))_x = \nu u_{xx}, \tag{4.2.1}$$

$$u|_{t=0} = u_0. \tag{4.2.2}$$

The associated inviscid problem is

$$u_t + (F(u))_x = 0, \tag{4.2.3}$$

$$u|_{t=0} = u_0. \tag{4.2.4}$$

It is a basic question that does the solution of (4.2.1) – (4.2.2) converge to that of (4.2.3) – (4.2.4)? This is the main problem for the method of vanishing viscosity.

In this section we are going to prove that the answer to this question is positive under some suitable assumptions. We shall make use of the method of matched asymptotic expansions and L^2-energy method.

4.2.1 Construction of approximate solutions

4.2.1.1 Outer and inner expansions

4.2.1.2 Matching conditions and approximations

4.2.2 Convergence

Chapter 5

An application to optimal control theory

5.1 Introduction

Optimal control for hyperbolic conservation laws requires a considerable analytical effort and computationally expansive in practice, is thus a difficult topic. Some methods have been developed in the last years to reduce the computational cost and to render this type of problems affordable. In particular, recently the authors of [11] have developed an alternating descent method that takes into account the possible shock discontinuities, for the optimal control of the inviscid Burgers equation in one space dimension. Further in [12] the vanishing viscosity method is employed to study this alternating descent method for the Burgers equation, with the aid of the Hopf-Cole formula which can be found in [23, 36], for instance. Most results in [12] are formal.

In the present chapter we will revisit this alternating descent method in the context of one dimensional viscous scalar conservation laws with a general nonlinearity. The vanishing viscosity method and the method of matched asymptotic expansions will be applied to study this optimal control problem and justify rigorously the results.

To be more precise, we state the optimal problem as follows. For a given $T > 0$, we study the following inviscid problem

$$u_t + (F(u))_x = 0, \text{ in } \mathbb{R} \times (0,T); \tag{5.1.1}$$

$$u(x,0) = u^I(x), \ x \in \mathbb{R}. \tag{5.1.2}$$

Here, $F : \mathbb{R} \to \mathbb{R}$, $u \mapsto F(u)$ is a smooth function, and f denotes its derivative in the following context. The case that $F(u) = \frac{u^2}{2}$ is studied in e.g. [11, 12].

Given a target $u^D \in L^2(\mathbb{R})$ we consider the cost functional to be minimized

$J : L^1(\mathbb{R}) \to \mathbb{R}$, defined by

$$J(u^I) = \int_{\mathbb{R}} \left| u(x,T) - u^D(x) \right|^2 dx,$$

where $u(x,t)$ is the unique entropy solution to problem (5.1.1) – (5.1.2).

We also introduce the set of admissible initial data $\mathcal{U}_{ad} \subset L^1(\mathbb{R})$, that we shall define later in order to guarantee the existence of the following optimization problem:

Find $u^{I,\min} \in \mathcal{U}_{ad}$ such that

$$J(u^{I,\min}) = \min_{u^I \in \mathcal{U}_{ad}} J(u^I).$$

This is one of the model optimization problems that is often addressed in the context of optimal aerodynamic design, the so-called inverse design problem, see e.g. [18].

The existence of minimizers has been proved in [11]. From a practical point of view it is however more important to be able to develop efficient algorithms for computing accurate approximations of discrete minimizers. The most efficient methods to approximate minimizers are the gradient methods.

But for large complex systems, as Euler equations in higher dimensions, the existing most efficient numerical schemes (upwind, Godunov, etc.) are not differentiable. In this case, the gradient of the functional is not well defined and there is not a natural and systematic way to compute its variations. Due to this difficulty, it would be natural to explore the possible use of non-smooth optimization techniques. The following two approaches have been developed: The first one is based on automatic differentiation, and the second one is the so-called continuous method consisting of two steps as follows: One first linearizes the continuous system (5.1.1) to obtain a descent direction of the continuous functional J, then takes a numerical approximation of this descent direction with the discrete values provided by the numerical scheme. However this continuous method has to face another major drawback when solutions develop shock discontinuities, as it is the case in the context of the hyperbolic conservation laws like (5.1.1) we are considering here.

The formal differentiation of the continuous states equation (5.1.1) yields

$$\partial_t(\delta u) + \partial_x(f(u)\delta u) = 0, \text{ in } \mathbb{R} \times (0,T); \tag{5.1.3}$$

But this is only justified when the state u on which the variations are being computed, is smooth enough. In particular, it is not justified when the solutions are discontinuous since singular terms may appear on the linearization over the shock location. Accordingly in optimal control applications we also need to take into account the sensitivity for the shock location (which has been studied by many authors, see, e.g. [9, 19, 32]). Roughly speaking, the main conclusion of that analysis is that the classical linearized system for the variation of the solutions must be complemented with some new equations for the sensitivity of the shock position.

To overcome this difficulty, we naturally think of another way, namely, the vanishing viscosity method (as in [12], in which an optimal control problem for the Burgers equation is studied) and add an artificial viscosity term to smooth the state equation. Equations (5.1.1) with smoothed initial datum then turns out to be

$$u_t + (F(u))_x = \nu u_{xx}, \text{ in } \mathbb{R} \times (0,T), \tag{5.1.4}$$

$$u|_{t=0} = g^{\varepsilon}. \tag{5.1.5}$$

Note that the Cauchy problem (5.1.4) – (5.1.5) is of parabolic type, thus from the standard theory of parabolic equations (see, for instance, Ladyzenskaya et al [27]) we have that the solution $u_{\nu,\varepsilon}$ of this problem is smooth. So the linearized one of eq. (5.1.4) can be derived easily, which reads

$$(\delta u)_t + (f(u)\delta u)_x = \nu(\delta u)_{xx}, \text{ in } \mathbb{R} \times (0,T), \tag{5.1.6}$$

$$\delta u|_{t=0} = h^{\varepsilon}. \tag{5.1.7}$$

Here ν, ε are positive constants, δu denotes the variation of u. The initial data g^{ε}, h^{ε} will be chosen suitably in Section 3, so that the perturbations of initial data and shock position are taken into account, this renders us that we can select the alternating descent directions in the case of viscous conservation laws.

To solve the optimal control problem, we also need the following adjoint problem, which reads

$$-p_t - f(u)p_x = 0, \text{ in } \mathbb{R} \times (0,T); \tag{5.1.8}$$
$$p(x,T) = p^T(x), \ x \in \mathbb{R}, \tag{5.1.9}$$

here, $p^T(x) = u(x,T) - u^D(x)$. And we smooth equation (5.1.8) and initial data as follows

$$-p_t - f(u)p_x = \nu p_{xx}, \text{ in } \mathbb{R} \times (0,T); \tag{5.1.10}$$
$$p(x,T) = p^T_{\varepsilon}(x), \ x \in \mathbb{R}, \tag{5.1.11}$$

Since solutions $u = u(x,t;\nu,\varepsilon), \delta u = \delta u(x,t;\nu,\varepsilon)$ are smooth, shocks vanish, instead quasi-shock regions are formed. Natural questions are arising as follows: 1) How should ν, ε go to zero, more precisely, can ν, ε go to zero independently? Which one goes to zero faster, or the same? What happen if the two parameters $\nu, \varepsilon \to 0$? 2) What are the limit of equations (5.1.10), (5.1.6) and (5.1.4) respectively? 3) To solve the optimal control problem correctly, the states of system (5.1.3) should be understood as a pair $(\delta u, \delta\varphi)$, where $\delta\varphi$ is the variation of shock position. As $\nu, \varepsilon \to 0$, is there an equation for $\delta\varphi$ which determines the evolution of $\delta\varphi$ and complements equation (5.1.3)?

To answer these questions, we shall make use of the method of matched asymptotic expansions. Our main results are: the parameters ν, ε must satisfy

$$\varepsilon = \sigma\nu,$$

where σ is a given positive constant. This means that ν, ε must go to zero at the same order, but speeds may be different. We write $\frac{\varepsilon}{\nu} = \sigma$. Then we see that if $\sigma > 1$, ν goes to zero faster than ε, and vice versa.

We now fix ε which is assumed to be very small. As $\sigma \to \infty$, namely $\nu \to 0$, the equation of variation of shock position differs from the one derived by Bressan and Marson [9], etc., by a term which converges to zero as σ tends to infinity, however may be very large if σ is small enough. Thus we conclude that

1) The equation derived by Bressan and Marson is suitable for developing the numerical scheme when σ is sufficiently large. In this case, the perturbation of initial data plays a dominant role and the effect due to the artificial viscosity can be omitted;

2) However, if σ is small, then the effect of viscosity must be taken into account while the perturbation of initial data can be neglected, and a corrector should be added.

We shall prove that the solutions to problem (5.1.4) – (5.1.5), and problem (5.1.8) and (5.1.11) converge, respectively, to the entropy solution and the reversible solution of the corresponding inviscid problems, while the solution to problem (5.1.6) – (5.1.7) converges to the one that solves (5.1.3) in the sub-regions away from the shock, and is complemented by an equation which governs the evolution of the variation of shock position.

Furthermore, using the method of asymptotic expansions we also clarify some formal expansions used frequently in the theory of optimal control, that they are valid only away from shock and when some parameter is not too small. For example, for solution u_ν to problem (5.1.4) – (5.1.5) we normally expand it as

$$u_\nu = u + \nu\delta u_\nu + O(\nu^2), \tag{5.1.12}$$

where u is usually believed to be the entropy solution to problem (5.1.1) – (5.1.2), and δu is the variation of u_ν, the solution to (5.1.6) – (5.1.7). However (5.1.12) is not correct near shock provided that $\delta u_\nu(x, 0)$ is bounded, for instance. From this assumption, we have δu_ν is continuous and uniformly bounded by the theory of parabolic equations, moreover u_ν is continuous too, thus it follows from (5.1.12) that u should be continuous too, but u is normally discontinuous. Therefore, we should understand (5.1.12) as a multi-scale expansion, and assume that $u = u(x, x/\nu, t), \delta u = \delta u(x, x/\nu, t)$. We shall obtain such an expansion by the method of matched asymptotic expansions.

The new features to the method of asymptotic expansions in this chapeter are mainly as follows: Firstly, our expansions for $u_{\nu,\varepsilon}$ and $\delta u_{\nu,\varepsilon}$ are different from the standard ones due to the fact that equations (5.1.6) and (5.1.4) are not independent, (5.1.6) is the variation of (5.1.4), so when constructing asymptotic expansions we should take this fact into account and find some compatible conditions for the asymptotic expansions of $u_{\nu,\varepsilon}$ and $\delta u_{\nu,\varepsilon}$. Secondly, we derive the equation of variation of shock location from the outer expansions, but not from the inner expansions as usual, see, e.g. [14]. Our approach is based upon a key observation that outer expansions converge to their values at the shock and the quasi-shock region vanishes as $\nu \to 0$.

We need to introduce some

Notations: For any $t > 0$, we define $Q_t = \mathbb{R} \times (0, t)$. $C(t), C_a, \cdots$ denote, respectively, constants depending on $t, a, \cdots$, and C is a universal constant in Sub-section 4.2.

For a function $f = f(r, t)$ where $r = r_\nu(x, t; \nu)$: f_t denotes the partial derivative with respect to t while $(f)_t = f_t + f_r r_t$, and so on.

Let X be a Banach space endowed with a norm $\| \cdot \|_X$, and $f : [0, T] \to X$. For any fixed t the X-norm of f is denoted by $\|f(t)\|_X$, when $X = L^2(\mathbb{R})$, we write $\|f(t)\| = \|f(t)\|_X$, sometimes the argument t is omitted.

Landau symbols $O(1)$ and $o(1)$. A quantity $f(x, t; \nu) = o(1)$ means $\|f\|_{L^\infty(Q_t)} \to 0$ as $\nu \to 0$, and $g(t; \xi) = o(1)$ implies that $\|g\|_{L^\infty(0,t)} \to 0$ as $\xi \to \infty$. And $f(x, t; \nu) = O(1)$ means $\|f\|_{L^\infty(Q_t)} \leq C$ uniformly for $\nu \in (0, 1]$. We also use the standard notations: $BV(\mathbb{R})$ ($BV_{\text{loc}}(\mathbb{R})$), $Lip(\mathbb{R})$ ($Lip_{\text{loc}}(\mathbb{R})$), are the spaces of the functions of (locally) bounded variations, the (locally) Lipschitz continuous functions in $\mathbb{R}$, respectively.

The remaining parts of this chapter are as follows: In Section 2, we collect some preliminaries and explain furthermore the motivation of this chapter. In Section 3, employing the method of matched asymptotic expansions and taking into account the infinitesimal perturbations of the initial datum and the infinitesimal translations of the shock position, we shall construct, the inner and outer expansions, and obtain, by a suitable combination of the two expansions, the approximate solutions to problems (5.1.4)–(5.1.5), (5.1.6)–(5.1.7), and (5.1.10)–(5.1.11). Also the equations for the shock and its variation will be derived. In Section 4 we shall prove the approximate solutions satisfy the corresponding equations asymptotically, and converge, respectively, to those of the inviscid problems in a suitable sense. Finally we discuss the alternating descent method in the context of viscous conservation laws in Section 5, where the convergence results will be used.

5.2 Sensitivity analysis: the inviscid case

A solution to a hyperbolic equation may become singular after certain time, even the initial datum is smooth. Therefore in practical applications it is more interesting to consider optimal control problems in the case that shocks appear. We shall study the optimal control problem for the inviscid equation in the present of shocks, and we focus on the particular case that solutions have a single shock, however the analysis can be extended to consider more general one-dimensional systems of conservation laws with a finite number of non-interacting shocks, see [8].

To develop the alternating descent method for the optimal control problem in presence of shocks, we need to investigate the sensitivity of the states of the system, with respect to perturbations of the initial datum and infinitesimal translations of the shock position. we shall see that equations (5.1.3) and (5.1.8) will become much more complicated. This section is devoted to introducing some basic tools needed in the sensivity analysis.

5.2.1 Linearization of the inviscid equation

Let us firstly introduce the following hypothesis:

(H) Assume that u is a weak entropy solution to (5.1.1)–(5.1.2) with a discontinuity along a regular curve $\Sigma = \{(\varphi(t), t) \mid t \in [0, T)\}$ which is Lipschitz continuous outside Σ. The Rankine-Hugoniot condition is satisfied on Σ:

$$\varphi'(t)[u]_{\varphi(t)} = [F(u)]_{\varphi(t)} \,.$$

Hereafter, we denote the jump at $x = \varphi(t)$ of a piecewise smooth function $f = f(x, t)$ by $[f]_{\varphi(t)} = f(\varphi(t) + 0, t) - f(\varphi(t) - 0, t)$, for any fixed t.

Note that Σ divides $Q_T = \mathbb{R} \times (0, T)$ into two subdomains Q^- and Q^+, to the left and to the right of Σ respectively. see Figure 1.

As analyzed in [11], to deal correctly with optimal control and design problems, the state of the system should be viewed as that constituted by the pair (u, φ) consisting of the solution of (5.1.1) and the shock position φ. The pair

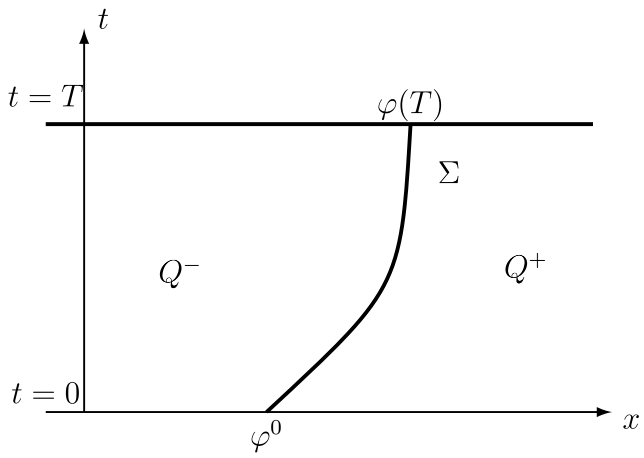

Figure 5.1: Subdomains Q^- and Q^+.

(u, φ) satisfies

$$
\begin{aligned}
u_t + (F(u))_x &= 0, \text{ in } Q^+ \cup Q^-, && (5.2.1)\\
\varphi'(t)[u]_{\varphi(t)} &= [F(u)]_{\varphi(t)}, \ t \in (0, T), && (5.2.2)\\
\varphi(0) &= \varphi^I, && (5.2.3)\\
u(x, 0) &= u^I(x), \ x \in \{x < \varphi^I\} \cup \{x > \varphi^I\}. && (5.2.4)
\end{aligned}
$$

We also need to analyze the sensitivity of (u, φ) with respect to perturbations of the initial datum, especially, with respect to δu^I and $\delta\varphi^I$ which are variations of the initial profile u^I and of the shock position φ^I, respectively. To be precise, we first need to introduce the functional framework based on the generalized tangent vectors introduced in [8].

Definition 2.1 Let $v : \mathbb{R} \to \mathbb{R}$ be a piecewise Lipschitz continuous function with a single discontinuity at $y \in \mathbb{R}$. We define Σ_v as the family of all continuous paths $\gamma : [0, \varepsilon_0] \to L^1(\mathbb{R})$ with

(1) $\gamma(0) = v$ and $\varepsilon_0 > 0$ possibly depending on γ.

(2) For any $\varepsilon \in [0, \varepsilon_0]$ the functions $u^\varepsilon = \gamma(\varepsilon)$ are piecewise Lipschitz with a single discontinuity at $x = y^\varepsilon$ depending continuously on ε and there exists a constant L independent of $\varepsilon \in [0, \varepsilon_0]$ such that

$$|v^\varepsilon(x) - v^\varepsilon(x')| \leq L|x - x'|,$$

whenever $y^\varepsilon \notin [x, x']$.

Furthermore, we define the set T_v of generalized tangent vectors of v as the space of $(\delta v, \delta y) \in L^1 \times R$ for which the path $\gamma_{(\delta v,\delta y)}$ given by

$$\gamma_{(\delta v,\delta y)}(\varepsilon) = \begin{cases} v + \varepsilon\delta v + [v]_y \chi_{[y+\varepsilon\delta y, y]} & \text{if} \quad \delta y < 0, \\ v + \varepsilon\delta v - [v]_y \chi_{[y, y+\varepsilon\delta y]} & \text{if} \quad \delta y > 0, \end{cases}$$

satisfies $\gamma_{(\delta v,\delta y)} \in \Sigma_v$.

Finally, we define the equivalence relation $\sim$ defined on Σ_v by

$$\gamma \sim \gamma' \text{ if and only if } \lim_{\varepsilon\to 0} \frac{\|\gamma(\varepsilon) - \gamma'(\varepsilon)\|_{L^1}}{\varepsilon} = 0,$$

and we say that a path $\gamma \in \Sigma_v$ generates the generalized tangent vector $(\delta v, \delta y) \in T_v$ if γ is equivalent to $\gamma_{(\delta v,\delta y)}$.

Remark 2.1. *The path $\gamma_{(\delta v,\delta y)} \in \Sigma_v$ represents, at first order, the variation of a function v by adding a perturbation function $\varepsilon\delta v$ and by shifting the discontinuity by $\varepsilon\delta y$.*

Note that, for a given v (piecewise Lipschitz continuous function with a single discontinuity at $y \in \mathbb{R}$) the associated generalized tangent vectors $(\delta v, \delta y) \in T_v$ are those pairs for which δv is Lipschitz continuous with a single discontinuity at $x = y$.

Now we assume that the initial datum u^I is Lipschitz continuous to both sides of a single discontinuity located at $x = \varphi^I$, and consider a generalized tangent vector $(\delta u^I, \delta\varphi^I) \in L^1(\mathbb{R}) \times \mathbb{R}$. Let $u^{I,\varepsilon} \in \Sigma_{u^I}$ be a path which generates $(\delta u^I, \delta\varphi^I)$. For sufficiently small ε the solution u^ε of problem (5.2.1) – (5.2.4) is Lipschitz continuous with a single discontinuity at $x = \varphi^\varepsilon(t)$, for all $t \in [0, T]$. Thus u^ε generates the generalized tangent vector $(\delta v, \delta\varphi) \in L^1(\mathbb{R}) \times \mathbb{R}$. Then it is proved in [9] that $(\delta u, \delta\varphi)$ satisfies the linearized system

$$\begin{aligned}
(\delta u)_t + (f(u)\delta u)_x &= 0, \text{ in } Q^+ \cup Q^-; && (5.2.5)\\
(\delta\varphi)'(t)[u]_{\varphi(t)} &+ \delta\varphi(t)\left(\varphi'(t)[u_x]_{\varphi(t)} - [f(u)u_x]_{\varphi(t)}\right) \\
&= [f(u)\delta u]_{\varphi(t)} - \varphi'(t)[\delta u]_{\varphi(t)},\ t \in (0,T), && (5.2.6)\\
\delta\varphi(0) &= \delta\varphi^I, && (5.2.7)\\
\delta u(x,0) &= \delta u^I(x), x \in \{x < \varphi^I\} \cup \{x > \varphi^I\}. && (5.2.8)
\end{aligned}$$

Remark 2.2. *In this way, we can obtain formally the expansion:*

$$(u^\varepsilon, \varphi^\varepsilon) = (u, \varphi) + \varepsilon(\delta u, \delta\varphi) + O(\varepsilon^2). \tag{5.2.9}$$

Unfortunately, this expansion is, in general, not true, as we explained in the introduction. For instance, suppose that δu is bounded and u^ε is continuous.

From (5.2.9) *we conclude that* u^ε *converges to* u *uniformly, whence* u *should be continuous too. But this is not true in general. Thus we should assume* $(u, \delta u) = (u, \delta u)(x, x/\varepsilon, t)$*, a multi-scale expansion.*

Remark 2.3. *In Section 3, we shall see that equation* (5.2.6) *can not be, as expected, recovered as the viscosity* ν *tends to zero. Instead, it is changed to*

$$\begin{aligned} [u]_{\varphi(t)}\,\delta\varphi'(t) &= \delta\varphi(t)\Big(-[u_x]_{\varphi(t)}\varphi'(t) + [f(u)u_x]_{\varphi(t)}\Big) \\ &\quad + \Big(-[\delta u]_{\varphi(t)}\varphi'(t) + [f(u)\delta u]_{\varphi(t)}\Big) \\ &\quad + \frac{1}{\sigma}\Big([u_x]_{\varphi(t)} - \big([w]_{\varphi(t)}\varphi'(t) - [f(u)w]_{\varphi(t)}\big)\Big), \quad (5.2.10) \end{aligned}$$

in which a corrector (the term involving σ*) is added. Here* w *is a function which will be constructed by asymptotic expansion and has limits as* $x \to \pm\varphi(t)$ *for* $t \in (0, T)$.

5.2.2 Sensitivity in presence of shocks

To study the sensitivity, in the presence of shocks, of J with respect to variations associated with the generalized tangent vector, we define an appropriate generalization of the Gateaux derivative.

Definition 2.2 (Ref. [8]) Let $J : L^1(\mathbb{R}) \to \mathbb{R}$ be a functional and $u^I \in L^1(\mathbb{R})$ be Lipschitz continuous with a discontinuity at $x = \varphi^I$ an initial datum for which the solution of (5.1.1) satisfies hypothesis (H). J is Gateaux differentiable at u^I in a generalized sense if for any generalized tangent vector $(\delta u^I, \delta\varphi^I)$ and any family $u^{I,\varepsilon} \in \Sigma_{u^I}$ associated to $(\delta u^I, \delta\varphi^I)$ the following limit exists,

$$\delta J = \lim_{\varepsilon \to 0} \frac{J(u^{I,\varepsilon}) - J(u^I)}{\varepsilon}.$$

Moreover, it depends only on (u^I, φ^I) and $(\delta u^I, \delta\varphi^I)$, i.e. it does not depend on the particular family $u^{I,\varepsilon}$ which generates $(\delta u^I, \delta\varphi^I)$. The limit is the generalized Gateux derivative of J in the direction $(\delta u^I, \delta\varphi^I)$.

Then we have the proposition which characterizes the generalized Gateaux derivative of J in terms of the solution to the associated adjoint system.

Proposition 2.1 *Assume that* u^D *is continuous at* $x = \varphi(T)$*. The Gateaux derivative of* J *can be written as follows:*

$$\delta J = \int_{\{x<\varphi^I\}\cup\{x>\varphi^I\}} p(x,0)\delta u^I(x)dx + q(0)[u^I]_{\varphi^I}\delta\varphi^I,$$

where the adjoint state pair (p, q) satisfies the system

$$-\partial_t p - f(u)\partial_x p = 0, \text{ in } Q^- \cup Q^+, \tag{5.2.11}$$
$$[p]_\Sigma = 0, \tag{5.2.12}$$
$$q(t) = p(\varphi(t), t), \; t \in (0, T), \tag{5.2.13}$$
$$q'(t) = 0, \; t \in (0, T), \tag{5.2.14}$$
$$p(x, T) = u(x, T) - u^D(x), \; x \in \{x < \varphi(T)\} \cup \{x > \varphi(T)\}, \tag{5.2.15}$$
$$q(T) = \frac{[F(u(x, T) - u^D(x))]_{\varphi(T)}}{[u]_{\varphi(T)}}. \tag{5.2.16}$$

Remark 2.4. *The backward system* (5.2.11)–(5.2.16) *has a unique solution. We can solve it in the following way: We first define the solution q on the shock Σ from the condition $q' = 0$, with the given final value $q(T)$. Then this determines the value of p along the shock. We then can propagate this information, together with the datum of p at time T to both sides of $\varphi(T)$, by characteristics. As both systems* (5.1.1) *and* (5.2.11) *have the same characteristics, any point $(x, t) \in \mathbb{R} \times (0, T)$ is reached backwards in time by a unique characteristics line coming either from the shock Σ or the final data at (x, T) (see Fig. 2). The solution obtained in this way coincides with the reversible solutions introduced in [6].*

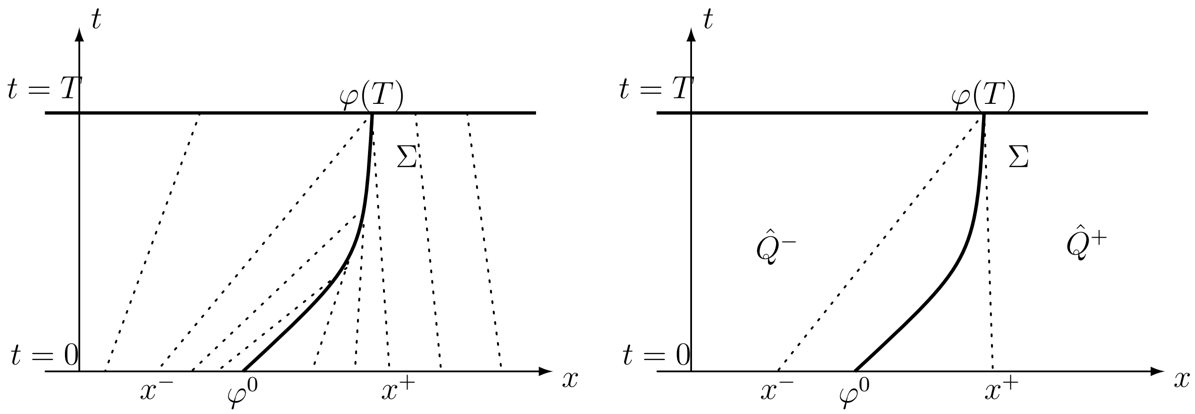

Figure 5.2: Characteristic lines entering on a shock(left) and subdomains $\hat{Q}^-$ and $\hat{Q}^+$ (right).

In Figure 2, we have used the following notations:

$$x^- = \varphi(T) - u^-(\varphi(T))T, \ x^+ = \varphi(T) - u^+(\varphi(T))T,$$

and

$$\hat{Q}^- = \{(x,t) \in \mathbb{R} \times (0,T) \text{ such that } x < \varphi(T) - u^-(\varphi(T))t\},$$
$$\hat{Q}^+ = \{(x,t) \in \mathbb{R} \times (0,T) \text{ such that } x > \varphi(T) - u^+(\varphi(T))t\}.$$

Remark 2.5. *We shall construct a solution to system* (5.2.11)–(5.2.16) *in the following manner: We approximate the initial datum* (5.1.11) *by functions* p_n^T *which satisfies*

p_n^T *are locally Lipschitz continuous, uniformly bounded in* $BV_{\text{loc}}(\mathbb{R})$ *such that*

$$p_n^T(\cdot,T) \to P^T(\cdot) = u(\cdot,T) - u^D(\cdot) \ \textit{in} \ L^1_{\text{loc}}(\mathbb{R}),$$

and

$$p_n^T(\varphi(T),T) = \frac{[F(u(x,T) - u^D(x))]_{\varphi(T)}}{[u]_{\varphi(T)}}.$$

Then we first take the limit of solutions $p_{\nu,n}$ *of* (5.1.10) *with initial data* p_n^T*, as* $\nu \to 0$ *to obtain the solution* p_n *of*

$$-\partial_t p - f(u)\partial_x p = 0, \text{ in } \mathbb{R} \times (0,T), \quad (5.2.17)$$
$$p(x,T) = p_n^T(x), \text{ in } \mathbb{R}, \quad (5.2.18)$$

the so-called reversible solution. These solutions can be characterized by the fact that they take the value $p_n(\varphi(T),T)$ *in the whole region occupied by the characteristics that meet the shock. Thus in particular they satisfy the equations* (5.2.12)–(5.2.14) *and* (5.2.16)*. Moreover,* $p_n \to p$ *as* $n \to \infty$*, and* p *takes a constant value in the region occupied by the characteristics that meet the shock. Note that by construction, this constant is the same for all* p_n *in this region. Thus this limit solution* p *coincides with the solution of system* (5.2.11)–(5.2.16)*.*

In this chapter, we shall apply the method of matched asymptotic expansions to justify these convergence results.

5.2.3 The method of alternating descent directions: Inviscid case

We shall present, in this sub-section, the main ideas of the alternating descent method which is introduced in [11] for the inviscid Burgers equation. The classification of the generalized tangent vectors into two cases is motivated by the following proposition.

Proposition 2.2 *Assume that we restrict the set of paths in Σ_{u^I} to those for which the associated generalized tangent vectors $(\delta u^I, \delta\varphi^I) \in T_{u^I}$ satisfy*

$$\delta\varphi^I = -\frac{\int_{x^-}^{\varphi^I} \delta u^I dx + \int_{\varphi^I}^{x^+} \delta u^I dx}{[u^I]_{\varphi^I}}. \tag{5.2.19}$$

Then, the solution $(\delta u, \delta\varphi)$ of problem (5.2.5)–(5.2.8) *satisfies $\delta\varphi(T) = 0$ and the general Gateaux derivative of J can be written as follows:*

$$\delta J = \int_{\{x<x^-\}\cup\{x>x^+\}} p(x,0)\delta u^I(x)dx, \tag{5.2.20}$$

where the adjoint state pair p satisfies the system

$$-\partial_t p - f(u)\partial_x p = 0, \text{ in } Q^- \cup Q^+, \tag{5.2.21}$$

$$p(x,T) = u(x,T) - u^D(x), \ x \in \{x < \varphi(T)\} \cup \{x > \varphi(T)\}. \tag{5.2.22}$$

Analogously, when considering paths in Σ_{u^I} for which the associated generalized tangent vectors $(\delta u^I, \delta\varphi^I) \in T_{u^I}$ satisfy that $\delta u^I = 0$, then $\delta u(x,T) = 0$ and the general Gateaux derivative of J in the direction $(\delta u^I, \delta\varphi^I)$ can be written as

$$\delta J = -\frac{[F(u(x,T) - u^D(x))]_{\varphi(T)}}{[u]_{\varphi(T)}}[u^I]_{\varphi^I}\delta\varphi^I. \tag{5.2.23}$$

Remark 2.6. *Formula* (5.2.20) *establishes a simplified expression for the generalized Gateaux derivative of J when considering directions $(\delta u^I, \delta\varphi^I)$ that do not move the shock position at $t = T$. These directions are characterized by formula* (5.2.19) *which determines the infinitesimal displacement of the shock position $\delta\varphi^I$ in terms of the variation of u^I to both sides of $x = \delta\varphi^I$. Note, in particular, that to any value δu^I to both sides of the jump φ^I, a unique infinitesimal translation $\delta\varphi^I$ corresponds of the initial shock position that does not move the shock at $t = T$. We see that formula* (5.2.20) *holds even if u^D is discontinuous at $x = \varphi(T)$, since we are dealing with a subspace of generalized tangent vectors satisfying $\delta\varphi(T) = 0$ and the Gateaux derivative of J, reduced to this subspace, is well defined.*

Note also that system (5.2.21)–(5.2.22) *does not allow to determine the function p outside the region $\hat{Q}^- \cup \hat{Q}^+$, i.e. in the region under the influence of the shock by the characteristic lines emanating from it. However, the value of p in this region is not required to evaluate the generalized Gateaux derivative in* (5.2.20). *Analogously, formula* (5.2.23) *provides a simplified expression of the generalized Gateaux derivative of J when considering directions $(\delta u^I, \delta\varphi^I)$*

that uniquely move the shock position at $t = T$ and which correspond to purely translating the shock.

Note that the results in Proposition 2.2 suggest the following decomposition of the set of generalized tangent vectors:

$$T_{u^I} = T^1_{u^I} \oplus T^2_{u^I}, \tag{5.2.24}$$

where $T^1_{u^I}$ contains those $(\delta u^I, \delta\varphi^I)$ for which identity (5.2.19) holds, and $T^1_{u^I}$ the ones for which $\delta u^I = 0$. Thus this provides *two classes* of descent directions for J at u^I. In principle they are not optimal in the sense that they are not the steepest descent directions but they both have three important properties:

(1) They are both descent directions.
(2) They allow to split the design of the profile and the shock location.
(3) They are true generalized gradients and therefore keep the structure of the data without increasing its complexity.

When considering generalized tangent vectors belonging to $T^1_{u^I}$ we can choose as descent direction,

$$\delta u^I = \begin{cases} -p(x,0) & \text{if } \ x < x^-, \\ -\lim_{x\to x^-,\ x<x^-} p(x,0) & \text{if } \ x^- < x < \varphi^I, \\ -\lim_{x\to x^+,\ x>x^+} p(x,0) & \text{if } \ \varphi^I < x < x^+, \\ -p(x,0) & \text{if } \ x^+ < x, \end{cases} \tag{5.2.25}$$

and

$$\delta\varphi^I = -\frac{\int_{x^-}^{\varphi^I} p(x,0)dx + \int_{\varphi^I}^{x^+} p(x,0)dx}{[u]_{\varphi^I}}, \tag{5.2.26}$$

while for $T^2_{u^I}$ a good choice is:

$$\delta u^I = 0,\ \delta\varphi^I = [F(u(x,T) - u^D(x))]_{\varphi(T)} \frac{[u(\cdot,T)]_{\varphi(T)}}{[u^I]_{\varphi^I}}. \tag{5.2.27}$$

In (5.2.26) the value of $\delta\varphi^I$ in the interval (x^-, x^+) does not affect the generalized Gateaux derivative in (5.2.20) under the condition that $\delta\varphi^I$ is chosen exactly as indicated (otherwise the shock would move and this would produce an extra term on the derivative of the functional J). We have chosen the simplest constant value that preserves the Lipschitz continuity of δu^I at $x = x^-$ and $x = x^+$, but not necessarily at $x = \delta\varphi^I$. Other choices would also provide descent directions for J at u^I, but would yield the same Gateaux derivative according to (5.2.20).

This allows us to define a strategy to obtain descent directions for J at u^I in T_{u^I}.

Based upon these studies, an alternating descent method for the optimal control of the Burgers equation can thus be developed, by applying in each step of the descent, the following two sub-steps:

1. Use generalized tangent vectors that move the shock to search its optimal placement.

2. Use generalized tangent vectors to modify the value of the solution at time $t = T$ to both sides of the discontinuity, leaving the shock location unchanged.

For more details, we refer the reader to [11], and a number of numerical experiments show that this method is much more robust and efficient than the usual ones.

In this chapter, motivated by the strategy for obtaining descent directions for J in the inviscid case, we will carry out, in Section 5, this procedure for viscous conservation laws provided that the parameter ν is sufficiently small.

We now turn to discuss the existence of minimizers of the functional J, depending on a small parameter that comes from the solutions to the viscous problems, over suitable admissible set and their limit as the small parameter tends to zero. Let us now introduce the set of admissible initial data $\mathcal{U}_{ad} \subset L^1(\mathbb{R})$, which is

$$\mathcal{U}_{ad} = \{f \in L^\infty(\mathbb{R}) \mid \text{supp}(f) \subset K, \|f\|_{L^\infty(\mathbb{R})}) \leq C\}, \tag{5.2.28}$$

where $K \subset \mathbb{R}$ is a bounded interval and $C > 0$ a constant. We shall see later this choice guarantees the existence of minimizers for the following optimization problem:

Find $u^{I,\min} \in \mathcal{U}_{ad}$ such that

$$J(u^{I,\min}) = \min_{u^I \in \mathcal{U}_{ad}} J(u^I). \tag{5.2.29}$$

To make the dependence on the viscosity parameter ν more explicit the functional J will be denoted by J_ν, although its definition is the same as that of J. Similarly we now consider the same minimization problem for the viscous model (5.1.4):

Find $u^{I,\min} \in \mathcal{U}_{ad}$ such that

$$J_\nu(u^{I,\min}) = \min_{u^I \in \mathcal{U}_{ad}} J_\nu(u^I). \tag{5.2.30}$$

For the existence of minimizers of the functionals J and J_ν, and the conclusion that the minimizers of the viscous problem ($\nu > 0$) converge to a minimizer of the inviscid problem as the viscosity goes to zero, we refer to a recent work [12] for the vanishing viscosity method for the inviscid Burgers equation. In that paper, the following theorems are proved.

Theorem 5.2.1 *(Existence of Minimizers) Assume that* $\mathcal{U}_{ad}$ *is defined in* (5.2.28) *and* $u^D \in L^2(\mathbb{R})$*. Then the minimization problem* (5.2.29) *and* (5.2.30) *have at least one minimizer* $u^{I,\min} \in \mathcal{U}_{ad}$*.*

Theorem 5.2.2 *Any accumulation point as* $\nu \to 0$ *of* $u_\nu^{I,\min}$*, the minimizers of* (5.2.30)*, with respect to the weak topology in* L^2*, is a minimizer of the continuous problem* (5.2.29)*.*

Note that for any positive ν solutions $\delta u,\ p$ of equations (5.1.6) and (5.1.10) are smooth, thus the Gateaux derivative of the functional J is as follows

$$\delta J = < \delta J(u^I), \delta u^I > = \int_{\mathbb{R}} p(x,0)\delta u^I(x) dx, \tag{5.2.31}$$

where the adjoint state $p = p_\nu$ is the solution to (5.1.10) with initial datum $p(x,T) = u(x,T) - u^D$.

Unlike in the inviscid one, the adjoint state now has only one component. Indeed, there is no adjoint shock variable since the state does not present shocks. Similarly, the derivative of J has also only one term. According to this, the straightforward application of a gradient method for the optimization of J would lead, in each step of the iteration, to make use of the variation pointing in the direction

$$\delta u^I = -p(x,0), \tag{5.2.32}$$

where $p = p_\nu$ is the solution to the viscous dual problem. So the alternating method is considerably simplified. But, when processing this way, we would not be exploiting the possibilities that the alternate descent method provides. Therefore, we take into account the effects of possible infinitesimal perturbations of initial datum and also infinitesimal translations, and use variations of the form

$$u_\varepsilon^I(x) = u_\varepsilon^I(x + \varepsilon\delta\varphi^I) + \varepsilon\delta u_\varepsilon^I(x), \tag{5.2.33}$$

where, φ^I stands for a reference point on the profile of u^I, not necessary a point of discontinuity. When u^I has a point of discontinuity, φ^I could be its location and $\delta\varphi^I$ an infinitesimal variation of it. However φ^I could also be another singular point on the profile of u^I, as, for instance, an extremal point or a point where the gradient of u^I is large, namely a smeared discontinuity. By a Taylor expansion, (5.2.33) can be rewritten in the following form

$$u_\varepsilon^I(x) = u^I(x) + \varepsilon\left(\delta\varphi^I u_x^I(x) + \delta u^I(x)\right) + O(\varepsilon^2). \tag{5.2.34}$$

This indicates that the result of these combined variations $(\delta u^I, \delta\varphi^I)$ is equivalent to a classical variation in the direction of $\delta\varphi^I u_x^I + \delta u^I$. We find that the effect of a small $\delta\varphi^I$ can be amplified by a large gradient δu^I.

As we will see in the next section, (5.2.33) and (5.2.34) give us some hints on how to construct outer expansions that include the effects of possible infinitesimal perturbations of initial datum and infinitesimal translations.

5.3 Matched asymptotic expansions and approximate solutions

In this section, we are going to apply the method of matched asymptotic expansions to construct inner and outer expansions. Then we get the approximate solutions by a suitable combination of them. Firstly, we derive the outer expansions. For the purpose of sensitivity analysis of the states of the system, we add infinitesimal perturbations to the initial datum and infinitesimal translations to the shock position, and make use of Taylor expansions. These operations make it possible to derive the equations for the shock and its variation.

Let $v = \delta u,\ \psi = \delta\varphi$ for simplicity. We consider asymptotic expansions of solutions to the following systems.

$$u_t + (F(u))_x = \nu u_{xx}, \text{ in } \mathbb{R} \times (0,T), \tag{5.3.1}$$

$$u|_{t=0} = g^\varepsilon, \tag{5.3.2}$$

$$v_t + (f(u)v)_x = \nu v_{xx}, \text{ in } \mathbb{R} \times (0,T), \tag{5.3.3}$$

$$v|_{t=0} = h^\varepsilon, \tag{5.3.4}$$

$$-p_t - f(u)p_x = \nu p_{xx}, \text{ in } \mathbb{R} \times (0,T), \tag{5.3.5}$$

$$p|_{t=T} = p^T_{n,\varepsilon}. \tag{5.3.6}$$

Equation (5.3.1) is the usual conservation law with viscosity, (5.3.3) is its linearized equation, and (5.3.5) is the dual equation of (5.3.3).

We make the following assumptions:
Assumptions. A1)

$$\text{The function } F \text{ is smooth.} \tag{5.3.7}$$

A2) Let $n \in \mathbb{N}$. $g_0,\ h_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $g_{0x} \in L^1(\mathbb{R})$ and $p_n^T \in L^1(\mathbb{R}) \cap Lip_{\text{loc}}(\mathbb{R}) \cap L^\infty(\mathbb{R})$. $g_{0x},\ h_0$ have only a shock at φ^I, respectively. There exist smooth functions $g^\varepsilon,\ h^\varepsilon,\ p_{n,\varepsilon}^T \in C_0^\infty(\mathbb{R})$, such that

$$g^\varepsilon,\ h^\varepsilon,\ p_{n,\varepsilon}^T \to g_0,\ h_0,\ p_n^T \tag{5.3.8}$$

in $L^2(\mathbb{R})$ as $\varepsilon \to 0$, respectively. Moreover, we assume that $\{p_n^T\}$ is bounded sequence in $BV_{\text{loc}}(\mathbb{R})$ such that as $n \to \infty$,

$$p_n^T \to p^T \quad \text{in } L^1_{\text{loc}}(\mathbb{R}). \tag{5.3.9}$$

A3) Assume further that Oleinik's one-sided Lipschitz condition (OSLC) is satisfied, i.e.

$$\left(f(u(x,t)) - f(u(y,t))\right)(x-y) \leq \alpha(t)(x-y)^2 \tag{5.3.10}$$

for almost every $x, y \in \mathbb{R}$ and $t \in (0,T)$, where $\alpha \in L^1(0,T)$.

5.3.1 Outer expansions

An outer expansion is valid outside the interfacial (or quasi-shock) region, it is a series of the following form

$$\eta = \eta(x,t,\nu) = \eta_0(x,t) + \nu\,\eta_1(x,t) + \nu^2\,\eta_2(x,t) + \cdots, \tag{5.3.11}$$

or

$$\eta = \eta(x,t,\nu) = \eta^0(x,t) + \nu\,\eta^1(x,t) + \nu^2\,\eta^2(x,t) + \cdots, \tag{5.3.12}$$

where $\eta,\ \eta_0,\ \eta^0,\ \eta_1,\ \eta^1, \cdots$ depend on variables t, x (but, do not depend on fast variable r_ν) and will be replaced by $u,\ u_0,\ u^0,\ u_1,\ u^1, \cdots$, respectively. So do the initial data $h,\ g,\ p_n^T$. Hereafter, we use $\cdots$ to denote the remainder which has higher order of the small parameter ν, thus can be omitted.

Step 1. Outer expansions of u. We start with the construction of outer expansion of u. The following *conventions* will apply throughout this subsection: u^i, u^i_j denote a function depending on i and i, j, respectively, where the i, j are non-negative integers. The k-th power of u^i is denoted by $(u^i)^k$. However, if ν is a parameter, ν^i still stands for the i-th power of ν.

We now expand the initial data as follows

$$h^\varepsilon = h_0(x) + \varepsilon h_1(x) + \varepsilon^2 h_2(x) + \cdots, \tag{5.3.13}$$

$$p_{n,\varepsilon}^T = p_{n0}^T(x) + \varepsilon p_{n1}^T(x) + \varepsilon^2 p_{n2}^T(x) + \cdots, \tag{5.3.14}$$

and the nonlinear terms as

$$F(\eta) = F(\eta_0) + \nu f(\eta_0)\eta_1 + \nu^2 \left(f(\eta_0)\eta_2 + \frac{1}{2} f(\eta_0)(\eta_1)^2 \right) + \cdots, \quad (5.3.15)$$

$$f(\eta) = f(\eta_0) + \nu f'(\eta_0)\eta_1 + \nu^2 \left(f'(\eta_0)\eta_2 + \frac{1}{2} f''(\eta_0)(\eta_1)^2 \right) + \cdots \quad (5.3.16)$$

However, the initial datum g^ε will be expanded in a special manner.

In order to analyze the sensitivity of the states with respect to perturbations of the initial data and the shock position, we construct the outer expansions in the following way: It is easy to see that the solution u to equation (5.3.1) with (5.3.13) depends on two parameters ν, ε, so we write

$$u = u(x, t; \nu, \varepsilon),$$

and define

$$x_\varepsilon = x - \varepsilon(\nu)\psi(t), \quad (5.3.17)$$

where $\varepsilon = \varepsilon(\nu)$ is a function to be determined. Then making a transformation of variable $x \to x_\varepsilon$, we obtain a new function

$$\hat{u} = \hat{u}(x_\varepsilon, t; \nu, \varepsilon) := u(x_\varepsilon + \varepsilon(\nu)\psi(t), t; \nu, \varepsilon) = u(x, t; \nu, \varepsilon).$$

For the initial data, we make the same transformation of variable, i.e. (5.3.17) with $t = 0$, and have

$$g^\varepsilon(x) = \hat{g}_0(x_\varepsilon) + \varepsilon \hat{g}_1(x_\varepsilon) + \varepsilon^2 \hat{g}_2(x_\varepsilon) + \cdots . \quad (5.3.18)$$

Here, we defined $\hat{g}_i(x_\varepsilon) = g_i(x_\varepsilon + \varepsilon\psi(0))$ with $i = 0, 1, 2$.

In order to get the outer expansion of u in the form (5.3.11), we need to do it in two steps. Firstly we expand $\hat{u}(x_\varepsilon, t; \nu, \varepsilon)$ in terms of ε and $(\eta, t) = (x_\varepsilon, t)$ (we now regard x_ε as an independent variable) as follows

$$\hat{u} = u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \cdots , \quad (5.3.19)$$

where $u^i = u^i(x_\varepsilon, t; \nu)$, for $i = 0, 1, 2, \cdots$, and we find, by testing different ansätze, that in order to get asymptotic expansions there must hold

$$\varepsilon = \sigma\nu. \quad (5.3.20)$$

Secondly we expand u^i in terms of ν and (x, t) as follows. For $i = 0$, invoking (5.3.17), by Taylor expansions and (5.3.20), one has

$$
\begin{aligned}
& u^0(x_\varepsilon, t; \nu) = u_0^0(x_\varepsilon, t) + \nu u_1^0(x_\varepsilon, t) + \nu^2 u_2^0(x_\varepsilon, t) + \cdots \\
= \;& u_0^0(x, t) + u_{0,x}^0(x, t)\varepsilon\psi + \frac{1}{2} u_{0,xx}^0(x, t)(\varepsilon\psi)^2 + \nu \left(u_1^0(x, t) + u_{1,x}^0(x, t)\varepsilon\psi \right) \\
& + \nu^2 u_2^0(x, t) + \cdots \\
= \;& u_0^0(x, t) + \nu \left(u_{0,x}^0(x, t)\sigma\psi + u_1^0(x, t) \right) \\
& + \nu^2 \left(\frac{1}{2} u_{0,xx}^0(x, t)(\sigma\psi)^2 + u_{1,x}^0(x, t)\sigma\psi + u_2^0(x, t) \right) + \cdots
\end{aligned}
\tag{5.3.21}
$$

In a similar manner, we obtain, for $i = 1, 2$, that

$$
\begin{aligned}
u^1(x_\varepsilon, t; \nu) &= u_0^1(x_\varepsilon, t) + \nu u_1^1(x_\varepsilon, t) + \cdots \\
&= u_0^1(x, t) + \nu \left(u_{0,x}^1(x, t)\sigma\psi + u_1^1(x, t) \right) + \cdots ,
\end{aligned}
\tag{5.3.22}
$$

and

$$
u^2(x_\varepsilon, t; \nu) = u_0^2(x_\varepsilon, t) + \cdots = u_0^2(x, t) + \cdots . \tag{5.3.23}
$$

Therefore, from (5.3.19) – (5.3.23) we get an ansatz for u as follows

$$
u = u_0 + \nu u_1 + \nu^2 u_2, \tag{5.3.24}
$$

here, u_0, u_1 and u_2 are defined by

$$
u_0 = u_0^0, \tag{5.3.25}
$$

$$
u_1 = u_1^0 + \sigma u_0^1, \tag{5.3.26}
$$

$$
u_2 = u_2^0 + \sigma u_1^1 + \sigma^2 u_0^2. \tag{5.3.27}
$$

Straightforward computations show that (5.3.1) can be written in terms of u^i and ε as

$$
\begin{aligned}
0 &= u_t + (F(u))_x - \nu u_{xx} \\
&= u_t^0 + (F(u^0))_x - \nu u_{xx}^0 + \varepsilon \left(u_t^1 + \left(f(u^0)u^1 \right)_x - \nu u_{xx}^1 \right) \\
&\quad + \varepsilon^2 \left(u_t^2 + (f(u^0)u^2 + \frac{f'(u^0)}{2}(u^1)^2)_x - \nu u_{xx}^1 \right) + \cdots .
\end{aligned}
\tag{5.3.28}
$$

Inserting (5.3.24) into (5.3.28), recalling $\varepsilon = \sigma\nu$, then equating the coefficients of ν^k (where $k = 0, 1, 2$) on both sides of the resulting equation, we thus obtain

$$
\nu^0 : \quad (u_0)_t + (F(u_0))_x = 0, \tag{5.3.29}
$$

$$
\nu^1 : \quad (u_1)_t + (f(u_0)u_1)_x = \mathcal{R}, \tag{5.3.30}
$$

$$
\nu^2 : \quad (u_2)_t + (f(u_0)u_2)_x = \mathcal{R}_1, \tag{5.3.31}
$$

where $\mathcal{R}$ and $\mathcal{R}_1$ are functions defined by

$$\mathcal{R} \quad := \quad u_{0,xx} - \sigma\Big(u_{0,x}\psi' + ((u_{0,x})_t + (f(u_0)u_{0,x})_x)\,\psi\Big)$$

and

$$\begin{aligned}
\mathcal{R}_1 \quad := \quad & \left(\frac{1}{2}u^0_{0,x}\sigma\psi + u^0_1\right)_{xx} - \left(\frac{1}{2}u^0_{0,xx}(\sigma\psi)^2 + u^0_{1,x}\sigma\psi\right)_t - \sigma\left(u^1_{0,x}\sigma\psi\right)_t \\
& - \left(f(u^0_0)\left(\frac{1}{2}u^0_{0,xx}(\sigma\psi)^2 + u^0_{1,x}\sigma\psi\right) + \frac{1}{2}f'(u^0_0)(u^0_{0,x}\sigma\psi + u^0_1)^2\right)_x \\
& -\sigma\left(f(u^0_0)u^1_{0,x}\sigma\psi + f'(u^0_0)(u^0_{0,x}\sigma\psi + u^0_1)u^1_0\right)_x + \sigma(u^1_0)_{xx} - \sigma^2\left(\frac{1}{2}f'(u^0_0)(u^1_0)^2\right)_x \\
= \quad & \left(\frac{1}{2}u_{0,x}\sigma\psi + u_1\right)_{xx} - \left(\frac{1}{2}u_{0,xx}(\sigma\psi)^2 + u_{1,x}\sigma\psi\right)_t \\
& - \left(f(u_0)\left(\frac{1}{2}u_{0,xx}(\sigma\psi)^2 + u_{1,x}\sigma\psi\right) + \frac{1}{2}f'(u_0)(u_{0,x}\sigma\psi + u_1)^2\right)_x \qquad (5.3.32)
\end{aligned}$$

from which we see that $\mathcal{R}$, $\mathcal{R}_1$ depend on constant σ and functions ψ, u_0 and u_1.

We now consider the expansion of the initial data. By using again Taylor expansions, (5.3.18) can be rewritten as

$$\begin{aligned}
g^\varepsilon(x) \quad = \quad & g_0(x) + \sigma\nu\left(\psi(0)g_0'(x) + g_1(x)\right) \\
& + \sigma^2\nu^2\left(\frac{1}{2}\psi(0)^2 g_0''(x) + \psi(0)g_1(x) + g_2(x)\right) + \cdots. \qquad (5.3.33)
\end{aligned}$$

Thus this expansion suggests us to choose the initial data of u_i $(i = 0, 1, 2)$ as follows

$$u_0|_{t=0} \quad = \quad \bar{g}_0, \qquad (5.3.34)$$

$$u_1|_{t=0} \quad = \quad \bar{g}_1, \qquad (5.3.35)$$

$$u_2|_{t=0} \quad = \quad \bar{g}_2. \qquad (5.3.36)$$

Here, $\bar{g}_0, \bar{g}_1, \bar{g}_2$ are defined by

$$\bar{g}_0 \quad = \quad g_0, \qquad (5.3.37)$$

$$\bar{g}_1 \quad = \quad \sigma\left(\psi(0)g_0' + \delta g_0\right), \qquad (5.3.38)$$

$$\bar{g}_2 \quad = \quad \sigma^2\left(\frac{1}{2}\psi(0)^2 g_0'' + \psi(0)g_1(x) + g_2(x)\right). \qquad (5.3.39)$$

From (5.3.38) it follows that $\nu\bar{g}_1$ is just the second term in (5.2.34). So the expansion of g^ε obtained here coincides with the one carried out formally in (5.2.34) in Section 2.

To solve the equations (5.3.29)–(5.3.31) with initial data (5.3.34)–(5.3.36), we need to get the equations for φ and ψ. So their solvability will be left to sub-section 3.3, where the interface equations will be derived.

Step 2. Outer expansions of v. Now we turn to derive the outer expansions for v. It is not necessary to take into account again the effects of infinitesimal perturbations of the initial datum and infinitesimal translations of the shock position, since we have had, in hand, the equations for shock and its perturbation. We can construct directly an ansatz for v, that has the same form of (5.3.24), as follows

$$v = v_0 + \nu v_1 + \nu^2 v_2. \tag{5.3.40}$$

Similarly, from equation (5.3.3) we then have

$$\nu^0: \quad (v_0)_t + (f(u_0)v_0)_x = 0, \tag{5.3.41}$$

$$\nu^1: \quad (v_1)_t + (f(u_0)v_1)_x = \delta_1, \tag{5.3.42}$$

$$\nu^2: \quad (v_2)_t + (f(u_0)v_2)_x = \delta_2, \tag{5.3.43}$$

and the initial data are

$$v_0|_{t=0} = h_0, \tag{5.3.44}$$

$$v_1|_{t=0} = h_1, \tag{5.3.45}$$

$$v_2|_{t=0} = h_2. \tag{5.3.46}$$

Here δ_1, δ_2 are given by

$$\begin{aligned} \delta_1 &= (v_0)_{xx} - (f'(u_0)u_1v_0)_x\,, \\ \delta_2 &= (v_1)_{xx} - \left(f'(u_0)u_2v_0 + \frac{1}{2}f''(u_0)(u_1)^2v_0\right)_x. \end{aligned} \tag{5.3.47}$$

By solving these problems (which are initial value problems of transport equations with discontinuous coefficients, for the well-posedness we refer to e.g. [6]), we then construct $v_0,\ v_1,\ v_2$ which are smooth up to the shock.

Step 3. Outer expansions of p. To get the outer expansions for p we repeat again the procedure performed for v, similarly, from equation (5.3.5) we then have

$$\nu^0: \quad (p_0)_t + f(u_0)(p_0)_x = 0, \tag{5.3.48}$$

$$\nu^1: \quad (p_1)_t + f(u_0)(p_1)_x = \beta_1, \tag{5.3.49}$$

$$\nu^2: \quad (p_2)_t + f(u_0)(p_2)_x = \beta_2, \tag{5.3.50}$$

and the initial data are

$$p_0|_{t=0} = p_0^T, \tag{5.3.51}$$
$$p_1|_{t=0} = p_1^T, \tag{5.3.52}$$
$$p_2|_{t=0} = p_2^T. \tag{5.3.53}$$

Here β_1, β_2 are given by

$$\beta_1 = -\left(f'(u_0)u_1(p_0)_x + (p_0)_{xx}\right), \tag{5.3.54}$$
$$\beta_2 = -\left(f'(u_0)u_1(p_1)_x + \left(f'(u_0)u_2 + \frac{f''(u_0)}{2}(u_1)^2\right)(p_0)_x + (p_1)_{xx}\right) \tag{5.3.55}$$

Thus we first solve p_0, then insert p_0 into equations p_1 and p_2, from the resulting linear equations we then can obtain p_1 and p_2.

We are now going to discuss the solvability of the problems (5.3.29) – (5.3.36). There are five unknowns, i.e. u_i $(i = 0, 1, 2)$, φ and ψ, but only three equations. So, to form a complete system, as the first step, we need to find the equations that φ and ψ satisfy.

5.3.2 Derivation of the interface equations

In this sub-section we shall derive the interface equations from the outer expansions of u, this derivation is based upon an important observation that the values of outer expansions tend to those at the shock $\varphi(t)$, as the thickness ν of the quasi-shock region goes to zero. This is different from the usual way that one derives such equations from inner expansions, see, e.g. [14, 16]. One advantage of this approach is that we can overcome the difficulty/limitation caused by the algebraically growth, as $r_\nu \to \infty$, which is required by the matching conditions, i.e. (5.3.73), of the second term i.e. $\tilde{u}_1$ of inner expansion of u, so how to define the jump (e.g. $\tilde{u}_1(\infty, t) - \tilde{u}_1(-\infty, t)$, but $\tilde{u}_1(\pm\infty, t)$ may not exsist) of the terms like $\tilde{u}_1$, of the inner expansion of u? We cannot define it in a usual way, some restrictions must be imposed, for instance, add an assumption that $\partial_x u_0(\varphi(t) \pm 0, t) = 0$ for $\tilde{u}_1$.

Assume that the limits of u_i (or its derivatives) exist for any $t \in [0, T]$, from the left and right side of the shock, i.e.

$$\partial_x^k u_0(\varphi(t) \pm 0, t), \ k = 0, 1, \text{ and } u_1(\varphi(t) \pm 0, t), \text{ exist.}$$

First of all, we derive the equation for the shock, and invoke equation (5.3.29) with initial datum (5.3.34) which is just a Cauchy problem for the unknown u_0. From the standard theory it then follows that

$$\varphi' [u_0]_{\varphi(t)} = [F(u_0)]_{\varphi(t)}, \tag{5.3.56}$$

which is the Rankine-Hugoniot condition.

Next we make use of (5.3.30) to find the equation for the variation of the shock position, i.e. ψ. We define, in a usual way, weak solution to problem (5.3.30) – (5.3.35), namely multiplying (5.3.30) by a test function ζ with compact support in $\mathbb{R} \times [0, T)$, and integrating it by parts. Then similar to the derivation of the Rankine-Hugoniot condition, we derive, as e.g. in Smoller [31], the equation of ψ from this weak formulation by choosing a test function ζ with compact support contained in a small neighborhood $D = D_1 \cup D_2$ of a fixed point (x, t), see Figure 3. Then applying integration by parts yield that

$$\begin{aligned} \sigma[u_0]_{\varphi(t)}\psi'(t) \quad &+ \quad \sigma\Big(\big(-\varphi'(t)\,[u_{0,x}]_{\varphi(t)} + [f(u_0)u_{0,x}]_{\varphi(t)}\big)\,\psi(t)\Big) \\ &= \quad \varphi'(t)\,[u_1]_{\varphi(t)} - [f(u_0)u_1]_{\varphi(t)} + [u_{0,x}]_{\varphi(t)}. \qquad (5.3.57) \end{aligned}$$

Here we used the fact that the integral vanishes over interior of D_1 and D_2, and that the values of ζ are equal to zero at the boundary of D.

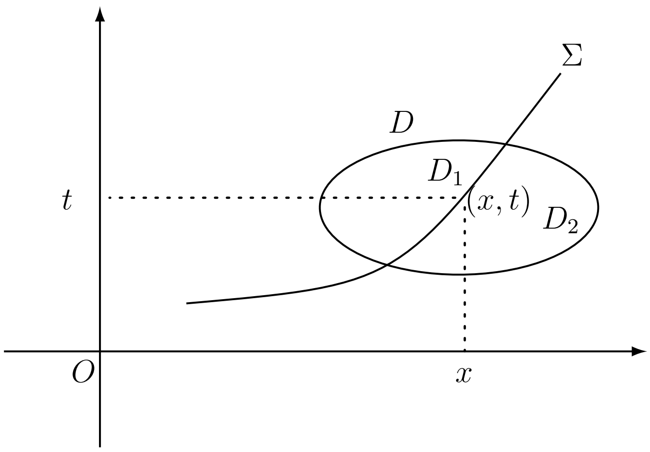

Figure 5.3: Subdomains D_1 and D_2.

Dividing equation (5.3.57) by σ we obtain

$$\begin{aligned} [u_0]_{\varphi(t)}\psi'(t) \quad &+ \quad \big(-\varphi'(t)\,[u_{0,x}]_{\varphi(t)} + [f(u_0)u_{0,x}]_{\varphi(t)}\big)\,\psi(t) \\ &= \quad \frac{1}{\sigma}\Big(\big(\varphi'(t)\,[u_1]_{\varphi(t)} - [f(u_0)u_1]_{\varphi(t)}\big) + [u_{0,x}]_{\varphi(t)}\Big). \qquad (5.3.58) \end{aligned}$$

Remark 3.1. *By* (5.3.26) *we can rewrite the right-hand side of* (5.3.58) *as*

$$\begin{aligned} &\frac{1}{\sigma}\Big(\big(\varphi'(t)\,[u_1]_{\varphi(t)} - [f(u_0)u_1]_{\varphi(t)}\big) + [u_{0,x}]_{\varphi(t)}\Big) \\ = \quad &\big(\varphi'(t)\,[u_0^1]_{\varphi(t)} - [f(u_0)u_0^1]_{\varphi(t)}\big) + \frac{1}{\sigma}\Big(\big(\varphi'(t)\,[u_0^1]_{\varphi(t)} - [f(u_0)u_0^1]_{\varphi(t)}\big) + [u_{0,x}]_{\varphi(t)}\Big) \\ \to \quad &\varphi'(t)\,[u_0^1]_{\varphi(t)} - [f(u_0)u_0^1]_{\varphi(t)}. \qquad (5.3.59) \end{aligned}$$

Thus we see that if $f(u) = u$, equation (5.3.58) *converges to the one derived in [8].*

We turn back to solvability of equation (5.3.58). To this end we need to determine a priori the quantity $[u_1]_{\varphi(t)}$. Invoking outer expansions of u_ν and v_ν, noting that v_ν is the variation of u_ν, we have two expansions of u_ν, which are valid up to the shock,

$$u(x,t;\nu,\sigma\nu) =: u_\nu \;=\; u_0 + \nu u_1 + \cdots \tag{5.3.60}$$

$$= \; u + \nu v_\nu + \cdots = u + \nu v_0 + \cdots . \tag{5.3.61}$$

Here u is believed to be the entropy solution to problem (5.1.1)–(5.1.2). So it is natural to assume that $u_0 = u$, whence $u_1 = v_0$. Thus, there must hold

$$\lim_{x\to\varphi(t)} u_1(x,t) = \lim_{x\to\varphi(t)} v_0(x,t) = v_0(\varphi(t),t)$$

for any $t \in [0,T]$.

Therefore, we can write $[u_1]_{\varphi(t)}$ in terms of $[v_0]_{\varphi(t)}$. From (5.3.56) we get $\varphi = \varphi(t)$ and insert it into (5.3.58). Hence from (5.3.58) one solves (5.3.58) and gets ψ. Furthermore, we can solve u_0 from equation (5.3.29) with (5.3.34) which form a Riemann problem. Once u_0 is obtained, we see that the right hand side of (5.3.30) and (5.3.31), namely $\mathcal{R}$, $\mathcal{R}_1$ are also known quantities, thus problem (5.3.30) with (5.3.35) and problem (5.3.31) with (5.3.36) are linear in u_1, u_2 respectively, and can be solved easily to get u_1, u_2.

Combining the results obtained in sub-sections 3.1 and 3.2, we then complete the construction of outer expansions of u, v and p. We also find that the first term v_0 of the outer expansion of $v(x,t;\nu,\varepsilon)$ can be expected to solve

$$(v_0)_t + (f(u_0)v_0)_x \;=\; 0, \text{ in } Q^+ \cup Q^-; \tag{5.3.62}$$

$$[u_0]_{\varphi(t)}\psi'(t) \;+\; \left(-\varphi'(t)\,[u_{0,x}]_{\varphi(t)} + [f(u_0)u_{0,x}]_{\varphi(t)}\right)\psi(t)$$

$$= \; \frac{1}{\sigma}\Big(\left(\varphi'(t)\,[v_0]_{\varphi(t)} - [f(u_0)v_0]_{\varphi(t)}\right) + [u_{0,x}]_{\varphi(t)} \Big) \tag{5.3.63}$$

$$\psi(0) \;=\; \delta\varphi^I, \tag{5.3.64}$$

$$v_0(x,0) \;=\; v^I(x), x \in \{x < \varphi^I\} \cup \{x > \varphi^I\}. \tag{5.3.65}$$

Note that (5.2.6) is replaced by (5.3.63).

Remark 3.2. *By the method of characteristics, we solve u_1 from problem* (5.3.30), (5.3.35), *and write it as*

$$u_1(x,t) = u_1(x(0),0)\int_0^t \mathcal{R}(x,s)\cdot \exp\left(\int_t^s (f(u_0(x,\tau)))_x d\tau\right) ds,$$

for $(x,t) \in Q^+ \cup Q^-$, and ψ from (5.3.58). Thus we see that ψ, u_1 depend on the parameter σ. On the other hand, under assumption that ψ depends on σ, checking the construction we find that functions u_1^0 and u_0^1 are still independent of σ. Therefore, by formula (5.3.26) we assert that

$$u_1^0 = u_1(x,t;\sigma)|_{\sigma=0} = u_1(x(0),0) \int_0^t u_{0,xx}(x,s) \exp\left(\int_t^s (f(u_0(x,\tau)))_x d\tau \right) ds,$$

whence

$$u_0^1 = \frac{1}{\sigma}(u_1 - u_1^0). \tag{5.3.66}$$

Note that u_1^0 is the second term of outer expansions suppose that there is no perturbation of the initial datum. It is possible that the right hand side of (5.3.66) is independent of σ.

5.3.3 Inner expansions

An inner expansion is the one that is valid inside the interfacial region, which can be written as a series consisting of terms in *fast* variable $r = r_\nu$ and time t

$$\tilde{\eta} = \tilde{\eta}(r_\nu, t, \nu) = \tilde{\eta}_0(r,t) + \nu\tilde{\eta}_1(r,t) + \nu^2\tilde{\eta}_2(r,t) + \cdots,$$

here $\tilde{\eta},\ \tilde{\eta}_0,\ \tilde{\eta}_1, \cdots$ will be replaced by $\tilde{u},\ \tilde{u}_0,\ \tilde{u}_1, \cdots,\ \tilde{v},\ \tilde{v}_0,\ \tilde{v}_1, \cdots$ and $\tilde{p},\ \tilde{p}_0,\ \tilde{p}_1, \cdots$, respectively.

The nonlinear terms can be expanded as

$$F(\tilde{\eta}) = F(\tilde{\eta}_0) + \nu f(\tilde{\eta}_0)\tilde{\eta}_1 + \nu^2 \left(f(\tilde{\eta}_0)\tilde{\eta}_2 + \frac{1}{2} f(\tilde{\eta}_0)\tilde{\eta}_1^2 \right) + \cdots, \tag{5.3.67}$$

$$f(\tilde{\eta}) = f(\tilde{\eta}_0) + \nu f'(\tilde{\eta}_0)\tilde{\eta}_1 + \nu^2 \left(f'(\tilde{\eta}_0)\tilde{\eta}_2 + \frac{1}{2} f''(\tilde{\eta}_0)\tilde{\eta}_1^2 \right) + \cdots. \tag{5.3.68}$$

In what follows, for implicity, we use the notation

$$\tilde{h}' = \frac{\partial}{\partial r_\nu}\tilde{h}$$

for a function $\tilde{h}$ in fast variable. Then we have

$$\frac{\partial \tilde{h}}{\partial d_\nu} = \frac{1}{\nu}\tilde{h}', \quad \frac{\partial^2 \tilde{h}}{\partial {d_\nu}^2} = \frac{1}{\nu^2}\tilde{h}''.$$

Therefore, we have

$$\nu^{-1}: \quad \tilde{u}_0'' + \dot{\varphi}\tilde{u}_0' - (F(\tilde{u}_0))' = 0, \tag{5.3.69}$$

$$\nu^{0}: \quad \tilde{u}_1'' + \dot{\varphi}\tilde{u}_1' - (f(\tilde{u}_0)\tilde{u}_1)' = -\sigma\dot{\psi}\tilde{u}_0' + \tilde{u}_{0t}, \tag{5.3.70}$$

$$\nu^{1}: \quad \tilde{u}_2'' + \dot{\varphi}\tilde{u}_2' - (f(\tilde{u}_0)\tilde{u}_2)' = -\sigma\dot{\psi}\tilde{u}_1' \left(\frac{f'(\tilde{u}_0)}{2}\tilde{u}_1^2\right)' + \tilde{u}_{1t}. \tag{5.3.71}$$

Then we use the matching conditions which make the inner and outer expansions coincide asymptotically in an intermediate region, say $\mathcal{M}$. Over this region, there holds

$$\tilde{u}_0(r_\nu,t)+\nu\tilde{u}_1(r_\nu,t)+\nu^2\tilde{u}_2(r_\nu,t) = u_0(x,t)+\nu u_1(x,t)+\nu^2 u_2(x,t)+O(\nu^3), \text{ in } \mathcal{M}.$$

By definition, we can rewrite $x = \nu(r_\nu + \sigma\psi(t)) + \varphi(t)$. Then using Taylor expansions we can obtain the matching conditions in the following form

$$\tilde{u}_0(r_\nu,t) = u_0(\varphi(t)\pm 0,t) + o(1), \tag{5.3.72}$$

$$\tilde{u}_1(r_\nu,t) = u_1(\varphi(t)\pm 0,t) + (r_\nu+\sigma\psi)\partial_x u_0(\varphi(t)\pm 0,t) + o(1), \tag{5.3.73}$$

$$\begin{aligned}\tilde{u}_2(r_\nu,t) &= u_2(\varphi(t)\pm 0,t) + (r_\nu+\sigma\psi)\partial_x u_1(\varphi(t)\pm 0,t)\\ &\quad + \frac{(r_\nu+\sigma\psi)^2}{2}\partial_x^2 u_0(\varphi(t)\pm 0,t) + o(1),\end{aligned} \tag{5.3.74}$$

for more details, we refer, e.g. to the book by Fife [14]. Later we will choose $o(1) = \exp(-c\xi^2)$, where c is a fixed positive number.

Therefore the problems (5.3.69) – (5.3.71) with corresponding boundary conditions (5.3.72) – (5.3.74), which are boundary value problems of ordinary differential equations of second order, have a unique solution, respectively, provided the orthogonality conditions are met:

$$\int_{-\infty}^{\infty}(-\sigma\dot{\psi}\tilde{u}_0' + \tilde{u}_{0t})p^* dr = 0 \text{ and } \int_{-\infty}^{\infty}\left(-\sigma\dot{\psi}\tilde{u}_1'\left(\frac{f'(\tilde{u}_0)}{2}\tilde{u}_1^2\right)' + \tilde{u}_{1t}\right)p^* dr \tag{5.3.75}$$

where p^* satisfied $L^*(p^*) = 0$, and L^* is the adjoint operator of L which is defined by

$$L(u_1) = u_1'' + \dot{\varphi}u_1' - (f(\tilde{u}_0)u_1)'.$$

We thus obtain $\tilde{u}_0, \tilde{u}_1, \tilde{u}_2$.

Straightforward computations yield the equations for the inner expansion of v

$$\nu^{-1}: \quad \tilde{v}_0'' + \dot{\varphi}\tilde{v}_0' - (f(\tilde{u}_0)\tilde{v}_0)' = 0, \tag{5.3.76}$$

$$\nu^{0}: \quad \tilde{v}_1'' + \dot{\varphi}\tilde{v}_1' - (f(\tilde{u}_0)\tilde{v}_1)' = f_1, \tag{5.3.77}$$

$$\nu^{1}: \quad \tilde{v}_2'' + \dot{\varphi}\tilde{v}_2' - (f(\tilde{u}_0)\tilde{v}_2)' = f_2, \tag{5.3.78}$$

where f_1 and f_2 are defined by

$$f_1 = \tilde{v}_{0t} - \sigma\dot{\psi}\tilde{v}_0' + \left(f'(\tilde{u}_0)\tilde{u}_1\tilde{v}_0\right)', \tag{5.3.79}$$

$$f_2 = \tilde{v}_{1t} - \sigma\dot{\psi}\tilde{v}_1' + \left(f'(\tilde{u}_0)\tilde{u}_1\tilde{v}_1 + \left(f'(\tilde{u}_0)\tilde{u}_2 + \frac{1}{2}f''(\tilde{u}_0)\tilde{u}_1^2\right)\tilde{v}_0\right)' \tag{5.3.80}$$

Using the matching conditions

$$\tilde{v}_0(r_\nu, t) = v_0(\varphi(t) \pm 0, t) + o(1), \tag{5.3.81}$$

$$\tilde{v}_1(r_\nu, t) = v_1(\varphi(t) \pm 0, t) + (r_\nu + \sigma\psi)\partial_x v_0(\varphi(t) \pm 0, t) + o(1), \tag{5.3.82}$$

$$\begin{aligned} \tilde{v}_2(r_\nu, t) &= v_2(\varphi(t) \pm 0, t) + (r_\nu + \sigma\psi)\partial_x v_1(\varphi(t) \pm 0, t) \\ &\quad + \frac{(r_\nu + \sigma\psi)^2}{2}\partial_x^2 v_0(\varphi(t) \pm 0, t) + o(1), \end{aligned} \tag{5.3.83}$$

Solving problems (5.3.76) – (5.3.78) with suitable orthogonality conditions and also boundary conditions (5.3.81) – (5.3.83), we then obtain $\tilde{v}_0,\ \tilde{v}_1,\ \tilde{v}_2$.

Finally we construct the inner expansion of p, the first three terms of it can be obtained by solving the following equations

$$\nu^{-1}: \quad \tilde{p}_0'' - \dot{\varphi}\tilde{p}_0' + f(\tilde{u}_0)\tilde{p}_0' = 0, \tag{5.3.84}$$

$$\nu^{0}: \quad \tilde{p}_1'' - \dot{\varphi}\tilde{p}_1' + f(\tilde{u}_0)\tilde{p}_1' = f_3, \tag{5.3.85}$$

$$\nu^{1}: \quad \tilde{p}_2'' - \dot{\varphi}\tilde{p}_2' + f(\tilde{u}_0)\tilde{p}_2' = f_4, \tag{5.3.86}$$

where f_3 and f_4 are defined by

$$f_3 = \dot{\psi}\tilde{p}_0' - \tilde{p}_{0t} - f'(\tilde{u}_0)\tilde{u}_1\tilde{p}_0', \tag{5.3.87}$$

$$f_4 = \dot{\psi}\tilde{p}_1' - \tilde{p}_{1t} - f'(\tilde{u}_0)\tilde{u}_1\tilde{v}_1 - \left(f'(\tilde{u}_0)\tilde{u}_2 + \frac{1}{2}f''(\tilde{u}_0)\tilde{u}_1^2\right)\tilde{p}_0'. \tag{5.3.88}$$

One can easily find that equations satisfied by $\tilde{p}_i$ $(i = 0, 1, 2)$, namely (5.3.84) – (5.3.86) are just the dual ones of those satisfied by v_i, i.e. (5.3.76) – (5.3.78).

From the following matching conditions

$$\tilde{p}_0(r_\nu, t) = p_0(\varphi(t) \pm 0, t) + o(1), \tag{5.3.89}$$

$$\tilde{p}_1(r_\nu, t) = p_1(\varphi(t) \pm 0, t) + (r_\nu + \sigma\psi)\partial_x p_0(\varphi(t) \pm 0, t) + o(1), \tag{5.3.90}$$

$$\begin{aligned} \tilde{p}_2(r_\nu, t) &= p_2(\varphi(t) \pm 0, t) + (r_\nu + \sigma\psi)\partial_x p_1(\varphi(t) \pm 0, t) \\ &\quad + \frac{(r_\nu + \sigma\psi)^2}{2}\partial_x^2 p_0(\varphi(t) \pm 0, t) + o(1), \end{aligned} \tag{5.3.91}$$

we solve uniquely, under suitable orthogonality conditions, equations (5.3.84) – (5.3.86) and get $\tilde{p}_0,\ \tilde{p}_1,\ \tilde{p}_2$.

Therefore, we have constructed the inner and outer expansions up to ν^2, for $u,\ v,\ p$ respectively, which can be written as follows

$$U_2(x,t) = u_0 + \nu u_1 + \nu^2 u_2, \tag{5.3.92}$$
$$\tilde{U}_2(r,t) = \tilde{u}_0 + \nu \tilde{u}_1 + \nu^2 \tilde{u}_2. \tag{5.3.93}$$

$$V_2(x,t) = v_0 + \nu v_1 + \nu^2 v_2, \tag{5.3.94}$$
$$\tilde{V}_2(r,t) = \tilde{v}_0 + \nu \tilde{v}_1 + \nu^2 \tilde{v}_2. \tag{5.3.95}$$

$$P_2(x,t) = p_0 + \nu p_1 + \nu^2 p_2, \tag{5.3.96}$$
$$\tilde{P}_2(r,t) = \tilde{p}_0 + \nu \tilde{p}_1 + \nu^2 \tilde{p}_2. \tag{5.3.97}$$

5.3.4 Approximate solutions

In this subsection we shall use a suitable cut-off function to combine together the outer and inner expansions derived in sub-sections 3.1, 3.3 and 3.4, whence the approximate solutions will be constructed.

We define a function $\chi = \chi(\xi) : \mathbb{R} \to \mathbb{R}^+$ which is smooth such that

$$\chi(\xi) = \begin{cases} 1 & \text{if } |\xi| \le 1, \\ 0 & \text{if } |\xi| \ge 2, \end{cases} \tag{5.3.98}$$

and $0 \le \chi(\xi) \le 1$ if $\xi \in [1,2]$. And let

$$\chi_\nu(x,t) = \chi(\nu^{1-\gamma} r_\nu), \tag{5.3.99}$$

which is easily seen from Figure 4 that

$$\text{supp}(\chi_\nu) \subset [0, 2\nu^\gamma], \quad \text{supp}(\chi_\nu'),\ \text{supp}(\chi_\nu'') \subset [\nu^\gamma, 2\nu^\gamma].$$

From the expansions (5.3.92) and (5.3.93), (5.3.94) and (5.3.95) and (5.3.96) and (5.3.97), we are in a position to construct, respectively, the approximate solutions $\hat{U}_2^\nu$, $\hat{V}_2^\nu$ and $\hat{P}_2^\nu$ as follows

$$\hat{U}_2^\nu(x,t) = \chi_\nu(x,t)\tilde{U}_2(r_\nu,t) + (1-\chi_\nu(x,t))\, U_2(x,t), \tag{5.3.100}$$

$$\hat{V}_2^\nu(x,t) = \chi_\nu(x,t)\tilde{V}_2(r_\nu,t) + (1-\chi_\nu(x,t))\, V_2(x,t), \tag{5.3.101}$$

$$\hat{P}_2^\nu(x,t) = \chi_\nu(x,t)\tilde{P}_2(r_\nu,t) + (1-\chi_\nu(x,t))\, P_2(x,t). \tag{5.3.102}$$

By definition, we find easily that if (x,t) is sufficiently close to the quasi-shock $x = \varphi(t)$, then $\chi_\nu(x,t) = 1$, thus the approximate solution $\hat{U}_2^\nu$ is equal to inner expansion $\tilde{U}_2$, namely

$$\hat{U}_2^\nu(x,t) = \tilde{U}_2(r_\nu,t),$$

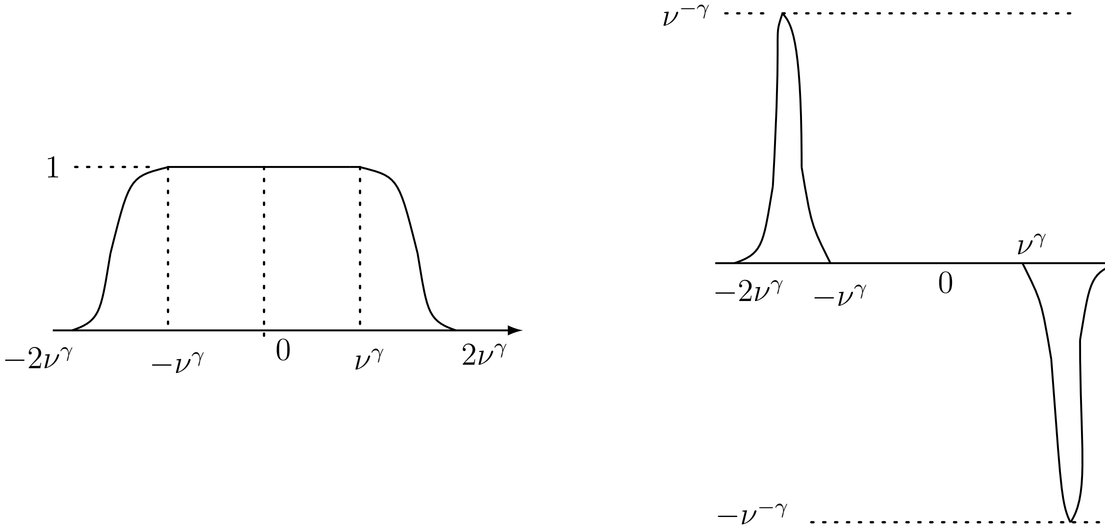

Figure 5.4: Typical shapes of functions χ_ν and χ'_ν.

on the other hand, suppose that if (x,t) is sufficiently away from the quasi-shock, then $1-\chi_\nu(x,t)=1$ which yields that the approximate solution is just the outer expansion, i.e.

$$\hat{U}_2^\nu(x,t) = U_2(x,t).$$

In the intermediate region, there hold $0 \le \chi_\nu(x,t) \le 1$ and $\tilde{U}_2(r_\nu,t), U_2(x,t)$ are equal asymptotically which can be seen from the matching conditions, thus we can replace one of the two expansions by one another with a small error, to be precise, we write

$$\hat{U}_2^\nu(x,t) = \chi_\nu(x,t)\tilde{U}_2(r_\nu,t) + (1-\chi_\nu(x,t))\,\tilde{U}_2(r_\nu,t) + o(1) = \tilde{U}_2(r_\nu,t) + o(1),$$

also there holds $\hat{U}_2^\nu(x,t) = U_2(x,t) + o(1)$. So the approximate solution is a good combination of inner and outer expansions.

In what follows we will omit arguments (x,t), (r_ν,t) and so on, for simplicity.

Theorem 5.3.1 *Suppose that the condition* (5.3.75) *and assumptions* (5.3.7) – (5.3.10) *are satisfied and that* $\varepsilon = \sigma\nu$ *with* σ *being a positive constant.*

Then the approximate solutions $\hat{U}_2^\nu, \hat{V}_2^\nu, \hat{P}_2^\nu$ *satisfy respectively, equations* (5.3.1), (5.3.3) *and* (5.3.5), *in the following sense*

$$(\hat{U}_2^\nu)_t - \nu(\hat{U}_2^\nu)_{xx} + (F(\hat{U}_2^\nu))_x = O(\nu^\alpha), \qquad (5.3.103)$$

$$(\hat{V}_2^\nu)_t - \nu(\hat{V}_2^\nu)_{xx} + (f(\hat{U}_2^\nu)\hat{V}_2^\nu)_x = O(\nu^\alpha), \qquad (5.3.104)$$

$$-(\hat{P}_2^\nu)_t - \nu(\hat{P}_2^\nu)_{xx} - f(\hat{U}_2^\nu)(\hat{P}_2^\nu)_x = O(\nu^\alpha), \qquad (5.3.105)$$

where $\alpha = 3\gamma - 1$ and $\gamma \in (\frac{1}{3}, 1)$.

5.4 Convergence of the approximate solutions

This section is devoted to the proof of following Theorem 4.1, which consists of two parts, one is to prove Theorem 3.1 which asserts that the equations are satisfied asymptotically, and another is to investigate the convergence rate.

Theorem 5.4.1 *Suppose that the assumptions in Theorem 3.1 are met. Let u, p be, respectively, the unique entropy solution with only one shock, the reversible solution to problems (5.1.1) – (5.1.2) and (5.1.8) – (5.1.9), and let v be the unique solutions to problem (5.3.62) – (5.3.65), such that*

$$\int_0^T \int_{\{x\neq\varphi(t),\, t\in[0,T]\}} \sum_{i=1}^{6} \left(|\partial_x^i u(x,t)|^2 + |\partial_x^i v(x,t)|^2 + |\partial_x^i p(x,t)|^2\right) dxdt \leq C. \tag{5.4.1}$$

Then the solutions (u^ν, v^ν) of problems (5.3.1) – (5.3.2) and (5.3.3) – (5.3.4) converge, respectively, to (u, v) in $L^\infty(0,T; L^2(\mathbb{R})) \times L^\infty(0,T; L^2(\mathbb{R}))$, and the following estimates hold

$$\sup_{0\leq t\leq T} \|u^\nu(t) - u(t)\| + \sup_{0\leq t\leq T} \|v^\nu(t) - v(t)\| \leq C_\eta \nu^\eta. \tag{5.4.2}$$

The solution $p_{\nu,n}$ of problem (5.3.5) – (5.3.6) converges to p in $L^\infty(0,T; L^1_{\text{loc}}(\mathbb{R}))$, namely,

$$\sup_{0\leq t\leq T} \|p_{\nu,n}(t) - p(t)\|_{L^1_{\text{loc}}(\mathbb{R})} \to 0, \tag{5.4.3}$$

as first $\nu \to 0$, then $n \to \infty$. Moreover, we also have in a sub-domain the following estimate

$$\sup_{0\leq t\leq T} \|p_{\nu,n} - p_n\|_{L^\infty(\Omega_h)} \leq C\nu \to 0, \tag{5.4.4}$$

as $\nu \to 0$. Here η is a constant defined by

$$\eta = \min\left\{\frac{3}{2}\gamma, \frac{1+\gamma}{2}\right\}, \text{ where } \gamma \text{ is the same as in Theorem 3.1,} \tag{5.4.5}$$

and C_η denotes a constant depending only on parameters η. The domain Ω_h is defined, for any positive constant h, by

$$\Omega_h = \{(x,t) \in Q_T \mid |x - \varphi(t)| > h\}.$$

Remark 4.1. *Combination of* (5.4.4) *and a stability theorem of the reversible solutions (see Theorem 4.1.10 in Ref. [6])) yields that*

$$\sup_{0\leq t\leq T} \|p_{\nu,n} - p\|_{L^\infty(\Omega_h\cap[-R,R])} \to 0, \tag{5.4.6}$$

as $\nu \to 0$*, then* $n \to \infty$*, where* R *is any positive constant. This ensures us to select alternating descent directions.*

5.4.1 The equations are satisfied asymptotically

In this sub-section, we are going to prove that the approximate solutions $\hat{U}_2^\nu,\ \hat{V}_2^\nu,\ \hat{P}_2^\nu$ satisfy asymptotically, the corresponding equations (5.3.1), (5.3.3) and (5.3.5) respectively. For simplicity, in this sub-section we omit the superscript ν and write the approximate solutions as $\hat{U}_2,\ \hat{V}_2,\ \hat{P}_2$.

Proof of Theorem 3.1. We divide the proof into three parts.

Part 1. We first investigate the convergence of $\hat{U}_2$. Straightforward computations yield

$$\begin{aligned}
(\hat{U}_2)_t &= \frac{d_t}{\nu^\gamma}\chi_\nu'\left(\tilde{U}_2 - U_2\right) + \chi_\nu\cdot(\tilde{U}_2)_t + (1-\chi_\nu)(U_2)_t, & (5.4.7)\\
(\hat{U}_2)_x &= \frac{1}{\nu^\gamma}\chi_\nu'\left(\tilde{U}_2 - U_2\right) + \chi_\nu\cdot(\tilde{U}_2)_x + (1-\chi_\nu)(U_2)_x, & (5.4.8)\\
(\hat{U}_2)_{xx} &= \frac{1}{\nu^{2\gamma}}\chi_\nu''\left(\tilde{U}_2 - U_2\right) + \frac{2}{\nu^\gamma}\chi_\nu'\left(\tilde{U}_2 - U_2\right)_x \\
&\quad +\chi_\nu\cdot(\tilde{U}_2)_{xx} + (1-\chi_\nu)(U_2)_{xx}. & (5.4.9)
\end{aligned}$$

Hereafter, to write the derivatives of $\hat{U}_2$ in the form, like the first term in the right hand side of (5.4.7), we changed the arguments t, x of $U_2,\ V_2,\ P_2$ to t, d, where $d = d(x,t)$ is defined by

$$d(x,t) = x - \varphi(t).$$

However, risking abuse of notations we still denote $U_2(d,t),\ V_2(d,t),\ P_2(d,t)$ by $U_2,\ V_2,\ P_2$ for the sake of simplicity. After such a transformation of arguments, the terms are easier to deal with, as we shall see later on.

Therefore, we find that $\hat{U}_2$ satisfies

$$(\hat{U}_2)_t - \nu\,(\hat{U}_2)_{xx} + (F(\hat{U}_2))_x = I_1 + I_2 + I_3, \tag{5.4.10}$$

where I_k $(k = 1,2,3)$ are the collections of like-terms according to whether or not their supports are contained in a same sub-domain of $\mathbb{R}$, more precisely, they are defined by

$$I_1 = \chi_\nu\Big((\tilde{U}_2)_t - \nu(\tilde{U}_2)_{xx} + f(\hat{U}_2)\cdot(\tilde{U}_2)_x\Big), \tag{5.4.11}$$

$$I_2 = (1-\chi_\nu)\left((U_2)_t - \nu(U_2)_{xx} + f(\hat{U}_2)(U_2)_x\right), \tag{5.4.12}$$

$$I_3 = (\tilde{U}_2 - U_2)\left(\frac{d_t\chi'_\nu}{\nu^\gamma} - \frac{\chi''_\nu}{\nu^{2\gamma-1}} + \frac{\chi'_\nu f(\hat{U}_2)}{\nu^\gamma}\right) - \frac{2\chi'_\nu}{\nu^{\gamma-1}}\left(\tilde{U}_2 - U_2\right)_x \tag{5.4.13}$$

It is easy to see that the support of $I_1,\ I_2$ is, respectively, a subset of $[0, 2\nu^\gamma]$ and a subset of $[\nu^\gamma, \infty)$, while the support of I_3 is a subset of $[\nu^\gamma, 2\nu^\gamma] \cup [-2\nu^\gamma, -\nu^\gamma]$.

Now we turn to estimate $I_1,\ I_2,\ I_3$. Firstly we handle I_3. In this case one can apply the matching conditions (5.3.72) – (5.3.74) and use Taylor expansions to obtain

$$\partial_x^l(\tilde{U}_2 - U_2)(x,t) = O(1)\nu^{(3-l)\gamma} \tag{5.4.14}$$

on the domain $\{(x,t) \mid \nu^\gamma \le |x-\varphi(t)| \le 2\,\nu^\gamma,\ 0 \le t \le T\}$ and $l = 0,1,2,3$. From these estimates (5.4.14), which can also be found, e.g. in Goodman and Xin [20], the following assertion follows easily that

$$I_3 = O(1)\nu^{2\gamma} \text{ as } \nu \to 0. \tag{5.4.15}$$

Moreover, we have

$$\int_{\mathbb{R}} |I_3(x,t)|^2 dx = \int_{\{\nu^\gamma \le |x-\varphi(t)| \le 2\,\nu^\gamma\}} |I_3(x,t)|^2 dx \le C\nu^{5\gamma}. \tag{5.4.16}$$

To deal with $I_1,\ I_2$ we rearrange the terms of $I_1,\ I_2$ as follows

$$\begin{aligned} I_1 &= \chi_\nu\Big((\tilde{U}_2)_t - \nu(\tilde{U}_2)_{xx} + (F(\tilde{U}_2))_x\Big) + \chi_\nu\cdot\Big(f(\hat{U}_2) - f(\tilde{U}_2)\Big)(\tilde{U}_2)_x, \\ &= I_{1a} + I_{1b}, \end{aligned} \tag{5.4.17}$$

$$\begin{aligned} I_2 &= (1-\chi_\nu)\Big((U_2)_t - \nu(U_2)_{xx} + (F(U_2))_x\Big) + (1-\chi_\nu)\Big(f(\hat{U}_2) - f(U_2)\Big)(U_2)_x, \\ &= I_{2a} + I_{2b}. \end{aligned} \tag{5.4.18}$$

Moreover, I_{1b} can be rewritten as

$$I_{1b} = \chi_\nu \int_0^1 f'(s\hat{U}_2 + (1-s)\tilde{U}_2)ds\cdot(\hat{U}_2 - \tilde{U}_2)(\tilde{U}_2)_x. \tag{5.4.19}$$

Note that $\operatorname{supp} I_{1b} \subset \{(x,t) \in Q_T \mid |x-\varphi(t)| \le 2\,\nu^\gamma\}$ and $\hat{U}_2(x,t) = \tilde{U}_2(x,t)$ if $\{(x,t) \in Q_T \mid |x-\varphi(t)| \le \nu^\gamma\}$. Therefore, from (5.4.14) and (5.4.19) we obtain

$$|I_{1b}| = \frac{C}{\nu}|(\hat{U}_2 - \tilde{U}_2)(\tilde{U}_2)'| = O(\nu^{3\gamma-1}), \tag{5.4.20}$$

where we choose $\gamma > \frac{1}{3}$ so that $3\gamma - 1 > 0$. Recalling the construction of $\tilde{U}_2$, from equations (5.3.69) – (5.3.71), we can rewrite I_{1a} as

$$\begin{aligned} I_{1a} &= \chi_\nu \left((\tilde{U}_2)_t - \nu(\tilde{U}_2)_{xx} + (F(\tilde{u}_0) + \nu f(\tilde{u}_0)\tilde{u}_1 + \nu^2(f(\tilde{u}_0)\tilde{u}_2 + \frac{1}{2}f'(\tilde{u}_0)\tilde{u}_1^2) + R_{\tilde{u}})_x \right) \\ &= \chi_\nu (R_{\tilde{u}})_x, \end{aligned} \tag{5.4.21}$$

where the remainder $R_{\tilde{u}}$ is defined by

$$R_{\tilde{u}} = F(\tilde{U}_2) - \left(F(\tilde{u}_0) + \nu f(\tilde{u}_0)\tilde{u}_1 + \nu^2(f(\tilde{u}_0)\tilde{u}_2 + \frac{1}{2}f'(\tilde{u}_0)\tilde{u}_1^2) \right) = O(\nu^3).$$

Thus

$$|I_{1a}| \leq |(R_{\tilde{u}})_x| = \frac{1}{\nu}|(R_{\tilde{u}})'| = O(\nu^2). \tag{5.4.22}$$

In a similar manner, we now handle I_2, and rewrite I_{2b} as

$$I_{2b} = (1-\chi_\nu) \int_0^1 f'(s\hat{U}_2 + (1-s)U_2)ds \cdot (\hat{U}_2 - U_2)(U_2)_x. \tag{5.4.23}$$

It is easy to see that $\operatorname{supp} I_{2b} \subset \{|d| \geq \nu^\gamma\}$ and $\hat{U}_2 = U_2$ if $|d| \geq 2\nu^\gamma$. From the fact that $\hat{U}_2 - U_2 = \chi_\nu(\tilde{U}_2 - U_2)$ and (5.4.14) it follows that

$$|I_{2b}| \leq C\,\chi_\nu |(\tilde{U}_2 - U_2)(U_2)_x| = O(\nu^{3\gamma}). \tag{5.4.24}$$

As for I_{2a}, invoking equations (5.3.29) and (5.3.30) we assert that there holds

$$\begin{aligned} I_{2a} &= (1-\chi_\nu)\Big((U_2)_t - \nu(U_2)_{xx} + (F(u_0) + \nu f(u_0)u_1 + R_u)_x\Big) \\ &= (1-\chi_\nu)(R_u)_x + O(\nu^2). \end{aligned} \tag{5.4.25}$$

and the remainder R_u is given by

$$R_u = F(U_2) - (F(u_0) + \nu f(u_0)u_1) = O(\nu^2),$$

hence

$$I_{2a} = O(\nu^2). \tag{5.4.26}$$

On the other hand, we have

$$\begin{aligned} \int_{\mathbb{R}} |I_1(x,t)|^2 dx &= \int_{\{|x-\varphi(t)|\leq 2\,\nu^\gamma\}} |I_1(x,t)|^2 dx \\ &\leq \int_{\{|x-\varphi(t)|\leq \nu^\gamma\}} |I_{1a}(x,t)|^2 dx + \int_{\{\nu^\gamma \leq |x-\varphi(t)|\leq 2\,\nu^\gamma\}} |I_{1b}(x,t)|^2 dx \\ &\leq C(\nu^{2*2+1} + \nu^{6\gamma-2+\gamma}) \leq C\,\nu^\gamma. \end{aligned} \tag{5.4.27}$$

Here we used the simple inequalities: $6\gamma - 2 + \gamma < 5$ and $6\gamma - 2 > 0$ since we assume $\gamma \in (\frac{1}{3}, 1)$.

Similarly, one can obtain

$$\int_{\mathbb{R}} |I_2(x,t)|^2 dx = \int_{\{|x-\varphi(t)| \geq \nu^\gamma\}} |I_2(x,t)|^2 dx \leq C(\nu^{2*2+1} + \nu^{6\gamma+\gamma}) \leq C\,\nu^{7\gamma} \quad (5.4.28)$$

In conclusion, from (5.4.10), (5.4.15), (5.4.20), (5.4.22), (5.4.24) and (5.4.26) we are in a position to assert that $\hat{U}_2^\nu$ satisfies the equation in the following sense

$$(\hat{U}_2^\nu)_t - \nu(\hat{U}_2^\nu)_{xx} + (F(\hat{U}_2^\nu))_x = O(\nu^\alpha),$$

as $\nu \to 0$. Here $\alpha = 3\gamma - 1$ and we used the fact that $3\gamma - 1 < 2\gamma < 2$ by assumption $\gamma < 1$. Furthermore, from construction we see easily that the initial data is satisfied asymptotically too.

Part 2. We now turn to investigate the convergence of $\hat{V}_2$. Similar computations show that $\hat{V}_2$ can be written in terms of $\tilde{V}_2$, V_2 as

$$(\hat{V}_2)_t = \frac{d_t}{\nu^\gamma}\chi'_\nu \left(\tilde{V}_2 - V_2\right) + \chi_\nu \cdot (\tilde{V}_2)_t + (1-\chi_\nu)(V_2)_t, \quad (5.4.29)$$

$$(\hat{V}_2)_x = \frac{1}{\nu^\gamma}\chi'_\nu \left(\tilde{V}_2 - V_2\right) + \chi_\nu \cdot (\tilde{V}_2)_x + (1-\chi_\nu)(V_2)_x, \quad (5.4.30)$$

$$(\hat{V}_2)_{xx} = \frac{1}{\nu^{2\gamma}}\chi''_\nu \left(\tilde{V}_2 - V_2\right) + \frac{2}{\nu^\gamma}\chi'_\nu \left(\tilde{V}_2 - V_2\right)_x + \chi_\nu \cdot (\tilde{V}_2)_{xx} + (1-\chi_\nu)(V_2)_{xx}, \quad (5.4.31)$$

and $\hat{V}_2$ satisfies the following equation

$$(\hat{V}_2)_t - \nu\,(\hat{V}_2)_{xx} + (f(\hat{U}_2)\hat{V}_2)_x = J_1 + J_2 + J_3, \quad (5.4.32)$$

where J_k $(k = 1,2,3)$ are given, according to their supports, by

$$J_1 = \chi_\nu\Big((\tilde{V}_2)_t - \nu(\tilde{V}_2)_{xx} + \Big(f(\hat{U}_2)\tilde{V}_2\Big)_x\Big), \quad (5.4.33)$$

$$J_2 = (1-\chi_\nu)\left((V_2)_t - \nu(V_2)_{xx} + \left(f(\hat{U}_2)V_2\right)_x\right), \quad (5.4.34)$$

$$J_3 = \left(\frac{d_t\chi'_\nu}{\nu^\gamma} - \frac{\chi''_\nu}{\nu^{2\gamma-1}} + \frac{f(\hat{U}_2)\chi'_\nu}{\nu^\gamma}\right)(\tilde{V}_2 - V_2) - \frac{2\chi'_\nu}{\nu^{\gamma-1}}(\tilde{V}_2 - V_2)_x. \quad (5.4.35)$$

Since for $\hat{V}_2$, we also have the same estimate (5.4.14) which is valid for $\hat{U}_2$, namely we have

$$\partial_x^l(\tilde{V}_2 - V_2) = O(1)\nu^{(3-l)\gamma} \quad (5.4.36)$$

on the domain $\{(x,t) \mid \nu^\gamma \le |x-\varphi(t)| \le 2\,\nu^\gamma,\ 0 \le t \le T\}$ and $l = 0, 1, 2, 3$. Thus it follows from (5.4.36) and the uniform boundedness of $\hat{U}_2$ that

$$J_3 = O(1)\nu^{2\gamma}, \text{ as } \nu \to 0. \tag{5.4.37}$$

The investigation of convergence for $J_1,\ J_2$ is more technically complicated than $I_1,\ I_2$. Rewrite $J_1,\ J_2$ as

$$\begin{aligned} J_1 &= \chi_\nu\Big((\tilde{V}_2)_t - \nu(\tilde{V}_2)_{xx} + (f(\tilde{U}_2)\tilde{V}_2)_x\Big) + \chi_\nu \cdot \Big((f(\hat{U}_2) - f(\tilde{U}_2))\,\tilde{V}_2\Big)_x \\ &= J_{1a} + J_{1b}, \end{aligned} \tag{5.4.38}$$

and

$$\begin{aligned} J_2 &= (1-\chi_\nu)\Big((V_2)_t - \nu(V_2)_{xx} + (f(U_2)V_2)_x\Big) + (1-\chi_\nu)\,\Big((f(\hat{U}_2) - f(U_2)\,V_2\Big)_x \\ &= J_{2a} + J_{2b}. \end{aligned} \tag{5.4.39}$$

We now deal with J_{1b} which can be changed to

$$\begin{aligned} J_{1b} &= \chi_\nu\left(\int_0^1 f'(s\hat{U}_2 + (1-s)\tilde{U}_2)ds(\hat{U}_2 - \tilde{U}_2)\,\tilde{V}_2\right)_x \\ &= \chi_\nu\left(\int_0^1 f'(s\hat{U}_2) + (1-s)\tilde{U}_2)ds\,\tilde{V}_2\right)_x(\hat{U}_2 - \tilde{U}_2) \\ &\quad + \chi_\nu \int_0^1 f'(s\hat{U}_2) + (1-s)\tilde{U}_2)ds\,\tilde{V}_2\left(\hat{U}_2 - \tilde{U}_2\right)_x \\ &= O(\nu^{3\gamma-1}) + O(\nu^{2\gamma}) = O(\nu^{3\gamma-1}), \end{aligned} \tag{5.4.40}$$

here we used that $3\gamma - 1 < 2\gamma$ since $\gamma < 1$.

Rewriting J_{1a} as

$$\begin{aligned} J_{1a} &= \chi_\nu\left((\tilde{V}_2)_t - \nu(\tilde{V}_2)_{xx} + \left((f(\tilde{u}_0) + f'(\tilde{u}_0)(\nu\tilde{u}_1 + \nu^2\tilde{u}_2) + \frac{1}{2}f''(\tilde{u}_0)(\nu\tilde{u}_1)^2)\tilde{V}_2\right)_x\right) \\ &\quad + \chi_\nu(R_{\tilde{v}}\tilde{V}_2)_x \\ &= \chi_\nu\Big(O(\nu^2) + (R_{\tilde{v}}\tilde{V}_2)_x\Big). \end{aligned} \tag{5.4.41}$$

Here, equations (5.3.76) - (5.3.78) were used. And the remainder $R_{\tilde{v}}$ is

$$R_{\tilde{v}} = f(\tilde{U}_2) - (f(\tilde{u}_0) + f'(\tilde{u}_0)(\nu\tilde{u}_1 + \nu^2\tilde{u}_2) + \frac{1}{2}f''(\tilde{u}_0)(\nu\tilde{u}_1)^2) = O(\nu^3)$$

Therefore, from (5.4.41) one has

$$J_{1a} = \chi_\nu\Big(O(\nu^2) + (R_{\tilde{v}}\tilde{V}_2)_x\Big) = O(\nu^2). \tag{5.4.42}$$

The terms $J_{2a},\ J_{2b}$ can be estimated in a similar way and we obtain

$$J_{2b} = O(\nu^{3\gamma}) + O(\nu^{2\gamma}) = O(\nu^{2\gamma}), \tag{5.4.43}$$

and

$$\begin{aligned} J_{2a} &= (1-\chi_\nu)\Big((V_2)_t - \nu(V_2)_{xx} + \big((f(u_0)+\nu f'(u_0)u_1 + R_v)V_2\big)_x\Big) \\ &= (1-\chi_\nu)\Big(O(\nu^2) + (R_v V_2)_x\Big), \end{aligned}$$

where R_v is given by $R_v = f(U_2) - (f(u_0) + \nu f'(u_0)u_1) = O(\nu^2)$. It is easy to see that

$$J_{2a} \;=\; O(\nu^2). \tag{5.4.44}$$

On the other hand, we have the following estimates of integral type

$$\begin{aligned} \int_{\mathbb{R}} |J_1(x,t)|^2 dx &= \int_{\{|x-\varphi(t)| \le 2\,\nu^\gamma\}} |J_1(x,t)|^2 dx \\ &\le \int_{\{|x-\varphi(t)| \le \nu^\gamma\}} |J_{1a}(x,t)|^2 dx + \int_{\{\nu^\gamma \le |x-\varphi(t)| \le 2\,\nu^\gamma\}} |J_{1b}(x,t)|^2 dx \\ &\le C(\nu^{2*2+1} + \nu^{6\gamma-2+\gamma}) \le C\,\nu^\gamma. \end{aligned} \tag{5.4.45}$$

and

$$\int_{\mathbb{R}} |J_2(x,t)|^2 dx = \int_{\{|x-\varphi(t)| \ge \nu^\gamma\}} |J_2(x,t)|^2 dx \;\le\; C(\nu^{2*2+1} + \nu^{4\gamma+\gamma}) \le C\,\nu^\gamma \tag{5.4.46}$$

Therefore, it follows from (5.4.32), (5.4.37), (5.4.40), (5.4.42), (5.4.43) and (5.4.44) that $\hat{V}_2^\nu$ satisfies the equation in the following sense

$$(\hat{V}_2^\nu)_t - \nu(\hat{V}_2^\nu)_{xx} + (f(\hat{U}_2^\nu)\hat{V}_2^\nu)_x = O(\nu^{3\gamma-1}),$$

as $\nu \to 0$. By construction, the initial data is satisfied asymptotically as well.

Part 3. Finally we turn to investigate the convergence of $\hat{P}_2$. Computations show that the derivatives of $\hat{P}_2$ can be written in terms of $\tilde{P}_2,\ P_2$ as

$$(\hat{P}_2)_t = \frac{d_t}{\nu^\gamma}\chi_\nu'\left(\tilde{P}_2 - P_2\right) + \chi_\nu\cdot(\tilde{P}_2)_t + (1-\chi_\nu)(P_2)_t, \tag{5.4.47}$$

$$(\hat{P}_2)_x = \frac{1}{\nu^\gamma}\chi_\nu'\left(\tilde{P}_2 - P_2\right) + \chi_\nu\cdot(\tilde{P}_2)_x + (1-\chi_\nu)(P_2)_x, \tag{5.4.48}$$

$$\begin{aligned} (\hat{P}_2)_{xx} &= \frac{1}{\nu^{2\gamma}}\chi_\nu''\left(\tilde{P}_2 - P_2\right) + \frac{2}{\nu^\gamma}\chi_\nu'\left(\tilde{P}_2 - P_2\right)_x \\ &\quad + \chi_\nu\cdot(\tilde{P}_2)_{xx} + (1-\chi_\nu)(P_2)_{xx}, \end{aligned} \tag{5.4.49}$$

and $\hat{P}_2$ satisfies the following equation

$$-(\hat{P}_2)_t - \nu\,(\hat{P}_2)_{xx} - f(\hat{U}_2)(\hat{P}_2)_x = K_1 + K_2 + K_3, \tag{5.4.50}$$

where K_i $(i = 1, 2, 3)$ are given, according to their supports, by

$$K_1 = -\chi_\nu\Big((\tilde{P}_2)_t + \nu(\tilde{P}_2)_{xx} + f(\hat{U}_2)(\tilde{P}_2)_x\Big), \tag{5.4.51}$$

$$K_2 = -(1-\chi_\nu)\left((P_2)_t + \nu(P_2)_{xx} + f(\hat{U}_2)\,(P_2)_x\right), \tag{5.4.52}$$

$$K_3 = -\left(\frac{d_t\chi'_\nu}{\nu^\gamma} + \frac{\chi''_\nu}{\nu^{2\gamma-1}} + \frac{f(\hat{U}_2)\chi'_\nu}{\nu^\gamma}\right)(\tilde{P}_2 - P_2) - \frac{2\chi'_\nu}{\nu^{\gamma-1}}(\tilde{P}_2 - P_2)_x. \tag{5.4.53}$$

By similar arguments as done for $\hat{U}_2^\nu$, we can prove that

$$-(\hat{P}_2^\nu)_t - \nu(\hat{P}_2^\nu)_{xx} - f(\hat{U}_2^\nu)(\hat{P}_2^\nu)_x = O(\nu^{3\gamma-1}),$$

as $\nu \to 0$.

5.4.2 Proof of the convergence

This sub-section is devoted to the proof of Theorem 4.1. Since this sub-section is concerned with the proof of convergence as $\nu \to 0$, we denote $\hat{U}_2^\nu,\ \hat{V}_2^\nu,\ \hat{P}_2^\nu$ by $\hat{U}^\nu,\ \hat{V}^\nu,\ \hat{P}^\nu$, respectively, for the sake of simplicity. We begin with the following lemma.

Lemma 5.4.2 *For* η *defined in* (5.4.5)*,*

$$\sup_{0\le t\le T}\|u^\nu(\cdot,t) - u(\cdot,t)\| + \sup_{0\le t\le T}\|v^\nu(\cdot,t) - v(\cdot,t)\| \le C\nu^\eta, \tag{5.4.54}$$

$$\sup_{0\le t\le T}\|p_{\nu,n}(\cdot,t) - p(\cdot,t)\|_{L^1_{\rm loc}(\mathbb{R})} \to 0, \tag{5.4.55}$$

as $\nu \to 0$, then $n \to \infty$. *Here* $p_{\nu,n}$ *denotes the solution to smoothed adjoint problem* (5.3.5) – (5.3.6)*.*

Proof. Firstly, by construction of the approximate solutions, we conclude that i) for $(x, t) \in \{|x - \varphi(t)| \ge 2\nu^\gamma, 0 \le t \le T\}$,

$$\hat{U}_2^\nu(x,t) = u(x,t) + O(1)\nu,$$

where $O(1)$ denotes a function which is square-integrable over outer region by the argument in sub-section 4.1.
ii) for $(x, t) \in \{|x - \varphi(t)| \le \nu^\gamma\}$,

$$\hat{U}_2^\nu(x,t) = \tilde{u}_0(x,t) + O(1)\nu^\gamma,$$

iii) and for $(x,t) \in \{\nu^\gamma \le |x-\varphi(t)| \le 2\nu^\gamma\}$,

$$\hat{U}_2^\nu(x,t) = U_2^\nu(x,t) + \chi_\nu(x,t)(\tilde{U}_2^\nu(x,t) - U_2^\nu(x,t)) \sim U_2^\nu(x,t) + O(1)\nu^{3\gamma}.$$

Here we have used again the estimate $\tilde{U}_2^\nu(x,t) - U_2^\nu(x,t) = O(1)\nu^{3\gamma}$. We can also establish similar estimates for $\hat{V}_2^\nu$, $\hat{P}_2^\nu$. Therefore, one can obtain

$$\sup_{0\le t\le T} \|u(\cdot,t) - \hat{U}^\nu(\cdot,t)\|^2 + \sup_{0\le t\le T} \|v(\cdot,t) - \hat{V}^\nu(\cdot,t)\|^2 \le C\nu^{\min\{3\gamma,2\}} \quad (5.4.56)$$

$$\sup_{0\le t\le T} \|p_n(\cdot,t) - \hat{P}^\nu(\cdot,t)\|^2 \le C\nu^{\min\{3\gamma,2\}} \quad (5.4.57)$$

where p_n is the reversible solution to the inviscid adjoint equation $-\partial_t p - f(u)\partial_x p = 0$ with final data $p(x,T) = p_n^T(x)$.

By Theorem 4.1.10 which is concerned with the stability (with respect to coefficient and final data) of reversible solutions in [6] (see also Theorem 5.1 in the appendix of this chapter), and the assumptions of final data p_n and the one-sided Lipschitz condition we conclude that the reversible solution p_n of the backward problem with initial data p_n satisfies

$$p_n \to p \text{ in } C([0,T]\times[-R,R])$$

for any $R > 0$, here p is the reversible solution to $-p_t - f(u)\partial_x p = 0$, in $\mathbb{R}\times(0,T)$ with finial data $p(x,T) = p^T(x)$ for $x \in \mathbb{R}$. Therefore, we have

$$\sup_{0\le t\le T} \|p_n - p\|_{L^1_{\text{loc}}(\mathbb{R})} \to 0, \quad (5.4.58)$$

as $n \to \infty$.

Secondly, we need to estimate $\sup_{0\le t\le T}\|u^\nu(\cdot,t) - \hat{U}^\nu(\cdot,t)\|^2$, etc. Suppose that we have obtained

$$\sup_{0\le t\le T} \|u^\nu(\cdot,t) - \hat{U}^\nu(\cdot,t)\|^2 \le C\nu^{\eta_1} \quad (5.4.59)$$

where $\eta_1 = \gamma + 1$. Then we arrive at (5.4.54) by using the triangle inequality, the estimate (5.4.56) and the fact that $\min\left\{\frac{3}{2}\gamma, \frac{1+\gamma}{2}, 1\right\} = \min\left\{\frac{3}{2}\gamma, \frac{1+\gamma}{2}\right\}$ (since $\frac{1}{3} < \gamma < 1$)). So we conclude for v, p.

Part 1. Now we prove (5.4.59). To this end, we define

$$\bar{w}(x,t) = u^\nu(x,t) - \hat{U}^\nu(x,t),$$

Then $\bar{w}(x,t)$ satisfies

$$\bar{w}_t - \nu\bar{w}_{xx} + f(u^\nu)\bar{w}_x = \mathcal{F}, \quad (5.4.60)$$

$$\bar{w}(x,0) = \bar{w}_0(x). \quad (5.4.61)$$

Here

$$\mathcal{F} := -\sum_{i=1}^{3} I_i - Q(x,t) - f'(\hat{U}^\nu)\bar{w}\hat{U}_x^\nu \tag{5.4.62}$$

$$Q := \Big(f(u^\nu) - f(\hat{U}^\nu) - f'(\hat{U}^\nu)\bar{w}\Big)\,\hat{U}_x^\nu. \tag{5.4.63}$$

Rescaling as follows

$$\bar{w}(x,t) = \nu w(y,\tau), \text{ where } y = \frac{x-\varphi(t)}{\nu}, \text{ and } \tau = \frac{t}{\nu},$$

one has

$$\bar{w}_t(x,t) = w_\tau(y,\tau) - \varphi'(\nu\tau)w_y, \quad \bar{w}_x(x,t) = w_y(y,\tau), \quad \bar{w}_{xx}(x,t) = \frac{1}{\nu}w_{yy}(y,\tau),$$

and problem (5.4.60) – (5.4.61) turns out to be

$$w_\tau - w_{yy} + \left(f(u^\nu) - \varphi'(\nu\tau)\right) w_y = \mathcal{F}(\nu y + \varphi, \nu\tau), \tag{5.4.64}$$

$$w(x,0) = w_0(x). \tag{5.4.65}$$

Here the initial data w_0 can be chosen very small such that

$$\|w_0\|^2_{H^1(\mathbb{R})} \le C\nu \le C\nu^\gamma,$$

provided that ν is suitably small.

The existence of solution $w \in C([0,T/\nu]; H^1(\mathbb{R}))$ to problem (5.4.64) – (5.4.65) follows from the method of continuation of local solution, which is based on local existence of solution and *a priori* estimates stated in the following proposition

Proposition 5.4.3 *(A priori estimates) Suppose that problem* (5.4.64) – (5.4.65) *has a solution $w \in C([0,\tau_0]; H^1(\mathbb{R}))$ for some $\tau_0 \in (0, T/\nu]$. There exists positive constants μ_1, ν_1 and C, which are independent of ν, τ_0, such that if*

$$\nu \in (0,\nu_1], \ \sup_{0\le\tau\le\tau_0} \|w(\tau,\cdot)\|_{H^1(\mathbb{R})} + \mu_0 \le \mu_1, \tag{5.4.66}$$

then

$$\sup_{0\le\tau\le\tau_0} \|w(\tau,\cdot)\|^2_{H^1(\mathbb{R})} + \int_0^{\tau_0} \|w(\tau,\cdot)\|^2_{H^2(\mathbb{R})} d\tau \le C\nu^\gamma.$$

Proof. **Step 1.** By the maximum principle and construction of approximate solution $\hat{U}^\nu$, we have

$$\|u^\nu\|_{L^\infty(Q_{\tau_0})} \le C, \quad \|\hat{U}^\nu\|_{L^\infty(Q_{\tau_0})} \le C. \tag{5.4.67}$$

Whence, from the smallness assumption (5.4.66) and definition of Q it follows that

$$|Q(x,t)| \le C|\bar{w}^2 \hat{U}_x^\nu|. \tag{5.4.68}$$

Step 2. Multiplying eq. (5.4.64) by w and integrating the resulting equation with respect to y over $\mathbb{R}$ we obtain

$$\frac{1}{2}\frac{d}{d\tau}\|w\|^2 + \|w_y\|^2 + \int_{\mathbb{R}} \left(f(u^\nu) - \varphi'(\nu\tau)\right) w_y w dy \;=\; \int_{\mathbb{R}} \mathcal{F}(\nu y + \varphi, \nu\tau) w dy \tag{5.4.69}$$

We first deal with the term

$$\int_{\mathbb{R}} f'(\hat{U}^\nu)\bar{w}\hat{U}_x^\nu w dy = \int_{\mathbb{R}} f'(\hat{U}^\nu) w^2 \nu \hat{U}_x^\nu dy.$$

From the property of profile $\hat{U}$ we have

$$\nu \hat{U}_x^\nu = \hat{U}_y^\nu \to 0, \quad \text{as } \nu \to 0.$$

Thus

$$\left|\int_{\mathbb{R}} f'(\hat{U}^\nu)\bar{w}\hat{U}_x^\nu w dy\right| \le C\|w\|^2.$$

For the term of Q, it is easier to obtain that

$$\left|\int_{\mathbb{R}} Q(y,\tau)\hat{U}_x^\nu w dy\right| \le C\int_{\mathbb{R}} |\bar{w}^2\hat{U}_x^\nu w| dy \le C\|w\|^2.$$

It remains to deal with the term $\int_{\mathbb{R}} Iwdy$ where $I = \sum_{i=1}^3 I_i$. We invoke the L^2-norm estimates for I, i.e. (5.4.27) and (5.4.28), which have been obtained in Subsection 3.1, and get

$$\begin{aligned}\left|\int_0^\tau \int_{\mathbb{R}} Iwd\tau dy\right| \;&\le\; C\left(\int_0^\tau \int_{\mathbb{R}} |I|^2 d\tau dy + \int_0^\tau \int_{\mathbb{R}} |w|^2 d\tau dy\right) \\ &\le\; C\nu^\gamma + C\int_0^\tau \int_{\mathbb{R}} |w|^2 d\tau dy. \end{aligned} \tag{5.4.70}$$

Finally, by the Young inequality one gets

$$\left|\int_{\mathbb{R}} \left(f(u^\nu) - \varphi'(\nu\tau)\right) w_y w dy\right| \le \frac{1}{2}\|w_y\|^2 + C\|w\|^2.$$

Therefore, the above arguments and integrating (5.4.69) with respect to τ yield

$$\frac{1}{2}\|w(\tau)\|^2 + \frac{1}{2}\int_0^\tau \|w_y(s)\|^2 ds \;\le\; C\int_0^\tau \|w(s)\|^2 ds + C\nu^\gamma, \tag{5.4.71}$$

from which and the Gronwall inequality in the integral form we then arrive at

$$\|w(\tau)\|^2 \le C\nu^\gamma.$$

So we also obtain

$$\|u^\nu(\cdot,t) - \hat{U}^\nu(\cdot,t)\|^2 = \nu\|w(\tau)\|^2 \le C\nu^{1+\gamma}.$$

Step 3. Next we multiply eq. (5.4.65) by $-w_{yy}$ and integrate the resultant with respect to y to get

$$\begin{aligned}\frac{1}{2}\frac{d}{d\tau}\|w_y\|^2 &+ \|w_{yy}\|^2 - \int_{\mathbb{R}} (f(u^\nu) - \varphi'(\nu\tau))\, w_{yy}w_y dy \\ &= -\int_{\mathbb{R}} \mathcal{F}(\nu y + \varphi, \nu\tau) w_{yy} dy, \end{aligned} \tag{5.4.72}$$

Using (5.4.67) and estimates on I_i ($I = 1,2,3$), from the Young inequality we then arrive at

$$\frac{1}{2}\|w_y(\tau)\|^2 + \frac{1}{2}\int_0^\tau \|w_{yy}(s)\|^2 ds \le C\int_0^\tau \|w_y(s)\|^2 ds + C\nu^\gamma, \tag{5.4.73}$$

making use of Gronwall's inequality again one has

$$\|w_y(\tau)\|^2 \le C\nu^\gamma. \tag{5.4.74}$$

Part 2. In this part we are in a position to prove the convergence of $p_{\nu,n} - \hat{P}^\nu$, namely to prove

$$\sup_{0\le t\le T} \|p_{\nu,n}(\cdot,t) - \hat{P}^\nu(\cdot,t)\|^2 \le C\nu^\eta. \tag{5.4.75}$$

Let

$$\bar{q} = p_{\nu,n} - \hat{P}^\nu,$$

then computations yield that $\bar{q}$ satisfies

$$-\bar{q}_t - \nu\bar{q}_{xx} - f(u^\nu)\bar{q}_x = \mathcal{G}, \tag{5.4.76}$$

$$\bar{q}(x,0) = \bar{q}_0(x). \tag{5.4.77}$$

Here

$$\mathcal{G} := -\sum_{i=1}^{3} K_i - Q_1(x,t) - f'(\hat{U}^\nu)\bar{w}\hat{P}^\nu_x \tag{5.4.78}$$

$$Q_1 := \left(f(u^\nu) - f(\hat{U}^\nu) - f'(\hat{U}^\nu)\bar{w}\right)\hat{P}^\nu_x, \tag{5.4.79}$$

and K_i (with $i = 1, 2, 3$) can be defined in a slightly different way as in the arguments for $\hat{V}^\nu$.

Rescaling again as follows

$$\bar{q}(x,t) = \nu q(y,\tau), \text{ where } y = \frac{x-\varphi(t)}{\nu}, \text{ and } \tau = \frac{T-t}{\nu},$$

one has

$$\bar{q}_t(x,t) = -q_\tau(y,\tau) - \varphi'(\nu\tau)q_y, \quad \bar{q}_x(x,t) = q_y(y,\tau), \quad \bar{q}_{xx}(x,t) = \frac{1}{\nu}q_{yy}(y,\tau),$$

and problem (5.4.76) – (5.4.77) can be rewritten as

$$q_\tau - q_{yy} - \left(f(u^\nu) - \varphi'(\nu\tau)\right) q_y = \mathcal{G}(\nu y + \varphi, \nu\tau), \tag{5.4.80}$$

$$q(x,0) = q_0(x). \tag{5.4.81}$$

Employing again the method of continuation of a local solution based upon a priori estimates, we prove easily the existence of solution $q \in C([0,\tau_0]; H^2(\mathbb{R}))$, also we have

$$\|p_{\nu,n}(\cdot,t) - \hat{P}^\nu(\cdot,t)\|^2 = \nu\|q(\tau)\|^2 \leq C\nu^{1+\gamma},$$

here we rewrite p_ν as $p_{\nu,n}$ to indicate that solution depends on n too. This estimate implies that

$$\sup_{0\leq t\leq T} \|p_{\nu,n}(\cdot,t) - p_n(\cdot,t)\|_{L^1_{\mathrm{loc}}(\mathbb{R})} \to 0,$$

invoking (5.4.58) we obtain

$$\sup_{0\leq t\leq T} \|p_{\nu,n}(\cdot,t) - p(\cdot,t)\|_{L^1_{\mathrm{loc}}(\mathbb{R})} \to 0,$$

when $\nu \to 0$ firstly, then $n \to \infty$.

Furthermore, we assume that the initial $p|_{t=0}$ is bounded in $H^2(\mathbb{R})$, similar to the argument in Goodman and Xin [20], we can prove that for any constant $h > 0$

$$\sup_{0\leq t\leq T} \|p_{\nu,n} - p_n\|_{L^\infty(\Omega_h)} \to 0, \tag{5.4.82}$$

as $\nu \to 0$, here Ω_h is defined by

$$\Omega_h = \{(x,t) \in Q_T\} \mid |x - \varphi(t)| > h\}.$$

On the other hand, from the stability (of the reversible solution) theorem 4.1.10 in [6], we have

$$\sup_{0\leq t\leq T} \|p_n - p\|_{L^\infty(\Omega_h)} \to 0, \tag{5.4.83}$$

as $n \to \infty$. Thus there holds

$$\sup_{0 \le t \le T} \|p_{\nu,n} - p\|_{L^\infty(\Omega_h)} \to 0, \tag{5.4.84}$$

as $\nu \to 0$, then $n \to \infty$.

Part 3. To prove the convergence of $\bar{O} := v^\nu - \hat{V}_2^\nu$, we rewrite equations (5.3.3) and (5.3.104) as follows

$$\begin{aligned} v_t^\nu - \nu v_{xx}^\nu + f(u^\nu)(v^\nu)_x + (f(u^\nu))_x v^\nu &= 0, & (5.4.85)\\ (\hat{V}_2^\nu)_t - \nu(\hat{V}_2^\nu)_{xx} + f(\hat{U}_2^\nu)(\hat{V}_2^\nu)_x + (f(\hat{U}_2^\nu))_x \hat{V}_2^\nu &= \sum_{i=1}^{3} J_i. & (5.4.86)\end{aligned}$$

Then we find $\bar{O}$ satisfies

$$\bar{O}_t - \nu \bar{O}_{xx} + f(u^\nu)\bar{O}_x + (f(u^\nu))_x \bar{O} = \mathcal{H}, \tag{5.4.87}$$

and

$$\mathcal{H} = -\Big(f(u^\nu) - f(\hat{U}_2^\nu)\Big)_x \hat{V}_2^\nu - \Big(f(u^\nu) - f(\hat{U}_2^\nu)\Big)(\hat{V}_2^\nu)_x - \sum_{i=1}^{3} J_i.$$

We use again the rescaling technique as follows

$$\bar{O}(x,t) = \nu O(y,\tau), \text{ where } y = \frac{x - \varphi(t)}{\nu}, \text{ and } \tau = \frac{t}{\nu},$$

To overcome the difficulty, due to the last term in the left hand side of (5.4.87), in the proof of the convergence of $\bar{O}$, we make use of the convergence result in Part 1, the estimate $\nu \int_0^t \|u_x^\nu(\tau)\|^2 d\tau \le C$, the interpolation inequality in the following form

$$\|f\|_{L^4(\mathbb{R})} \le C\|f_x\|^{\frac{1}{4}} \|f\|^{\frac{3}{4}} + C'\|f\|,$$

and the Young inequality of the form: $abc \le \varepsilon a^4 + C_\varepsilon(b^4 + c^2)$. We then estimate

$$\begin{aligned} \left|\int_{\mathbb{R}} (f(u^\nu))_x \bar{O} O dx\right| &\le C\nu \int_{\mathbb{R}} |u_x^\nu| O^2 dx \\ &\le C\nu \|u_x^\nu\| \|O\|_{L^4}^2 \\ &\le C\nu \|u_x^\nu\| (\|O_x\|^{\frac{2}{4}} \|O\|^{\frac{3}{2}} + C'\|O\|^2) \\ &\le C(\nu^2 \|u_x^\nu\|^2 \|O\|^2 + \|O\|^2) + \frac{1}{2}\|O_x\|^2. \quad (5.4.88)\end{aligned}$$

The last term in the right hand side of (5.4.88) can be absorbed by the left hand side. The other terms in (5.4.87) and $\mathcal{H}$ can be treated in a similar way as in Part 1, and we omit the details.

Therefore, the proof of Theorem 4.1 is complete.

5.5 The method of alternating descent directions: Viscous case

This section is concerned with the extension of the arguments in sub-section 2.3 on the choices of alternating descent directions, to the viscous problem. We consider the case that the initial data u^I satisfies

$$\left.\begin{array}{l} u^I \text{ is continuous and smooth up to the shock, and } u^I_x \text{ has only a} \\ \text{discontinuity at } \varphi^I. \text{ Moreover, } u^I,\ u^I_x \text{ are integrable over } Q_T\backslash\Sigma. \end{array}\right\} \quad (5.5.1)$$

As pointed out in the end of Section 1, for any positive ν solutions $\delta u,\ p$ of equations (5.1.6) and (5.1.10) are smooth, thus the Gateaux derivative of the functional J is as follows

$$\delta J = <\delta J(u^I), \delta u^I> = \int_{\mathbb{R}} p(x,0)\delta u^I(x)dx, \quad (5.5.2)$$

where the adjoint state $p = p_\nu$ is the solution to (5.1.10) with initial datum $p(x,T) = u(x,T) - u^D(x)$.

To exploit the possibilities that the alternate descent method provides, we take into account, as in [12], the effects of possible infinitesimal perturbations of initial datum and also infinitesimal translations, and choose the initial data u^I_ε of the form

$$u^I_\varepsilon(x) = u^I_\varepsilon(x + \varepsilon\delta\varphi^I) + \varepsilon\delta u^I_\varepsilon(x), \quad (5.5.3)$$

By a Taylor expansion, (5.5.3) can be rewritten in the following form

$$u^I_\varepsilon(x) = u^I(x) + \varepsilon\left(\delta\varphi^I u^I_x(x) + \delta u^I(x)\right) + O(\varepsilon^2). \quad (5.5.4)$$

Correspondingly we formulate the linearized problem as follows

$$\begin{aligned} (\delta u)_t + (f(u)\delta u)_x &= \nu(\delta u)_{xx}, & (5.5.5)\\ \delta u(x,0) &= \delta\varphi^I u^I_x(x) + \delta u^I(x), & (5.5.6)\end{aligned}$$

and its adjoint problem is

$$\begin{aligned} -p_t - f(u)p_x &= \nu p_{xx}, & (5.5.7)\\ p(x,0) &= p^T_n(x). & (5.5.8)\end{aligned}$$

But by doing this way it leads to some difficulties: A Dirac delta appears in the Taylor expansion (5.5.4) and the initial data (5.5.6) in the case $u^I(x)$ has a jump. How to understand this expansion and how to solve problem (5.5.5) – (5.5.6)? This initial value problem is difficult even (5.5.5) is parabolic for fixed ν. There are not too many references related to this topic: some authors investigate generalized solutions to Burgers' equation with singular data see, e.g. [4], while in [10, 5, 13] the authors studied parabolic equations with a Dirac delta as an initial datum. However the solution exists only for some special cases. Another difficulty is that we need more regular initial data for our construction of asymptotic expansions, moreover, we also consider the limit as $\nu \to 0$, the limit equation of (5.5.5) has discontinuous coefficient, which leads to a term like $(f(u)\delta u)_x$ where $f(u)$ and δu may be discontinuous.

Therefore, we don't expand $u^I(x + \varepsilon\delta\varphi^I)$ directly as done in (5.5.4). To overcome the above difficulties we approximate the initial datum as follows: We use again the cut-off function χ_h for $h > 0$, define $\xi = \frac{x-\varphi^I}{h}$ and choose U^I a smooth function in ξ satisfying the matching conditions

$$\lim_{\xi\to\pm\infty} U^I(\xi) = \lim_{\xi\to\varphi^I\pm 0} u^I(x),$$

then by a Taylor expansion, we obtain

$$\begin{aligned} u^I_{\varepsilon,h}(x) &= \chi_h(x)U^I(\xi) + (1-\chi_h(x))u^I(x+\varepsilon\delta\varphi^I) + \varepsilon\delta u^I_\varepsilon(x) \\ &= \chi_h(x)U^I(\xi) + (1-\chi_h(x))u^I(x) \\ &\quad +\varepsilon\left((1-\chi_h(x))u^I_x(x)\delta\varphi^I + \delta u^I_\varepsilon(x)\right) + O(\varepsilon^2), \end{aligned} \tag{5.5.9}$$

Letting $h \to 0$ we see that there is no any Dirac delta appearing in (5.5.9) any more. The corresponding linearized problem turn out to

$$(\delta u)_t + (f(u)\delta u)_x = \nu(\delta u)_{xx}, \tag{5.5.10}$$

$$\delta u(x,0) = \delta\varphi^I(1-\chi_h(x))u^I_x(x) + \delta u^I(x), \tag{5.5.11}$$

Moreover for any fixed ν we can easily pass the solution $\delta u_{\nu,h}$ of problem (5.5.10) – (5.5.11) to its limit δu_ν as $h \to 0$.

We assume, to begin with, that u^I satisfies (5.5.1). We shall make use of the convergence result (5.4.84), from which one concludes that the smooth solution $p_{\nu,n}$ of problem (5.5.7)–(5.5.8) is very close to its limit p, provided that ν is very small and n is very large. To determine the alternating descent directions, the first thing to be done is to identify the region of influence $[x^-, x^+]$ of the inner boundary of the inviscid adjoint system. Similar to [12], we can compute x^-, x^+, so the region of influence is thus defined. Then we need to identify the variations $(\delta u^I, \delta\varphi^I)$ such that

$$\int_{x^-}^{x^+} p_{\nu,n}(x,0)\left(\delta\varphi^I(1-\chi_h(x))u^I_x(x) + \delta u^I(x)\right)dx = 0. \tag{5.5.12}$$

It is easy to see that we can rewrite (5.5.13) as

$$\int_{[x^-,x^+]\backslash\{\varphi^I\}} p_{\nu,n}(x,0)\left(\delta\varphi^I u_x^I(x)+\delta u^I(x)\right)dx = o_h(1), \tag{5.5.13}$$

where, $o_h(1)$ denotes a small quantity such that $o_h(1)\to 0$ as $h\to 0$.

In [12], it is argued that if $p_{\nu,n}(x,0)$ were constant within the interval $[x^-,x^+]$ as in the inviscid case, this would amount to consider variations such that

$$\delta\varphi^I = -\frac{\int_{x^-}^{x^+}\delta u^I(x)dx}{u^I(x^+)-u^I(x^-)}. \tag{5.5.14}$$

One possibility would be to consider variations δu^I in $[x^-,x^+]$ such that $\int_{x^-}^{x^+}\delta u^I(x)dx = 0$ and $\delta\varphi^I = 0$. The variation of the functional J would then be

$$\delta J = \int_{\{x>x^+\}\cup\{x<x^-\}} p_{\nu,n}(x,0)\delta u^I(x)dx,$$

and the optimal descent direction

$$\delta u^I(x) = -p_{\nu,n}(x,0), \text{ in } \{x>x^+\}\cup\{x<x^-\}.$$

However, the assumption that $p_{\nu,n}(x,0)$ is constant within the interval $[x^-,x^+]$, is *not* true, in general, in the viscous case, and is only true in the inviscid case. Invoking (5.4.84), we find $p_{\nu,n}(x,0)$ in (5.5.13) is close to a constant over Ω_μ, provided that ν is small and n is large. Thus we rewrite (5.5.13) as follows

$$\begin{aligned}
o_h(1) &= \int_{x^-}^{x^+} p_{\nu,n}(x,0)\left(\delta\varphi^I u_x^I(x)+\delta u^I(x)\right)dx \\
&= \int_{\varphi^I-\mu}^{\varphi^I+\mu} + \int_{[x^-,x^+]\backslash[\varphi^I-\mu,\varphi^I+\mu]} p_{\nu,n}(x,0)\left(\delta\varphi^I u_x^I(x)+\delta u^I(x)\right)dx \\
&= I_1 + I_2.
\end{aligned} \tag{5.5.15}$$

Here, μ is a small positive number. Recalling $\delta u^I \in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$, by assumptions (5.5.1) and the results $p_{\nu,n}\in L^\infty(Q_T)$, we see that the integrand is integrable over $\mathbb{R}\backslash\{\varphi^I\}$, then I_1 is small and depends on the small parameter μ (also n,ν, but they are assumed temporarily to be fixed). While for I_2 we can replace $p_{\nu,n}$ by p, however a small error, depending on ν and n, appears. Then (5.5.15) can be rewritten as

$$\begin{aligned}
0 &= C_{h,\nu,\mu,n} + \int_{[x^-,x^+]\backslash[\varphi^I-\mu,\varphi^I+\mu]} p(x,0)\left(\delta\varphi^I u_x^I(x)+\delta u^I(x)\right)dx \\
&= C_{h,\nu,\mu,n} + p(x,0)\left(u^I(x^+)-u^I(x^-)-\left(u^I(\varphi^I+\mu)-u^I(\varphi^I-\mu)\right)\right)\delta\varphi^I \\
&\quad + p(x,0)\left(\int_{[x^-,x^+]\backslash[\varphi^I-\mu,\varphi^I+\mu]} \delta u^I(x)dx\right).
\end{aligned} \tag{5.5.16}$$

Here $C_{h,\nu,\mu,n}$ denotes a small quantity depending on h, ν, μ, n. But by assumption (5.5.1), $u^I(\varphi^I + \mu) - u^I(\varphi^I - \mu) \to [u^I]_{\varphi^I}$ as $\mu \to 0$. Therefore,

$$\begin{aligned} \delta\varphi^I &= -\frac{C_{h,\nu,\mu,n}/p(x,0) + \int_{[x^-,x^+]\setminus[\varphi^I-\mu,\varphi^I+\mu]} \delta u^I(x)dx}{u^I(x^+) - u^I(x^-) - [u^I]_{\varphi^I}} \\ &\sim -\frac{\int_{[x^-,x^+]} \delta u^I(x)dx}{u^I(x^+) - u^I(x^-) - [u^I]_{\varphi^I}}. \end{aligned} \tag{5.5.17}$$

This implies that we can choose a descent direction as the case that $p_{\nu,n}(x,0)$ is a constant, at least for numerical simulation since there are some errors when we compute any quantity. Moreover, we can extend δu^I to the sub-domain $[x^-, x^+]$ such that

$$\int_{x^-}^{x^+} \delta u^I(x)dx = 0,$$

whence

$$\delta\varphi^I = 0.$$

The second class of variations is the one that takes advantage of the infinitesimal translations $\delta\varphi^I$. We can then set $\delta u^I \equiv 0$ and choose $\delta\varphi^I$ such that

$$\delta\varphi^I = -\int_{\mathbb{R}\setminus\{\varphi^I\}} p(x,0)u_x^I(x)dx - [u^I]_{\varphi^I} p(\varphi^I, 0).$$

As mentioned above, we could consider slightly different variations of the initial data of the form

$$\delta\varphi^I = -[u^I]_{\varphi^I} p(\varphi^I, 0)$$

as in [11].

In this way, we have identified two classes of variations and its approximate values inspired in the structure of the state and the adjoint state in the inviscid case, allowing to implement the method of alternating descent in the inviscid case when u^I is discontinuous.

The efficiency of the method discussed here has been illustrated by several numerical experiments in the case of the Burgers equation in reference [12], where an implicit assumption that σ is large is assumed due to the use of equation (5.2.6) (corresponding to (5.3.63) in the case that $\sigma = \infty$) with $f(u) = u$. However, with the help of the modified equation (5.3.63), we can carry out simulations for optimal control problems of nonlinear conservation laws, in the case that σ is not too large.

From the above arguments we can draw the following conclusion:

Conclusion: There exists a number ν_0 such that for any $\nu \in (0, \nu_0]$, $p(x, 0)$

can be used to replace the exact solution $p_{\nu,n}(x,0)$ (as in (5.5.16)) with a small error which is much smaller than the mesh size. Thus this error can be omitted, and the algorithm (see Algorithm 6 in [11]) of the alternating descent method for inviscid Burgers equation is applicable to the viscous problem with small viscosity, and this method is efficient.

If $\nu \in (\nu_0, \infty)$, then the solutions $u^\nu, \delta u^\nu$ are smooth for any $t > 0$ and $p_{\nu,n}$ is smooth for $t < T$. In this case, if we replace the exact solution $p_{\nu,n}(x,0)$ by $p(x,0)$, the error is probably not sufficiently small, so the alternating descent method is not efficient in this case, instead the classical descent method is applicable.

Appendix

For the convenience of the reader, we record the definition of the reversible solutions to linear transport equation and Theorem 4.1.10 in [6].

Let $\mathcal{S}_{Lip} = Lip_{\mathrm{loc}}([0,T] \times \mathbb{R})$. Denote by $\mathcal{L}$ the solutions $p \in \mathcal{S}_{Lip}$ to

$$\partial_t p + a\partial_x p = 0.$$

We study solutions to the backward problem, consisting of all $p \in \mathcal{L}$ such that $p(\cdot,T) = p^T$ for any given $p^T \in Lip_{\mathrm{loc}}(\mathbb{R})$.

Definitions (Nonconservative reversible solutions) *i) We call exceptional solution any function $p_e \in \mathcal{L}$ such that $p_e(\cdot,T) = 0$. We denote by $\mathcal{E}$ the vector space of exceptional solutions.*

ii) We call domain of support of exceptional solutions the open set

$$\mathcal{V}_e = \{(x,t) \in \mathbb{R} \times (0,T) \mid \exists p_e \in \mathcal{E}, p_e(x,t) \neq 0\}.$$

iii) Any $p \in \mathcal{L}$ is called reversible if p is locally constant in $\mathcal{V}_e$.

The following is an important feature of reversible solutions, namely the stability with respect to coefficient and final data:

Theorem 5.5.1 (Stability) *Let (a_n) be a bounded sequence in $L^\infty(\mathbb{R}\times(0,T))$ such that $a_n \rightharpoonup^* a$ in $L^\infty(\mathbb{R} \times (0,T))$. Assume that $\partial_x a_n \leq \alpha_n(t)$, where (α_n) is bounded in $L^1((0,T))$, $\partial_x a \leq \alpha$ where $\alpha \in L^1((0,T))$. Let (p_n^T) be a bounded sequence in $Lip_{\mathrm{loc}}(\mathbb{R})$, $p_n^T \to p^T$, and denote by p_n the reversible solution to*

$$\begin{aligned} \partial_t p_n + a_n\partial_x p_n &= 0, \text{ in } \mathbb{R} \times (0,T), \\ p_n(x,T) &= p_n^T(x). \end{aligned}$$

Then $p_n \to p$ *in* $C([0,T]\times[-R,R])$ *for any* $R>0$*, where* p *is the reversible solution to*

$$\begin{aligned} \partial_t p + a\partial_x p &= 0, \text{ in } \mathbb{R}\times(0,T), \\ p(x,T) &= p^T(x). \end{aligned}$$

Bibliography

[1] Bardos, C. and Pironneau, O. (2002) A formalism for the differentiation of conservation laws, *C. R. Acad. Sci.*, Paris, Ser. I **335**, 839–845.

[2] Bardos, C. and Pironneau, O. (2003) Derivatives and control in presence of shocks, *Compu. Fluid Dyna. J.*, **11** No. 4, 383–392.

[3] Berger, M. and Fraenkel, L. E. (1970) On the asymptotic solution of a nonlinear Dirichlet problem, *Journal of Math. mech*, **19** No. 7, 553–585.

[4] Biagionit, H., and Obergugenberger, M. (1997) Generalized solutions to Burgers' equation, *J. Diff. Eq.*, **97**, 263–287.

[5] Biagionit, H., Cadeddu, L. and Cramchevs, T. (1997) Parabolic equations with conservative nonlinear term and singular initial data, *Nonlinear Analysis TMA*, **30**, No. 4, 2489–2496.

[6] Bouchut, F. and James, F. (1998) One-dimensional transport equations with discontinuous coefficients, *Nonlinear Anal. Th. Appl.* **32**, 891–933.

[7] Bouchut, F., James, F. and Mancini, S. (2005) Uniqueness and weak stability for multi-dimensional transport equations with one-sided Lipschitz coefficient, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **4**, 1–25.

[8] Bressan, A. and Marson, A. (1995) A variational calculus for discontinuous solutions of systems of conservation laws, *Commun. Partial Diff. Eqns.* **20**, 1491–1552.

[9] Bressan, A. and Marson, A. (1995) A maximum principle for optimally controlled systems of conservation laws, *Rend. Sem. Mat. Univ. Padova* **94**, 79–94.

[10] Brezis, H. and Friedman, A. (1983) Nonlinear parabolic equations involving measures as initial conditions, *J. Math. Pures Appl.* **62**, 73–97.

[11] Castro, C., Palacios, F. and Zuazua, E. (2008) An alternating descent method for the optimal control of the inviscid Burgers equation in the presence of shocks, *Math Models and Meth in Appl Sci* **18** No. 3, 369–416.

[12] Castro, C., Palacios, F. and Zuazua, E. (2009) Optimal control and vanishing viscosity for the Burgers equation. Preprint.

[13] Colombeau, J. and Langlais, M. (1990) An existence-uniqueness result for a nonlinear parabolic equation with Cauchy data distribution, *J. Math. Anal. Appl.* **145** No. 1, 186–196.

[14] Fife, P. (1988) *Dynamics of Internal Layers and Diffusive Interfaces*, CBMS-NSF regional conference series in applied mathematics, Society for Industrial and Applied Mathematics, **Vol 53**.

[15] Fraenkel, L. E. (1969) On the method of matched asymptotic expansions, *Proc. Camb. Phil. Soc.*, **65**, Part I. A matching principle, 209–231, Part II. Some applications of the composite series, 233–261, Part III. Two boundary-value problems, 263–284

[16] Fried, E. and Gurtin, M. (1994) Dynamic solid-solid transitions with phase characterized by an order parameter, *Physica D* **72**, 287–308.

[17] Friedman, A. (1964) *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Inc., Englewood Cliffs, New Jersy.

[18] Giles, M. and Pierce, N. (2001) Analytic adjoint solutions for the quasi one-dimensional Euler equations, *J. Fluid Mech.* **426**, 327–345.

[19] Godlewski, E. and Raviart, P. (1999) On the linearization of hyperbolic systems of conservation laws. A general nummerical approach, *Math. Comput. Simul.* **50**, 77–95.

[20] Goodman, J. and Xin, Z. (1992) Viscous limits for piecewise smooth solutions to systems of conservation laws, *Arch. Rational Mech. Anal.* **121**, 235–265.

[21] Hinch, E. J. (1991) *Perturbation Methods*, Cambridge University Press.

[22] Holmes M. (1995) *Introduction to Perturbation Methods*, Springer-Verlag, New York.

[23] Hopf, E. (1950) The partial differential equation $u_t + uu_x = \mu u_{xx}$, *Comm. Pure Appl. Math.* **3**, 201–230.

[24] Il'in A. M. (1992) *Matching of Asymptotic Expansions of Solutins of Boundary Value Problems*, Translations of Mathematical monographs, Vol. 102, American Math. Society, Providence, RI.

[25] James, F. and Sepúlveda, M. (1999) Convergence results for the flux identification in a scalar conservation law, *SIAM J. Contr. Opti.* **37** No.3, 869–891.

[26] Kevorkian J. and Cole, J. (1996) *Multiple Scale and Singular Perturbation Methods*, Springer-Verlag, New York.

[27] Ladyzenskaya, O., Solonnikov, V. and Uralceva, N. (1968) *Linear and Quasilinear Equations of Parabolic type*, Trans. Math. Monographs, Vol. 23, American Math. Soc., Providence.

[28] LeVeque, R. (2002) *Finite Volume Methods for Hyperbolic Problems*, Combridge Univ. Press.

[29] Metivier, G. (2003) Stability of Multidimensional Shocks. Manuscript, Univ. de Rennes I.

[30] Pego, R. (1989) Front migration in the nonlinear Cahn-Hilliard equation, *Proc. R. Soc. Lond.* **422A**, 261–278.

[31] Smoller, J. (1983) *Shock waves and reaction-diffusion equations*, Springer Verlag, New York.

[32] Ubrich, S. (2003) Adjoint-based derivative computations for the optimal control of discontinuous solutions of hyperbolic systems of conservation laws, *Syst. Cont. Lett.* **48**, 313–328.

[33] Van Dyke, M. (1964) *Perturbation methods in fluid mechanics*, Academic Press. Annotated version (1975) Parabolic Press.

[34] Van Dyke, M. (1974) Analysis and improvement of perturbation series, *Q. J. Mech. Appl. Math.*, **27**, 423–450.

[35] Van Dyke, M. (1975) Computer extension of perturbation series in fluid mechanics, *SIAM J. Appl. Math.*, **28**, 720–734.

[36] Whitham, G. (1974) *Linear and nonlinear waves*, John Wiley & Sons.

Index